Proceedings of the 2007 IEEE/RSJ International
Conference on Intelligent Robots and Systems
San Diego, CA, USA, Oct 29 - Nov 2, 2007

Kinematic and Dynamic Control of a Wheeled Mobile Robot

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Abstract— **This paper considers the problem of stabilizing a unicycle-type mobile robot using a time-invariant, discontinuous control law. In order to simplify the control design, most previous approaches neglect second-order system dynamics (compensating for them later using techniques such as partial feedback linearization). This paper shows that an approach based on invariant manifold theory can be extended to account for these dynamics. The performance of the resulting control law is demonstrated in simulation.**

I. INTRODUCTION

IN THIS paper, we consider the problem of stabilizing a unicycle-type mobile robot subject to the nonholonomic constraint that it rolls without slipping. This problem has received considerable attention (for example, see [9]), and a variety of control laws have been proposed. In general, these control laws are either time-varying (such as [10]–[12], [17]) or discontinuous (such as [1]–[4], [6], [8], [13]–[15]), since it is impossible to stabilize the unicycle using a feedback control law that is both smooth and time-invariant [5].

In order to simplify the design of a control law, most previous approaches neglect second-order system dynamics. Such approaches initially assume that the unicycle is driven by system velocities rather than by wheel torques. The feedback control is designed for the resulting first-order kinematic model, in which the forward speed and turning rate are determined that will steer the unicycle to a desired state. The control law is then augmented using techniques such as partial feedback linearization (in particular, computed torque methods) to generate wheel torques.

In fact, some of these previous approaches can readily be extended to consider second-order system dynamics. We focus in particular on a time-invariant, discontinuous control law that was designed using invariant manifold theory [13]–[15]. This control law drives the state of the system into a provably invariant set, within which the system is asymptotically stable about the origin (or more generally, any desired state). We show that simply by adding two terms—one proportional to the rate of change of forward speed, the other proportional to the rate of change of turning rate—we can extend this control law to directly generate stabilizing wheel torques. Because no additional feedback linearization is required, our extension may be more robust to model uncertainty (this potential benefit is still under investigation). The resulting controller is similar to the hybrid one presented by [1]-[2], which is also time-invariant and discontinuous. But where the hybrid controller of [1]-[2] required switching between three control laws, ours requires switching between only two. Moreover, because our approach is based on invariant manifold theory, we can guarantee that our controller will switch, at most, once.

In this paper, we first describe both a first-order kinematic and a second-order dynamic model of one unicycle-type mobile robot (Section II). Then, we present both the control law from [15] that stabilizes the first-order system and our extension that stabilizes the second-order system (Section III). Finally, we compare the performance of these two approaches in simulation (Section IV).

II. A UNICYCLE-TYPE MOBILE ROBOT

Consider the unicycle-type mobile robot in Fig. 1. The state of the system is parameterized by $(x, y, \theta) \in \mathbb{R}^2 \times S^1$, where x and y are the position coordinates of the center of the rear wheel axis and θ is the angle between the center line of the vehicle (direction) and the x-axis. The velocity of the robot's center of mass is orthogonal to the rear wheels axis. We assume the robot rolls without slipping, so

$$\dot{x}\sin\theta - \dot{y}\cos\theta = 0.$$

In this section we describe both a first-order kinematic and a second-order dynamic model of this robot.

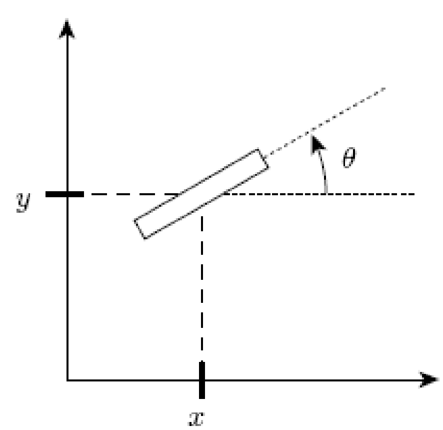

Fig. 1. Unicycle-type wheeled mobile robot

A. Model

Under the rolling without slipping constraint, the dynamical model of the wheeled mobile robot is given by

$$\dot{x} = v\cos\theta \qquad \dot{y} = v\sin\theta \qquad \dot{\theta} = \omega \tag{1a}$$

$$m\dot{v} = F \qquad I\dot{\omega} = N, \tag{1b}$$

where the model parameters are the robot mass m and the robot inertia I, and the control inputs are the pushing force F and the steering torque N. A purely kinematic model of the robot is given by (1a), where the control inputs are simply the forward velocity v and the angular velocity ω. Notice that with a suitable input transformation, a differential-drive robot could be modeled in the same way.

