# SKELETAL CONFIGURATIONS OF RIBBON TREES

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Abstract. The straight skeleton construction creates a straight-line tree from a polygon. Motivated by moduli spaces from algebraic geometry, we consider the inverse problem of constructing a polygon whose straight skeleton is a given tree. We prove there exists only a finite set of planar embeddings of a tree appearing as straight skeletons of convex polygons. The heavy lifting of this result is performed by using an analogous version of Cauchy's arm lemma. Computational issues are also considered, uncovering ties to a much older angle bisector problem.

## 1. Straight Skeletons

This paper is interested in the relationship between polygons and their underlying tree structures. The *medial axis* of a simple polygon $P$ is the closure of the set of points in its interior having more than one closest point to the boundary of $P$. If the polygon is convex, the medial axis is a straight-line tree [8, Chapter 5], where the leaves are the vertices of the polygon, and the internal nodes are equidistant to three or more of its sides. If the polygon has a reflex vertex, however, the medial axis can have a parabolic arc.

The *straight skeleton* of a polygon is a related structure to the medial axis, which constructs a straight-line metric tree for any simple polygon. It was introduced to computational geometry by Aichholzer et al. [1], and appears in areas ranging from automated designs of roofs to origami folding problems. To construct the straight skeleton, start moving the sides of the polygon inward at equal velocity, parallel to themselves. These lines, at each point of time, bound a polygon with equal corresponding angles to the original one. Continue until the topology of the polygon traced out by this process changes. One of two events occur:

1. *Shrink event*: When one side of the polygon shrinks to a point, two non-adjacent sides become adjacent. Continue moving all the sides inward, parallel to themselves again.
2. *Split event*: When a reflex vertex touches a side of the shrinking polygon, the polygon is split into two. Continue the inward line movement in each of them.

The straight skeleton is the straight-line tree traced out by the vertices of the shrinking polygons. Figure 1 shows the (a) medial axis and (b) straight skeleton of a nonconvex polygon. For convex polygons, the medial axis and the straight skeleton coincide; in this case, only shrink events occur.

This paper considers the inverse problem to this construction: Given a metric tree, construct a polygon whose straight skeleton is the tree. Section 2 provides some preliminary definitions and observations, extending the examples constructed in [2]. The notion of velocity in capturing the skeleton of a polygon is introduced in Section 3, and the main theorem is given in Section 5: For

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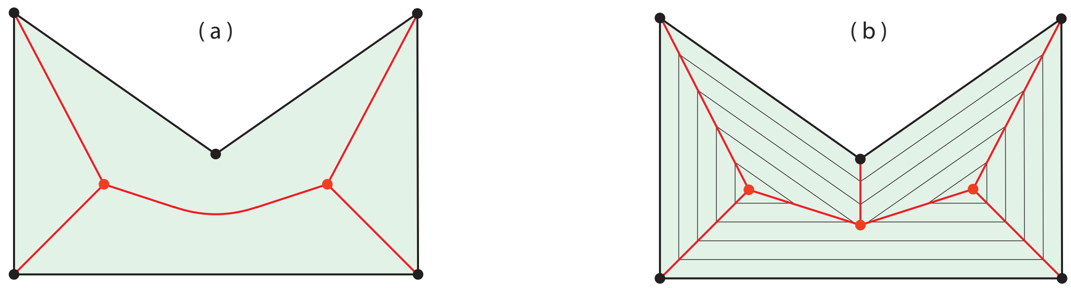


FIGURE 1. (a) Medial axis and (b) straight skeleton.

a ribbon tree $T$ with $n$ leaves, there exist at most $2n - 5$ configurations of $T$ which appear as straight skeletons of convex polygons. Section 4 contains the lemma which does the heavy lifting, analogous to Cauchy's arm lemma used in the rigidity of convex polyhedra [8]. Finally, Section 6 closes with computational issues related to constructing a polygon, uncovering ties to a much older angle bisector problem.

*Remark.* There has been tremendous interest recently in mathematical biology, including the fields of phylogenetics and genomics. The work by Boardman [6] on the language of trees from the homotopy viewpoint has kindled numerous structures of tree spaces. The most notable could be that of Billera, Holmes, and Vogtmann [5] on a space of metric trees. Another construction involving planar trees is given in [7], where a close relationship (partly using origami foldings) is given to $\overline{\mathcal{M}}_{0,n}(\mathbb{R})$, the real points of moduli spaces of stable genus zero algebraic curves marked with families of distinct smooth points. One can understand them as spaces of rooted metric trees with labeled leaves, resolving the singularities studied in [5] from the phylogenetic point of view.

As there exist spaces of planar metric trees, there are moduli spaces of planar polygons: Given a collection of positive real numbers $\bar{r} = \langle r_1, \ldots, r_n \rangle$, consider the moduli space of polygons in the plane with consecutive side lengths as given by $\bar{r}$, viewed as equivalence classes of planar linkages. A complex-analytic structure can be given to this space, defined by Deligne-Mostow weighted quotients [11]. From a high level, our inverse problem provides a rudimentary map between moduli spaces of metric trees and moduli spaces of polygons.

