Problem 1: MENTAL MATH (NO CALCULATORS ALLOWED)

Example:

Question 1:

Question 2:

Question 3:

Question 4:

Question 5:

Question 6:

Question 7:

Question 8:

Question 9:

Question 10:

Problem 2: ALGEBRA Egyptian Fractions

(NO CALCULATORS ALLOWED.)

You might find it surprising to learn that the ancient Egyptians did not do arithmetic in the way we do it now. We think of fractions as a ratio $\frac{p}{q}$, such as $\frac{3}{4}$ or $\frac{7}{122}$. However, the ancient Egyptians did not think of fractions in this way: instead, they would write fractions as a sum of distinct unit fractions (a "unit fraction" being a fraction whose numerator is 1, and "distinct" meaning that they wouldn't duplicate a fraction).

For example, where we would write $\frac{3}{4}$, the Egyptians would instead write $\frac{1}{2}+\frac{1}{4}$, because

$$\frac{3}{4}=\frac{1}{2}+\frac{1}{4}.$$

We define an **Egyptian Fraction Decomposition** (EFD) to be a sum of distinct unit fractions.

Thus, $\frac{1}{2}+\frac{1}{4}$ is an Egyptian Fraction Decomposition for $\frac{3}{4}$. Another Egyptian Fraction Decomposition for $\frac{3}{4}$ is $\frac{1}{2}+\frac{1}{6}+\frac{1}{12}$.

However, $\frac{1}{2}+\frac{1}{8}+\frac{1}{8}$ would NOT be an Egyptian Fraction Decomposition, because the fraction $\frac{1}{8}$ appears twice. It can be shown that every positive fraction has an Egyptian Fraction Decomposition.

Part 1: Calculations

Question 2.1: (1 points) $\frac{1}{2}+\frac{1}{5}$ is an Egyptian Fraction Decomposition for what number?

Question 2.2: (1 points) What term is missing, if $\frac{1}{3}+\boxed{}$ is an Egyptian Fraction Decomposition for $\frac{7}{12}$?

Part 2: Decompositions

Question 2.3: (1 points) Find an EFD of $\frac{8}{15}$ into two terms.

Question 2.4: (2 points) Find two EFDs of $\frac{2}{9}$ into two terms.

Question 2.5: (2 points) Find two EFDs of $\frac{7}{8}$ into three terms.

Part 3: Maximizations

Question 2.6: (1 points) If an EFD has 3 terms, what is the largest possible value it could have? (Note that $\frac{1}{1}$ is not allowed as a term.)

Question 2.7: (1 points) If an EFD has 4 terms, what is the largest possible value it could have? (Note that $\frac{1}{1}$ is not allowed as a term.)

Question 2.8: (1 points) If an EFD has 3 terms, what is the largest possible value it could have that is **strictly less than 1**?

Problem 3: GEOMETRY Prisms (and pools!)

(NO CALCULATORS ALLOWED.)

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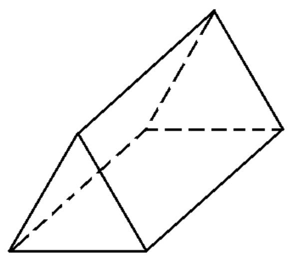
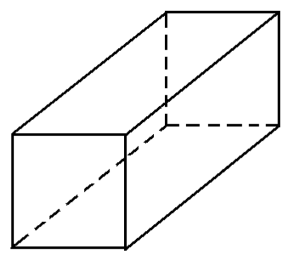
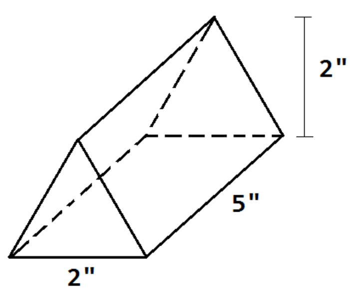
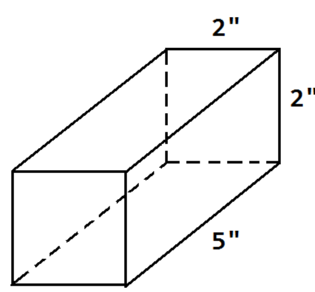
A **prism** is a geometric solid. Its ends are made up of two parallel copies of some polygon. You can think of it this way: start with one polygon, and then make a copy of it. Move the copy somewhere else, and then paste it without twisting or turning it. Then the corners of the polygons are connected with **straight lines**. For example, these are both prisms.





