Helices, relative equilibria, and optimality on the special Euclidean group

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***Abstract*— We consider a left-invariant optimal control problem on the six-dimensional special Euclidean group. Helical trajectories are extremals of the optimal control problem, and we derive an explicit parameterization of these helices. Using this parameterization of the helical extremals, we compute the relative equilibria of the Hamiltonian system associated with the optimal control problem. For a particular choice of system parameters, we use Jacobi's sufficient condition to determine which of the relative equilibria correspond to local optima of the optimal control problem. We show that the optimality of a relative equilibrium is completely determined by the curvature and torsion of the helix that the trajectory traces.**

I. Introduction

In this paper, we consider an optimal control problem with historical roots tracing back to Kirchhoff's analysis of elastic wires. Kirchhoff considered the problem of finding equilibrium shapes of a thin inextensible wire that minimizes elastic potential energy [1]. This same problem can be formulated as an optimal control problem for an oriented vehicle on the special Euclidean group $SE(3)$ [2]. In both the elastic wire problem and the optimal control problem, helices are stationary solutions of the objective function, i.e., helices are equilibrium shapes of the elastic wire and helical trajectories satisfy the necessary conditions for the optimal control problem. The analysis in this paper extends previous characterizations of these helical extremals. Furthermore, for a special class of helices, we provide a characterization of the helical trajectories that are local optima, and not just extrema, of the optimal control problem.

In the optimal control problem that we consider, the cost function is a weighted sum of the turning rates of an oriented vehicle traveling at a constant speed. The optimal control problem is left-invariant, and we can therefore derive a reduced Hamiltonian system that governs the costate trajectory of the optimal control problem. We initially consider a cost function in which two of the three turning rates of the vehicle are given the same weight. As mentioned earlier, helical trajectories of the oriented vehicle are extremals of this optimal control problem. These helices correspond to a particular set of solutions of the reduced Hamiltonian system, and these solutions were previously shown to correspond to the roots of a cubic polynomial [2]. In this paper, we give an explicit parameterization of these helices. Our analysis of the helical extremals also leads to an explicit parameterization of the relative equilibria of the Hamiltonian system associated with the optimal control problem.

These relative equilibria correspond to extremals of the optimal control problem, but they do not necessarily correspond to local optima. We consider the question of optimality in the special case when all three turning rates of the vehicle are weighted equally in the cost function. For this choice of system parameters, we determine which of the relative equilibria correspond to locally optimal solutions of the optimal control problem. We do this by applying Jacobi's sufficient condition for optimality and computing conjugate points along the helices corresponding to the relative equilibria. The sufficient conditions we use are particularly suited for analyzing relative equilibria because they only require solving a linear time-invariant system, whereas Jacobi's condition generally requires solving a linear time-varying system. Our results show that the optimality of a helical trajectory corresponding to a relative equilibrium is completely determined by the curvature and torsion of the helical trajectory.

Previous work in the optimal control literature has mainly focused on the necessary conditions for optimality for the problem that we consider, and on showing that helices satisfy these conditions. As already discussed, Biggs et al. showed that helical extremals of this optimal control problem correspond to the roots of a cubic polynomial [2]. They later used these helical trajectories in path planning problems for rigid bodies and autonomous underwater vehicles [3], [4]. Walsh et al. studied the same optimal control problem in the context of optimally landing an airplane, and gave particular attention to the helical solutions [5]. Justh and Krishnaprasad considered similar optimal control problems on the higher dimensional special Euclidean groups $SE(n)$ [6]. Necessary conditions for the optimal control problem in this paper were also analyzed by Jurdjevic [7].

As discussed earlier, local solutions of the optimal control problem that we consider are stable equilibrium configurations of a thin inextensible elastic wire. Previous work from the mechanics literature can therefore be applied to the problem in this paper. Helical equilibria of an elastic wire were analyzed by Chouaieb et al. using a noncanonical Hamiltonian formulation [1], [8]. Shi and Hearst used helical configurations of an elastic wire to study supercoiling of DNA [9]. Stability results for helical elastic wires were derived by Goriely and Tabor [10] using a dynamic model of the elastic wire. Majumdar and Raisch [11] also derived stability bounds for helices based on a direct analysis of the

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second variation of the elastic potential energy.

