Problem 1: Mental Math
NO CALCULATORS.

Example:	20
Question 1.1:	2
Question 1.2:	5182
Question 1.3:	333
Question 1.4:	735
Question 1.5:	28
Question 1.6:	14
Question 1.7:	32
Question 1.8:	181
Question 1.9:	81
Question 1.10:	53

Problem 2: Knip, Pickle, Scallion

On the faraway Horizont Isle, people use a money system consisting of Knips, Pickles, and Scallions. There are seventeen Knips in a Pickle and twenty-two Pickles in one Scallion. When exchanging money, people always prefer to use as few coins as possible. For example, two Pickles are worth the same amount as thirty-four Knips, but we would always use two Pickles given the choice.

For example, Anglefracc Gakenberg is a banker on Horizont Isle. A new customer deposits two Scallions, and asks for five Pickles and two Knips back. Anglefracc knows that there will be 1 Scallion, 16 Pickles, and 15 Knips left in the account.

Part 1:

Question 2.1: (1 points) Skallagrim Gonfle-Rognons purchased three raw fish for a total of eight Pickles and two Knips. If she pays with a Scallion, how many Knips will she get in change?

15

Question 2.2: (1 points) Sinsikolo Augenblick has been saving the family Knips in a jar. When the jar is full, he goes to the bank to trade them for Scallions and Pickles. Of course, there might not be exactly the right number of Knips to convert them all to bigger coins, so there may be some Knips left over. If there are 1718 Knips in the jar, how many Knips will be left after the trip to the bank?

1

Question 2.3: (2 points) Morgenrote Irrlicht is quite wealthy and only carries Scallions. She is going to start a band. In the music shop there are trumpets for eight Pickles and drums for twelve Pickles. Morgenrote wants to buy at least one of each and she doesn't want change. How many instruments will she end up with if she spends as little money as possible?

1 trumpets

3 drums

Question 2.4: (2 points) One Fola nut costs three Pickles. How many Fola nuts can Rogaroot Stufensus purchase if he has ten Knips, one hundred and one Pickles, and one thousand Scallions?

7367 nuts

Part 2:

Question 2.5: (2 points) If Jemanina Haggorstrom trades 2^6 Pickles for as many Scallions as possible, how many Pickles will be left over?

20 pickles

Question 2.6: (2 points) Recently, Horizont Isle introduced a new type of money called a Curd. A Curd is worth 10 Scallions. Brackenhood Smelterine is very rich, so she decides to convert his money to Curds. If she converts 2^{2015} Scallions into Curds, how many Scallions will be left over?

8 Scallions

Problem 3: Descent to Landing

You are in an airplane, flying toward your home airport, and you wish to calculate your descent. You have various instruments that tell you:

- How fast you are going horizontally
- How high off the ground you are
- How fast you are descending
- How far you are from the runway horizontally

Part 1:

Without worrying about the descending part, you will first calculate how long your trip home will take. The formula for this is, as always:

$$\text{speed} = \frac{\text{distance}}{\text{time}}$$

So if you know how far you've gone and how long it took, you can find how fast you're going. This formula can also be rearranged to

$$\text{time} = \frac{\text{distance}}{\text{speed}}$$

So that if you know how far you're going and how fast, you can find how long it will take.

Question 3.1: (1 points) You are currently 150 miles from the airport, going 100 miles per hour. How many hours will it take before you're over the airport?

1.5 hours

Question 3.2: (1 points) You are now 20 miles from the airport, going 100 miles per hour. How many *minutes* will it take before you're over the airport?

12 minutes

Question 3.3: (2 points) Suppose your speed-measuring instrument has failed. So you look outside and you notice you are flying over a barn. 15 seconds later (according to your watch) you are flying over a TV tower. Looking at your map, you observe that they are exactly a half-mile apart. How fast are you going, in miles per hour?

120 mph

Part 2:

Next, we can figure out how much time will pass until we reach the ground based on how fast we are descending.

Question 3.4: (1 points) Suppose we are at 4500 feet off the ground and are descending at 500 feet per minute. How many minutes will it take to reach the ground.