We can find the **volume** of a prism by multiplying the **area of an end** by the **length** of the sides. Let's find the volume of these prisms. Remember that the " sign tells us that the units for the given lengths are in inches.





First, we need to find the area of one of the ends. In this case, the ends are triangles. The base and height are both 2", so the area of the end is...

$$\textit{base} \times \textit{height} \times \tfrac{1}{2} = 2 \times 2 \times \tfrac{1}{2} = 2 \text{ in}^2.$$

Since the length of this prism is 5in, the volume is $2 \times 5 = 10\text{in}^3$. Now, take a minute to find the volume of the other prism.

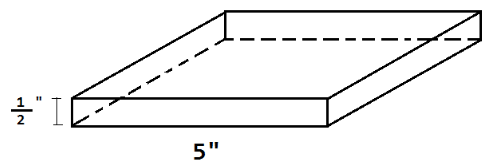
You should get 20 in^3. Which sides did you use as the ends?

Part 1:

Let's practice with prisms!

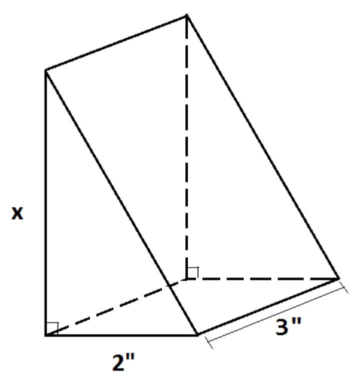
Question 3.1: (1 points) What is the volume of the prism below, given that the top and bottom sides are squares?

in^3



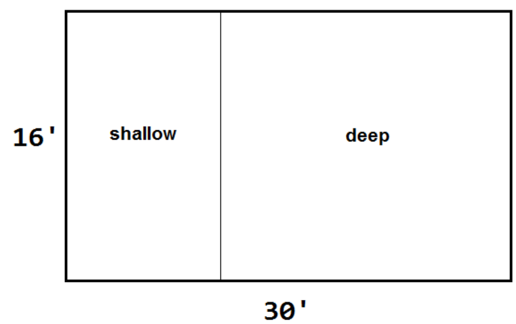
Question 3.2: (1 points) Suppose I know that the volume of this prism is 13 in^3. Find x. (The triangles are right triangles.)

in



Part 2:

Donald just bought a new house. There's an old pool dug into the backyard that he wants to fix up. Looking at the pool from above, it is a rectangle with the following lengths (the $'$ symbol means the units are in feet):



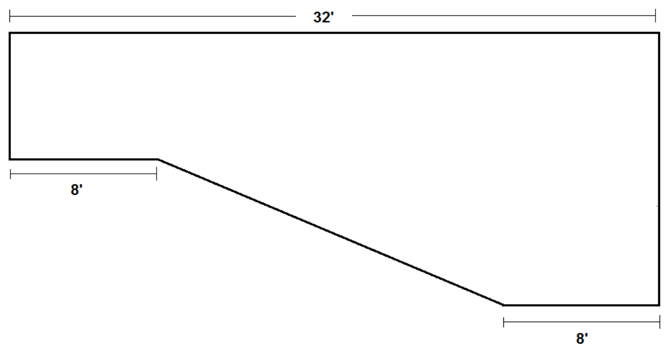
As shown in the picture, for one third of its length, the pool is a shallow end that's 3 feet deep. The rest of the pool immediately drops off to 10 feet deep.

