Equilibrium Configurations of a Kirchhoff Elastic Rod under Quasi-static Manipulation

Timothy Bretl and Zoe McCarthy

Abstract. Consider a thin, flexible wire of fixed length that is held at each end by a robotic gripper. The curve traced by this wire can be described as a local solution to a geometric optimal control problem, with boundary conditions that vary with the position and orientation of each gripper. The set of all local solutions to this problem is the configuration space of the wire under quasi-static manipulation. We will show that this configuration space is a smooth manifold of finite dimension that can be parameterized by a single chart. Working in this chart—rather than in the space of boundary conditions—makes the problem of manipulation planning very easy to solve. Examples in simulation illustrate our approach.

1 Introduction

Figure 1 shows a thin, flexible wire of fixed length that is held at each end by a robotic gripper. Our basic problem of interest is to find a path of each gripper that causes the wire to move between start and goal configurations while remaining in static equilibrium and avoiding self-collision. As will become clear, it is useful to think about this problem equivalently as finding a path *of the wire* through its set of equilibrium configurations (i.e., the set of all configurations that would be in equilibrium if both ends of the wire were held fixed).

There are two reasons why this problem seems hard to solve. First, the configuration space of the wire has infinite dimension. Elements of this space are framed

Timothy Bretl
Department of Aerospace Engineering,
University of Illinois at Urbana-Champaign, Urbana, IL, 61801
e-mail: tbretl@illinois.edu

Zoe McCarthy
Department of Electrical and Computer Engineering,
University of Illinois at Urbana-Champaign, Urbana, IL, 61801
e-mail: zoemccarthy12@gmail.com

E. Frazzoli et al. (Eds.): *Algorithmic Foundations of Robotics X*, STAR 86, pp. 71–87.
DOI: 10.1007/978-3-642-36279-8_5

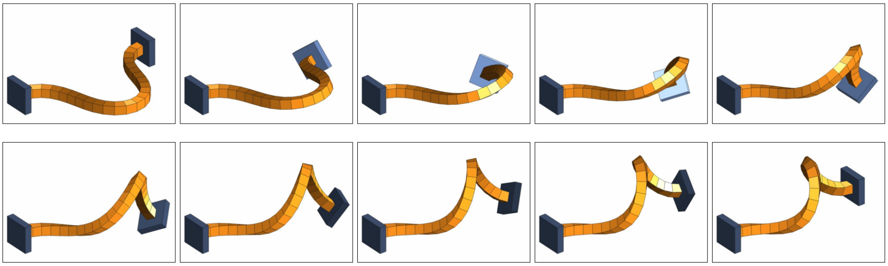

Fig. 1 Quasi-static manipulation of an elastic rod (orange) by robotic grippers (blue). Notice that the grippers begin and end in the same position and orientation. Remarkably, this motion corresponds to a straight-line path in the global coordinate chart we derive in this paper.

curves, i.e., continuous maps $q\colon [0,1] \to SE(3)$, the shape of which in general must be approximated. Second, a countable number of configurations may be in static equilibrium for given placements of each gripper, none of which (typically) can be computed in closed form. For these two reasons, the literature on manipulation planning suggests exploring the set of equilibrium configurations indirectly, by sampling displacements of each gripper and using numerical simulation to approximate their effect on the wire. This approach was developed in the seminal work of Lamiraux and Kavraki [21] and was applied by Moll and Kavraki [29] to manipulation of elastic "deformable linear objects" like the flexible wire we consider here.

Our contribution in this paper is to show that the set of equilibrium configurations for the wire is a smooth manifold of finite dimension that can be parameterized by a single (global) coordinate chart. We model the wire as a Kirchhoff elastic rod [8]. The framed curve traced by this elastic rod in static equilibrium can be described as a local solution to a geometric optimal control problem, with boundary conditions that vary with the position and orientation of each gripper [41, 8]. Coordinates for the set of *all* local solutions over *all* boundary conditions are provided by the initial value of costates that arise in necessary and sufficient conditions for optimality. These coordinates describe all possible configurations of the elastic rod that can be achieved by quasi-static manipulation, and make manipulation planning—the seemingly "hard problem" described above—very easy to solve.

Our approach builds on a long history in analysis of elasticity [3]. Recent work gives a more or less complete picture of planar elastic rods [32, 31], and this work rests on similar foundations as our own [1]. We have also been influenced by analysis of conjugate points in elastic filament models of DNA [14] and by an earlier sequence of papers initiated by Langer and Singer [22]. In addition, we note the emergence of new approaches to dynamic simulation of elastic rods based on discrete geometry [7], which has started to find application in robotics [18]. However, none of this previous work answers our questions about the set of equilibrium configurations: is it a finite-dimensional manifold, what are its coordinate charts, etc. These questions are the foundation of our approach to manipulation planning.

We are motivated by applications that require manipulation of deformable objects: knots and suturing [15,38,40,33,5], cable routing [16], folding clothes [6,43], compliant parts handling [26, 13] and assembly [4], surgical retraction of tissue [17], and protein folding [2]. Related applications include haptic exploration with "whisker" sensors, often modeled as elastic rods [36, 12]. We are also motivated by the link, pointed out by Tanner [39], between manipulation of deformable objects and control of hyper-redundant [10] and continuum [30, 42] robots.

Section 2 establishes our theoretical framework. The two key parts of this framework are optimal control on manifolds and Lie-Poisson reduction. We derive coordinate formulae for necessary and sufficient conditions—in the former case these formulae are well known, but in the latter case they are not. Section 3 shows how our framework applies to the elastic rod. We prove that the set of equilibrium configurations for this rod is a smooth manifold of finite dimension that can be parameterized by a single chart, and we explain why this result makes the problem of manipulation planning easy to solve. We note in particular that the computations required for planning are trivial to implement—the example of Figure 1 was generated by about a dozen lines of code. Section 4 identifies several research directions that are enabled by our analysis of the elastic rod. Our ideas follow from but significantly extend earlier work on a simpler model (a planar elastic kinematic chain [28]).

2 Theoretical Framework

We will see in Section 3 that the framed curve traced by an elastic rod in equilibrium is a local solution to a geometric optimal control problem. Here, we provide the framework to characterize this solution. Section 2.1 gives our notation for smooth manifolds. It is not a review (for this, see [25]), and is included only because notation varies widely in the literature. Section 2.2 states necessary and sufficient conditions for optimality on manifolds in a form that is useful for us. Section 2.3 derives coordinate formulae to compute these necessary and sufficient conditions. Most of these results are a translation of [1] in a style more consistent with [25, 27]. We conclude with coordinate formulae to test sufficiency for left-invariant systems on Lie groups (Theorem 4), an important result that is not in [1] and is hard to find elsewhere.

2.1 Smooth Manifolds

Let M be a smooth manifold. The space of smooth real-valued functions on M is $C^\infty(M)$. The space of smooth vector fields on M is $\mathfrak{X}(M)$. The action of $v \in T_mM$ on $f \in C^\infty(M)$ is $v \cdot f$. The action of $w \in T_m^*M$ on $v \in T_mM$ is $\langle w, v \rangle$. The action of $X \in \mathfrak{X}(M)$ on $f \in C^\infty(M)$ produces the function $X[f] \in C^\infty(M)$ satisfying $X[f](m) = X(m) \cdot f$ for all $m \in M$. The Jacobi-Lie bracket of $X, Y \in \mathfrak{X}(M)$ is the vector field $[X,Y] \in \mathfrak{X}(M)$ satisfying $[X,Y][f] = X[Y[f]] - Y[X[f]]$ for all $f \in C^\infty(M)$. If $F : M \to N$ is a smooth map between manifolds M and N, then the pushforward of F at $m \in M$ is the linear map $T_mF : T_mM \to T_{F(m)}N$ satisfying $T_mF(v) \cdot f = v \cdot (f \circ F)$ for all $v \in T_mM$ and $f \in C^\infty(N)$. The pullback of F at $m \in M$ is the dual map