In Section II, we state the optimal control problem on $SE(3)$, and we give necessary and sufficient conditions for optimality for the problem. In Section III, we compute the relative equilibria of the optimal control problem, and along the way, we derive an explicit parameterization of the helical extremals of the problem. In Section IV, we determine which relative equilibria correspond to locally optimal solutions of the optimal control problem for a particular choice of system parameters. Concluding remarks concerning generalizations of the results in this paper are given in Section V.

II. The optimal control problem

In this section, we state an optimal control problem on the matrix Lie group $SE(3)$, and we give necessary and sufficient optimality conditions for the problem. Let $c > 0$ be a constant, and consider the optimal control problem

$$\begin{aligned} \underset{q,u}{\text{minimize}} \quad & \frac{1}{2}\int_0^1 cu_1^2 + u_2^2 + u_3^2 \; dt \\ \text{subject to} \quad & \dot{q} = q\xi(u) \\ & q(0) = I, \quad q(1) = q_1, \end{aligned} \tag{1}$$

where $u \colon [0,1] \to \mathbb{R}^3$ is the control input, $q \colon [0,1] \to SE(3)$ is the system state and is given by

$$q(t) = \begin{bmatrix} R(t) & r(t) \\ 0 & 1 \end{bmatrix}, \tag{2}$$

where $R(t) \in SO(3)$ and $r(t) \in \mathbb{R}^3$, and $q_1 \in SE(3)$ is fixed. The function $\xi \colon \mathbb{R}^3 \to \mathfrak{se}(3)$ is defined by

$$\xi(u) = \begin{bmatrix} \widehat{u} & v \\ 0 & 0 \end{bmatrix},$$

where $v^T = [1 \; 0 \; 0]$, and the function $\widehat{\cdot}$ is defined by $\widehat{a}b = a \times b$ for all $a, b \in \mathbb{R}^3$. The optimal control problem (1) is left-invariant, so there is no loss of generality by assuming $q(0) = I$ [12]. Further, there is no loss of generality by setting the final time to be 1, since a problem with a different final time can be non-dimensionalized to have the form of (1).

In Section II-A, we state necessary conditions for the problem (1) based on the Pontryagin maximum principle. In Section II-B, Jacobi's condition is used to state sufficient conditions for the problem (1).

A. Necessary conditions for optimality

Necessary conditions for (q, u) to be a local optimum of the problem (1) are derived in Theorem 5 of [12] using the Pontryagin maximum principle [13], and we summarize them here. Extremal controls are given by

$$u_1 = \frac{1}{c}\mu_1 \qquad u_2 = \mu_2 \qquad u_3 = \mu_3, \tag{3}$$

where μ is a solution of system

$$\begin{aligned} \dot{\mu}_1 &= 0 & \dot{\mu}_4 &= \mu_3\mu_5 - \mu_2\mu_6 \\ \dot{\mu}_2 &= k\mu_1\mu_3 + \mu_6 & \dot{\mu}_5 &= c^{-1}\mu_1\mu_6 - \mu_3\mu_4 \\ \dot{\mu}_3 &= -k\mu_1\mu_2 - \mu_5 & \dot{\mu}_6 &= \mu_2\mu_4 - c^{-1}\mu_1\mu_5, \end{aligned} \tag{4}$$

where $k = (c^{-1} - 1)$. The system (4) is obtained by applying Lie-Poisson reduction to the Hamiltonian system associated with the optimal control problem (1), and the function μ is the costate trajectory for the problem (1), after left-translation to the identity element in $SE(3)$ [14].

A solution μ of (4) and the corresponding control u defined in (3) are normal when the initial condition $\mu(0)$ for the system (4) is in the set

$$\mathcal{A} = \{a \in \mathbb{R}^n \colon (a_2, a_3, a_5, a_6) \neq (0,0,0,0)\} \tag{5}$$

and are abnormal otherwise. Relative equilibria of the Hamiltonian system associated with the optimal control problem (1) are fixed points of the system (4).

B. Sufficient conditions for optimality

If u is a normal control input, we can determine if (q, u) is a local optimum of (1) by applying Jacobi's condition, i.e., testing for the absence of conjugate points [15]. Theorem 7 of [12] provides a method for computing conjugate points in the problem (1), which we now summarize.