Question 3.3: (2 points) What is the volume of Donald's pool?

ft^3

Part 3:

Donald's neighbors the Jacksons like his pool so much that they decide to put in a pool too. Their pool also has a shallow end that's 3 feet deep and a deep end that's 10 feet deep, but it has a sloped floor between the two. If we could look at the pool from the side, it would look like this:



As labeled, the pool is 32 feet long. If we looked at the pool from above, it looks like a rectangle that is 32 feet by 20 feet.

Question 3.4: (3 points) How many ft^3 of water will the Jacksons need to fill their pool?

ft^2

Question 3.5: (3 points) The Jacksons would like to paint the walls of their new pool (not including any of the bottom). Each quart of this paint covers 100 ft^2. How many quarts of paint will they need to buy? Round your answer **up** to the next whole quart.

quarts

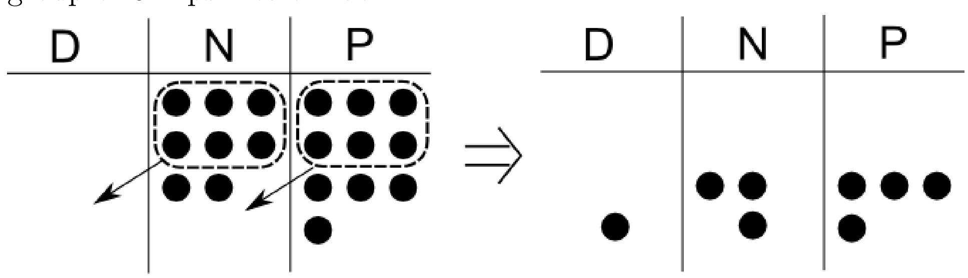
Problem 4: Money on Mars

(NO CALCULATORS ALLOWED.)

On Mars, they don't use dollars and cents for money like we do. Instead, they use Pips, Nobs, Dabs, and Quids. 1 Nob is worth 6 Pips, 1 Dab is worth 6 Nobs, and 1 Quid is worth 6 Dabs.

Martians are not good at using big numbers, so they only use at most 5 of each coin. Martians would **never** say that something costs 11 Pips! Since 6 Pips is worth 1 Nob, they would that say it costs 1 Nob and 5 Pips.

Everyday amounts of Martian money can be expressed by at most 5 of each coin. For example, if you had 8 Nobs and 10 Pips in your pocket, you can convert a group of 6 Nobs into a Dab and convert a group of 6 Pips into a Nob.



So your money is worth 1 Dab, 3 Nobs, and 4 Pips.

Part 1:

Julia is on vacation on Mars and she wants to buy souvenirs. Items at the souvenir shop have the following prices:

	Q	D	N	P
T-shirt	3	5	0	4
Keychain	0	3	2	2
4 Postcards	0	2	0	4
Magnet	0	1	2	3

This means that 1 T-shirt costs 3 Quids, 5 Dabs, 0 Nobs, and 4 Pips. For the questions below, give your answers in Martian prices (no more than 5 of each coin).

Question 4.1: (1 points) How much do a T-shirt and a magnet cost together?

Question 4.2: (1 points) How much do 6 keychains cost?

Question 4.3: (1 points) Which costs more, 8 postcards or 3 magnets? Circle one:

8 postcards 3 magnets

Question 4.4: (2 points) If Julia pays 2 Dabs for a magnet, how much should she receive in change?

Question 4.5: (2 points) The price of 4 postcards is given above. If each postcard costs the same amount, how much does 1 postcard cost?

Part 2:

Travelers between Mars and Earth have to convert their money to the local currency. The exchange rate between US Dollars and Martian money is \$0.01 = 1 Pip. For example, if you want to convert 1 Nob and 2 Pips to US Dollars, 1 Nob and 2 Pips = 8 Pips, so you will get \$0.08.

Question 4.6: (1 points) If you convert 1 Quid to US Dollars, how much do you get?

Question 4.7: (2 points) If you convert \$1 to Martian money, how much do you get? (Answer should have no more than 5 of each coin!)

Problem 5: Team Problem Criminal Logic

The Mob of Miscreants (or MOM, for short) is an organization of evil-doers on a mission to make mischief in Madison. Luckily for the citizens of Madison, the Posse of Protectors (or POP) is a counter-organization that exists to protect the populous from the Mob of Miscreants.