B. Nonholonomic integrator

By considering the coordinate transformation

$$\begin{aligned} z_1 &= \theta \\ z_2 &= x\cos\theta + y\sin\theta \\ z_3 &= x\sin\theta - y\cos\theta, \end{aligned} \tag{2}$$

with the input transformation

$$u_1 = \omega \qquad u_2 = v - z_3 u_1,$$

the robot kinematics (1a) can be expressed in power form as

$$\dot{z}_1 = u_1 \qquad \dot{z}_2 = u_2 \qquad \dot{z}_3 = z_2 u_1$$

By defining new state variables as

$$x_1 = z_1 \qquad x_2 = z_2 \qquad x_3 = -2z_3 + z_1 z_2, \tag{3}$$

the system takes the form of the nonholonomic integrator, given by

$$\begin{aligned} \dot{x}_1 &= u_1 \\ \dot{x}_2 &= u_2 \\ \dot{x}_3 &= x_1 u_2 - x_2 u_1. \end{aligned} \tag{4}$$

The nonholonomic integrator, also known as the Brockett integrator, is considered a benchmark for controller designs due to the simple form that exhibits the basic properties of nonholonomic systems. The nonholonomic integrator is a third-order driftless system, which is not linearly controllable at any equilibrium point, and no continuous control law can globally stabilize the system. Although this problem has been widely researched, the nonholonomic integrator fails to capture both the kinematics and dynamics. The complete kinematics and dynamics must be considered for most practical systems.

C. Nonholonomic double integrator

By applying the coordinate transformation (2) and input transformation

$$u_1 = \frac{N}{I} \qquad u_2 = \frac{F}{m} - \frac{N}{I}z_3 - \omega^2 z_2,$$

the robot dynamics (1) can be expressed in extended power form as

$$\ddot{z}_1 = u_1 \qquad \ddot{z}_2 = u_2 \qquad \dot{z}_3 = z_2\dot{z}_1.$$

In the new coordinates (3), the system takes the form of the extended nonholonomic double integrator [1]-[2], given by

$$\begin{aligned} \ddot{x}_1 &= u_1 \\ \ddot{x}_2 &= u_2 \\ \dot{x}_3 &= x_1\dot{x}_2 - x_2\dot{x}_1. \end{aligned} \tag{5}$$

The nonholonomic double integrator can be viewed as an extension of the nonholonomic integrator. The nonholonomic double integrator models both the system kinematics and dynamics as a fifth-order system with drift. There are three states and two (first-order) dynamic control inputs, which are the pushing force and steering torque for the wheeled mobile robot.

III. CONTROL

The control strategy follows general ideas from previous research into stabilizing both the nonholonomic integrator and double integrator. In considering the states in (4) or (5), the difficulty arises in stabilizing the system about the origin. When the states x_1 and x_2 are zero, the state x_3 will remain constant (with $\dot{x}_3 = 0$). Thus, a commonly proposed strategy is to make x_3 and $\dot{x}_3$ converge to zero while keeping the remaining two states away from the axis $x_1 = x_2 = 0$. Once x_3 and $\dot{x}_3$ converge to zero, the remaining states x_1 and x_2 are stabilized [1]–[4], [8], [14].

For both the kinematic and dynamic case, a time-invariant, discontinuous controller design is considered for stabilizing the transformed nonholonomic system, which is equivalent to stabilizing the original system (1). The invariant approach presented by Tsiotras [8, 14], which is also applied to nonholonomic chained systems in Tayebi [13], is used to develop a kinematic switching controller to stabilize the nonholonomic integrator (4). Then, the invariant approach is extended in designing a dynamic controller for the nonholonomic double integrator (5).

A. Kinematic control

The objective is to asymptotically stabilize the nonholonomic kinematic model (4) about the origin. The time-invariant, discontinuous control proposed by Tsiostras and Kim [8, 14] is considered. Let $\mathcal{M}$ be a manifold in $\mathbb{R}^3$ described by $\mathcal{M} = \{x \in \mathbb{R}^3 \mid x_3 = 0\}$. The strategy is find a control law to make $\mathcal{M}$ invariant such that x converges to the origin. If $x \notin \mathcal{M}$, the control law results in x_3 converging to the invariant manifold, $\mathcal{M}$. Once on the manifold, the remaining two states, x_1 and x_2, converge to the origin. This motivates the following proposition.