## 2. Preliminary Properties

This section provides definitions and preliminary results, and the reader is encouraged to consult [2], where some constructive examples are provided in detail.

**Definition.** A *ribbon tree* is a tree (a connected graph with no cycles) for which each edge is assigned a nonnegative length, each internal vertex has degree at least three, and the edges incident to each vertex are cyclically ordered.[1] A *drawing* of a ribbon tree $T$ is a planar straight-line embedding of $T$, respecting its cyclic orderings.

**Definition.** A polygon is *suitable* for a ribbon tree $T$ if its straight skeleton is a drawing of $T$. Such a drawing is called a *skeletal configuration* of $T$.

[1] This is sometimes called a *fatgraph* as well [12].

Consider two natural classes of trees: stars, and more generally, caterpillars. A *star* $S_n$ has $n+1$ vertices, with one vertex of degree $n$ connecting to $n$ leaves. A *caterpillar* becomes a path if all its leaves are deleted.

**Proposition 1.** *There exist ribbon caterpillars without suitable polygons.*

*Proof.* First consider the special case of ribbon stars. Consider a ribbon star $S_{3n}$ with edges $e_0, e_1, \ldots, e_{3n-1}$ in clockwise order. We set the edge length of $e_i$ to be equal to $x$ if $i \equiv 0 \bmod 3$, and equal to $y$ otherwise. We claim that if $n \geq 3$ and the ratio $x/y$ is sufficiently small, then this tree cannot be the straight skeleton of any polygon; see Figure 2(a) for the case $n = 3$.

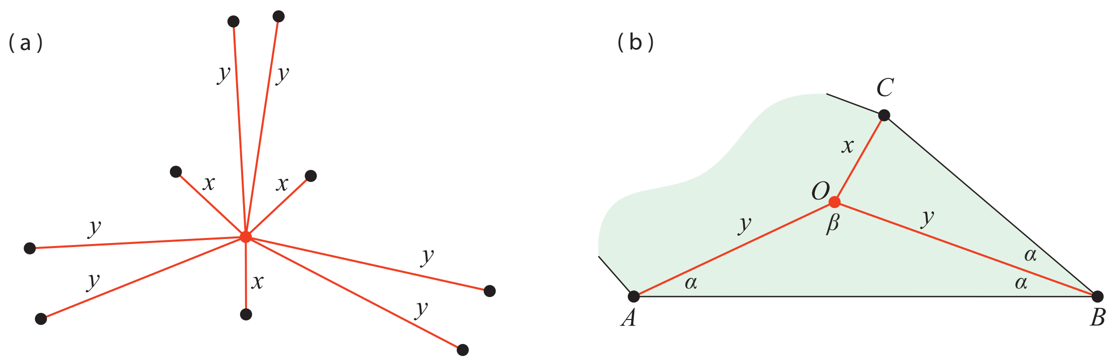


FIGURE 2. Ribbon stars without suitable polygons.

Let $O$ be the center of the star, and $A, B, C$ denote leaves with edges $AO$ and $BO$ of length $y$ and $CO$ of length $x$; see Figure 2(b). In order to have a straight skeleton, edge $BO$ must bisect $\angle ABC$ into two angles of measure $\alpha$. Defining $\beta := \angle AOB$, note that since triangle $AOB$ is isosceles, $\beta = \pi - 2\alpha$. As the length of $x$ decreases relative to $y$, $\alpha$ becomes arbitrarily close to zero. Then $\beta$ must approach $\pi$; specifically, $\beta$ can be made greater than $2\pi/3$. Since star $S_{3n}$ has $n \geq 3$ groups of three consecutive $\{x, y, y\}$ edges, then at least three such angles $\beta$ around the center $O$ are greater than $2\pi/3$, giving a contradiction.

The stars above can be tweaked to show ribbon caterpillars without suitable polygons, as given in Figure 3(a). The proof is nearly identical, replacing the center of the star with the backbone of the caterpillar, where the internal backbone edges are sufficiently small. □

**Proposition 2.** *There exist ribbon caterpillars with multiple suitable polygons.*

*Proof.* Constructive examples are given as Lemmas 5 and 10 in [2], respectively, along with detailed proofs; we include them here for the sake of completeness: For stars, there are at least two different suitable polygons for $S_5$, with lengths 100, 100, 60, 79, and 75. For caterpillars, there exist multiple orthogonal polygons with identical straight skeletons, as in Figure 3(b). Embed the backbone of the caterpillar as a orthogonal path, and at each internal vertex, make a left or right turn. For each choice of a turn, a different configuration is created. The polygon associated to a configuration is formed as the rectilinear hose (a mitered offset curve) around the embedding. □