Let μ be a normal solution of (4), let u be the control defined in (3), and let q be the solution of $\dot{q} = q\xi(u)$ with the initial condition $q(0) = I$. Let $\mathbf{F}$ be the linearization of the system (4), i.e., $\mathbf{F}$ is the time-varying matrix

$$\mathbf{F} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ k\mu_3 & 0 & k\mu_1 & 0 & 0 & 1 \\ -k\mu_2 & -k\mu_1 & 0 & 0 & -1 & 0 \\ 0 & -\mu_6 & \mu_5 & 0 & \mu_3 & -\mu_2 \\ c^{-1}\mu_6 & 0 & -\mu_4 & -\mu_3 & 0 & c^{-1}\mu_1 \\ -c^{-1}\mu_5 & \mu_4 & 0 & \mu_2 & -c^{-1}\mu_1 & 0 \end{bmatrix}.$$

Define the matrix $\mathbf{G}$ and the time-varying matrix $\mathbf{H}$ by

$$\mathbf{G} = \operatorname{diag}(c^{-1}, 1, 1, 0, 0, 0) \qquad \mathbf{H} = \begin{bmatrix} -\widehat{u} & 0 \\ -\widehat{v} & -\widehat{u} \end{bmatrix}.$$

Now solve the system of linear time-varying matrix differential equations

$$\dot{\mathbf{M}} = \mathbf{F}\mathbf{M} \qquad \dot{\mathbf{J}} = \mathbf{H}\mathbf{J} + \mathbf{G}\mathbf{M} \tag{6}$$

with the initial conditions $\mathbf{M} = I$ and $\mathbf{J} = 0$. The matrix $\mathbf{J}(t)$ describes the first-order change in $q(t)$ with respect to a change in $\mu(0)$. When a nonzero change in $\mu(0)$ produces no change (to first order) in $q(t)$ for some $t \in (0, 1]$, t is called a conjugate time. The solution (q, u) is a local optimum of (1) for the boundary condition $q_1 = q(1)$ if it contains no conjugate times, i.e., if $\det(\mathbf{J}(t)) \neq 0$ for all $t \in (0, 1]$.

The sufficient condition outlined above is particularly useful for evaluating relative equilibria, since the matrices $\mathbf{F}$, $\mathbf{G}$, and $\mathbf{H}$ do not depend on the state q of the optimal control problem. At a relative equilibrium, these matrices are constant, and finding the matrix $\mathbf{J}$ only requires solving a linear time-invariant system.

III. Computation of the Relative Equilibria

In this section, we compute the fixed points of the system (4). We begin in Section III-A by analyzing the solutions of (4) that produce helical trajectories in $SE(3)$. In Section III-B, we compute the initial conditions for the system (4) that produce these helices. In Section III-C, we determine which of these initial conditions are fixed points of the system (4). In Section III-D, we compute the state trajectories corresponding to the fixed points of the system (4).

A. *Curves with constant curvature and torsion*

In this section, we analyze the solutions of (4) that produce helical extremals of the optimal control problem (1). We begin by defining the curvature and torsion of a solution of the differential equation $\dot{q} = q\xi(u)$, where u satisfies (3) for some normal solution of the system (4). Following the analysis in [2], the curvature κ and torsion τ can be defined in terms of the costate μ by

$$\kappa^2 = \mu_2^2 + \mu_3^2 \qquad \tau = \mu_1 - \frac{\mu_2\mu_5 + \mu_3\mu_6}{\kappa^2}. \tag{7}$$

Trajectories in $SE(3)$ that have constant curvature and torsion are helices. To find these helical trajectories, we must compute the solutions of (4) with constant curvature and torsion. Taking the time derivative of κ^2 and simplifying using (4) gives

$$2\kappa\dot{\kappa} = -\dot{\mu}_4.$$

The curvature κ is therefore constant if and only if μ_4 is constant. Taking the time derivative of τ and simplifying using (4) and (7) gives

$$\dot{\tau} = \dot{\mu}_4 \frac{2\tau - \mu_1}{\kappa^2}.$$

We see that the torsion is also constant if μ_4 is constant. We therefore see that the curve traced by the trajectory q in $SE(3)$ is a helix if and only if μ_4 is constant, as was previously shown in [2].