Proposition 3.1: For any $x \notin \mathcal{D}_\infty$, where the set $\mathcal{D}_\infty$ is defined as $\mathcal{D}_\infty = \{x \in \mathbb{R}^3 \mid x_1^2 + x_2^2 = 0,\ x_3 \neq 0\}$, the kinematic control law [8, 14]

$$u_1 = -k_1 x_1 + \frac{k_2 x_3}{(x_1^2 + x_2^2)}x_2 \tag{6a}$$

$$u_2 = -k_1 x_2 - \frac{k_2 x_3}{(x_1^2 + x_2^2)}x_1, \tag{6b}$$

where $k_1 > 0$ and $k_2 > 0$ are positive constants, stabilizes the nonholonomic integrator (4) about the origin.

Proof: The proof follows from Lyapunov Theory. Consider a candidate Lyapunov function $V(x) = \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2 + \frac{1}{2}x_3^2$, where $V(x) > 0$, $\forall\ x \neq 0$. The control law (6) gives $\dot{V} = -k_1 x_1^2 - k_1 x_2^2 - k_2 x_3^2 < 0$, $\forall\ x \neq 0$. Thus, the derivative is negative along the state trajectories. Since $V(x)$ is radially unbounded (i.e. $V(x) \to \infty$ as $\|x\| \to \infty$), the origin is asymptotically stable for all $x \notin \mathcal{D}_\infty$ [7]. Moreover, the control law (6) yields $\dot{x}_3 = -k_2 x_3$, with the solution $x_3(t) = x_3(0)e^{-k_2 t}$. Thus, x_3 is exponentially stable for all $x \notin \mathcal{D}_\infty$. This gives convergence to $\mathcal{M}$ and shows $\mathcal{M}$ is invariant (i.e. $\dot{\mathcal{M}} = 0$). Invariance of $\mathcal{M}$ implies that once the states enter $\mathcal{M}$, the states never leave $\mathcal{M}$ for all future time. ■

When $x \notin \mathcal{M}$, the control law is undefined for $x_1^2 + x_2^2 = 0$. To avoid this singularity, an alternative control law is used to steer away from the singularity condition. Define the function [8, 14]

$$\eta := \frac{x_3}{\sqrt{x_1^2 + x_2^2}}$$

with the set $\mathcal{D}_{\bar{\eta}} = \left\{x \in \mathbb{R}^3 \mid |\eta| \geq \bar{\eta}\right\}$, where $\bar{\eta} > 0$ is a positive constant that determines how close to the singularity the switching between the controllers occurs.

Proposition 3.2: With the condition $k_2 - k_1 > 0$, the kinematic control law (6) makes the set $\mathcal{D} = \mathbb{R}^3 / \mathcal{D}_{\bar{\eta}}$ invariant.

Proof: The proof follows from Lyapunov Theory. Consider the candidate Lyapunov function $V(\eta) = \frac{1}{2}\eta^2 > 0$, $\forall\ \eta \neq 0$. The control law (6) gives $\dot{V} = -(k_2 - k_1)\eta^2 < 0$ $\forall\ \eta \neq 0$. Since $V(\eta)$ is radially unbounded (i.e. $V(\eta) \to \infty$ as $\|\eta\| \to \infty$), $\eta = 0$ is asymptotically stable for all $\eta \in \mathcal{D}$ [7]. This implies that, if $x \in \mathcal{D}$, the states x remain in $\mathcal{D}$ for all $t > 0$. Moreover, the states converge to the origin, as shown in Proposition 3.1. Hence, $\mathcal{D}$ is invariant. ■

From Proposition 3.1, the manifold $\mathcal{M}$ is invariant and $\mathcal{M} \in \mathcal{D}$. The states converge to the invariant set $\mathcal{M}$ and remain in $\mathcal{M}$ for all t. With the value of η decreasing, the singularity condition is never met. Hence, the states asymptotically converge to zero, with no switching of the controller. If the state trajectories are in $\mathcal{D}_{\bar{\eta}}$, the controller switches to a singularity control law that drives the states out of $\mathcal{D}_{\bar{\eta}}$ to the invariant region $\mathcal{D}$.

Proposition 3.3: The singularity control law

$$u_1 = k_s x_1 + k_p \qquad \text{if } |\eta| \geq \bar{\eta} \tag{7a}$$
$$u_2 = k_s x_2 + k_p \qquad \text{if } |\eta| \geq \bar{\eta} \tag{7b}$$

where $k_s > 0$ and $k_p > 0$ are positive constants, yields the region $\mathcal{D}_{\bar{\eta}}$ unstable with finite escape for all $x \in \mathcal{D}_{\bar{\eta}}$.