$T_m^*F\colon T_{F(m)}^*N \to T_m^*M$ satisfying $\langle T_m^*F(w), v\rangle = \langle w, T_mF(v)\rangle$ for all $v \in T_mM$ and $w \in T_{F(m)}^*N$. We say F is degenerate at $m \in M$ if there exists non-zero $v \in T_mM$ such that $T_mF(v) = 0$. It is equivalent that the Jacobian matrix of any coordinate representation of F at m have zero determinant. The Poisson bracket generated by the canonical symplectic form on T^*M is $\{\cdot,\cdot\}\colon C^\infty(T^*M) \times C^\infty(T^*M) \to C^\infty(T^*M)$. The cotangent bundle T^*M together with the bracket $\{\cdot,\cdot\}$ is a Poisson manifold. The Hamiltonian vector field of $H \in C^\infty(T^*M)$ is the unique vector field $X_H \in \mathfrak{X}(T^*M)$ satisfying $X_H[K] = \{K, H\}$ for all $K \in C^\infty(T^*M)$. We use this same notation when H is time-varying. Finally, let $\pi\colon T^*M \to M$ satisfy $\pi(m,w) = m$ for all $v \in T_m^*M$.

2.2 Optimal Control on Manifolds

Let $U \subset \mathbb{R}^m$ for some $m > 0$. Assume $g\colon M \times U \to \mathbb{R}$ and $f\colon M \times U \to TM$ are smooth maps. Consider the optimal control problem

$$\begin{aligned} \underset{q,u}{\text{minimize}} \quad & \int_0^1 g(q(t), u(t))\, dt \\ \text{subject to} \quad & \dot{q}(t) = f(q(t), u(t)) \text{ for all } t \in [0,1] \\ & q(0) = q_0, \qquad q(1) = q_1, \end{aligned} \tag{1}$$

where $q_0, q_1 \in M$ and $(q,u)\colon [0,1] \to M \times U$. Define the parameterized Hamiltonian $\widehat{H}\colon T^*M \times \mathbb{R} \times U \to \mathbb{R}$ by $\widehat{H}(p,q,k,u) = \langle p, f(q,u)\rangle - kg(q,u)$, where $p \in T_q^\star M$.

Theorem 1 (Necessary Conditions). *Suppose $(q_{opt}, u_{opt})\colon [0,1] \to M \times U$ is a local optimum of* (1). *Then, there exists $k \geq 0$ and an integral curve $(p,q)\colon [0,1] \to T^*M$ of the time-varying Hamiltonian vector field X_H, where $H\colon T^*M \times \mathbb{R} \to \mathbb{R}$ is given by $H(p,q,t) = \widehat{H}(p,q,k,u_{opt}(t))$, that satisfies $q(t) = q_{opt}(t)$ and*

$$H(p(t), q(t), t) = \max_{u \in U} \widehat{H}(p(t), q(t), k, u) \tag{2}$$

for all $t \in [0,1]$. Furthermore, if $k = 0$, then $p(t) \neq 0$ for all $t \in [0,1]$.

Proof. See Theorem 12.10 of [1]. □

We call the integral curve (p,q) in Theorem 1 an *abnormal extremal* when $k = 0$ and a *normal extremal* otherwise. As usual, when $k \neq 0$ we may simply assume $k = 1$. We call (q,u) *abnormal* if it is the projection of an abnormal extremal. We call (q,u) *normal* if it is the projection of a normal extremal and it is not abnormal.

Theorem 2 (Sufficient Conditions). *Suppose $(p,q)\colon [0,1] \to T^*M$ is a normal extremal of* (1). *Define $H \in C^\infty(T^*M)$ by*

$$H(p,q) = \max_{u \in U} \widehat{H}(p,q,1,u), \tag{3}$$

assuming the maximum exists and $\partial^2\widehat{H}/\partial u^2 < 0$. Define $u\colon [0,1] \to U$ so $u(t)$ is the unique maximizer of (3) *at $(p(t), q(t))$. Assume that X_H is complete and that there* *exists no other integral curve* (p',q') *of* X_H *satisfying* $q(t) = q'(t)$ *for all* $t \in [0,1]$. *Let* $\varphi\colon \mathbb{R}\times T^*M \to T^*M$ *be the flow of* X_H *and define the endpoint map* $\phi_t\colon T^*_{q(0)}M \to M$ *by* $\phi_t(w) = \pi \circ \varphi(t,w,q(0))$. *Then,* (q,u) *is a local optimum of* (1) *if and only if there exists no* $t \in (0,1]$ *for which* ϕ_t *is degenerate at* $p(0)$.

Proof. See Theorem 21.8 of [1]. □

2.3 Lie-Poisson Reduction

Let G be a Lie group with identity element $e \in G$. Let $\mathfrak{g} = T_eG$ and $\mathfrak{g}^* = T_e^*G$. For any $q \in G$, define the left translation map $L_q\colon G \to G$ by $L_q(r) = qr$ for all $r \in G$. A function $H \in C^\infty(T^*G)$ is left-invariant if $H\left(T_r^*L_q(w), r\right) = H(w,s)$ for $w \in T_s^*G$ and $q,r,s \in G$ satisfying $s = L_q(r)$. For $\zeta \in \mathfrak{g}$, let X_ζ be the vector field that satisfies $X_\zeta(q) = T_eL_q(\zeta)$ for all $q \in G$. Define the Lie bracket $[\cdot,\cdot]\colon \mathfrak{g}\times\mathfrak{g} \to \mathfrak{g}$ by $[\zeta,\eta] = \left[X_\zeta, X_\eta\right](e)$ for all $\zeta,\eta \in \mathfrak{g}$. For $\zeta \in \mathfrak{g}$, the adjoint operator $\mathrm{ad}_\zeta\colon \mathfrak{g} \to \mathfrak{g}$ satisfies $\mathrm{ad}_\zeta(\eta) = [\zeta,\eta]$ and the coadjoint operator $\mathrm{ad}^*_\zeta\colon \mathfrak{g}^* \to \mathfrak{g}^*$ satisfies $\left\langle \mathrm{ad}^*_\zeta(\mu), \eta \right\rangle = \left\langle \mu, \mathrm{ad}_\zeta\, \eta \right\rangle$ for all $\eta \in \mathfrak{g}$ and $\mu \in \mathfrak{g}^*$. The functional derivative of $h \in C^\infty(\mathfrak{g}^*)$ at $\mu \in \mathfrak{g}^*$ is the unique element $\delta h/\delta\mu$ of $\mathfrak{g}$ that satisfies the following for all $\delta\mu \in \mathfrak{g}^*$:

$$\lim_{s\to 0} \frac{h(\mu + s\delta\mu) - h(\mu)}{s} = \left\langle \delta\mu, \frac{\delta h}{\delta\mu} \right\rangle$$

Theorem 3 (Reduction of Necessary Conditions). *Suppose* $H\colon T^*G\times[0,1] \to \mathbb{R}$ *is both smooth and left-invariant for all* $t \in [0,1]$. *Denote the restriction of* H *to* $\mathfrak{g}^*$ *by* $h = H|_{\mathfrak{g}^*\times[0,1]}$. *Given* $p_0 \in T^*_{q_0}G$, *let* $\mu\colon [0,1] \to \mathfrak{g}^*$ *be the solution of*

$$\dot{\mu} = \mathrm{ad}^*_{\delta h/\delta\mu}(\mu) \tag{4}$$

with initial condition $\mu(0) = T_e^*L_{q_0}(p_0)$. *The integral curve* $(p,q)\colon [0,1] \to T^*G$ *of* X_H *with initial condition* $p(0) = p_0$ *satisfies* $p(t) = T^*_{q(t)}L_{q(t)^{-1}}(\mu(t))$ *for all* $t \in [0,1]$, *where* q *is the solution of* $\dot{q} = X_{\delta h/\delta\mu}(q)$ *with initial condition* $q(0) = q_0$.

