Communication

Comments on "An Optimality Principle Governing Human Walking"

T. Bretl, *Member, IEEE*, G. Arechavaleta,
A. Akce, *Student Member, IEEE*, and J.-P. Laumond, *Fellow, IEEE*

***Abstract*—The paper in question [G. Arechavaleta, J. P. Laumond, H. Hicheur, and A. Berthoz, "An optimality principle governing human walking," *IEEE Trans. Robot.*, vol. 24, no. 1, pp. 5—14, Feb. 2008] suggested that human-walking paths minimize variation in curvature and hence can be approximated by the solution to an optimal control problem. This conclusion was reached by analysis of experimental data based on the maximum principle. We correct two errors in this analysis and consider their consequences.**

***Index Terms*—Biological system modeling, humanoid robots, optimal control.**

I. Correction

A. Two Mistakes That Were Made

In [1], it was suggested that human-walking paths can be approximated by solutions to the optimal control problem

$$\begin{aligned} \text{minimize} \quad & \frac{1}{2}\int_0^T \left(u_1^2 + u_2^2\right) dt \\ \text{subject to} \quad & \dot{x}_1 = u_1 \cos x_3 \\ & \dot{x}_2 = u_1 \sin x_3 \\ & \dot{x}_3 = u_1 x_4 \\ & \dot{x}_4 = u_2 \end{aligned} \tag{1}$$

together with the constraints

$$u_1 \in [a, b], \qquad u_2 \in [-c, c] \tag{2}$$

for $a, b, c > 0$ and with the initial and final conditions

$$x(0) = x_{\text{start}}, \qquad x(T) = x_{\text{goal}}. \tag{3}$$

The final time T was assumed given, but this assumption is critical neither to the original argument nor to ours. The center of the torso, as viewed from above, is located at the point (x_1, x_2). The path traced by this point has tangent angle x_3 and curvature x_4. The inputs are the forward speed u_1 and the time derivative of curvature u_2. The model states that, for goal-directed motion, the torso follows a trajectory that minimizes the norm of these two inputs. Predicted trajectories were found to match well the experimental data presented in [1].

In developing its argument for (1)–(3) as a useful model of human walking, the authors of [1] made two errors that were later pointed out by Akce and Bretl [2]:

1) By applying the maximum principle [3], it was found that solutions to (1)–(3) must locally satisfy

$$u_1^2(t) + u_2^2(t) = \text{constant} \tag{4}$$

for all $t \in [0, T]$. Then, through statistical analysis of experimental data, it was found that u_1 may be assumed piecewise constant. It was concluded that u_2 must also be piecewise constant (i.e., that optimal trajectories consist of clothoid arcs). This conclusion is false (see Section I-B).

2) By applying a method of numerical optimization [4], it was found that solutions to particular instances of (1)–(3) may exhibit constant u_1 but piecewise constant u_2 (for example, see [1, Fig. 7]). Again, this result cannot possibly be correct (see Section I-B). It was later shown by Arechavaleta and Laumond [5] that if we fix $u_1 = 1$ and consider the resultant problem

$$\begin{aligned} \text{minimize} \quad & \frac{1}{2}\int_0^T \left(1 + u_2^2\right) dt \\ \text{subject to} \quad & \dot{x}_1 = \cos x_3 \\ & \dot{x}_2 = \sin x_3 \\ & \dot{x}_3 = x_4 \\ & \dot{x}_4 = u_2 \end{aligned} \tag{5}$$

together with the constraint $u_2 \in [-c, c]$ for some $c > 0$ and with the same initial and final conditions (3), then solutions may exhibit piecewise constant u_2. However, this case corresponds to control saturation, and the only values that can possibly be attained by constant u_2 are either c or $-c$, which is a condition that is not satisfied by the results of [1, Fig. 7]. Hence, we must conclude either that [1, Fig. 7] corresponds to some optimal control problem other than (1) and (5)—i.e., to some problem that was not presented in [1]—or that the numerical method used to generate [1, Fig. 7] did not produce a good approximation to the optimal trajectory (see Section I-C).

We will address these errors in the following two sections.

B. Correcting the First Mistake

The Hamiltonian associated with (1) is

$$H(p, x, u) = \frac{1}{2}\left(u_1^2 + u_2^2\right) + p_1 u_1 \cos x_3 + p_2 u_1 \sin x_3 + p_3 u_1 x_4 + p_4 u_2 \tag{6}$$

where p is the costate. The maximum principle tells us that along optimal trajectories $(p^*\, x^*\, u^*)$, we must have

$$-\dot{p}^* = \nabla_x H(p^*, x^*, u^*) \tag{7}$$

and

$$u^* \leq \arg\min_u H(p^*, x^*, u). \tag{8}$$

Manuscript received August 18, 2010; revised September 22, 2010; accepted September 24, 2010. Date of publication November 9, 2010; date of current version December 8, 2010. This work was supported by the National Science Foundation under Grant 0931871, Grant 0956362, and Grant 0955088. This paper was recommended for publication by Editor K. Lynch upon evaluation of the reviewers' comments.

T. Bretl is with the Department of Aerospace Engineering, University of Illinois, Urbana-Champaign, Urbana, IL 61801 USA (e-mail: tbretl@illinois.edu).
G. Arechavaleta is with the Centro de Investigaciones y Estudios Avanzados del IPN, 25900 Mexico City, Mexico (e-mail: garechav@cinvestav.edu.mx).
A. Akce is with the Department of Computer Science, University of Illinois, Urbana-Champaign, Urbana, IL 61801 USA (e-mail: akce2@illinois.edu).
J.-P. Laumond is with the Laboratoire d'Analyse et d'Architectures des Systèmes, Centre National de la Recherche Scientique, Toulouse University, 31077 Toulouse, France (e-mail: jpl@laas.fr).