Now suppose μ is a normal solution of (4) such that μ_4 is constant. Denote the initial condition for μ by $\mu(0) = a$ so that the components of μ satisfy $\mu_1(t) = a_1$ and $\mu_4(t) = a_4$ for all $t \in [0, 1]$. Since κ is constant, we have

$$\mu_2 = \kappa\cos\phi \qquad \mu_3 = \kappa\sin\phi$$

for some function $\phi\colon [0,1] \to [0, 2\pi)$. We also have that

$$\phi = \arctan\left(\frac{\mu_3}{\mu_2}\right)$$

and

$$\dot{\phi} = \frac{\mu_2\dot{\mu}_3 - \mu_3\dot{\mu}_2}{\kappa^2}.$$

Simplifying this expression using (4), (7), and $\mu_1 = a_1$ gives

$$\dot{\phi} = \tau - c^{-1}a_1,$$

which is a constant that we will denote by $\gamma = \tau - c^{-1}a_1$. The function ϕ is therefore given by

$$\phi(t) = \gamma t + \phi_0$$

for some $\phi_0 \in [0, 2\pi)$. The expressions for μ_5 and μ_6 are found from (4) to be

$$\mu_5 = -k\mu_1\mu_2 - \dot{\mu}_3 = \mu_2(a_1 - \tau)$$
$$\mu_6 = -k\mu_1\mu_3 + \dot{\mu}_2 = \mu_3(a_1 - \tau).$$

Summarizing the results in this section, solutions of (4) with constant μ_4 have the form

$$\begin{aligned} \mu_1 &= a_1 & \mu_4 &= a_4 \\ \mu_2 &= \kappa\cos(\gamma t + \phi_0) & \mu_5 &= \mu_2(a_1 - \tau) \\ \mu_3 &= \kappa\sin(\gamma t + \phi_0) & \mu_6 &= \mu_3(a_1 - \tau), \end{aligned} \tag{8}$$

where κ, τ, γ, and ϕ_0 are determined from the initial condition $\mu(0) = a$.

B. *Initial conditions corresponding to helices*

In the previous section, we analyzed the helical extremals of the problem (1) by assuming that we know a solution of (4) with constant μ_4. We now find the initial conditions $\mu(0)$ that produce solutions of (4) with constant μ_4. In [2], it was shown that these solutions correspond to the roots of a cubic polynomial. Below, we give an explicit parameterization of these solutions.

We begin by taking the time derivative of $\dot{\mu}_4$ and simplifying using (4) and (7), which gives

$$\ddot{\mu}_4 = \kappa^2\left(\mu_1^2 - \mu_1\tau - \mu_4\right) - \mu_5^2 - \mu_6^2. \tag{9}$$

Taking another time derivative and simplifying using (4) and (7) gives

$$\dddot{\mu}_4 = -\left(\mu_1^2 + \kappa^2 - 4\mu_4\right)\dot{\mu}_4. \tag{10}$$

For μ_4 to be constant, we need the solution of (10) to satisfy $\dot{\mu}_4 = 0$, which only happens when $\dot{\mu}_4(0) = \ddot{\mu}_4(0) = 0$. Solutions of (4) with constant μ_4 therefore correspond to choosing initial conditions $\mu(0) = a$ that satisfy

$$\dot{\mu}_4(0) = a_3a_5 - a_2a_6 = 0 \tag{11}$$

and

$$\ddot{\mu}_4(0) = a_1\left(a_2a_5 + a_3a_6\right) - a_4\left(a_2^2 + a_3^2\right) - a_5^2 - a_6^2 = 0,$$

where we computed $\ddot{\mu}_4(0)$ by expanding (9) using (7).

We now determine the initial conditions $\mu(0)$ of the system (4) that satisfy the two conditions $\dot{\mu}_4(0) = \ddot{\mu}_4(0) = 0$. Suppose μ is a normal solution of (4) that produces a helical trajectory q in $SE(3)$ with curvature κ and torsion τ. From (7), we have that $a = \mu(0)$ satisfies

$$a_2 = \kappa\cos\phi_0 \qquad a_3 = \kappa\sin\phi_0 \tag{12}$$

for some $\phi_0 \in [0, 2\pi)$. From (8), we must have

$$a_5 = a_2(a_1 - \tau) \qquad a_6 = a_3(a_1 - \tau) \tag{13}$$

Using (12)-(13), it is easy to see from (11) that $\dot{\mu}_4(0) = 0$.

Using the expressions (13) for a_5 and a_6 in the condition $\ddot{\mu}_4(0) = 0$ gives

$$a_4 = \tau(a_1 - \tau),$$

Summarizing the results in this section, suppose $\kappa > 0$, $\tau \in \mathbb{R}$, $a_1 \in \mathbb{R}$, and $\phi_0 \in [0, 2\pi)$ are given. Define

$$\begin{gathered} a_2 = \kappa\cos\phi_0 \qquad a_3 = \kappa\sin\phi_0 \qquad a_4 = \tau(a_1 - \tau) \\ a_5 = a_2(a_1 - \tau) \qquad a_6 = a_3(a_1 - \tau). \end{gathered} \tag{14}$$

Then the solution of (4) with $\mu(0) = a$ is given by (8), the control u defined in (3) is normal, and the resulting trajectory q in $SE(3)$ is a helix with curvature κ and torsion τ.