Proof. See Theorem 13.4.4 of [27]. □

It is convenient for us to introduce coordinates on $\mathfrak{g}$ and $\mathfrak{g}^*$. Let $\{X_1,\ldots,X_n\}$ be a basis for $\mathfrak{g}$ and let $\{P_1,\ldots,P_n\}$ be the dual basis for $\mathfrak{g}^*$ that satisfies $\left\langle P_i, X_j\right\rangle = \delta_{ij}$, where δ_{ij} is the Kronecker delta. We write ζ_i to denote the ith component of $\zeta \in \mathfrak{g}$ with respect to this basis, and so forth. Define the structure constants $C^k_{ij} \in \mathbb{R}$ associated with our choice of basis by

$$[X_i, X_j] = \sum_{k=1}^{n} C^k_{ij} X_k \tag{5}$$

for $i,j \in \{1,\ldots,n\}$. We require two lemmas before our main result (Theorem 4).

Lemma 1. *Let $q\colon U \to G$ be a smooth map, where $U \subset \mathbb{R}^2$ is simply connected. Denote its partial derivatives $\zeta : U \to \mathfrak{g}$ and $\eta : U \to \mathfrak{g}$ by*

$$\zeta(t,\varepsilon) = T_{q(t,\varepsilon)}L_{q(t,\varepsilon)^{-1}}\left(\frac{\partial q(t,\varepsilon)}{\partial t}\right) \qquad \eta(t,\varepsilon) = T_{q(t,\varepsilon)}L_{q(t,\varepsilon)^{-1}}\left(\frac{\partial q(t,\varepsilon)}{\partial \varepsilon}\right). \tag{6}$$

Then,

$$\partial\zeta/\partial\varepsilon - \partial\eta/\partial t = [\zeta,\eta]. \tag{7}$$

Conversely, if there exist smooth maps ζ and η satisfying (7)*, then there exists a smooth map q satisfying* (6)*.*

Proof. See Proposition 5.1 of [9]. □

Lemma 2. *Let $\alpha,\beta,\gamma \in \mathfrak{g}$ and suppose $\gamma = [\alpha,\beta]$. Then $\gamma_k = \sum_{r=1}^n \sum_{s=1}^n \alpha_r\beta_s C_{rs}^k$.*

Proof. This result is easily obtained from the definition (5). □

Theorem 4 (Reduction of Sufficient Conditions). *Suppose that $H \in C^\infty(T^*G)$ is left-invariant and that X_H is complete. Let $h = H|_{\mathfrak{g}^*}$ be the restriction of H to $\mathfrak{g}^*$ and let $\varphi\colon \mathbb{R}\times T^*G \to T^*G$ be the flow of X_H. Given $q_0 \in G$, define the endpoint map $\phi_t : T^*_{q_0}G \to G$ by $\phi_t(p) = \pi\circ\varphi(t,p,q_0)$. Given $p_0 \in T^*_{q_0}G$, let $a \in \mathbb{R}^n$ be the coordinate representation of $T^*_e L_{q_0}(p_0)$, i.e.,*

$$T^*_e L_{q_0}(p_0) = \sum_{i=1}^n a_i P_i. \tag{8}$$

Solve the ordinary differential equations

$$\dot{\mu}_i = -\sum_{j=1}^n \sum_{k=1}^n C_{ij}^k \frac{\delta h}{\delta \mu_j}\mu_k \qquad i \in \{1,\dots,n\} \tag{9}$$

with the initial conditions $\mu_i(0) = a_i$. Define matrices $\mathbf{F},\mathbf{G},\mathbf{H} \in \mathbb{R}^{n\times n}$ as follows:

$$[\mathbf{F}]_{ij} = -\frac{\partial}{\partial \mu_j}\sum_{r=1}^n\sum_{s=1}^n C_{ir}^s \frac{\delta h}{\delta \mu_r}\mu_s \qquad [\mathbf{G}]_{ij} = \frac{\partial}{\partial \mu_j}\frac{\delta h}{\delta \mu_i}. \qquad [\mathbf{H}]_{ij} = -\sum_{r=1}^n \frac{\delta h}{\delta \mu_r} C_{rj}^i$$

Solve the (linear, time-varying) matrix differential equations

$$\dot{\mathbf{M}} = \mathbf{F}\mathbf{M} \tag{10}$$

$$\dot{\mathbf{J}} = \mathbf{G}\mathbf{M} + \mathbf{H}\mathbf{J} \tag{11}$$

with initial conditions $\mathbf{M}(0) = I$ and $\mathbf{J}(0) = 0$. The endpoint map ϕ_t is degenerate at p_0 if and only if $\det(\mathbf{J}(t)) = 0$.

Proof. Define the smooth map $\rho\colon \mathbb{R}^n \to T^*_{q_0}G$ by $\rho(a) = T^*_{q_0}L^{-1}_{q_0}\left(\sum_{i=1}^n a_i P_i\right)$. This same expression defines $\rho\colon \mathbb{R}^n \to T_{p_0}(T^*_{q_0}G)$ if we identify $T^*_{q_0}G$ with $T_{p_0}(T^*_{q_0}G)$ in the usual way. Given $p_0 = \rho(a)$ for some $a \in \mathbb{R}^n$, there exists non-zero

$\lambda \in T_{p_0}(T^*_{q_0}G)$ satisfying $T_{p_0}\phi_t(\lambda) = 0$ if and only if there exists non-zero $s \in \mathbb{R}^n$ satisfying $T_{\rho(a)}\phi_t(\rho(s)) = 0$. Define $q\colon [0,1] \times \mathbb{R}^n \to G$ by $q(t,a) = \phi_t \circ \rho(a)$. Noting that $\partial q(t,a)/\partial a_j = T_{\rho(a)}\phi_t(T^*_{q_0}L_{q_0^{-1}}(P_j))$ for $j \in \{1,\ldots,n\}$, we have

$$T_{\rho(a)}\phi_t(\rho(s)) = \sum_{j=1}^{n} s_j \frac{\partial q(t,a)}{\partial a_j}.$$

By left translation, $T_{\rho(a)}\phi_t(\rho(s)) = 0$ if and only if

$$0 = \sum_{j=1}^{n} s_j T_{q(t,a)} L_{q(t,a)^{-1}} \left(\frac{\partial q(t,a)}{\partial a_j} \right). \tag{12}$$

Let $\eta^j(t,a) = T_{q(t,a)}L_{q(t,a)^{-1}}(\partial q(t,a)/\partial a_j)$ for $j \in \{1,\ldots,n\}$. Define $\mathbf{J}\colon [0,1] \to \mathbb{R}^{n\times n}$ so that $\mathbf{J}(t)$ has entries $[\mathbf{J}]_{ij} = \eta_i^j(t,a)$, i.e., the jth column of $\mathbf{J}(t)$ is the coordinate representation of $\eta^j(t,a)$ with respect to $\{X_1,\ldots,X_n\}$. Then, (12) holds for some $s \neq 0$ if and only if $\det(\mathbf{J}(t)) = 0$. We conclude that ϕ_t is degenerate at p_0 if and only if $\det(\mathbf{J}(t)) = 0$. It remains to show that $\mathbf{J}(t)$ can be computed as described in the theorem. Define $\zeta(t,a) = T_{q(t,a)}L_{q(t,a)^{-1}}(\partial q(t,a)/\partial t)$. Taking $\mu_1(t),\ldots,\mu_n(t)$ as coordinates of $\mu(t)$, it is easy to verify that (4) and (9) are equivalent (see [27]). We extend each coordinate function in the obvious way to $\mu_i\colon [0,1] \times \mathbb{R}^n \to \mathbb{R}$, so $\mu_i(t,a)$ solves (9) with initial condition $\mu_i(0,a) = a_i$. Define $\mathbf{M}\colon [0,1] \to \mathbb{R}^{n\times n}$ by $[\mathbf{M}(t)]_{ij} = \partial\mu_i/\partial a_j$. Differentiating (9), we compute

$$\begin{aligned}[\dot{\mathbf{M}}]_{ij} = \frac{\partial}{\partial t}\frac{\partial \mu_i}{\partial a_j} = \frac{\partial}{\partial a_j}\frac{\partial \mu_i}{\partial t} &= \frac{\partial}{\partial a_j}\left(-\sum_{r=1}^{n}\sum_{s=1}^{n} C_{ir}^{s} \frac{\delta h}{\delta \mu_r}\mu_s \right) \\ &= \sum_{k=1}^{n} -\frac{\partial}{\partial \mu_k}\left(\sum_{r=1}^{n}\sum_{s=1}^{n} C_{ir}^{s} \frac{\delta h}{\delta \mu_r}\mu_s \right)\frac{\partial \mu_k}{\partial a_j} = \sum_{k=1}^{n} [\mathbf{F}]_{ik}[\mathbf{M}]_{kj}.\end{aligned}$$