9 minutes

Question 3.5: (1 points) Suppose we are also 15 miles from the airport going at 90 miles per hour (horizontally). How many minutes will it take before we reach the airport?

10 minutes

Question 3.6: (2 points) In this case did we overshoot the airport (i.e. reach the ground after we passed the runway) or undershoot (i.e. reach the ground before we got to the runway? Or did we land on the runway exactly?

undershoot

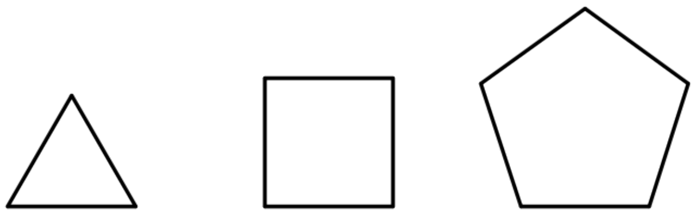
Part 3:

Question 3.7: (2 points) We are currently going 100 miles per hour and we are 10 miles from the airport. We are 3600 feet off the ground. How many feet per minute should we set for our descent in order to reach the ground exactly when we reach the airport?

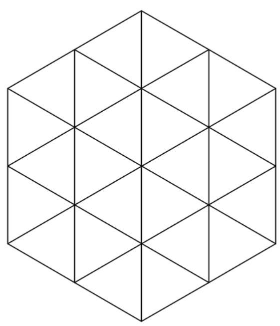
600 fpm

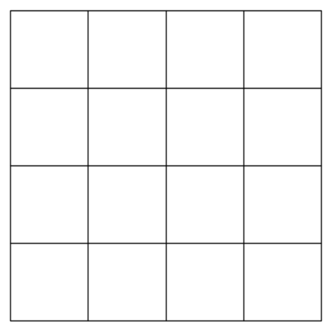
Problem 4: Tyler's Tiles

Tyler the tile maker owns a store that sells tiles in the shape of regular polygons, where each side of every tile is 1 foot long. If a tile has n sides, he calls it an n-gon, and he sells n-gons with $n = 3, 4, 5, 6, 7, 8$. The 3-gon, 4-gon, and 5-gon tiles are shown below:



Tyler's customers use these tiles to tile floors so that there are no gaps between the tiles and no overlaps. Some examples of this are shown below:

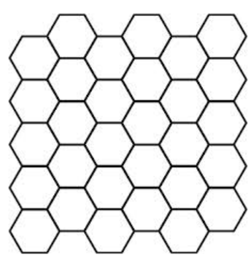




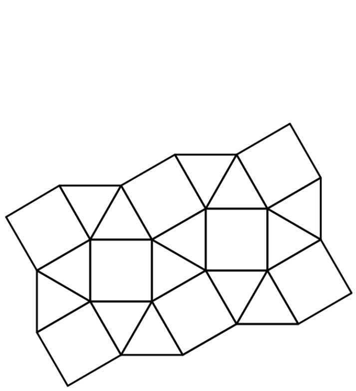
The example on the left uses only the 3-gon tile, and the example on the right uses only the 4-gon tile.

Example: Which other type of tile can be used on its own to create a tiling.

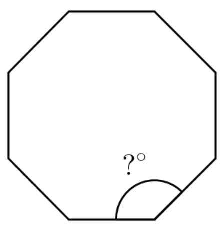
6-gon



Many of Tyler's customers like to make more complex tiling patterns by using several different types of tiles, but they always make sure that every corner is shared by the same types of tiles in the same order. Some examples of this are shown below:



Question 4.1: (2 points) In order to use several different n-gons in a tiling, the interior angles of the n-gons used must add up to 360°. If the interior angles of an n-gon are given by the formula $\frac{180(n-2)}{n}$, what is the interior angle of an 8-gon?