Proof: The proof follows from directly solving the resulting differential equations. The singularity control law yields the state equations

$$\dot{x}_1 = k_s x_1 + k_p \qquad \dot{x}_2 = k_s x_2 + k_p \qquad \dot{x}_3 = k_p(x_1 - x_2)$$

The solutions of these equations are

$$\begin{aligned} x_1(t) &= a_1 e^{k_s t} - k_p/k_s \\ x_2(t) &= a_2 e^{k_s t} - k_p/k_s \\ x_3(t) &= b_1 e^{k_s t} + b_2, \end{aligned}$$

where $a_{1,2} = x_{1,2}(0) + k_p/k_s$, $b_1 = k_p/k_s\,(x_1(0) - x_2(0))$, and $b_2 = x_3(0) - b_1$. Using the solutions of the state equations, the singularity measure can be written as

$$\eta(t) = \frac{b_1 + b_2 e^{-k_s t}}{\sqrt{(a_1^2 + a_2^2) - 2(a_1 + a_2)\frac{k_p}{k_s}e^{-k_s t} + 2\frac{k_p^2}{k_s^2}e^{-2k_s t}}}.$$

The rate of asymptotic convergence is k_s; therefore, η exponentially converges towards $\frac{b_1}{\sqrt{a_1^2 + a_2^2}}$ as $t \to \infty$. With $\bar{\eta}$ chosen under the condition

$$\bar{\eta} > \frac{|b_1|}{\sqrt{a_1^2 + a_2^2}},$$

the system leaves the region $\mathcal{D}_{\bar{\eta}}$ in finite time. Hence, the trajectories push the states away from the singularity condition such the stabilizing controller can then be used. ■

Eqns. (6) and (7) represent the discontinuous controller for stabilizing the origin of the nonholonomic kinematic system.

B. Dynamic control

Consider the nonholonomic dynamic model as described by (5). The objective is to asymptotically stabilize the system about the origin. Let $\mathcal{M}_d$ be a manifold in $\mathbb{R}^5$ described by $\mathcal{M}_d = \left\{\rho = (x_1, x_2, x_3, \dot{x}_1, \dot{x}_2) \in \mathbb{R}^5 \mid x_3 = 0\right\}$. The strategy is find a control law to make $\mathcal{M}_d$ invariant such that ρ converges to the origin. If $\rho \notin \mathcal{M}_d$, the control law results in x_3 converging to the invariant manifold, $\mathcal{M}_d$. This yields the following proposition.

Proposition 3.4: For any $\rho \notin \mathcal{G}_\infty$, where $\mathcal{G}_\infty$ is defined as $\mathcal{G}_\infty = \left\{\rho \in \mathbb{R}^5 \mid x_1^2 + x_2^2 = 0,\ x_3 \neq 0\right\}$, the control law

$$u_1 = -k_1 x_1 - k_2 \dot{x}_1 + \frac{k_3 x_3}{(x_1^2 + x_2^2)} x_2 \tag{8a}$$
$$u_2 = -k_1 x_2 - k_2 \dot{x}_2 - \frac{k_3 x_3}{(x_1^2 + x_2^2)} x_1, \tag{8b}$$

with the conditions $k_2 > 0$, $\frac{k_2^2}{4} > k_1 > 0$, and $\frac{k_2^2}{4} > k_3 > 0$, stabilizes the nonholonomic double integrator (5) about the origin.

Proof: The proof follows from directly solving the resulting differential equations. Let $\rho \notin \mathcal{G}_\infty$. The control law (8) yields the second-order differential equation, $\ddot{x}_3 + k_2 \dot{x}_3 + k_3 x_3 = 0$. The solution is $x_3(t) = c_1 e^{\lambda_{3_1} t} + c_2 e^{\lambda_{3_2} t}$, with the constants

$$c_1 = x_3(0) - \frac{\dot{x}_3(0) - x_3(0)\lambda_{3_1}}{\sqrt{k_2^2 - 4k_3}}$$