It is clear that $[\mathbf{M}(0)]_{ij} = \delta_{ij}$, so we have verified (10). Next, we have

$$\dot{\eta}^j = \frac{\partial \zeta}{\partial a_j} - \left[\zeta, \eta^j\right] = \frac{\partial}{\partial a_j}\frac{\delta h}{\delta \mu} - \left[\frac{\delta h}{\delta \mu}, \eta^j\right]$$

from Lemma 1 and Theorem 3. We write this in coordinates by Lemma 2:

$$\begin{aligned}[\dot{\mathbf{J}}]_{ij} = \dot{\eta}_i^j &= \sum_{k=1}^{n}\left(\frac{\partial}{\partial \mu_k}\frac{\delta h}{\delta \mu_i} \right)\frac{\partial \mu_k}{\partial a_j} + \sum_{k=1}^{n}\left(-\sum_{r=1}^{n}\frac{\delta h}{\delta \mu_r} C_{rk}^{i} \right)\eta_k^j \\ &= \sum_{k=1}^{n}[\mathbf{G}]_{ik}[\mathbf{M}]_{kj} + \sum_{k=1}^{n}[\mathbf{H}]_{ik}[\mathbf{J}]_{kj}.\end{aligned}$$

It is clear that $[\mathbf{J}(0)]_{ij} = 0$, so we have verified (11). □

3 Mechanics and Manipulation of an Elastic Rod

The previous section derived coordinate formulae to compute necessary and sufficient conditions for a particular class of optimal control problems on manifolds. Here, we apply these results to a Kirchhoff elastic rod. Section 3.1 recalls that the framed curve traced by the rod in static equilibrium is a local solution to a geometric optimal control problem [8,41]. Section 3.2 proves that the set of all trajectories that are normal with respect to this problem is a smooth manifold of finite dimension that can be parameterized by a single chart (Theorem 6). Section 3.3 proves that the set of all normal trajectories that are also local optima is an open subset of this smooth manifold, and provides a computational test for membership in this subset (Theorem 7). Together, these two results suffice to describe all possible configurations of the elastic rod that can be achieved by quasi-static manipulation. Section 3.4 explains why these results make the problem of manipulation planning easy to solve.

3.1 Model

We refer to the object in Figure 1 as a *rod*. Assuming that it is thin, inextensible, and of unit length, we describe the shape of this rod by a continuous map $q\colon [0,1] \to G$, where $G = SE(3)$. Abbreviating $T_eL_q(\zeta) = q\zeta$, we require this map to satisfy

$$\dot{q} = q(u_1X_1 + u_2X_2 + u_3X_3 + X_4) \tag{13}$$

for some $u\colon [0,1] \to U$, where $U = \mathbb{R}^3$ and

$$\{X_1,\ldots,X_6\} = \left\{ \begin{bmatrix} 0&0&0&0\\ 0&0&-1&0\\ 0&1&0&0\\ 0&0&0&0 \end{bmatrix}, \begin{bmatrix} 0&0&1&0\\ 0&0&0&0\\ -1&0&0&0\\ 0&0&0&0 \end{bmatrix}, \begin{bmatrix} 0&-1&0&0\\ 1&0&0&0\\ 0&0&0&0\\ 0&0&0&0 \end{bmatrix}, \begin{bmatrix} 0&0&0&1\\ 0&0&0&0\\ 0&0&0&0\\ 0&0&0&0 \end{bmatrix}, \begin{bmatrix} 0&0&0&0\\ 0&0&0&1\\ 0&0&0&0\\ 0&0&0&0 \end{bmatrix}, \begin{bmatrix} 0&0&0&0\\ 0&0&0&0\\ 0&0&0&1\\ 0&0&0&0 \end{bmatrix} \right\}$$

is a basis for $\mathfrak{g}$. Denote the dual basis for $\mathfrak{g}^*$ by $\{P_1,\ldots,P_6\}$. We refer to q and u together as $(q,u)\colon [0,1] \to G \times U$ or simply as (q,u). Each end of the rod is held by a robotic gripper, which we assume fix arbitrary $q(0)$ and $q(1)$. We further assume, without loss of generality, that $q(0) = e$. We denote the space of all $q(1)$ by $\mathscr{B} = G$. Finally, we assume that the rod is elastic in the sense of Kirchhoff [8], so has total elastic energy $\frac{1}{2}\int_0^1 \left(c_1u_1^2 + c_2u_2^2 + c_3u_3^2\right) dt$ for given constants $c_1, c_2, c_3 > 0$. For fixed endpoints, the wire is motionless only if its shape locally minimizes energy. In particular, we say that (q,u) is in static equilibrium if it is a local optimum of

$$\begin{aligned} \underset{q,u}{\text{minimize}} \quad & \frac{1}{2}\int_0^1 \left(c_1u_1^2 + c_2u_2^2 + c_3u_3^2\right) dt \\ \text{subject to} \quad & \dot{q} = q(u_1X_1 + u_2X_2 + u_3X_3 + X_4) \\ & q(0) = e, \qquad q(1) = b \end{aligned} \tag{14}$$

for some $b \in \mathscr{B}$.

3.2 Necessary Conditions for Static Equilibrium

Theorem 5. *A trajectory* (q,u) *is normal with respect to* (14) *if and only if there exists* $\mu\colon [0,1] \to \mathfrak{g}^*$ *that satisfies*

$$\begin{aligned} \dot{\mu}_1 &= u_3\mu_2 - u_2\mu_3 & \dot{\mu}_4 &= u_3\mu_5 - u_2\mu_6 \\ \dot{\mu}_2 &= \mu_6 + u_1\mu_3 - u_3\mu_1 & \dot{\mu}_5 &= u_1\mu_6 - u_3\mu_4 \\ \dot{\mu}_3 &= -\mu_5 + u_2\mu_1 - u_1\mu_2 & \dot{\mu}_6 &= u_2\mu_4 - u_1\mu_5, \end{aligned} \tag{15}$$

$$\dot{q} = q(u_1X_1 + u_2X_2 + u_3X_3 + X_4), \tag{16}$$

$$u_i = c_i^{-1}\mu_i \quad \text{for all } i \in \{1,2,3\}, \tag{17}$$

with initial conditions $q(0) = e$ *and* $\mu(0) = \sum_{i=1}^{6} a_iP_i$ *for some* $a \in \mathscr{A}$, *where*

$$\mathscr{A} = \left\{a \in \mathbb{R}^6 \colon (a_2,a_3,a_5,a_6) \neq (0,0,0,0)\right\}.$$