All helical extremals of the problem (1) correspond to solutions of (4) with initial conditions of the form (14). The expressions in (14) therefore give an explicit parameterization of the helical extremals in terms of the parameters κ, τ, a_1, and ϕ_0. The implicit parameterization described in [2], which is given in terms of the roots of a cubic polynomial, is equivalent to the explicit parameterization given above, in the sense that both parameterizations describe all the possible helical extremals of the problem (1). Further, the explicit parameterization in (14) complements similar characterizations of helices in the mechanics literature (cf. Lemma 4.1 in [1]).

C. Fixed points

We now compute the relative equilibria of the Hamiltonian system associated with the optimal control problem (1), i.e., the fixed points of the system (4). These fixed points correspond to a subset of the helical trajectories that we found in the previous section. In particular, fixed points are solutions of the form (8) with $\gamma = 0$. The condition $\gamma = 0$ gives $a_1 = c\tau$. Using $a_1 = c\tau$ in (14) gives

$$\begin{gathered} a_2 = \kappa\cos\phi_0 \qquad a_3 = \kappa\sin\phi_0 \qquad a_4 = \tau^2(c-1) \\ a_5 = \tau a_2(c-1) \qquad a_6 = \tau a_3(c-1). \end{gathered}$$

We now have an explicit parameterization of the fixed points of the system (4) in terms of the parameters κ, τ, and ϕ_0, given by

$$\begin{aligned} \mu_1 &= c\tau & \mu_4 &= \tau^2(c-1) \\ \mu_2 &= \kappa\cos\phi_0 & \mu_5 &= \kappa\tau(c-1)\cos\phi_0 \\ \mu_3 &= \kappa\sin\phi_0 & \mu_6 &= \kappa\tau(c-1)\sin\phi_0. \end{aligned} \tag{15}$$

D. Integrating the state equations

We now compute the state trajectory q corresponding to a relative equilibrium. At the fixed points (15), the control input u is constant, and we can easily integrate the state equation $\dot{q} = q\xi(u)$. The state equation can be decomposed using (2) to give

$$\dot{R} = R\widehat{u} \qquad \dot{r} = Rv \tag{16}$$

The solution of the system (16) is

$$R(t) = I + \frac{\widehat{u}}{\eta}\sin\eta t + \frac{\widehat{u}^2}{\eta^2}(1 - \cos\eta t) \tag{17}$$

$$r(t) = \begin{bmatrix} \frac{\tau^2}{\eta^2}t + \frac{\kappa^2}{\eta^3}\sin\eta t \\ \frac{\tau u_2}{\eta^2}\left(t - \frac{1}{\eta}\sin\eta t\right) - \frac{u_3}{\eta^2}\left(\cos\eta t - 1\right) \\ \frac{\tau u_3}{\eta^2}\left(t - \frac{1}{\eta}\sin\eta t\right) + \frac{u_2}{\eta^2}\left(\cos\eta t - 1\right) \end{bmatrix}$$

where $\eta = ||u|| = \sqrt{\tau^2 + \kappa^2}$.

IV. Optimality of the relative equilibria

In this section, we determine which of the relative equilibria defined in (15) correspond to local optima of the problem (1) in the case when $c = 1$. In this case, the fixed points of (4) are given by

$$\begin{aligned} \mu_1 &= \tau & \mu_4 &= 0 \\ \mu_2 &= \kappa\cos\phi_0 & \mu_5 &= 0 \\ \mu_3 &= \kappa\sin\phi_0 & \mu_6 &= 0, \end{aligned} \tag{18}$$

for some choice of $\kappa > 0$, $\tau \in \mathbb{R}$, and $\phi_0 \in [0, 2\pi)$.

We begin in Section IV-A by integrating the system of matrix differential equations (6). Then, in Section IV-B, we find the determinant of the matrix $\mathbf{J}$, which was defined in Section II-B. In Section IV-C, we determine the optimality properties of the fixed points (18).