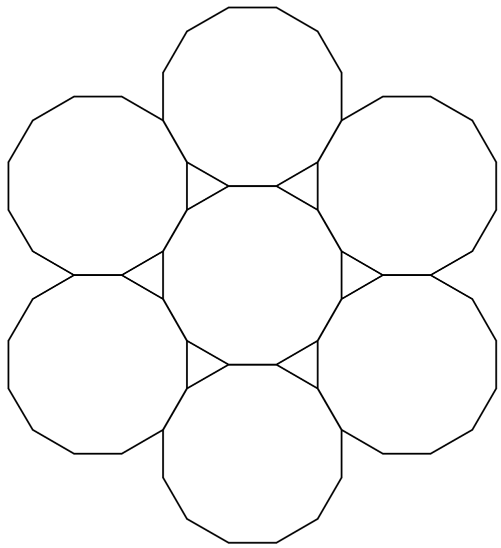


135°

Question 4.2: (3 points) Tyler discovers that his 8-gons are always purchased with one other type of tile. Which n-gon can be paired with the 8-gons to create a tiling pattern?

4-gon

Question 4.3: (3 points) Tyler decides to start selling a new 12-gon tile. He knows that there are exactly two tile patterns that use a 12-gon. One of these is shown below:



The other tiling pattern uses two types of tiles in addition to the 12-gon. Which two other n-gons can together be used with a 12-gon to make a tiling pattern?

4 and 6 -gons

Question 4.4: (2 points) Some of Tyler's tiles can't be used for any tilings of the types described above. Name *all* the tiles that cannot be used to make a tiling.

5 & 7-gons

Problem 5: Sequences

A **sequence** is an infinite list of whole numbers, like this one:

$$1, 3, 5, 7, 9, 11, \cdots$$

The symbol $\cdots$ means that the list continues and you should figure out what the pattern is.

If we want to be more clear, we can also be define a sequence using **closed form**, an equation which tells the reader exactly what all of the terms are without making them guess the pattern. For the odd numbers above, we would write the closed form equation

$$a_n = 2 \times n - 1$$

Here a_n means "the n-th number in the list." Notice that the third number in the list above was 5, and if we use replace n with 3 in the formula we get $a_3 = 2 \times 3 - 1 = 5$.

A third way to describe a sequence is to use a **recurrence relation**, an equation that defines each term of a sequence using its previous terms. For example, we could declare that the first number in the list is 1 and then each new number is two more than the last one: $a_1 = 1$ and $a_n = 2 + a_{n-1}$. Here a_{n-1} means "the number in the list right before the n-th number."

Part 1:

Consider the sequence

$$2, 5, 10, 17, 26, 37, 50, 65, 82, 101, \cdots$$

Question 5.1: (5 points) Here, $a_2 = 5$. What are the values of a_5 and a_{12}?

26$= a_5$

145$= a_{12}$

Question 5.2: (5 points) Can you guess the closed-form formula for a_n?

$n^2 + 1 = a_n$

Part 2:

Now let's look at a recurrence relation. We'll declare that the first two numbers of a sequence are $a_1 = 10$ and $a_2 = 4$, and we'll use the formula

$$a_n = 3 \times a_{n-1} + a_{n-2}.$$

To figure out a_3 we would put $n = 3$ in the last line to get

$$a_3 = 3 \times a_2 + a_1 = 3 \times 4 + 10 = 22.$$

Question 5.3: (5 points) What are the values of a_4 and a_6?

$70 = a_4$

$766 = a_6$

This is an example of a special type of recurrence relation that we will call "awesome." Awesome recurrence relations look like this:

$$a_n = Ba_{n-1} + Ca_{n-2}$$

where B and C are numbers.

Question 5.4: (7 points) For our recurrence relation, $a_n = 3a_{n-1} + a_{n-2}$, what is B? What is C?

$3 = B$

$1 = C$

The "roots" of an awesome recurrence relation are given by the formula

$$r_1 = \frac{B + \sqrt{B^2 + 4C}}{2}, \quad r_2 = \frac{B - \sqrt{B^2 + 4C}}{2}.$$

Question 5.5: (8 points) For the recurrence relation $a_n = 3a_{n-1} + a_{n-2}$ what are r_1 and r_2?