Proof. We begin by showing that (q,u) is abnormal if and only if $u_2 = u_3 = 0$. Theorem 1 tells us that (q,u) is abnormal if and only if it is the projection of an integral curve (p,q) of X_H that satisfies (2), where $H(p,q,t) = \widehat{H}(p,q,0,u(t))$ and

$$\widehat{H}(p,q,0,u) = \langle p, q(u_1X_1 + u_2X_2 + u_3X_3 + X_4)\rangle.$$

For any $g, r \in G$ satisfying $q = gr$, we compute

$$\begin{aligned} H(T_r^*L_g(p),r,t) &= \left\langle T_r^*L_g(p), g^{-1}q(u_1X_1 + u_2X_2 + u_3X_3 + X_4)\right\rangle \\ &= \left\langle p, g\left(g^{-1}q(u_1X_1 + u_2X_2 + u_3X_3 + X_4)\right)\right\rangle \\ &= \langle p, q(u_1X_1 + u_2X_2 + u_3X_3 + X_4)\rangle = H(p,q,t), \end{aligned} \tag{18}$$

so H is left-invariant. Then, the existence of (p,q) satisfying Theorem 1 is equivalent to the existence of μ satisfying Theorem 3: $\dot{\mu} = \mathrm{ad}^*_{\delta h/\delta\mu}(\mu)$ and $\dot{q} = q(\delta h/\delta\mu)$, where $h = H|_{\mathfrak{g}^*}$. Application of (9) produces the formulae (15)-(16), where we require $\mu_1 = \mu_2 = \mu_3 = 0$ to satisfy (2). We therefore have $\dot{\mu}_2 = \mu_6$ and $\dot{\mu}_3 = -\mu_5$, hence $\mu_5 = \mu_6 = 0$. Applying this result again to (15), we find $\dot{\mu}_5 = -u_3\mu_4 = 0$ and $\dot{\mu}_6 = u_2\mu_4 = 0$. Since μ cannot vanish when $k = 0$, we must have $\mu_4 \neq 0$, hence $u_2 = u_3 = 0$, with u_1 an arbitrary integrable function. Our result follows.

Now, we return to the normal case. As before, Theorem 1 tells us that (q,u) is normal if and only if it is not abnormal and it is the projection of an integral curve (p,q) of X_H that satisfies (2), where $H(p,q,t) = \widehat{H}(p,q,1,u(t))$ and

$$\widehat{H}(p,q,1,u) = \langle p, q(u_1X_1 + u_2X_2 + u_3X_3 + X_4)\rangle - \left(c_1u_1^2 + c_2u_2^2 + c_3u_3^2\right)/2.$$

By a computation identical to (18), H is left-invariant. Application of (9) to the conditions of Theorem 3 produces the same formulae (15)-(16), where (17) follows from (2) because $\widehat{H}$ is quadratic in u. It remains to show that trajectories produced by (15)-(17) are not abnormal if and only if $a \in \mathscr{A}$. We prove the converse.

First, assume $a \in \mathbb{R}^6 \backslash \mathscr{A}$, so $(a_2, a_3, a_5, a_6) = (0,0,0,0)$. From (15) and (17), we see that $u_2 = u_3 = 0$, hence (q,u) is abnormal. Now, assume (q,u) is abnormal, so $u_2 = u_3 = 0$. From (17), we therefore have $\mu_2 = \mu_3 = 0$, and in particular $a_2 = a_3 = 0$. Plugging this result into (15), we see that $\dot{\mu}_2 = \mu_6$ and $\dot{\mu}_3 = -\mu_5$, hence also that $\mu_5 = \mu_6 = 0$, i.e., that $a_5 = a_6 = 0$. So, $a \in \mathbb{R}^6 \backslash \mathscr{A}$. Our result follows. □

Theorem 5 provides a set of candidates for local optima of (14), which we now characterize. Denote the set of all smooth maps $(q,u)\colon [0,1] \to G \times U$ under the smooth topology by $C^\infty([0,1], G \times U)$. Let $\mathscr{C} \subset C^\infty([0,1], G \times U)$ be the subset of all (q,u) that satisfy Theorem 5. Any such $(q,u) \in \mathscr{C}$ is completely defined by the choice of $a \in \mathscr{A}$, as is the corresponding μ. Denote the resulting map by $\Psi(a) = (q,u)$ and $\Gamma(a) = \mu$. We require three lemmas before our main result (Theorem 6).

Lemma 3. *If $\Psi(a) = \Psi(a')$ for some $a, a' \in \mathscr{A}$, then $a = a'$.*

Proof. Suppose $(q,u) = \Psi(a)$ and $\mu = \Gamma(a)$ for some $a \in \mathscr{A}$. It suffices to show that a is uniquely defined by u (and its derivatives, since u is clearly smooth). From (17), we have $a_i = c_i u_i(0)$ for $i \in \{1,2,3\}$. From (15), we have

$$a_5 = -c_3 \dot{u}_3(0) + a_1 a_2 (c_2^{-1} - c_1^{-1}) \qquad a_6 = c_2 \dot{u}_2(0) - a_1 a_3 (c_1^{-1} - c_3^{-1}). \tag{19}$$

It is now possible to compute $\dot{\mu}_i(0)$ and $\ddot{\mu}_i(0)$ for $i \in \{4,5,6\}$ by differentiation of

$$\dot{\mu}_4 = u_3 \mu_5 - u_2 \mu_6 \qquad \mu_5 = -\dot{\mu}_3 + u_2 \mu_1 - u_1 \mu_2 \qquad \mu_6 = \dot{\mu}_2 - u_1 \mu_3 - u_3 \mu_1. \tag{20}$$

Based on these results, we differentiate (15) again to produce

$$\begin{aligned}
(c_3^{-1} a_3) a_4 &= c_1^{-1} a_1 a_6 - \dot{\mu}_5(0) \\
(c_2^{-1} a_2) a_4 &= c_1^{-1} a_1 a_5 + \dot{\mu}_6(0) \\
(-a_5 + a_1 a_2 (c_2^{-1} - c_1^{-1})) a_4 &= c_3 (c_1^{-1} (\dot{\mu}_1(0) a_6 + a_1 \dot{\mu}_6(0)) - \ddot{\mu}_5(0)) - a_3 \dot{\mu}_4(0) \\
(a_6 + a_1 a_3 (c_1^{-1} - c_3^{-1})) a_4 &= c_2 (c_1^{-1} (\dot{\mu}_1(0) a_5 + a_1 \dot{\mu}_5(0)) + \ddot{\mu}_6(0)) - a_2 \dot{\mu}_4(0).
\end{aligned} \tag{21}$$

At least one of these four equations allows us to compute a_4 unless $(a_2, a_3, a_5, a_6) = (0,0,0,0)$, which would violate our assumption that $a \in \mathscr{A}$. Our result follows. □

Lemma 4. *The map $\Psi\colon \mathscr{A} \to \mathscr{C}$ is a homeomorphism.*

Proof. The map Ψ is clearly a bijection—it is well-defined and onto by construction, and is one-to-one by Lemma 3. Continuity of Ψ also follows from Theorem 5. It remains to show that $\Psi^{-1}\colon \mathscr{C} \to \mathscr{A}$ is continuous. This result is a corollary to the proof of Lemma 3. It is immediate that a_1, a_2, a_3 depend continuously on $u(0)$. From (19), we see that a_5, a_6 depend continuously on $a_1, a_2, a_3, \dot{u}(0)$, hence on $u(0), \dot{u}(0)$. From (20), we see in the same way that $\dot{\mu}_4(0), \dot{\mu}_5(0), \dot{\mu}_6(0), \ddot{\mu}_5(0), \ddot{\mu}_6(0)$ depend continuously on $u(0), \dot{u}(0), \ddot{u}(0)$. Hence, all of the quantities in (21) depend continuously on $u(0), \dot{u}(0), \ddot{u}(0)$, so a_4 does as well. Our result follows. □

Lemma 5. *If the topological n-manifold M has an atlas consisting of the single chart (M,α), then $N=\alpha(M)$ is a topological n-manifold with an atlas consisting of the single chart (N,id_N), where id_N is the identity map. Furthermore, both M and N are smooth n-manifolds and $\alpha\colon M\to N$ is a diffeomorphism.*