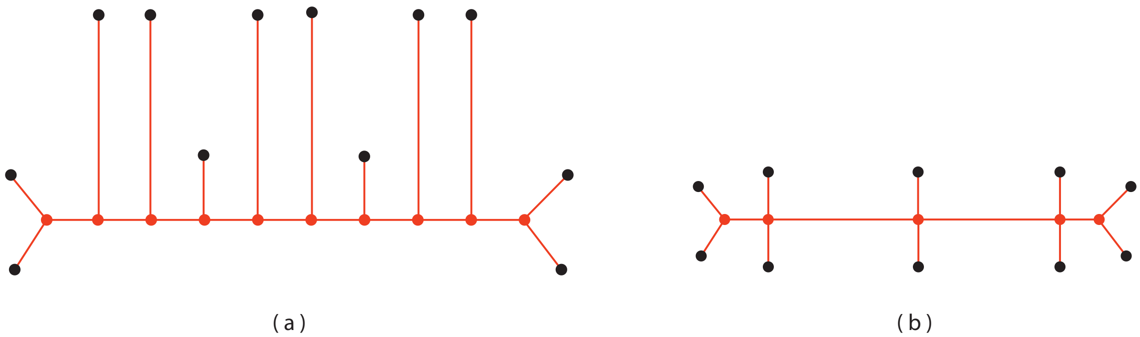


FIGURE 3. Ribbon caterpillars (a) without suitable polygons and (b) with multiple ones.

## 3. VELOCITY FRAMEWORK

The straight skeleton is defined in terms of a continuously shrinking polygon. Parameterize this shrinking process by time so that each edge moves inward with unit velocity. Then each vertex $v$ will also move inward at some velocity $\nu$. We derive a relationship between the velocity and the angle subtended at a vertex and use this in the proof of our results.

Let the angle at vertex $v$ be $2\alpha$ (see Figure 4), and first consider the convex case, where $2\alpha \leq \pi$. If the sides incident at $v$ have been moving for time $t$ at unit speed, they will have traversed $t$ units, with vertex $v$ reaching another point $w$. It follows that $t(\csc \alpha)$ is the length of $vw$, implying $\alpha = \arcsin(1/\nu)$, where $\nu$ is the velocity at vertex $v$. For the reflex case, when $2\alpha > \pi$, a similar argument applied to the exterior angle at $w$ shows that $\alpha = \pi - \arcsin(1/\nu)$.

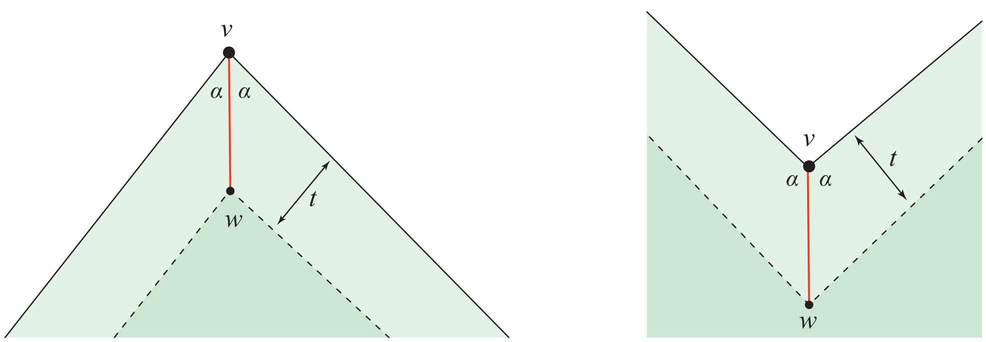


FIGURE 4. Relationship between angle and speed.

*Remark.* Since, a priori, we do not know the nature of these vertex angles, alleviate notational hassle by denoting $\arcsin^* x$ to be either $\arcsin x$ or $\pi - \arcsin x$, based on the situation.

**Proposition 3.** *A star has a finite number of suitable polygons.*

*Proof.* Let star $T$ consist of a center vertex $O$ and edges $e_i$ incident to leaves $v_i$. If $l(e_i)$ is the length of edge $e_i$, the amount of time vertex $v_i$ requires to traverse this edge is $l(e_i)/\nu_i$. Since $T$ is a star, all vertices start and end their movements simultaneously, meeting at the center $O$. Thus

$$\frac{l(e_i)}{\nu_i} = \frac{l(e_1)}{\nu_1}, \tag{3.1}$$

for each $2 \le i \le n$, establishing the relative velocities of all the vertices.

The sum of the internal angles of a polygon with $n$ vertices is $(n-2)\pi$. So if the angle subtended at vertex $v_i$ for a suitable polygon of $T$ is $2\alpha_i$, then

$$\alpha_1 + \cdots + \alpha_n = \frac{(n-2)\cdot\pi}{2}.$$

In light of Eq. (3.1), rewrite as

$$\arcsin^*\left(\frac{1}{\nu_1}\right) + \arcsin^*\left(\frac{l(e_1)}{l(e_2)\cdot\nu_1}\right) + \cdots + \arcsin^*\left(\frac{l(e_1)}{l(e_n)\cdot\nu_1}\right) = \frac{(n-2)\pi}{2}. \tag{3.2}$$

Since the lengths of the star edges are fixed, this can be viewed as an equation in $1/\nu_1$. We show this equation, for each possible choice for $\arcsin^*$, leads to at most a finite number of solutions.

