**Supplementary Material**

**Mood as representation of momentum**

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**Supplementary Material Inventory:**

1. Supplementary Note S1

**Supplementary Note S1. Derivation of optimal learning algorithms.**

Here we will derive the optimal learning algorithm for the models in Box 2 by casting each model as a Kalman filter model [S1], and then using the standard solution for Kalman filters.

A Kalman filter model consists of an unobservable (hidden) state \(\mathbf{x}\) and state covariance matrix \(\mathbf{P}\), which need to be inferred from a sequence of observations \(\mathbf{r}\), and a set of assumptions about how observations are generated. These assumptions include the state transition matrix \(\mathbf{F}\), which determines how states change over time; the state-to-observation transformation matrix \(\mathbf{H}\), which determines how observations are generated given the current state; and the state noise covariance matrix \(\mathbf{Q}\) and observation noise covariance matrix \(\mathbf{C}\), which introduce stochasticity into state transitions and the generation of observations.

Given observation \(\mathbf{r}_t\) at time step \(t\), an optimal learning algorithm updates the estimates of the current state and the state covariance matrix as follows:

\[
\mathbf{x}_t = \mathbf{F}\mathbf{x}_{t-1} + \mathbf{K}_t(\mathbf{r}_t - \mathbf{H}\mathbf{x}_{t-1})
\]  

(1)

\[
\mathbf{P}_t = \mathbf{F}\mathbf{P}_{t-1}\mathbf{F}^T + \mathbf{Q} - \mathbf{K}_t\mathbf{H}(\mathbf{F}\mathbf{P}_{t-1}\mathbf{F}^T + \mathbf{Q}),
\]  

(2)

where

\[
\mathbf{K}_t = (\mathbf{F}\mathbf{P}_{t-1}\mathbf{F}^T + \mathbf{Q})\mathbf{H}^T(\mathbf{H}(\mathbf{F}\mathbf{P}_{t-1}\mathbf{F}^T + \mathbf{Q})\mathbf{H}^T + \mathbf{C})^{-1}
\]  

(3)

**Model I: Independent states**

In this baseline model, described in Box 2 (Figure IA), each state corresponds to an independent one-dimensional Kalman filter model with the following properties:

\[
\mathbf{x}_t = [v_t] \quad \mathbf{F} = [1] \quad \mathbf{H} = [1] \quad \mathbf{Q} = [q]
\]

Substituting the above terms into equations (1) and (3) we get:

\[
v_t = v_{t-1} + K_t(r_t - v_{t-1})
\]  

(4)

where
\[ K_t = \frac{P_{t-1} + q}{P_{t-1} + q + C} \]  

(5)

Thus, the estimated state is updated by the difference between the observation and the previous estimate (i.e., the prediction error; \( r_t - v_{t-1} \)) multiplied by an adaptive learning rate \( (K_t) \) that reflects the relationship between the state and observation variance.

**Model II: States with independent and shared variance**

The next model described in Box 2 (Figure IB) consists of multiple states, all of which are affected by the same general source of variance. For the case of three states, the model corresponds to a three-dimensional Kalman filter whose three dimensions represent three states with both state-specific \( (q_1:q_3) \) and shared \( (q_g) \) variance:

\[
\begin{bmatrix}
v_{t,1} \\ v_{t,2} \\ v_{t,3}
\end{bmatrix} = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix} \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix} + \mathbf{Q} = \begin{bmatrix}
q_g + q_1 & q_g & q_g \\
q_g & q_g + q_2 & q_g \\
q_g & q_g & q_g + q_3
\end{bmatrix}
\]

Substituting the above terms into equations (1) and (3) we get:

\[
\begin{bmatrix}
v_{t,1} \\ v_{t,2} \\ v_{t,3}
\end{bmatrix} = \begin{bmatrix}
v_{t-1,1} \\ v_{t-1,2} \\ v_{t-1,3}
\end{bmatrix} + K_t \begin{bmatrix}
K_t,1 & K_t,2 & K_t,3
\end{bmatrix} \begin{bmatrix}
r_t - v_{t-1,1} \\ r_t - v_{t-1,2} \\ r_t - v_{t-1,3}
\end{bmatrix},
\]

(6)

where

\[
K_t = (P_{t-1} + Q)(P_{t-1} + Q + C)^{-1}
\]

(7)

Thus, the estimated states are updated by the vector of prediction errors \( (r_t - x_{t-1}) \) multiplied by the matrix \( K_t \), which integrates the different types of covariance. Importantly, due to the variance that is shared between the states (i.e., the off-diagonal entries in \( Q \)), the Kalman gain matrix \( K_t \) has nonzero off-diagonal values, and thus each state is updated not only by its own prediction error, but also by the prediction errors of the other states.

**Model III: A state with momentum**

The final model described in Box 2 (Figure IC) corresponds to a two-dimensional Kalman filter, consisting of a state \( (v_t) \) and its momentum \( (m_t) \). Observations are only directly dependent on the state \( v_t \), but \( v_t \) is updated at each time step by addition of the momentum \( m_t \) (as indicated by the triangular state-transition matrix \( F \)):

\[
\begin{bmatrix}
m_t \\ v_t
\end{bmatrix} = \begin{bmatrix}
1 & 0 \\ 1 & 1
\end{bmatrix} + \begin{bmatrix}
0 \\ 1
\end{bmatrix} = \begin{bmatrix}
q_m \\ 0
\end{bmatrix}
\]

Substituting the above terms into equations (1) and (3) we get:

\[
m_t = m_{t-1} + K_{t,1}(r_t - v_{t-1}) - m_{t-1}
\]

(8)

\[
v_t = v_{t-1} + K_{t,2}(K_{t,3}m_{t-1} + r_t - v_{t-1})
\]

(9)
where $K_{t,1} = \frac{P_{t,1} + P_{t,2}}{q + \sum_{i,j} P_{i,j}}$ and $K_{t,2} = \frac{q + \sum_{i,j} P_{i,j}}{q + c + \sum_{i,j} P_{i,j}}$ are adaptive learning rates whose value is between 0 and 1, and $K_{t,3} = \frac{1}{K_{t,2}} - 1$ is an adaptive scaling factor.

Thus, the momentum estimate consists of a running average of recent outcome prediction errors, and the state estimate update includes a bonus that is proportional to the momentum estimate.

References