Proof. Since (M,α) is chart, then N is an open subset of $\mathbb{R}^n$ and α is a bijection. Hence, our first result is immediate and our second result requires only that both α and α^{-1} are smooth maps. For every $p\in M$, the charts (M,α) and (N,id_N) satisfy $\alpha(p)\in N$, $\alpha(M)=N$, and $\mathrm{id}_N\circ\alpha\circ\alpha^{-1}=\mathrm{id}_N$, so α is a smooth map. For every $q\in N$, the charts (N,id_N) and (M,α) again satisfy $\alpha^{-1}(q)\in M$, $\alpha^{-1}(N)=M$, and $\alpha\circ\alpha^{-1}\circ\mathrm{id}_N=\mathrm{id}_N$, so α^{-1} is also a smooth map. Our result follows. □

Theorem 6. *$\mathscr{C}$ is a smooth 6-manifold with smooth structure determined by an atlas with the single chart $(\mathscr{C},\Psi^{-1})$.*

Proof. Since $\Psi\colon\mathscr{A}\to\mathscr{C}$ is a homeomorphism by Lemma 4 and $\mathscr{A}\subset\mathbb{R}^6$ is open, then $(\mathscr{C},\Psi^{-1})$ is a chart whose domain is $\mathscr{C}$. Our result follows from Lemma 5. □

3.3 Sufficient Conditions for Static Equilibrium

Theorem 7. *Let $(q,u)=\Psi(a)$ and $\mu=\Gamma(a)$ for some $a\in\mathscr{A}$. Define*

$$\mathbf{F}=\begin{bmatrix}0 & \mu_3(c_3^{-1}-c_2^{-1}) & \mu_2(c_3^{-1}-c_2^{-1}) & 0 & 0 & 0\\ \mu_3(c_1^{-1}-c_3^{-1}) & 0 & \mu_1(c_1^{-1}-c_3^{-1}) & 0 & 0 & 1\\ \mu_2(c_2^{-1}-c_1^{-1}) & \mu_1(c_2^{-1}-c_1^{-1}) & 0 & 0 & -1 & 0\\ 0 & -\mu_6/c_2 & \mu_5/c_3 & 0 & \mu_3/c_3 & -\mu_2/c_2\\ \mu_6/c_1 & 0 & -\mu_4/c_3 & -\mu_3/c_3 & 0 & \mu_1/c_1\\ -\mu_5/c_1 & \mu_4/c_2 & 0 & \mu_2/c_2 & -\mu_1/c_1 & 0\end{bmatrix}$$

$$\mathbf{G}=\operatorname{diag}\left(c_1^{-1},c_2^{-1},c_3^{-1},0,0,0\right)$$

$$\mathbf{H}=\begin{bmatrix}0 & \mu_3/c_3 & -\mu_2/c_2 & 0 & 0 & 0\\ -\mu_3/c_3 & 0 & \mu_1/c_1 & 0 & 0 & 0\\ \mu_2/c_2 & -\mu_1/c_1 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & \mu_3/c_3 & -\mu_2/c_2\\ 0 & 0 & 1 & -\mu_3/c_3 & 0 & \mu_1/c_1\\ 0 & -1 & 0 & \mu_2/c_2 & -\mu_1/c_1 & 0\end{bmatrix}.$$

Solve the (linear, time-varying) matrix differential equations

$$\dot{\mathbf{M}}=\mathbf{F}\mathbf{M}\qquad\qquad \dot{\mathbf{J}}=\mathbf{G}\mathbf{M}+\mathbf{H}\mathbf{J}\tag{22}$$

with initial conditions $\mathbf{M}(0)=I$ and $\mathbf{J}(0)=0$. Then, (q,u) is a local optimum of (14) *for $b=q(1)$ if and only if* $\det(\mathbf{J}(t))\neq 0$ *for all $t\in(0,1]$.*

Proof. As we have already seen, normal extremals of (14) are derived from the parameterized Hamiltonian function

$$\widehat{H}(p,q,1,u)=\langle p,q(u_1X_1+u_2X_2+u_3X_3+X_4)\rangle-\left(c_1u_1^2+c_2u_2^2+c_3u_3^2\right)/2.$$

This function satisfies $\partial^2 \widehat{H}/\partial u^2 = -\operatorname{diag}(c_1, c_2, c_3) < 0$ and admits a unique maximum at $u_i = \langle p, qX_i \rangle / c_i$ for $i \in \{1,2,3\}$. The maximized Hamiltonian function is

$$H(p,q) = \langle p, qX_4 \rangle + \left(c_1^{-1} \langle p, qX_1 \rangle^2 + c_2^{-1} \langle p, qX_2 \rangle^2 + c_3^{-1} \langle p, qX_3 \rangle^2 \right) /2.$$

It is clear that X_H is complete. By Lemma 3, the mapping from (q,u) to a and hence to $\mu = \Gamma(a)$ is unique. By Theorem 3, it is equivalent that the mapping from (q,u) to (p,q) is unique. As a consequence, we may apply Theorem 2 to establish sufficient conditions for optimality. Since a computation identical to (18) shows that H is left-invariant, we may apply the equivalent conditions of Theorem 4. Noting that the restriction $h = H|_{\mathfrak{g}^*} \in C^\infty(\mathfrak{g}^*)$ is given by

$$h(\mu) = \mu_4 + \left(c_1^{-1} \mu_1^2 + c_2^{-1} \mu_2^2 + c_3^{-1} \mu_3^2 \right) /2$$

it is easy to verify that **F**, **G** and **H** take the form given above. Our result follows. □

Theorem 7 provides a computational test of which points $a \in \mathscr{A}$ actually produce local optima $\Psi(a) \in \mathscr{C}$ of (14). Let $\mathscr{A}_{\text{stable}} \subset \mathscr{A}$ be the subset of all a for which the conditions of Theorem 7 are satisfied and let $\mathscr{C}_{\text{stable}} = \Psi(\mathscr{A}_{\text{stable}}) \subset \mathscr{C}$. An important consequence of membership in $\mathscr{A}_{\text{stable}}$ is smooth local dependence of (14) on variation in b. Define $\mathscr{B}_{\text{stable}} = \{q(1) \in \mathscr{B} \colon (q,u) \in \mathscr{C}_{\text{stable}}\}$ and let $\Phi \colon \mathscr{C} \to \mathscr{B}$ be the map taking (q,u) to $q(1)$. Clearly $\mathscr{A}_{\text{stable}}$ is open, so $\Psi|_{\mathscr{A}_{\text{stable}}} \colon \mathscr{A}_{\text{stable}} \to \mathscr{C}_{\text{stable}}$ is a diffeomorphism. We arrive at the following result:

Theorem 8. *The map $\Phi \circ \Psi|_{\mathscr{A}_{stable}} \colon \mathscr{A}_{stable} \to \mathscr{B}_{stable}$ is a local diffeomorphism.*

Proof. The map $\Phi \circ \Psi|_{\mathscr{A}_{\text{stable}}}$ is smooth and by Theorem 7 has non-singular Jacobian $\mathbf{J}(1)$. Our result follows from the Implicit Function Theorem [25, Theorem 7.9]. □

3.4 Application to Manipulation Planning

Recall that we want to find a path of the gripper that causes the rod to move between given start and goal configurations while remaining in static equilibrium. It is equivalent to find a path *of the rod* through its set of equilibrium configurations. We showed that any equilibrium configuration can be represented by a point in $\mathscr{A}_{\text{stable}} \subset \mathscr{A} \subset \mathbb{R}^6$. Think of $\mathscr{A}$ as the "configuration space" of the rod during quasi-static manipulation and of $\mathscr{A}_{\text{stable}}$ as the "free space." Theorems 5-6 say how to map points in $\mathscr{A}$ to configurations of the rod. Theorem 7 says how to test membership in $\mathscr{A}_{\text{stable}}$, i.e., it provides a "collision checker." Theorem 8 says that paths in $\mathscr{A}_{\text{stable}}$ can be "implemented" by the gripper, by establishing a well-defined map between differential changes in the rod (represented by $\mathscr{A}_{\text{stable}}$) and in the gripper (represented by $\mathscr{B}_{\text{stable}}$). We have expressed the manipulation planning problem for an elastic rod as a standard motion planning problem in a configuration space of dimension 6, for which there are hundreds of solution approaches [23, 11, 24].