Reformulate Eq. (3.2) as

$$\phi := \arcsin(m_1 x) \pm \arcsin(m_2 x) \pm \ldots \pm \arcsin(m_n x) = c, \tag{3.3}$$

where all equal terms with opposite signs are cancelled out. First we assume the constant term $c$ is nonzero. Since the Maclaurin series expansion for arcsin is

$$\arcsin z = \sum_{k=0}^{\infty} \frac{(2k)!}{2^{2k}\ (k!)^2\ (2k+1)}\ z^{2k+1},$$

then $\phi$ also has an infinite series expansion when all of the $m_i x$ terms are between -1 and 1. Hence, if $\phi - c$ has an infinite number of solutions on a compact interval for $x$, it must be identically zero. But notice that $\arcsin(m_i x)$ has no constant term in the infinite series expansion. Because $c$ is nonzero, $\phi - c$ cannot be identically zero, a contradiction. Thus there are at most a finite number of solutions in this case.

Now assume the constant term $c$ vanishes. One can show that at least one $\arcsin(m_i x)$ term will remain in Eq. (3.3). Out of all such terms, consider the one with the maximum $m_i$ value, say $\widehat{m}$. Looking at the Maclaurin series expansion again, for sufficiently large $k$, the coefficient in front of $z^{2k+1}$ will be dominated by a constant factor of

$$\frac{(2k)!\ \widehat{m}^{2k+1}}{2^{2k}\ (k!)^2\ (2k+1)},$$

since $\widehat{m}$ must be positive and strictly larger than all other $m_i$ values. Thus, for sufficiently large $k$, the coefficient will not vanish, and therefore $\phi - c$ cannot be identically zero, a contradiction. □

## 4. Racing Lemma

Let $P$ be a convex polygon, and $P(t)$ denote the polygon formed by the edges of $P$ moving inward at unit speed at time $t$. Suppose that there are a total of $n$ events which occur at times $t_1 \le \ldots \le t_n$. Observe that $P(t_1), \ldots, P(t_{n-1})$ forms a sequence of polygons with a strictly decreasing number of vertices, resulting in the degenerate polygon $P(t_n)$. We call $P(t_n)$ the *chronological center* $cc(P)$ of polygon $P$. Figure 5(a) shows an example, where the direction on the edges of the tree is based on how the skeleton is constructed. The unique sink of this directed graph is the chronological center. We omit the proof of the following:

**Lemma 4.** *Let $P$ be a convex polygon. If $P$ is in general position (no simultaneous events), $cc(P)$ is a vertex of its straight skeleton; otherwise, $cc(P)$ is either a vertex or an edge.*

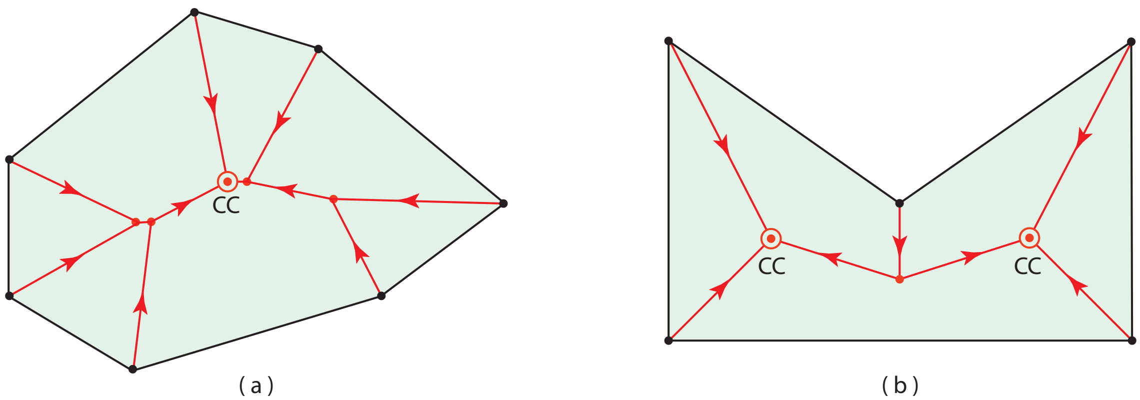


FIGURE 5. Chronological centers of polygons.

*Remark.* The chronological center can be generalized for nonconvex polygons: Here, multiple sinks will appear, one for each polygon that shrank to a point during an event, such as in Figure 5(b).