and

$$c_2 = \frac{\dot{x}_3(0) - x_3(0)\lambda_{3_1}}{\sqrt{k_2^2 - 4k_3}}.$$

The eigenvalues are $\lambda_{3_{1,2}} = -0.5k_2 \mp 0.5\sqrt{k_2^2 - 4k_3}$. If k_3 is such that $\frac{k_2^2}{4} > k_3 > 0$, then the quantity $\sqrt{k_2^2 - 4k_3}$ satisfies $k_2 > \sqrt{k_2^2 - 4k_3} > 0$. This yields $-k_2 \mp \sqrt{k_2^2 - 4k_3} < 0$, which gives $\lambda_1 < \lambda_2 < 0$. Thus, x_3 is exponentially stable for all $\rho \notin \mathcal{G}_\infty$. If $\rho \notin \mathcal{M}_d$, this gives convergence to $\mathcal{M}_d$ and shows $\mathcal{M}_d$ is invariant (i.e. $\dot{\mathcal{M}}_d = 0$) for all $\rho \notin \mathcal{G}_\infty$. This comes from x_3 converging towards zero and remaining zero for all time.

Next, assume $\rho \in \mathcal{M}_d$. The control dynamic law (8) yields $\ddot{x}_1 + k_2\dot{x}_1 + k_1 x_1 = 0$ and $\ddot{x}_2 + k_2\dot{x}_2 + k_1 x_2 = 0$. As before, the solutions are given by $x_i(t) = c_{i_1}e^{\lambda_1 t} + c_{i_2}e^{\lambda_2 t}$, for $i = 1, 2$. The constants $c_{i_{1,2}}$ are the same as before for the initial conditions of x_i and $\dot{x}_i$. With $\frac{k_2^2}{4} > k_1 > 0$, the eigenvalues are $\lambda_{1,2} = -0.5k_2 \mp 0.5\sqrt{k_2^2 - 4k_1} < 0$. The states x_1 and x_2 exponentially converge to the origin as $t \to \infty$. Since $\mathcal{M}_d$ is invariant with $x_3 = 0$ and $\dot{x}_3 = 0$, the origin is exponentially stable for all $\rho \in \mathcal{M}_d$. ■

When $\rho \notin \mathcal{M}_d$, the control law is undefined for $x_1^2 + x_2^2 = 0$. To avoid this singularity, an alternative control law is used to steer away from the singularity condition. With using the same singularity measure from the kinematic case, consider the set $\mathcal{G}_{\bar{\eta}} = \{\rho \in R^5 \mid |\eta| \geq \bar{\eta}\}$, where $\bar{\eta} > 0$.

Proposition 3.5: The singularity control law

$$u_1 = k_{s_2}\dot{x}_1 + k_{s_1}x_1 + k_d \quad \text{if } |\eta| \geq \bar{\eta} \quad (9a)$$
$$u_2 = k_{s_2}\dot{x}_2 + k_{s_1}x_2 + k_d \quad \text{if } |\eta| \geq \bar{\eta} \quad (9b)$$

with the conditions

$$k_{s_2} > 0 \qquad k_d > 0 \qquad \frac{k_{s_2}^2}{4} > k_{s_1} > 0$$
$$\frac{k_d}{k_{s_1}} > \max_{i=1,2}\left\{\frac{|\dot{x}_i(0) - x_i(0)(\lambda_1 - 1)|}{\lambda_1 + 1}\right\}$$

yields the set $\mathcal{G}_{\bar{\eta}}$ unstable with finite escape for all $x \in \mathcal{G}_{\bar{\eta}}$.

Proof: As before, the proof follows directly from solving the resulting state (differential) equations. The singularity control law yields $\ddot{x}_1 = k_{s_2}\dot{x}_1 + k_{s_1}x_1 + k_d$, $\ddot{x}_2 = k_{s_2}\dot{x}_2 + k_{s_1}x_2 + k_d$, and $\ddot{x}_3 = k_{s_2}\dot{x}_3 + k_d(x_1 - x_2)$. Solving for x_1 and x_2 yields

$$x_1(t) = d_{1_1}e^{\lambda_1 t} + d_{1_2}e^{\lambda_2 t} - k_d/k_{s_1}$$
$$x_2(t) = d_{2_1}e^{\lambda_1 t} + d_{2_2}e^{\lambda_2 t} - k_d/k_{s_1},$$

with the constants

$$d_{i_1} = x_i(0) + k_d/k_{s_1} - \frac{\dot{x}_i(0) - x_i(0)\lambda_1 - k_d\lambda_1/k_{s_1}}{\sqrt{k_{s_2}^2 - 4k_{s_1}}}$$

and

$$d_{i_2} = \frac{\dot{x}_i(0) - x_i(0)\lambda_1 - k_d\lambda_1/k_{s_1}}{\sqrt{k_{s_2}^2 - 4k_{s_1}}}.$$