$\frac{3+\sqrt{13}}{2} = r_1$

$\frac{3-\sqrt{13}}{2} = r_2$

Part 3:

The Fibonacci sequence is an important sequence that is often seen in nature. It is the awesome recurrence relation $a_n = a_{n-1} + a_{n-2}$ with $a_1 = 0$ and $a_2 = 1$.

Question 5.6: (10 points) What are the roots of a_n?

$\frac{1+\sqrt{5}}{2} = r_1$

$\frac{1-\sqrt{5}}{2} = r_2$

Question 5.7: (10 points) What is the closed form of the Fibonacci sequence?

$\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n-1} - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{n-1} = a_n$

To do the last problem, you can use the following neat fact: an awesome recurrence relation has the closed form

$$a_n = Mr_1^{n-1} + Nr_2^{n-1}$$

where M and N are numbers that can be found from knowing a_1 and a_2.

Problem 6: What are the odds?

Think about flipping coins. Do you think that if you sat around flipping coins for a day, you will ever flip ten heads in a row? That may seem unlikely to you. On the other hand, you could flip a lot of coins in a day. This means that there are a lot of chances for that unlikely event to happen. Maybe you should expect ten heads in a row.

Let's answer some questions about chance.

Part 1:

I am going to flip a coin 4 times. It seems pretty unlikely that heads will come up all four times. Indeed, we can calculate the chance of this using the formula:

$$\text{Probability of a particular result} = \frac{\text{Number of ways result can happen}}{\text{Number of possible results that can happen}}$$

When flipping a coin 4 times, there are 16 possible outcomes for the sequence of four flips. Explicitly, these are:

HHHH, HHHT, HHTH, HHTT, HTHH, HTHT, HTTH, HTTT, THHH, THHT, THTH, THTT, TTHH, TTHT, TTTH, TTTT

Out of these, only one has four heads, so the chance of getting four heads in four flips is $\frac{1}{16}$.

Question 6.1: (2 points) What is the probability of getting three heads in three flips?

$$\frac{1}{8}$$

Question 6.2: (3 points) What is the probability of getting three heads in four flips?

$$\frac{4}{16} = \frac{1}{4}$$

Question 6.3: (5 points) What is the probability of getting no heads in seven flips?

$$\frac{1}{2^7} = \frac{1}{128}$$

Part 2:

Suppose I start experimenting by flipping two coins and get the result HT. Then I perform the experiment of flipping two coins again and get the result TT. Then I perform the experiment 8 more times and I never see the result HH. Let's figure out how likely this is.

Question 6.4: (5 points) If I flip two coins, what are the chances that I do *not* get two heads?

$$\frac{3}{4}$$

Question 6.5: (7 points) If I perform the experiment four times, what are the chances that I never see HH?

$$\frac{3}{4} \times \frac{3}{4} \times \frac{3}{4} \times \frac{3}{4} = \frac{81}{256}$$

Question 6.6: (8 points) How many times do I need to flip my two coins until the probability that I have seen two heads is more than half?

3 Flips

The answer to the previous question is actually a relatively small number. So while two heads is a fairly unusual outcome, it is also very very unlikely that if you flip two coins enough, you will never see it happen.

Part 3:

I decide I've had enough of flipping coins and want to start rolling some dice. I want to roll enough dice so that the sum of the dice I roll is ten or greater. If I roll ten dice, then I know that their sum is at least ten. But I want to roll less than that.

Question 6.7: (5 points) If I roll 72 dice, I expect each number $1, 2, 3, 4, 5, 6$ to occur about equally. If that happens, what is the sum of all 72 rolls?

252

Question 6.8: (5 points) I want to know what number I should "expect" when I roll one die. Based on the previous question, what is the value of an average roll?

3.5

Question 6.9: (5 points) What is the probability that, when I roll one die, the number I roll will be what I should "expect" according to the previous question?

0

Question 6.10: (5 points) How many dice do I need to roll so that I can expect that the sum of the numbers rolled is at least 20?

6