Our key lemma is an analog of Cauchy's arm lemma in the theory of polyhedral reconstruction. The arm lemma states that if we increase one of the angles of a convex polygonal chain, then the distance between the endpoints will only increase. To parallel this, we show that increasing one of the velocities of the leaves causes all velocities in the tree to increase.

**Racing Lemma.** *Consider a ribbon tree $T$ with suitable convex polygons $P_1$ and $P_2$ in general position. If the speed of some leaf $v$ of $P_1$ is greater than its corresponding leaf at $P_2$, then the speed of any leaf of $P_1$ is greater than its corresponding leaf at $P_2$.*

*Proof.* Since we are given a skeletal configuration, the order of events is determined by the angles of the underlying suitable polygon $P$, dictating the location of the chronological center $cc(P)$. Root $T$ at its chronological center, and first treat the claim when $cc(P)$ is a vertex of $T$. We prove a slightly stronger claim, namely that for any subtree with a root distinct from the chronological center, increasing the velocity of one of the leaves in the subtree by an arbitrarily small amount forces the velocities of all nodes in the subtree to increase.

Proceed by induction on the height of the subtree, defined as the maximum topological length of a path from the root to a leaf in the subtree. If the height is one, we have a group of $k$ leaves with a common parent, with velocities $\nu_1, \ldots, \nu_k$. If the lengths of the edges from the parent to the leaves are $l_1, \ldots, l_k$, correspondingly, then $l_i/\nu_i = l_j/\nu_j$. Thus, if the velocity of one leaf increases, so must all others.

Now consider a subtree of height $n$ with root $O$, and let the children of $O$ be $O_1, \dots O_m$. Since $T$ is a skeletal configuration of $P$, the subtree of $O$ corresponds to $P$ being "chiseled out" by two supporting lines $AB$ and $AC$ as shown in Figure 6 (where points $B$ and $C$ do not need to be vertices of $P$). Note the sequence of edges lying between the edges of the polygon corresponding

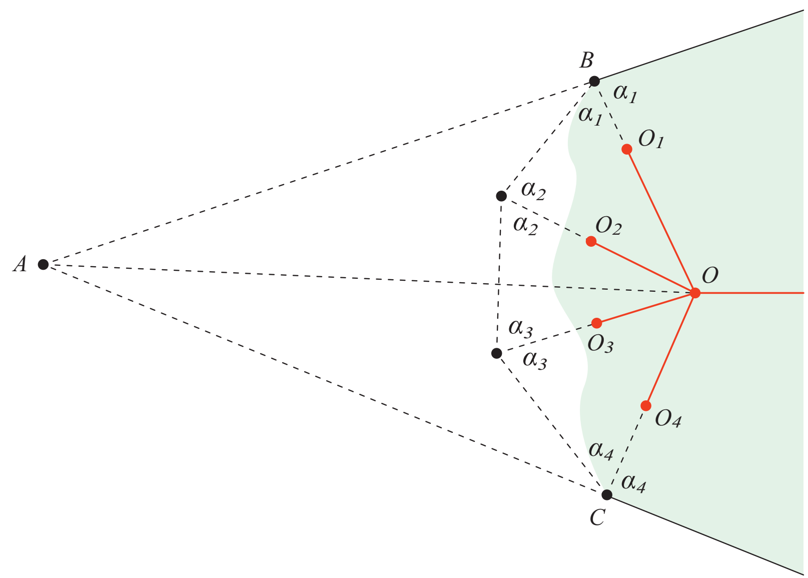


FIGURE 6. Tree to polygon perspective for the Racing Lemma.

to lines $AB$ and $AC$ will have shrunk before the event at node $O$, where the lines $OO_i$ are angle bisectors of some (possibly non-adjacent) edges of $P$. Because $AO$ is an angle bisector of the angle between lines $AB$ and $AC$, arithmetic shows that

$$\psi := \frac{\pi(m-1) - (\pi - 2\alpha_1) - (2\pi - 2\alpha_2) - \cdots - (2\pi - 2\alpha_{m-1}) - (\pi - 2\alpha_m)}{2}$$

equals $\angle BAO$ and $\angle CAO$. Since the angle at $O$ is $\psi$, this implies $O$ has velocity $(\sin\psi)^{-1}$. The convexity of the polygon forces the angles $\alpha_i$ (at vertices $O_i$) and the angle $\psi$ to be convex, where sin is monotonically increasing. Therefore, if the velocities $(\sin\alpha_i)^{-1}$ of nodes $O_i$ increase, so does the velocity of $O$.

Assume we increase the velocity of a leaf in the subtree of $O_1$. Then, by the inductive hypothesis, all of the vertices in $O_1$'s subtree (including $O_1$) will increase in velocity, so that $O_1$ finishes tracing out edge $O_1O$ faster than before. For any other child $O_i$ of $O$, since the edge $O_iO$ is traced out at the same time as $O_1O$, the velocity of $O_i$ also increases. □

*Remark.* This lemma can be strengthened to include convex polygons in *degenerate* positions, where from Lemma 4, the chronological center can be an edge, with endpoints $T_1$ and $T_2$. If we increase the velocity of a leaf in tree $T_1$, all of the vertices of the nodes of $T_1$ will increase. But since both endpoints will be reached at the same time, it follows that all nodes in $T_2$'s subtree must increase their velocities.