Part 1:

Winston the wicked is a member of MOM. He has a nemesis in POP named Horton the heroic. Both Winston and Horton, however wicked or heroic, obey the following ground rules for performing acts of wickedness or heroism.

- Winston is either out doing evil or he is at home.
- Horton is either out doing good or he is at home.
- Winston does evil if and only if he is wearing white or if he did not have any coffee that morning.
- If Horton is wearing hot pink, then he will do good.
- If it is a sunny day, then Horton will do good.
- If Winston is out doing evil and Horton is out doing good, and if the day of the week has an "n" in it, then Horton will succeed in stopping Winston.
- If Winston is out doing evil and Horton is out doing good, and if the day of the week begins with "t", then Winston will be successful in carrying out his evil endeavor.

Question 5.1: (2 points) Suppose that Winston is out doing evil and that he had a cup of coffee in the morning. Can we determine what color clothes Winston is wearing?

Question 5.2: (2 points) Suppose that Horton is out doing good. Can we determine what color clothes Horton is wearing?

Question 5.3: (2 points) True or false: If it is raining and Horton is wearing blue, then he will not be out doing good.

Question 5.4: (2 points) Suppose that Winston is wearing white and Horton is wearing hot pink on Tuesday. What can we be certain will happen?

Question 5.5: (10 points) Determine which of the following must be true from the above rules; there may be more than one answer.

(1) If it is sunny on Sunday, then there is no way for Winston to succeed in doing evil.
(2) If it is sunny on Saturday, then there is no way for Winston to succeed in doing evil.
(3) If Horton is out doing good, then it is either sunny or Horton is wearing hot pink.
(4) If Winston is wearing white, then he must not have had any coffee in the morning.
(5) If Winston succeeds in doing evil on a day where Horton wears hot pink, then it cannot be Wednesday.

Part 2:

Each member of MOM has a nemesis in POP who tries to thwart his/her evil plans. Here are the names of 8 individuals:

(1) Dr. Bad
(2) Dr. So-So
(3) Dr. Good (Female)
(4) Captain McAwesome
(5) Queen Carington (Female)
(6) Sparkles (Female)
(7) Catface (Female)
(8) Bob

In the above list, we know that we have four pairs consisting of one miscreant and one protector. We also have the following clues to help sort out the pairs.

- None of the doctors has a doctor as a nemesis.
- There is a pair of two women.
- Dr. Bad's nemesis and Bob's nemesis both have titles.
- Sparkles' nemesis is male.
- Captain McAwesome's nemesis does not have a title.
- Dr. So-So's nemesis is female.

Question 5.6: (16 points) What are the four pairs? Make no assumptions about the allegiance of a character based on name or gender.

Now that we have the four pairs of miscreants and protectors, we must determine who belongs to the Mob of Miscreants and who belongs to the Posse of Protectors. We have the following clues to help us figure it out.

- Two of the doctors are good.
- Queen Carington and Captain McAwesome are not in the same organization.
- The two male protectors have titles.

Question 5.7: (16 points) Who are the four Protectors?

Problem 6: Team Problem Hats!

In the kingdom of Zqthrrrbk, there is a small village called ttttpklk with its humble 12 inhabitants. Each year, a yearly challenge is posed to the residents of ttttpklk by the mayor of a rival village of the same size: "Ryybc ppklkl ttttpklk ww–". Er, let me activate a translator. One second... There:

"Greetings citizens of ttttpklk! Congregate at the village square and hear a challenge! A hat will be placed on each person's head on the top of which will be written either a 0 or a 1. You, as a wearer of a hat, cannot see your number, but you can see the number of everyone else. Then you will line up, left to right, and we'll go down the line with each person announcing a number in turn. For each of you, when it is your turn to say a number, if the number you say is the number on your hat, your village will gain a point."

