THE SHAPE OF ASSOCIATIVITY

SATYAN L. DEVADOSS

Associativity is ubiquitous in mathematics. Unlike commutativity, its more popular cousin, associativity has for the most part taken a backseat in importance. But over the past few decades, this concept has blossomed and matured. We start with a brief look at three fields of mathematics that find it very powerful.

An *elliptic curve* is a smooth, projective algebraic curve of genus one, a core object in fields such as cryptography or number theory, where it was used to prove Fermat's Last Theorem. The key property of an elliptic curve stems from the group law, a product structure defined in the curve's point set.

Theorem 1. *Under this product structure, the points on an elliptic curve form an abelian group, making the elliptic curve into an abelian variety.*

The proof is quite trivial except for showing *associativity* for the group law, which is laborious when tackled directly [5]. An insight to this phenomena comes from Riemann-Roch.

Associativity also plays a key role in enumerative geometry: a classic question is to find the number N_d of plane rational curves of degree d that pass through $3d-1$ points. The cases up to $d=4$ were known in the 19th century, but the full result was provided by Maxim Kontsevich in the mid-1990s, one reason why he was awarded the Fields Medal:

Theorem 2. $$N_d \;=\; \sum_{a+b=d} a^2\; b\; N_a\; N_b\; \left[b\binom{3d-4}{3a-2} - a\binom{3d-4}{3a-1} \right].$$

Remarkably, this formula expresses the *associativity* of a new product structure called the *quantum product* [3]. Developed in string theory, this product for quantum cohomology plays a crucial role in symplectic geometry, stable maps, and Gromov-Witten invariants.

Our final example comes from algebraic topology. Let X be a space and ΩX its loop space, i.e. the space of continuous maps from the unit circle to X. The *recognition question* is an archetype problem in the theory of homotopy invariant structures, asking when a space Y has the same homotopy type as the loop space of some X. The main result follows:

Theorem 3. *A connected space Y has the homotopy type of ΩX, for some X, if and only if Y admits the structure of an A_∞ space.*

We say Y is an A_∞ space if it has a family of maps $K_n \times Y^n \to Y$ that fit together nicely [6]. The heart of this construction is the space K_n, an object which *embodies* associativity.

Our story focuses on K_n, the famous *associahedron* polytope, born in 1951 due to Dov Tamari as a realization of the poset of bracketings on n letters. Independently, in his 1961 thesis, Jim Stasheff constructed a convex curvilinear version useful in homotopy theory relative to associativity properties of H-spaces, as described above. Figure 1(a) shows the 2D associahedron K_4 with a labeling of its faces, and (b) shows the 3D version, K_5.

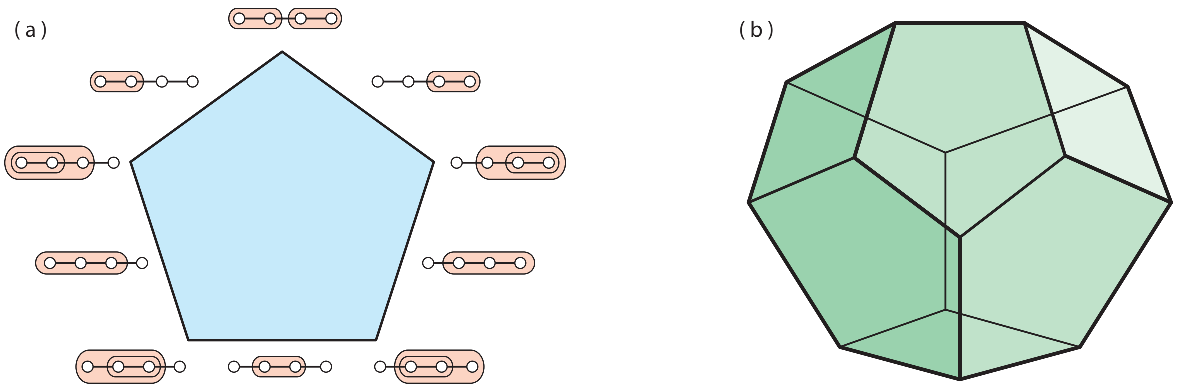

FIGURE 1. Associahedra K_4 and K_5.

Definition. Let $A(n)$ be the poset of all bracketings of n letters, ordered such that $a \prec a'$ if a is obtained from a' by adding new brackets. The *associahedron* K_n is a convex polytope of dimension $n-2$ whose face poset is isomorphic to $A(n)$.

From this we see that K_n does indeed capture the shape of associativity: the vertices correspond to all different ways n letters can be multiplied, each with a different associative grouping. The famous *Catalan numbers* enumerate the vertices, with over one hundred different combinatorial and geometric interpretations available. The beauty of this polytope is the multiplicity of areas in which it makes an appearance [4], a sampling of which include root systems, real algebraic geometry, computational geometry, phylogenetics, string theory, J-holomorphic curves, and hypergeometric functions. This should not surprise us since the underlying principle of associativity is a foundational concept.

Rather than a combinatorial framework discussed above, K_n also has a rich *geometric* perspective. There are numerous realizations of the associahedron, obtained by taking the convex hull of the vertices of K_n with integer coordinates. The most prominent is the elegant construction by Jean-Louis Loday based on the language of trees. There are also constructions of K_n obtained from truncations of cubes and simplices [1]. Figure 2 shows examples of the 4D polytope K_6, generated from iterated truncations of the 4-simplex and the 4-cube, respectively. These truncations make the associahedron inherit the algebraic structures inherent in the simplex and cube. Such an approach relates blowups of varieties (from algebraic geometry) to the world of operads and category theory.