## 5. Skeletal Rigidity

Two more results are needed in order to prove the main theorem. We begin with an algebraic proof concerning the trivalent stars of Proposition 3, helping pave the way for some discussion on the computational issues involved in the problem.

**Proposition 5.** *There is a unique suitable triangle for each ribbon star $S_3$ of degree three.*

*Proof.* Assume a skeletal configuration of $S_3$ is given, and let the angles of the suitable triangle be $2\alpha, 2\beta$, and $2\gamma$ at vertices $A, B$, and $C$ respectively, as in Figure 7.

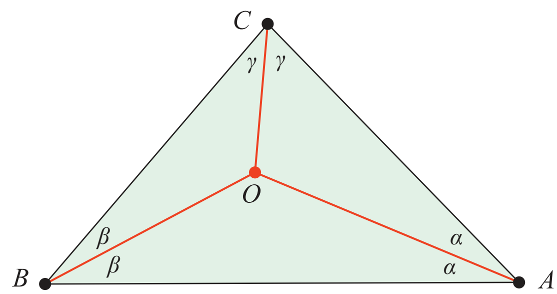


Figure 7. Suitable triangle of a trivalent star.

The law of sines applied to triangle $AOC$ yields

$$\frac{AO}{\sin\gamma} = \frac{CO}{\sin\alpha}.$$

Since $\gamma = \frac{\pi}{2} - \alpha - \beta$, it follows that

$$\frac{AO}{\sin\gamma} = \frac{AO}{\cos(\alpha+\beta)} = \frac{AO}{\cos\alpha\cos\beta - \sin\alpha\sin\beta} = \frac{AO}{\sqrt{1-\sin^2\beta}\sqrt{1-\sin^2\alpha} - \sin\alpha\sin\beta}.$$

Thus

$$\frac{AO}{\sqrt{1-\sin^2\beta}\sqrt{1-\sin^2\alpha} - \sin\alpha\sin\beta} = \frac{CO}{\sin\alpha}.$$

The law of sines applied to triangle $AOB$ results in

$$\frac{AO}{\sin\beta} = \frac{BO}{\sin\alpha}.$$

Setting $x = \sin\alpha$, together with the previous equation produces the following polynomial:

$$(2AO^2 \cdot BO \cdot CO)\, x^3 + (AO^2 \cdot CO^2 + AO^2 \cdot BO^2 + BO^2 \cdot CO^2)\, x^2 - (CO^2 \cdot BO^2) = 0 \tag{5.1}$$

Since the polynomial cannot vanish identically, we have three solutions, counting multiplicities. The discriminant of the equation is

$$\Delta \ = \ -27CO^4 \cdot BO^4 + (BO^2 + CO^2(1+BO^2))^3.$$

By the arithmetic mean–geometric mean inequality,

$$BO^2 + CO^2(1+BO^2) \ \geq \ 3(BO^2 \cdot CO^2 \cdot (CO^2 \cdot BO^2))^{\frac{1}{3}},$$

showing $\Delta \geq 0$. So the solutions of the equations are all real, say $x_1, x_2, x_3$. By Vieta's formulas, we have the following system,

$$\begin{aligned} x_1 + x_2 + x_3 &= -\frac{AO^2 \cdot CO^2 + AO^2 \cdot BO^2 + BO^2 \cdot CO^2}{2AO^2 \cdot BO \cdot CO} \\ x_1x_2 + x_1x_3 + x_2x_3 &= 0 \\ x_1x_2x_3 &= \frac{CO^2 \cdot BO^2}{2AO^2 \cdot BO \cdot CO} \end{aligned}$$

all of which show that there is exactly one positive root (say $x_1$). If $x_1 > 1$, then the left hand side of Eq. (5.1) is positive, which cannot be as $x_1$ is a root. Thus, $0 < x_1 \leq 1$, resulting in a unique value for $\alpha$. □

**Lemma 6.** *A ribbon tree with a fixed chronological center and an assignment of velocities at each leaf has at most one suitable convex polygon.*

*Proof.* This is based on induction on the number of edges of the ribbon tree $T$. The claim is true for three edges by Proposition 5, so the base case is covered. Now consider a ribbon tree with $k$ edges, and suppose the statement holds for ribbon trees with fewer than $k$ edges. We will use the relationship between the angles subtended at the vertices and their speed. Assume there are two distinct, convex polygons $P$ and $Q$ with the same angles (corresponding to velocity assignments), both with $T$ as their skeleton (with different embeddings). Since they have identical chronological centers, the event sequences for both polygons must be the same.