For the sake of completeness, here is one way to implement a sampling-based planning algorithm like PRM [19]:

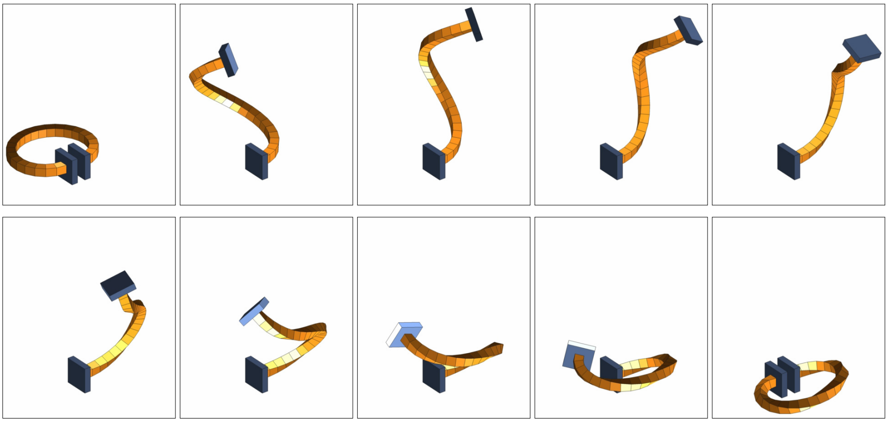

Fig. 2 A second example of quasi-static manipulation by robotic grippers (blue) of an elastic rod (orange). Again, the grippers begin and end in the same position and orientation. And again, this motion corresponds to a straight-line path in the global coordinate chart $\mathscr{A}$ that we derived.

- Sample points in $\mathscr{A}$, for example uniformly at random in $\{a \in \mathscr{A} : \|a\|_\infty \leq w\}$ for some $w > 0$. Note that it is possible to choose w in practice by taking advantage of the direct correspondence (which we do not discuss here) between a and forces/torques at the base of the elastic rod.
- Keep points that are in $\mathscr{A}_{\text{stable}}$ and add them as nodes in the roadmap. This test requires only solving the ordinary differential equations (15)-(17) in 6 variables and the matrix differential equations (22) in 72 variables.
- Try to connect each pair of nodes a and a' with a straight-line path in $\mathscr{A}$, adding this path as an edge in the roadmap if it lies entirely in $\mathscr{A}_{\text{stable}}$. This test can be approximated in the usual way by sampling points along the straight-line path at some resolution, again solving (15)-(17) and (22) for each point.
- Declare $a_{\text{start}}, a_{\text{goal}} \in \mathscr{A}_{\text{stable}}$ to be path-connected if they are connected by a sequence of nodes and edges in the roadmap. This sequence is a continuous and piecewise-smooth map $\alpha : [0,1] \to \mathscr{A}_{\text{stable}}$, where $\alpha(0) = a_{\text{start}}$ and $\alpha(1) = a_{\text{goal}}$.
- Move the robotic gripper along the path $\Phi \circ \Psi|_{\mathscr{A}_{\text{stable}}} \circ \alpha : [0,1] \to \mathscr{B}_{\text{stable}}$. This path is again continuous and piecewise-smooth, and can be evaluated at waypoints $s \in [0,1]$ by solving the matrix differential equation (16) on $SE(3)$.

Each step is trivial to implement using modern numerical methods. It is also easy to include other constraints, such as self-collision, within this basic framework.

We emphasize that the "start" and "goal" for the manipulation planning problem must be points in $\mathscr{A}_{\text{stable}}$, or equivalently points in $\mathscr{C}_{\text{stable}}$ through the diffeomorphism Ψ. It is insufficient to specify start and goal by points in $\mathscr{B}_{\text{stable}}$, since these points do not uniquely define configurations of the elastic rod.

Figure 2 shows another example result. The start and goal configurations are both associated with the same boundary conditions, each one being a different local minimum of total elastic energy. The motion of the rod therefore could not possibly correspond to a single straight-line path in $\mathscr{B}_{\text{stable}}$, where planning has traditionally been done (e.g., [21, 29]). However, this motion does indeed correspond to a single straight-line path in $\mathscr{A}_{\text{stable}}$ and was trivial to generate with our approach. We have constructed many similar examples, all of which point to favorable visibility properties of $\mathscr{A}_{\text{stable}}$ and lead us to expect standard motion planning algorithms to perform well in this context [19, 11, 24]. We note further that a number of planning heuristics like lazy collision-checking [34]—which bring huge speed-ups in practice—are easy to apply when planning in $\mathscr{A}_{\text{stable}}$ but hard to apply when planning in $\mathscr{B}_{\text{stable}}$. Finally, should we still want to plan in $\mathscr{B}_{\text{stable}}$ (i.e., to connect nearby configurations by straight-line paths in $\mathscr{B}_{\text{stable}}$ rather than in $\mathscr{A}_{\text{stable}}$), it is now easy to do so by using the Jacobian matrix $\mathbf{J}(1)$, which is non-singular in $\mathscr{B}_{\text{stable}}$ by construction. In particular, we have the relationship $\delta b = \mathbf{J}(1)\delta a$, which can be inverted to move along straight lines in $\mathscr{B}_{\text{stable}}$. Without this relationship, we would be forced to apply gradient descent in the infinite-dimensional space of inputs $u\colon [0,1] \to U$, prompting methods of approximation like the one described in [29].

4 Conclusion

Our contribution in this paper was to show that the set of equilibrium configurations for a Kirchhoff elastic rod held at each end by a robotic gripper is a smooth manifold of finite dimension that can be parameterized by a single (global) coordinate chart. The fact that we ended up with a finite-dimensional smooth manifold is something that might have been guessed in hindsight (it's dimension—six—is intuitive given that the grippers move in $SE(3)$), but the fact that this manifold admitted a global chart is something that we find remarkable. Our results led to a simple algorithm for manipulation planning, which at the outset had seemed very hard to solve.

A straightforward extension is to implement a sampling-based planner as described in Section 3.4 and perform experiments that compare our approach to others (e.g., [29]) in terms of standard metrics like running time, failure probability, etc. This implementation requires consideration of certain details that we did not address explicitly. For example, to verify static equilibrium, Theorem 7 requires a check that $\det(\mathbf{J}(t))$ does not vanish on $(0,1]$. We can approximate this check by sampling t, but would prefer an approach with guarantees (as in "exact" collision checking [35]). This is problematic since $\det(\mathbf{J}(t))$ and all its derivatives vanish at $t = 0$.

There are several other opportunities for future work. First, the coordinates we derive can be interpreted as forces and torques at the base of the elastic rod, so $\mathscr{A}$ is exactly the space over which to perform inference in state estimation with a force/torque sensor. Second, our model of an elastic rod depends on three physical parameters $c_1, c_2, c_3 > 0$. Finding these parameters from observations of equilibrium configurations can be cast as an inverse optimal control problem [18]. The structure established by Theorem 6 allows us to define a notion of orthogonal distance between $\mathscr{C}$ and these observations, similar to [20], and may lead to an efficient method of solution. Third, we note that an elastic inextensible strip (or "ribbon") is a developable surface whose shape can be reconstructed from its centerline [37]. This centerline conforms to a similar model as the elastic rod and is likely amenable to similar analysis, which may generalize to models of other developable surfaces.