A. Integrating the sufficient conditions

In this section, we solve the linear matrix differential equations in (6). With μ given by (18), the matrices $\mathbf{F}$, $\mathbf{G}$, and $\mathbf{H}$ defined in II-B now have the form

$$\mathbf{F} = \begin{bmatrix} 0 & -\widehat{v} \\ 0 & -\widehat{u} \end{bmatrix} \quad \mathbf{G} = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \quad \mathbf{H} = \begin{bmatrix} -\widehat{u} & 0 \\ -\widehat{v} & -\widehat{u} \end{bmatrix}$$

We can decompose the 6×6 matrices $\mathbf{M}$ and $\mathbf{J}$ into 3×3 blocks according to

$$\mathbf{M} = \begin{bmatrix} \mathbf{M}_{11} & \mathbf{M}_{12} \\ \mathbf{M}_{21} & \mathbf{M}_{22} \end{bmatrix} \qquad \mathbf{J} = \begin{bmatrix} \mathbf{J}_{11} & \mathbf{J}_{12} \\ \mathbf{J}_{21} & \mathbf{J}_{22} \end{bmatrix}.$$

The matrix differential equation $\dot{\mathbf{M}} = \mathbf{FM}$, which is now time-invariant since u is constant, can be written as

$$\begin{aligned} \dot{\mathbf{M}}_{11} &= -\widehat{v}\mathbf{M}_{21} & \dot{\mathbf{M}}_{12} &= -\widehat{v}\mathbf{M}_{22} \\ \dot{\mathbf{M}}_{21} &= -\widehat{u}\mathbf{M}_{21} & \dot{\mathbf{M}}_{22} &= -\widehat{u}\mathbf{M}_{22} \end{aligned}$$

with the initial conditions $\mathbf{M}_{11} = \mathbf{M}_{22} = I$ and $\mathbf{M}_{12} = \mathbf{M}_{21} = 0$.

We can immediately conclude that

$$\mathbf{M}_{21} = 0 \qquad \mathbf{M}_{11} = I \tag{19}$$

Next, we note that

$$\dot{\mathbf{M}}_{22}^T = \mathbf{M}_{22}^T\widehat{u}$$

This equation is identical to the differential equation we solved to find $R(t)$ in Section III-D, and we therefore have

$$\mathbf{M}_{22}(t) = R^T(t)$$

We then have

$$\mathbf{M}_{12}(t) = -\widehat{v}\int_0^t R^T(s)\,ds \tag{20}$$

Next, the matrix differential equation $\dot{\mathbf{J}} = \mathbf{HJ} + \mathbf{GM}$, which is time-invariant, can be written as

$$\begin{aligned} \dot{\mathbf{J}}_{11} &= -\widehat{u}\mathbf{J}_{11} + \mathbf{M}_{11} & \dot{\mathbf{J}}_{12} &= -\widehat{u}\mathbf{J}_{12} + \mathbf{M}_{12} \\ \dot{\mathbf{J}}_{21} &= -\widehat{u}\mathbf{J}_{21} - \widehat{v}\mathbf{J}_{11} & \dot{\mathbf{J}}_{22} &= -\widehat{u}\mathbf{J}_{22} - \widehat{v}\mathbf{J}_{12} \end{aligned} \tag{21}$$

with the initial conditions $\mathbf{J}_{11} = \mathbf{J}_{12} = \mathbf{J}_{21} = \mathbf{J}_{22} = 0$.

As was the case for $\mathbf{M}_{22}$, the homogeneous solution for each of the four differential equations in (21) is R^T. Using (19), we can now solve for $\mathbf{J}_{11}$ to get

$$\begin{aligned}\mathbf{J}_{11}(t) &= R^T(t)\int_0^t R(x)\mathbf{M}_{11}(x)\,dx \\ &= R^T(t)\int_0^t R(x)\,dx\end{aligned} \tag{22}$$

Next, using (20), we can solve for $\mathbf{J}_{12}$ to get

$$\begin{aligned}\mathbf{J}_{12}(t) &= R^T(t)\int_0^t R(x)\mathbf{M}_{12}(x)\,dx \\ &= -R^T(t)\int_0^t R(x)\,\widehat{v}\int_0^x R^T(y)\,dy\,dx\end{aligned} \tag{23}$$

Using (22), $\mathbf{J}_{21}$ is found to be

$$\begin{aligned}\mathbf{J}_{21}(t) &= -R^T(t)\int_0^t R(x)\,\widehat{v}\,\mathbf{J}_{11}(x)\,dx \\ &= -R^T(t)\int_0^t R(x)\,\widehat{v}\,R^T(x)\int_0^x R(y)\,dy\,dx\end{aligned} \tag{24}$$