Let the first event in $P$ (and $Q$) be a shrink event where the edges $A_1A_2, \ldots, A_{m-1}A_m$ disappear. Here, $A_1, \ldots, A_m$ denote the vertices of these edges in cyclic order, and $O$ is their parent node in $T$; see Figure 8. Notice that all of the triangles $OA_iA_{i+1}$ must be congruent in $P$ and $Q$.

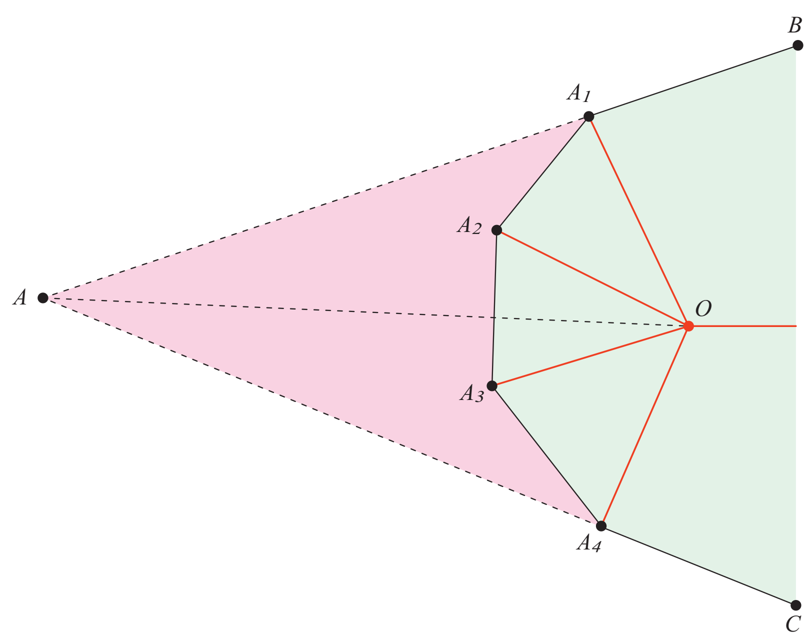


FIGURE 8. Construction of congruent polygons.

Let $B$ and $C$ be vertices of the polygons adjacent to $A_1$ and $A_m$, respectively. Let the intersection of lines $A_1B$ and $A_mC$ be at point $A$. Consider the polygons in $P$ and $Q$ with vertices $AA_1 \dots A_m$ (shaded red in Figure 8): the edges $A_iA_{i+1}$ are all equal, as are the angles subtended at vertices $A_i$ by the congruence of the triangles $OA_iA_{i+1}$. So these polygons must be congruent in $P$ and $Q$, implying the angle $A_1AA_m$ and the length $OA$ be identical in both $P$ and $Q$ as well. Deleting all $A_i$ vertices and adding vertex $A$ creates new convex polygons $P'$ and $Q'$ with new sides $BA$ and $AC$, both with identical angles at all leaves, with the same underlying skeleton consisting of $k-m+1$ edges. By the induction hypothesis, $P'$ and $Q'$ are congruent polygons, implying that $P$ and $Q$ cannot be distinct. □

**Theorem 7.** *A ribbon tree with $n$ leaves has at most $2n-5$ suitable convex polygons.*

*Proof.* It is sufficient to prove that each possible choice of a chronological center for ribbon tree $T$ yields at most one suitable convex polygon. Since $T$ has at most $n-3$ interior edges and $n-2$ interior vertices, there are $2n-5$ possible chronological center choices. Fix the chronological center in an edge or vertex of $T$. If there is an assignment of velocities $\nu_i$, which may be induced by (the angles of) some convex polygon, then no other convex polygon can correspond to this assignment by Lemma 6. We now show that no other assignment exists to which any convex polygon would correspond.

The angle $\alpha_i$ of the convex polygon subtended at vertex $v_i$ is $2\arcsin(1/\nu_i)$, resulting in

$$\sum_{i=1}^{n} \arcsin\left(\frac{1}{\nu_i}\right) = \frac{(n-2)\pi}{2}.$$

Assume there is another set of velocities $\nu_i'$ resulting in a valid skeletal configuration with the same chronological center. Without loss of generality, assume $\nu_1' > \nu_1$. Then by the Racing Lemma, $\nu_i' \geq \nu_i$ for all $i$, implying

$$\frac{(n-2)\pi}{2} = \sum_{i=1}^{n} \arcsin\left(\frac{1}{\nu_i'}\right) < \sum_{i=1}^{n} \arcsin\left(\frac{1}{\nu_i}\right) = \frac{(n-2)\pi}{2},$$

a contradiction. □

**Corollary 8.** *Every metric tree has a finite number of suitable convex polygons.*

*Proof.* Each metric tree has a finite number of ribbon tree representations, based on different cyclic orderings placed around each of its internal nodes. The result follows from the theorem above. □

## 6. Construction Results

Thus far, not much has been said about constructing a suitable polygon given a ribbon tree. The approach from Proposition 5 can be developed to connect our problem and the angle bisector problem, dating back to Euler and thoroughly studied in [4]:

**Angle Bisector Problem.** *Construct a triangle given the lengths of the angle bisectors.*

In the work of Zajic [13], this is proven impossible with ruler and compass, and the nature of the polynomial equations given the triangle lengths is explored. Our problem for the case of the triangle is slightly different, since the incenter is given as well. But considering the straight skeletons of arbitrary polygons can be viewed as a generalization of the angle bisector problem.