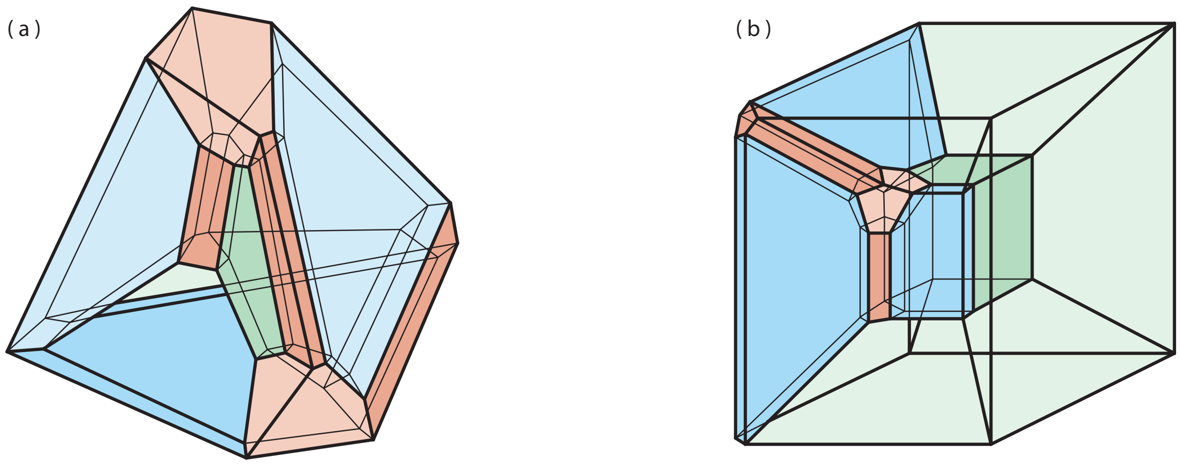

FIGURE 2. K_6 from truncations of (a) the 4-simplex and (b) the 4-cube.

One measure of an object's importance is the number of its generalizations. In our case, there are a plethora of associahedral siblings, including generalized associahedra coming from cluster algebras, Coxeter-associahedra from root systems, multiplihedra from category theory, polytopes from pseudotriangulations, and permutoassociahedra. In line with associativity, one generalization is close to our heart: the *graph associhedron*, a polytope that captures the shape of associativity on a *graph*, whereas the classic associahedron considered letters arranged on a path [2].

Let G be a connected simple graph. A *tube* is a set of nodes of G whose induced graph is a connected proper subgraph of G. Two tubes u_1 and u_2 may interact on the graph as follows:

(1) Tubes are *nested* if $u_1 \subset u_2$.
(2) Tubes *intersect* if $u_1 \cap u_2 \neq \emptyset$ and $u_1 \not\subset u_2$ and $u_2 \not\subset u_1$.
(3) Tubes are *adjacent* if $u_1 \cap u_2 = \emptyset$ and $u_1 \cup u_2$ is a tube in G.

Tubes are *compatible* if they do not intersect and are not adjacent. A *tubing* U of G is a set of tubes of G such that every pair of tubes in U is compatible.

Theorem 4. *For a graph G with n nodes, the* graph associahedron K_G *is a simple convex polytope of dimension $n-1$ whose face poset is isomorphic to the set of tubings of G, ordered such that $U \prec U'$ if U is obtained from U' by adding tubes.*

These graph associahedra are starting to appear in fields such as Floer homology, moduli space of punctured surfaces, tropical geometry, gene sequencing, Coxeter complexes, and Bergman complexes. Figure 3 gives examples of graphs and their corresponding graph associahedra. When G is a path with n nodes, K_G becomes the associahedron K_{n+1}. When G is a cycle, the resulting polytope K_G is the *cyclohedron*, a very close and natural kin of the classical associahedron, first making its appearance in the knot invariant work of Raoul

Bott and Clifford Taubes. Most interesting is when G is a complete graph, when K_G be-

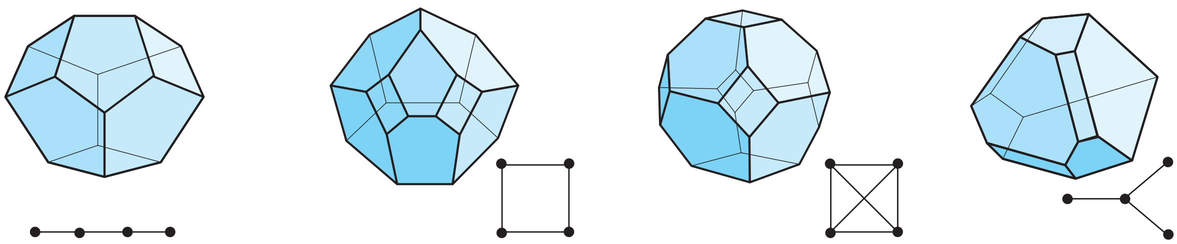

FIGURE 3. Examples of graph associahedra.

comes the *permutohedron*. The permutohedron P_n is a classic polytope, studied by Schoute in the early 20th century, defined as the convex hull of all vectors obtained by permuting the coordinates of $\langle 1, 2, \cdots, n\rangle$ in $\mathbb{R}^n$. The polytope P_n has dimension $n - 1$ with $n!$ vertices, one for each element of the permutation group of n letters. Indeed, as the associahedron captures associativity, the permutohedron encapsulates commutativity. Not only does the graph associahedron generalize both, but Alex Postnikov has generalized graph associahedra into some beautiful polytopes called *generalized permutohedra*.

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S. DEVADOSS: WILLIAMS COLLEGE, WILLIAMSTOWN, MA 01267
E-mail address: satyan.devadoss@williams.edu