The eigenvalues are $\lambda_{1,2} = 0.5k_{s_2} \mp 0.5\sqrt{k_{s_2}^2 + 4k_{s_1}}$. If k_{s_1} satisfies $\frac{k_{s_2}^2}{4} > k_{s_1} > 0$, then $k_{s_2} > \sqrt{k_{s_2}^2 - 4k_{s_1}} > 0$. This implies that $\lambda_2 > \lambda_1 > 0$, and the conditions $2k_{s_2} > 2\lambda_2 > k_{s_2} > 0$ and $k_{s_2} > 2\lambda_1 > 0$. With the solutions of x_1 and x_2, x_3 can be written as

$$x_3(t) = \sigma_1 e^{\lambda_1 t} + \sigma_2 e^{\lambda_2 t} + \sigma_3 e^{(\lambda_1+\lambda_2)t} + \sigma_4,$$

where $\sigma_{1,2,3} \in \mathbb{R}$. The quantity σ_3 can be found to be

$$\sigma_3 = \frac{\dot{x}_3(0)}{k_{s_2}} + \frac{k_d(d_{1_1} - d_{2_1})}{k_{s_2} - \lambda_1} + \frac{k_d(d_{1_2} - d_{2_2})}{k_{s_2} - \lambda_2}.$$

The objective is to show that the singularity measure is decreasing and converging towards a constant as time increases. Hence, the singularity measure can be written as

$$\eta(t) = \frac{x_3}{\sqrt{x_1^2 + x_2^2}}\frac{e^{-k_{s_2}t}}{e^{-k_{s_2}t}} = \frac{f(t)}{\sqrt{g(t)}}$$

with $e^{-(\lambda_1+\lambda_2)t} = e^{-k_{s_2}t}$ and

$$f(t) = \sigma_1 e^{-\lambda_2 t} + \sigma_2 e^{-\lambda_1 t} + \sigma_3 + \sigma_4 e^{-k_{s_2}t}$$
$$g(t) = \alpha_1 e^{-2k_{s_2}t} + \alpha_2 e^{-(2\lambda_1+\lambda_2)t} + \alpha_3 e^{-(\lambda_1+2\lambda_2)t} + \alpha_4 e^{-2\lambda_1 t} + \alpha_5 e^{-2\lambda_2 t} + \alpha_6,$$

where $\alpha_i \in \mathbb{R}$ for $i = (1, \ldots, 6)$. Under the condition

$$\frac{k_d}{k_{s_1}} > \max_{i=1,2}\left\{\frac{|\dot{x}_i(0) - x_i(0)(\lambda_1 - 1)|}{\lambda_1 + 1}\right\},$$

the quantity α_6 can be found to be

$$\alpha_6 = 2(d_{1_1}d_{1_2} + d_{2_1}d_{2_2}) > 0$$

The rate of asymptotic convergence of η is k_{s_2}; therefore, η exponentially converges towards $\frac{\sigma_3}{\sqrt{\alpha_6}}$ as $t \to \infty$. With $\bar{\eta}$ chosen under the condition

$$\bar{\eta} > \frac{|\sigma_3|}{\sqrt{\alpha_6}},$$

the system leaves the region $\mathcal{D}_{\bar{\eta}}$ in finite time. The state trajectories leave the singularity space such the stabilizing dynamic controller may be applied. ■

Eqns. (8) and (9) represent the discontinuous controller for stabilizing the origin of both the nonholonomic double integrator and (dynamic) wheeled mobile robot.

IV. Simulation

Consider a typical "parallel parking" maneuver with the initial conditions $(x_0, y_0, \theta_0) = (0, 2, 0)$. We assume the system is noiseless and that all parameters are known exactly. Our goal is to drive the robot from its initial position (where it starts at rest) to the origin.

A. System Dynamics

The complete robot dynamical model (1) is considered in the simulation. The control inputs for the system are the steering torque N and pushing force F. The dynamic controller directly yields these inputs, but the kinematic controller only yields the forward velocity v and the angular velocity ω. So in the latter case, we must apply an additional feedback law to track v and ω. In our simulation, we use high-gain proportional feedback, as is typical for simple robots with relatively low inertia [16]. Our results show some of the drawbacks of this approach, as compared to a dynamic controller. For more complex systems (robots with high inertia, high operating speeds, significant unmodeled dynamics, or high system noise), using a dynamic controller may be even more advantageous [8].

