Deriving Range and Cross-range Distances in Spherical Coordinates

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Consider a missile or hypersonic glider being launched north from the point $P_0 = (\theta, \phi) = (0, 0)$. The range direction is along the great circle defined by the initial direction of motion—in this case the range direction is a line of longitude. If the missile’s trajectory deviates laterally from that direction, you can also define its cross-range at any time.

Assume at some time the missile has reach the point $P_1 = (\theta_1, \phi_1)$. The cross-range is measured along a great circle that passes through $P_1$ and is perpendicular to the great circle that lies along the range direction. Call $Q$ the point where the range and cross-range great circles cross. Then the range is the ground distance along the great circle between $P_0$ and $Q$, and the cross-range is the ground distance along the great circle between $P_1$ and $Q$.

Figure 1: $P_0$ is off the bottom of the figure, lying on the yellow longitude line.

These distances can be calculated from the coordinates of $P_1$ using spherical geometry.

Calculating Cross-Range:

Consider a spherical triangle with sides made of segments of great circles as in Figure 2. Take $A$ to be at the north pole, and angles $B$ and $C$ to be the located where a great circle crosses the circle of latitude $\theta_1$, so that $C$ is at the point $P_1$ in Figure 1.
Figure 2: All the arcs in this figure are segments of great circles and the red dot marks the center of the sphere, so the straight lines are radii to points on the sphere. The letters $A$, $B$, $C$, $a$, $b$, $c$ label the angles shown. The points labeled correspond to points in Figure 1.

Then angle $A = 2\phi_1$, and angles $b$ and $c$ are $90^0 - \theta_1$.

Then the **spherical law of cosines**:

$$\cos a = \cos b \cos c + \sin b \sin c \cos A$$  \hspace{1cm} (1)

gives:

$$\cos a = \sin^2 \theta_1 + \cos^2 \theta_1 \cos(2\phi_1)$$  \hspace{1cm} (2)

and

$$Crossrange = \frac{a}{2r_e}$$  \hspace{1cm} (3)

Calculating Range:

The range angle corresponding to a missile at the point $P_1$ can be found in a similar way.

Consider the spherical triangle in Figure 3, which now represents the right half of the triangle in Figure 2; the point $Q$ from Figure 1 is now at the left end of the lower arc rather than in the middle. The arc between the north pole and $Q$ is part of the line of longitude = 0, which is the range direction. That arc bisects the angles $A$ and $a$ in Figure 2. Since the missile is assumed to be launched from the equator, the range angle is given by $90^0 - d$. 


Using the spherical law of sines:

\[ \frac{\sin \frac{A}{2}}{\sin \frac{a}{2}} = \frac{\sin D}{\sin d} \]  

(4)

So that

\[ \sin d = \sin D \frac{\sin \frac{a}{2}}{\sin \frac{A}{2}} \]  

(5)

From Figure 2:

\[ \frac{\sin A}{\sin a} = \frac{\sin C}{\sin c} = \frac{\sin D}{\sin c} \]  

(6)

where the last equality comes from the fact that angles C and D are the same in Figures 2 and 3.

Combining these and using \( A = 2\phi_1 \) and \( c = 90^\circ - \theta_1 \) gives:

\[ \sin d = \left( \frac{\sin \frac{a}{2}}{\sin a} \right) \left( \frac{\sin 2\phi_1}{\sin \phi_1} \right) \cos \theta_1 = \frac{\cos \phi_1 \cos \theta_1}{\cos \frac{a}{2}} \]  

(7)

or

\[ \sin d = \frac{\cos \phi_1 \cos \theta_1}{\sqrt{\frac{1 + \cos a}{2}}} \]  

(8)
where \( \cos a \) is given by Equation 2. The range is then given by:

\[
\text{Range} = \left(\frac{\pi}{2} - d\right) r_e \quad (9)
\]

So for a missile at point \((\theta_1, \phi_1)\), the range and cross-range distances are given by Equations (3) and (9).

Letting \( \varepsilon = a/2 \), so that \( \varepsilon \) is the cross-range angle, these equations can be written in an alternate form as:

\[
\cos^2 \varepsilon = \left(1 + \sin^2 \theta_1 + \cos^2 \theta_1 \cos(2\phi_1)\right)/2 \quad (10)
\]

\[
\text{crossrange} = \varepsilon r_e \quad (11)
\]

\[
\text{range} = r_e \cos^{-1}\left(\frac{\cos \theta_1 \cos \phi_1}{\cos \varepsilon}\right) \quad (12)
\]

To check these equations in one specific case, consider the case \((\theta_1, \phi_1) = (85^0, 90^0)\) (see Figure 4).

Notice that for a missile on this trajectory, the latitude angle \( \theta \) never gets larger than 85\(^0\), which makes clear that it cannot be treated as the range angle. In this case:

Eqs. 2 and 3 give: \[ \cos a = 0.984908 \rightarrow a/2 = \varepsilon = 5^0 \]
\text{Cross-range} = 0.08727 \text{ } r_e = 556 \text{ km}

Eqs. 8 and 9 give: \[ \sin d = 0 \rightarrow d = 0^0 \rightarrow \text{range angle} = 90^0 \]
\text{Range} = 1.571 \text{ } r_e = 10,006 \text{ km}

Eqs. (10) – (12) give the same results.
Figure 4. This figure shows the final part of a trajectory (in red) starting at (0, 0) and lying past the north pole. In this case it is clear that the range angle is $90^0$ and the cross-range angle is $5^0$. 