**Lemma 9.** *There is a star $S_3$ with integer edge lengths for which any suitable polygon cannot be constructed with ruler and compass.*

*Proof.* Let $O$ be the center of the star, and $A$, $B$, $C$ be the leaves, again as in Figure 7. Consider the star edge lengths where $AO = r$, $BO = 2r$, $CO = 3r$, for $r \in \mathbb{N}$. Then Eq. (5.1) becomes

$$12x^3 + 49x^2 - 36 = 0.$$

Any rational root $a/b$ of this equation is such that $a$ divides 36 and $b$ divides 12; by exhaustion, no such combination succeeds. But if a polynomial of degree three with rational coefficients has no rational root, then it is irreducible.

It is well-known that the set of numbers constructible by ruler and compass is exactly the quadratic closure of the rationals. Since $\sin\alpha$ is a root of an irreducible cubic, it must not be constructible. Thus, if the triangle was constructible, then clearly $\alpha$, as well as $\sin\alpha$, are constructible as numbers, a contradiction. □

We close with discussing unsolvability issues of this problem when constrained to the algebraic model: only the arithmetic operations $+, -, \times, /$ along with $\sqrt[k]{\ }$ of rational numbers are allowed.

**Lemma 10.** *There is a star $S_5$ with rational edge lengths for which the side lengths of any suitable convex polygon cannot be expressed by radicals over $\mathbb{Q}$.*

*Proof.* Consider a star $S_5$ with edge lengths $r, r, r, 10r/11, 10r/12$, for $r \in \mathbb{N}$. For a suitable convex polygon $P$, the sum of the interior angles yields

$$3 \cdot \arcsin(x) + \arcsin\left(\frac{11x}{10}\right) + \arcsin\left(\frac{12x}{10}\right) = \frac{3\pi}{2}, \tag{6.1}$$

where $x^{-1}$ equals the velocity $\nu$ at the leaf with edge length $r$. It is not hard to see that if Eq. (6.1) has a solution, there exists a suitable polygon $P$ for $S_5$.

Now we show that the edge lengths of $P$ are not expressible as radicals over $\mathbb{Q}$. By rewriting $\arcsin x$ in its logarithmic form as $-i\ln(ix+\sqrt{1-x^2})$, and substituting into Eq. (6.1), *Mathematica* shows that $x$ becomes the square root of one of the zeros of the polynomial

$$\begin{aligned} p(x) \quad = \quad & 1 - 2330x + 1837225x^2 - 653926400x^3 + 111607040000x^4 \\ & -8795136000000x^5 + 256000000000000x^6. \end{aligned}$$

The discriminant of polynomial $p$ is

$$\Delta(p) = 2^{74} \cdot 3^6 \cdot 5^{30} \cdot 11^6 \cdot 23^2 \cdot 29 \cdot 31 \cdot 79^2 \cdot 43151 \cdot 2626069.$$

Since $p$ is irreducible modulo 13, and since 13 does not divide 1 (the constant term), $p$ is irreducible. Considering $p$ modulo 7, 13, and 17 (the "good" primes), we get a (2+3)-permutation, a 5-cycle and a 6-cycle as the Galois groups, respectively, showing the Galois group of $p$ is the symmetric group $\mathbb{S}_6$ on six letters [3, Lemma 8]. Since $\mathbb{S}_6$ is not solvable [10], then $x = \frac{1}{\nu} = \sin\alpha$ is not expressible using arithmetic operations and $\sqrt[k]{\ }$ of rational numbers, where $\alpha$ is the angle subtended at the vertex with velocity $\nu$ [9]. But since $\sin\alpha$ can be expressed using arithmetic operations on radicals of the side lengths of polygon $P$ and the edge lengths of $S_5$, the claim holds. □

*Remark.* Of course, the fact that side lengths are not expressible via radicals over $\mathbb{Q}$ implies that the coordinates of the vertices of the polygon must also not be expressible via radicals over $\mathbb{Q}$.

## 7. Conclusion

Finding an algorithm approximately calculating a suitable polygon from a given ribbon tree remains open, along with the major question concerning rigidity: What is the test needed to decide which trees have suitable polygons? Moreover, extending our work from convex to arbitrary suitable polygons will require an entirely new approach and argumentation. From an alternate perspective, we hope a further understanding of this inverse straight skeleton problem will lead to developing and exploiting the relationship between the moduli spaces of trees and planar polygons.

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