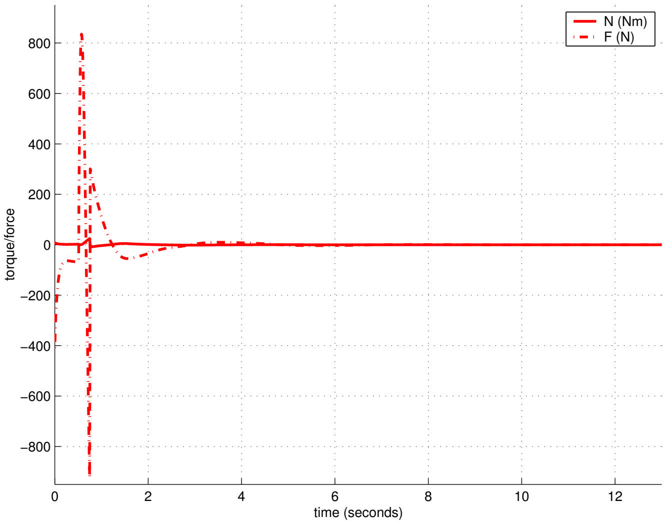

Fig. 2. Force and torque for the kinematic controller (red).

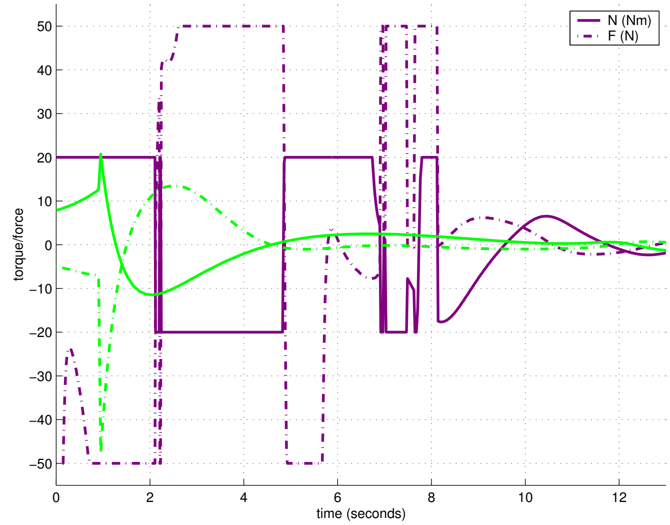

Fig. 3. Force and torque for the dynamic and constrained kinematic controller: constrained kinematic control (purple), and dynamic control (green)

B. Results

We used the system parameters and control gains shown in Table 1. The dynamic control gains were heuristically determined for the system to have a satisfactory settling time (approximately 13 seconds), while maintaining acceptable control torques. The kinematic gains were chosen to achieve a similar settling time (without constraint on the resultant control torques). The results of the simulation are shown in the figures. We chose a high proportional feedback gain to track v and ω accurately for the kinematic controller, but doing so requires very high forces and torques (see Fig. 2).

In practice, these forces and torques are often limited. Figure 3 shows the response of the kinematic controller, as compared to the dynamic one, when the control torques are limited. The resulting trajectories all three controllers (unconstrained kinematic, constrained kinematic, and dynamic) are shown in Fig. 4. The unconstrained kinematic control and dynamic control laws both have similar settling times, but the dynamic controller has much lower control torques. The robot angular velocity ω and forward velocity v are shown in Fig. 5.

C. Switching

As discussed in the previous sections, when the initial conditions do not violate the singularity condition, the system state converges to an invariant set under the invariant control law. Once on the invariant set, no switching will occur. If the initial conditions do violate the singularity conditions, the singularity control drives the states out of the singularity condition so the invariant control law can then stabilize the system.

Table 1.

Kinematic				Dynamic		
k_1	k_2	k_s	k_p	k_1	k_2	k_3
0.6	0.9	2	0.9	0.25	0.75	0.25
m = 10 kg		I = 15 kg m^2		k_{s_1}	k_{s_2}	k_d
feedback gain = 200				.3	.4	0.5

Hence, the most each controller will switch is only once, when the initial conditions violate the singularity condition. Note that switching in the control law does not necessarily correspond to "cusps" in the robot trajectory.