Part 1:

They meet together the day before and decide to pair up. The first person to guess will say the number on the last person's hat. So the first person might get it wrong, but now the last person to guess will definitely get it right, so they will be guaranteed at least one point for the pair (and maybe two if they are lucky). They make the same arrangement for the second person pairing with the second-to-last person.

Come challenge day, and the hats are arranged in the following sequence:

1, 1, 0, 0, 0, 1, 1, 1, 0, 1, 0, 1

Question 6.1: (3 points) Using this strategy, what number does the third person say?

Question 6.2: (3 points) How many points does tttpklk get in the challenge in all?

Question 6.3: (4 points) The challenger learns of their strategy and wishes to place the hats next year so that as few people as possible get it right. What is one arrangement he might use?

Part 2:

They did OK that first year. Many villages guessed randomly and only half got it right, but in the village of rkvk, every person out of their 12 got it right. 'Maybe they just got lucky?' it was suggested, but the next year, only one from rkvk got it wrong, whereas in that second year, ttttpklk only scored 6 points (due to the conniving challenger). So the villagers of ttttpklk convene again to try to improve their strategy.

They decide that the first person to guess should, after looking at everyone else's hat, try to say a number that will help everyone else guess correctly. In the end they conclude that he should add all the numbers he sees (that is, the numbers of everyone else's hat but his). If it is odd, he will say '1', and if even, he will say '0'.

Then suppose the hats are arranged like they were the first year:

$$1, 1, 0, 0, 0, 1, 1, 1, 1, 0, 1, 0, 1$$

He adds all the numbers he can see, which are

$$1, 0, 0, 0, 1, 1, 1, 0, 1, 0, 1$$

and gets 6, which is even, so he says '0'.

Now person 2's turn comes. He looks at all the hats that person 1 saw (except his own) and sees

$$x, 0, 0, 0, 1, 1, 1, 0, 1, 0, 1$$

(where x was his own hat, which he cannot see). He tries to add them all up and he gets $x + 5$. He doesn't know x, because he cannot see his own hat, but he knows it is either a 0 or a 1, and he knows that $x + 5$ is even, so he concludes he is wearing a 1.

Question 6.4: (4 points) Come the next year, the hats are arranged like

$$1, 1, 0, 0, 0, 1, 1, 0, 1, 0, 0$$

What should person 1 say?

Question 6.5: (4 points) You are person 5, and the hat arrangement is as follows:

$$1, 1, 1, 1, x, 1, 1, 0, 0, 1, 1$$

(There is an x in your spot because you cannot see your own hat.) Person 1 says '1'. What is the number on your hat?

Part 3:

It turns out that many villages had figured out some kind of improved strategy, so the following year the challenge is made more complex: Now the hats will say either 0, 1, or 2.

Question 6.6: (7 points) Devise an ordering of the hats for which, if the villagers use their previous strategy (from part 2) every person will guess incorrectly, so the village will receive zero points.

Part 4:

Dismayed, the ttttpklkites must attempt to adapt their strategy. They realise that 'odd' just means '1 above a multiple of 2' and 'even' just means '0 above a multiple of 2'.

Instead of using multiples of 2, they decide to use multiples of 3. That is, the first person will again add all the numbers he sees. But now instead of saying '1' if the sum is odd–i.e. is 1 above a multiple of 2–he will say '1' if the sum is 1 above a multiple of 3.

You are the second person, and you see the ordering of the hats is now:

$$1, x, 2, 2, 1, 0, 0, 1, 2, 2, 2, 1$$

(Once again, there is an x because you cannot see your own hat.)

Question 6.7: (5 points) If your hat was a 1, what would the first person say?

Question 6.8: (5 points) The first person said '0'. What is the number on your hat?

Part 5:

Next year, the challenge is upped once more so that the hats may say anything from 0-9. The village adopts its strategy accordingly.

You are the third person, and you see the ordering of the hats as:

$$5, 6, x, 1, 3, 9, 2, 9, 3, 1, 1, 0$$

Question 6.9: (7 points) If your hat were a 1, what would the first person say?

Question 6.10: (8 points) If the first person says '7', what should you say?