Digital Object Identifier 10.1109/TRO.2010.2082110

Condition (7) implies that

$$\begin{aligned} \dot{p}_1 &= 0 \\ \dot{p}_2 &= 0 \\ \dot{p}_3 &= p_1 u_1 \sin x_3 - p_2 u_1 \cos x_3 \\ \dot{p}_4 &= -p_3 u_1 \end{aligned} \tag{9}$$

while—in the absence of control saturation—condition (8) implies that $\nabla_u H = 0$, hence

$$\begin{aligned} 0 &= u_1 + p_1 \cos x_3 + p_2 \sin x_3 + p_3 x_4 \\ 0 &= u_2 + p_4. \end{aligned} \tag{10}$$

Differentiating (10) and plugging in (1) and (9), we find that

$$\begin{aligned} \dot{u}_1 &= -p_3 u_2 \\ \dot{u}_2 &= p_3 u_1. \end{aligned} \tag{11}$$

It is at this point that the authors of [1] make a mistake. It is indeed true that (4) follows by direct integration of (11). However, if u_1 is constant along an optimal trajectory, then $\dot{u}_1 = 0$; hence, either $p_3 = 0$, or $u_2 = 0$ from (11). Each case implies that the resulting trajectory is a straight line segment (i.e., that x_3 is constant):

1) If $p_3 = 0$, then $\dot{p}_3 = 0$, and so

$$u_1 \left(p_1 \sin x_3 - p_2 \cos x_3\right) = 0$$

from (9). If $u_1 = 0$, then $\dot{x_3} = 0$ from (1) and so x_3 is constant. If $u_1 \neq 0$, then $p_1 \sin x_3 - p_2 \cos x_3 = 0$, which has countable solutions for constant p_1 and p_2 and so x_3 is again constant.

2) If instead $p_3 \neq 0$ (hence, $u_2 = 0$), then $\dot{u}_2 = 0$ and so (11) tells us that $u_1 = 0$. From (1), x_3 is constant.

To summarize, the only solutions to (1)–(3) for which u_1 is constant are straight line segments, which does not match the experimental data in [1]. Similarly, should [1, Fig. 7] show the numerical solution to a particular instance of (1)–(3), then this numerical solution must not be a good approximation to the optimal trajectory.

C. *Correcting the Second Mistake*

The Hamiltonian associated with (5) is

$$\begin{aligned} H(p, x, u) = \frac{1}{2}\left(1 + u_2^2\right) + p_1 \cos x_3 \\ + p_2 \sin x_3 + p_3 x_4 + p_4 u_2 \end{aligned} \tag{12}$$

where p is the costate. The maximum principle again provides the necessary conditions (7) and (8). Condition (7) implies that

$$\begin{aligned} \dot{p}_1 &= 0 \\ \dot{p}_2 &= 0 \\ \dot{p}_3 &= p_1 \sin x_3 - p_2 \cos x_3 \\ \dot{p}_4 &= -p_3 \end{aligned} \tag{13}$$

while condition (8) implies that

$$u_2 = \begin{cases} -c, & p_4 \geq c \\ -p_4, & -c < p_4 < c \\ c, & p_4 \leq -c. \end{cases} \tag{14}$$

It is clear, therefore, that solutions to (5) may exhibit piecewise constant u_2. However, assume there exists a nonempty interval $[t_1, t_2] \subset [0, T]$ on which $u_2 =$ constant and $|u_2| < c$. We must have $\dot{p}_4 = 0$ on this interval. From (13), this condition implies that $p_3 = 0$, hence $\dot{p}_3 = 0$, and so finally

$$p_1 \sin x_3 - p_2 \cos x_3 = 0. \tag{15}$$

There are countable solutions to (15), so it must also be the case that x_3 is constant along $[t_1, t_2]$, hence, $u_2 = 0$. To summarize, the only piecewise constant values of u_2 that can possibly be exhibited by solutions to (5) are $-c$, 0, and c. Therefore, should [1, Fig. 7] show the numerical solution to a particular instance of (5), then this numerical solution must not be a good approximation to the optimal trajectory.

II. Conclusion

Papers like [1] suggest that optimal control is a good framework in which to characterize human motion and that the results of this analysis have implications to planning and control of robots. This framework must be applied with care, since the error between trajectories predicted by numerical computation and trajectories observed in experiment may be insufficient to detect problems in the underlying approach. As we have seen, the predictions in [1] matched well the experimental data, despite two mistakes made in the analysis.

Acknowledgment

The authors would like to thank F. Jean and K. Mombaur for useful discussions. This note is published with the permission of A. Berthoz and H. Hicheur.

References

[1] G. Arechavaleta, J. P. Laumond, H. Hicheur, and A. Berthoz, “An optimality principle governing human walking,” *IEEE Trans. Robot.*, vol. 24, no. 1, pp. 5–14, Feb. 2008.

[2] A. Akce and T. Bretl, “Reflections on ‘An optimality principle governing human walking’,” May 2009, personal communication.

[3] L. S. Pontryagin, V. G. Boltyanskii, R. V. Gamkrelidze, and E. F. Mishchenko, *The Mathematical Theory of Optimal Processes*. New York: Wiley, 1962.

[4] C. Fernandes, L. Gurvits, and Z. Li, “Near-optimal nonholonomic motion planning for a system of coupled rigid bodies,” *IEEE Trans. Autom. Control*, vol. 39, no. 3, pp. 450–463, Mar. 1994.

[5] G. Arechavaleta and J.-P. Laumond, “A note on the shape of minimum curvature variation curves and human walking paths,” Apr. 2010, personal communication.