When the force and torque inputs are constrained, the system can not completely track the reference velocities given by the kinematic controller. This implies that the system may not remain on the invariant set, which results in switching from the invariant controller to the singularity controller. In fact, the system may switch more than once, degrading the overall performance. Figure (6) shows the switching curve for each controller. The invariant controller is marked as "1" and the singularity controller as "0". Since the control torques are not limited for the unconstrained kinematic controller, and since the control torques do not reach their limits for the dynamic controller, both of these controllers switch only once. The torque-constrained kinematic controller switches multiple times. This yields poor performance compared to the unconstrained case, as shown in Fig. (4). The system converges, but with a longer settling time (due to the states having to move back to the invariant set after each switch).

V. Conclusion

The problem of stabilizing a mobile wheeled robot was considered. The control strategy was to reach an invariant set, from which the system states are asymptotically stable. A stabilizing kinematic controller was extended to the dynamic case. The dynamic controller was shown to stabilize the system, while simulations showed good performance with manageable force and torque inputs. Increasing the velocity feedback gain improved the response of the kinematic controller, but yielded larger control torques, which may be impractical for some systems. Future work might extend our approach to include noise cancellation and adaptive control.

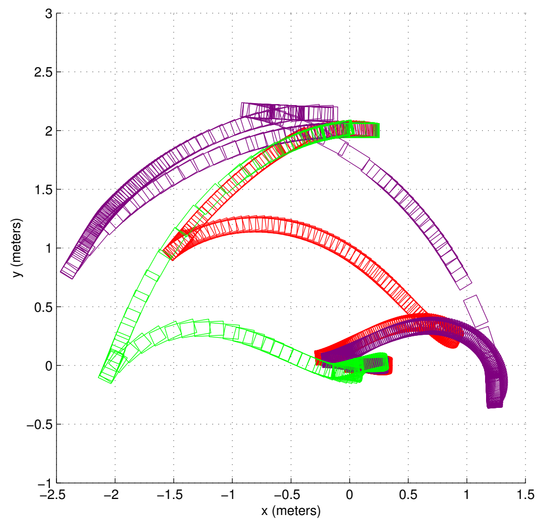

Fig. 4. Robot (x,y)-trajectory: kinematic control (red), constrained kinematic control (purple), and dynamic control (green)

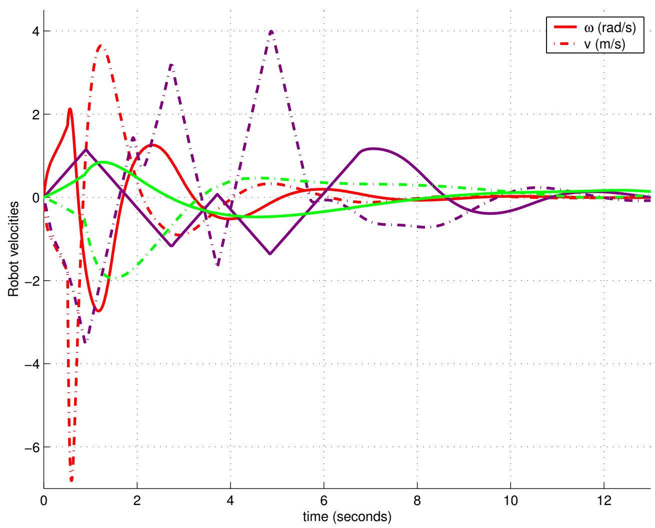

Fig. 5. Steering rate ω and forward velocity v: kinematic control (red), constrained kinematic control (purple), and dynamic control (green)

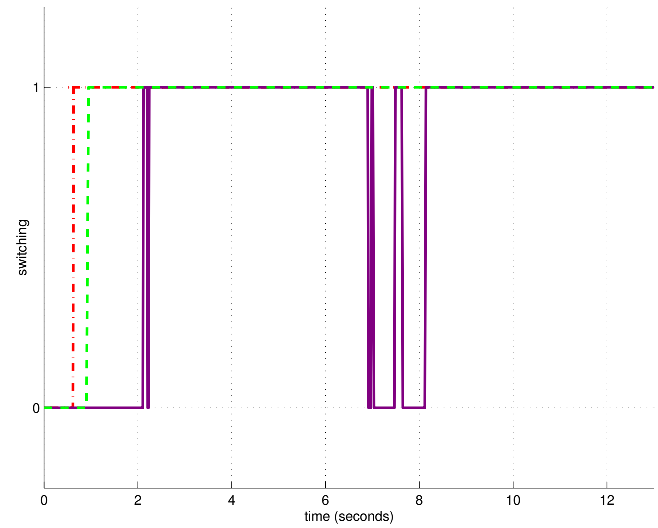

Fig. 6. Switching between controllers: kinematic control (red), constrained kinematic control (purple), and dynamic control (green)

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