Finally, using (23), $\mathbf{J}_{22}$ is found to be

$$\begin{aligned}\mathbf{J}_{22}(t) &= -R^T(t)\int_0^t R(x)\,\widehat{v}\,\mathbf{J}_{12}(x)\,dx \\ &= R^T(t)\int_0^t R(x)\,\widehat{v}\,R^T(x)\ldots \\ &\qquad \int_0^x R(y)\widehat{v}\int_0^y R^T(z)\,dz\,dy\,dx\end{aligned} \tag{25}$$

We have an explicit expression for $R(t)$, given in (17). We can therefore evaluate the integrals in (22)-(25) to find the elements of the matrix $\mathbf{J}$.

B. Evaluation of the determinant

We now use the expressions in (22)-(25) to compute the determinant of the matrix $\mathbf{J}$, which we will use in the next section to evaluate the optimality of the fixed points in (18). We begin by noting that a helix transitions from being locally optimal to being non-optimal when a conjugate point appears at $t = 1$, since conjugate points cannot appear within the time interval $(0, 1)$ without passing through $t = 1$ (see Corollary 2.2 in [16]). Since we are interested in finding the boundary between optimal and non-optimal helices, we will evaluate the determinant of the matrix $\mathbf{J}(t)$ at $t = 1$.

The determinant of the matrix $\mathbf{J}(1)$ can be written as a function of κ and $\eta = \sqrt{\tau^2+\kappa^2}$, and has the form

$$\Lambda(\kappa,\eta) = \frac{\kappa^2}{16\eta^{13}}\left(A(\eta)\kappa^4 + B(\eta)\kappa^2 + C(\eta)\right), \tag{26}$$

where

$$\begin{aligned}A(\eta) = \frac{1}{\eta^3}\big(&8\eta^4 + 48\eta^2 - 12 \\ &+ 2\left(8\eta^4 - 20\eta^2 + 3\right)\cos\eta \\ &+ \eta\left(4\eta^4 - 60\eta^2 + 19\right)\sin\eta \\ &- 4\left(2\eta^2 - 3\right)\cos 2\eta - 8\eta\sin 2\eta \\ &- 6\cos 3\eta - \eta\sin 3\eta\big)\end{aligned}$$

$$\begin{aligned}B(\eta) = \frac{8}{\eta^2}\big(&-2\eta^5 - 17\eta^3 + 6\eta \\ &- \eta\left(5\eta^4 - 16\eta^2 + 8\right)\cos\eta \\ &- \left(\eta^6 - 17\eta^4 + 4\eta^2 - 5\right)\sin\eta \\ &+ \eta\left(\eta^2 + 2\right)\cos 2\eta + 2\left(\eta^2 - 2\right)\sin 2\eta + \sin 3\eta\big) \\ C(\eta) = 4\eta\big(&2\eta^4 + 24\eta^2 - 24 + 2\left(3\eta^4 - 12\eta^2 + 16\right)\cos\eta \\ &+ \eta\left(\eta^4 - 20\eta^2 + 8\right)\sin\eta - 8\cos 2\eta - 4\eta\sin 2\eta\big).\end{aligned}$$

The leading coefficient in (26) is always positive. The determinant $\Lambda(\kappa,\eta)$ therefore vanishes when the fourth order polynomial with coefficients $A(\eta)$, $B(\eta)$, and $C(\eta)$ is zero.

While the fixed points in (18) depend upon the parameters τ, κ, and ϕ_0, the determinant $\Lambda(\kappa,\eta)$ only depends upon τ and κ, and is independent of ϕ_0. We can therefore visualize the boundary between locally optimal and non-optimal solutions by partitioning the half-plane (τ,κ) with $\kappa > 0$ into optimal and non-optimal regions, as shown in the next section.

C. Optimality of the fixed points

We now determine which fixed points in (18) correspond to locally optimal solutions of the optimal control problem (1). The boundary between optimal and non-optimal points in the (τ,κ) half-plane corresponds to points where $\Lambda(\kappa,\eta) = 0$. When $\Lambda(\kappa,\eta)$ vanishes, we have

$$\kappa^2 = \frac{-B(\eta) \pm \sqrt{B(\eta)^2 - 4A(\eta)C(\eta)}}{2A(\eta)} \tag{27}$$

The right-hand side of (27) depends only on $\eta = \sqrt{\tau^2+\kappa^2}$. Therefore, for each $\eta > 0$, we can compute κ using (27) (if the right-hand side of (27) is real and positive), and then compute τ from

$$\begin{aligned}\tau^2 &= \eta^2 - \kappa^2 \\ &= \eta^2 - \frac{-B(\eta) \pm \sqrt{B(\eta)^2 - 4A(\eta)C(\eta)}}{2A(\eta)}\end{aligned} \tag{28}$$

The expressions (27) and (28) give a parameterization in terms of the parameter $\eta > 0$ of the points in the (τ,κ) half-plane that satisfy $\Lambda(\kappa,\eta) = 0$.