Acknowledgements. This material is based upon work supported by the National Science Foundation under CPS-0931871 and CMMI-0956362. The authors would like to thank Don Shimamoto for helpful comments, leading to the current proof of Lemma 4. Thanks also to the organizers, anonymous reviewers, and audience at WAFR 2012, whose suggestions improved this paper.

References

1. Agrachev, A.A., Sachkov, Y.L.: Control theory from the geometric viewpoint, vol. 87. Springer, Berlin (2004)
2. Amato, N.M., Song, G.: Using motion planning to study protein folding pathways. Journal of Computational Biology 9(2), 149–168 (2002)
3. Antman, S.S.: Nonlinear Problems of Elasticity, 2nd edn. Applied Mathematical Sciences, vol. 107. Springer, New York (2005)
4. Asano, Y., Wakamatsu, H., Morinaga, E., Arai, E., Hirai, S.: Deformation path planning for manipulation of flexible circuit boards. In: IEEE/RSJ Int. Conf. Int. Rob. Sys. (2010)
5. Bell, M., Balkcom, D.: Knot tying with single piece fixtures. In: Int. Conf. Rob. Aut. (2008)
6. van den Berg, J., Miller, S., Goldberg, K., Abbeel, P.: Gravity-based robotic cloth folding. In: WAFR (2011)
7. Bergou, M., Wardetzky, M., Robinson, S., Audoly, B., Grinspun, E.: Discrete elastic rods. ACM Trans. Graph. 27(3), 1–12 (2008)
8. Biggs, J., Holderbaum, W., Jurdjevic, V.: Singularities of optimal control problems on some 6-d lie groups. IEEE Trans. Autom. Control 52(6), 1027–1038 (2007)
9. Bloch, A., Krishnaprasad, P., Marsden, J., Ratiu, T.: The Euler-Poincaré equations and double bracket dissipation. Communications In Mathematical Physics 175(1), 1–42 (1996)
10. Chirikjian, G.S., Burdick, J.W.: The kinematics of hyper-redundant robot locomotion. IEEE Trans. Robot. Autom. 11(6), 781–793 (1995)
11. Choset, H., Lynch, K., Hutchinson, S., Kanto, G., Burgard, W., Kavraki, L., Thrun, S.: Principles of Robot Motion: Theory, Algorithms, and Implementations. MIT Press (2005)
12. Clements, T.N., Rahn, C.D.: Three-dimensional contact imaging with an actuated whisker. IEEE Trans. Robot. 22(4), 844–848 (2006)
13. Gopalakrishnan, K., Goldberg, K.: D-space and deform closure grasps of deformable parts. International Journal of Robotics Research 24(11), 899–910 (2005)
14. Hoffman, K.A.: Methods for determining stability in continuum elastic-rod models of dna. Philosophical Transactions of the Royal Society of London. Series A: Mathematical, Physical and Engineering Sciences 362(1820), 1301–1315 (2004)
15. Hopcroft, J.E., Kearney, J.K., Krafft, D.B.: A case study of flexible object manipulation. The International Journal of Robotics Research 10(1), 41–50 (1991)
16. Inoue, H., Inaba, H.: Hand-eye coordination in rope handling. In: ISRR, pp. 163–174 (1985)

17. Jansen, R., Hauser, K., Chentanez, N., van der Stappen, F., Goldberg, K.: Surgical retraction of non-uniform deformable layers of tissue: 2d robot grasping and path planning. In: IEEE/RSJ Int. Conf. Int. Rob. Sys., pp. 4092–4097 (2009)
18. Javdani, S., Tandon, S., Tang, J., O'Brien, J.F., Abbeel, P.: Modeling and perception of deformable one-dimensional objects. In: Int. Conf. Rob. Aut., Shanghai, China (2011)
19. Kavraki, L.E., Svetska, P., Latombe, J.C., Overmars, M.: Probabilistic roadmaps for path planning in high-dimensional configuration spaces. IEEE Trans. Robot. Autom. 12(4), 566–580 (1996)
20. Keshavarz, A., Wang, Y., Boyd, S.: Imputing a convex objective function. In: IEEE Multi-Conference on Systems and Control (2011)
21. Lamiraux, F., Kavraki, L.E.: Planning paths for elastic objects under manipulation constraints. International Journal of Robotics Research 20(3), 188–208 (2001)
22. Langer, J., Singer, D.: The total squared curvature of closed curves. Journal of Differential Geometry 20, 1–22 (1984)
23. Latombe, J.C.: Robot Motion Planning. Kluwer Academic Publishers, Boston (1991)
24. LaValle, S.M.: Planning algorithms. Cambridge University Press, New York (2006)
25. Lee, J.M.: Introduction to smooth manifolds, vol. 218. Springer, New York (2003)
26. Lin, Q., Burdick, J., Rimon, E.: A stiffness-based quality measure for compliant grasps and fixtures. IEEE Trans. Robot. Autom. 16(6), 675–688 (2000)
27. Marsden, J.E., Ratiu, T.S.: Introduction to mechanics and symmetry: a basic exposition of classical mechanical systems, 2nd edn. Springer, New York (1999)
28. McCarthy, Z., Bretl, T.: Mechanics and manipulation of planar elastic kinematic chains. In: IEEE Int. Conf. Rob. Aut. St. Paul, MN (2012)
29. Moll, M., Kavraki, L.E.: Path planning for deformable linear objects. IEEE Trans. Robot. 22(4), 625–636 (2006)
30. Rucker, D.C., Webster, R.J., Chirikjian, G.S., Cowan, N.J.: Equilibrium conformations of concentric-tube continuum robots. Int. J. Rob. Res. 29(10), 1263–1280 (2010)
31. Sachkov, Y.: Conjugate points in the euler elastic problem. Journal of Dynamical and Control Systems 14(3), 409–439 (2008)
32. Sachkov, Y.: Maxwell strata in the euler elastic problem. Journal of Dynamical and Control Systems 14(2), 169–234 (2008)
33. Saha, M., Isto, P.: Manipulation planning for deformable linear objects. IEEE Trans. Robot. 23(6), 1141–1150 (2007)
34. Sánchez, G., Latombe, J.C.: On delaying collision checking in PRM planning: Application to multi-robot coordination. Int. J. Rob. Res. 21(1), 5–26 (2002)
35. Schwarzer, F., Saha, M., Latombe, J.C.: Exact collision checking of robot paths. In: WAFR, Nice, France (2002)
36. Solomon, J.H., Hartmann, M.J.Z.: Extracting object contours with the sweep of a robotic whisker using torque information. Int. J. Rob. Res. 29(9), 1233–1245 (2010)
37. Starostin, E.L., van der Heijden, G.H.M.: Tension-induced multistability in inextensible helical ribbons. Physical Review Letters 101(8), 084,301 (2008)
38. Takamatsu, J., Morita, T., Ogawara, K., Kimura, H., Ikeuchi, K.: Representation for knot-tying tasks. IEEE Trans. Robot. 22(1), 65–78 (2006)
39. Tanner, H.: Mobile manipulation of flexible objects under deformation constraints. IEEE Trans. Robot. 22(1), 179–184 (2006)
40. Wakamatsu, H., Arai, E., Hirai, S.: Knotting/unknotting manipulation of deformable linear objects. The International Journal of Robotics Research 25(4), 371–395 (2006)

41. Walsh, G., Montgomery, R., Sastry, S.: Optimal path planning on matrix lie groups. In: IEEE Conference on Decision and Control, vol. 2, pp. 1258–1263 (1994)
42. Webster, R.J., Jones, B.A.: Design and kinematic modeling of constant curvature continuum robots: A review. Int. J. Rob. Res. 29(13), 1661–1683 (2010)
43. Yamakawa, Y., Namiki, A., Ishikawa, M.: Motion planning for dynamic folding of a cloth with two high-speed robot hands and two high-speed sliders. In: Int. Conf. Rob. Aut., pp. 5486–5491 (2011)