A fixed value of η corresponds to a semicircle in the (τ,κ) half-plane. For each solution κ of (27), there are at most two choices of τ that satisfy (28). Since (27) can have at most two positive solutions, we conclude that each semicircle in the (τ,κ) half-plane intersects the curve satisfying $\Lambda(\kappa,\eta) = 0$ in at most four places.

The boundary between optimal and non-optimal points in the (τ,κ) half-plane is plotted in Figure 1. The equation $\Lambda(\kappa,\eta) = 0$ has multiple disconnected sets of solutions, and these are shown by the black curves in Figure 1. Moving outward from the origin, each of these curves corresponds to an increase in the number of conjugate points along the resulting helix. The green region corresponds to helices with no conjugate points, and therefore the green region corresponds to local optima of the problem (1) with $c = 1$. The blue region corresponds to helices with one conjugate

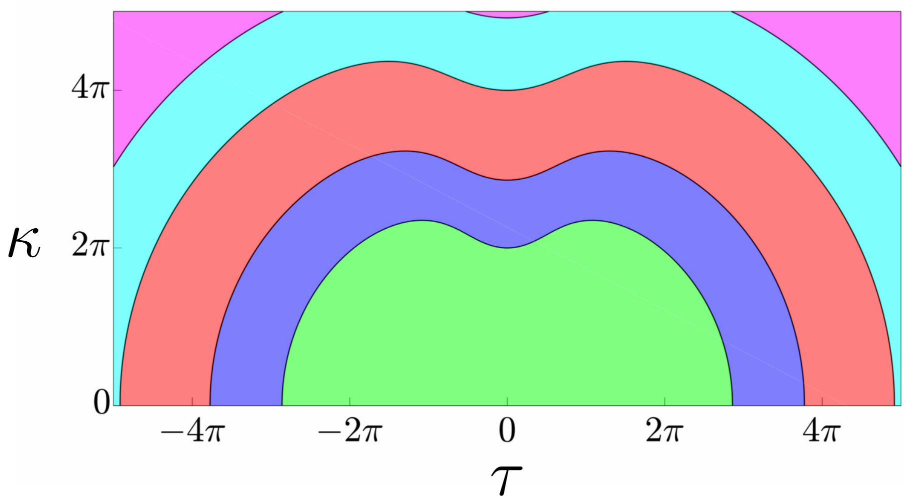

Fig. 1: Subsets of the (τ, κ) half-plane with different optimality properties. The green region corresponds to relative equilibria that are local optima of the optimal control problem (1) for $c = 1$, i.e., helices that do not have conjugate points. The blue, red, cyan, and magenta regions correspond to non-optimal helices with one, two, three, and four conjugate points, respectively.

point, the red region to helices with two conjugate points, and so on.

V. Conclusion

We have derived an explicit parameterization of the helical extremals and the relative equilibria of the optimal control problem (1). We also determined which relative equilibria correspond to local optima of the problem (1) in the case when each term in the cost function is weighted equally. We found that the optimality properties of the relative equilibria are completely characterized by the curvature and torsion of the corresponding helical trajectory in $SE(3)$.

In future work, optimality of the relative equilibria given in (15) could be determined when $c \neq 1$. As was the case in this paper, doing so would involve solving a linear time-invariant system. More generally, optimality of the helical solutions given in (8), which are not necessarily relative equilibria, could be analyzed. However, the linear system (6) would become time-varying in this case and may be difficult to solve analytically

Other generalizations include considering more general cost functions and system dynamics. A cost function of the form $u^T Q u$, where Q is a symmetric positive definite matrix, accounts for coupling between the turning rates. In the mechanics literature discussed in Section I, this cost function corresponds to an elastic wire with an anisotropic cross-section [8]. Similarly, an oriented vehicle with non-constant speed corresponds to an elastic wire that can stretch axially [1]. Determining the optimality properties of these more general systems would be of interest to both the control and mechanics communities.

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