A REALIZATION OF GRAPH-ASSOCIAHEDRA

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Abstract. Given any finite graph G, we offer a simple realization of the graph-associahedron $\mathcal{P}G$ using integer coordinates.

1. Introduction

Given a finite graph G, the graph associahedron $\mathcal{P}G$ is a simple, convex polytope whose face poset is based on the connected subgraphs of G. This polytope was first motivated by De Concini and Procesi in their work on "wonderful" compactifications of hyperplane arrangements [5]. In particular, if the hyperplane arrangement is associated to a Coxeter system, the graph associahedra $\mathcal{P}G$ appear as tiles in certain tilings of these spaces, where its underlying graph G is the Coxeter graph of the system [3] [4] [14]. These compactified arrangements are themselves natural generalizations of the Deligne-Knudsen-Mumford compactification $\overline{\mathcal{M}}_{0,n}(\mathbb{R})$ of the real moduli space of curves [6].

Graph associahedra have also appeared in several other contexts. From a combinatorics viewpoint, for example, they arise in relation to positive Bergman complexes of oriented matroids [1]. Recent work also include studies of their enumerative properties [11], as well as their generalization to a larger class of polytopes [10]. Most notably, graph associahedra emerge as graphical tests on ordinal data in statistics [9].

For special examples of graphs, the graph associahedra become well-known, sometimes classical, polytopes. For instance, when G is a set of vertices, $\mathcal{P}G$ is the simplex. Moreover, when G is a path, a cycle, or a complete graph, $\mathcal{P}G$ results in the associahedron, cyclohedron, and permutohedron, respectively. Loday [8] provides an elegant formula for the coordinates of the vertices of the associahedron which contains the classical realization of the permutohedron. Based on Loday's work, Hohlweg and Lange [7] offer different realizations of the associahedron and cyclohedron. Recently, based on Minkowski sums, Postnikov [10] constructs realizations of *generalized permutohedra*, a large family of polytopes encompassing graph associahedra. This paper offers a simple realization of graph associahedra, based on truncations of the simplex, and compares it to other realizations.

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2. Convex Hull

We begin with definitions; the reader is encouraged to see [3, Section 1] for details.

Definition. Let G be a finite graph. A *tube* is a proper, nonempty set of nodes of G whose induced graph is a proper, connected subgraph of G. There are three ways that two tubes u_1 and u_2 may interact on the graph.

(1) Tubes are *nested* if $u_1 \subset u_2$.
(2) Tubes *intersect* if $u_1 \cap u_2 \neq \emptyset$ and $u_1 \not\subset u_2$ and $u_2 \not\subset u_1$.
(3) Tubes are *adjacent* if $u_1 \cap u_2 = \emptyset$ and $u_1 \cup u_2$ is a tube in G.

Tubes are *compatible* if they do not intersect and they are not adjacent. A *tubing* U of G is a set of tubes of G such that every pair of tubes in U is compatible. A *k-tubing* is a tubing with k tubes.

When G is a disconnected graph with connected components $G_1, \ldots, G_k$, an additional condition is needed: If u_i is the tube of G whose induced graph is G_i, then any tubing of G cannot contain all of the tubes $\{u_1, \ldots, u_k\}$. Thus, for a graph G with n nodes, a tubing of G can at most contain $n - 1$ tubes. Figure 1 shows examples of (a) valid tubings and (b) invalid tubings.

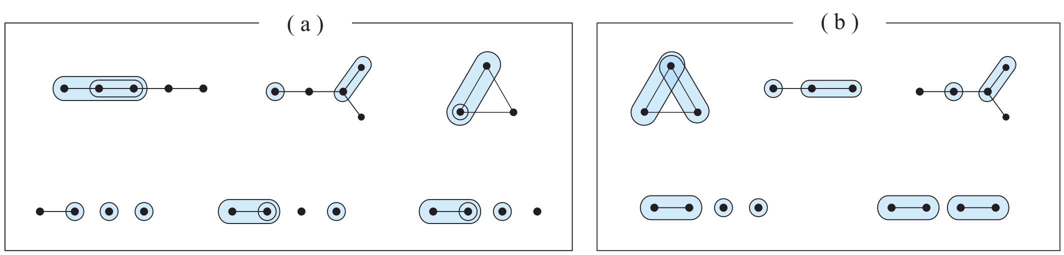

Figure 1. (a) Valid tubings and (b) invalid tubings.

Definition. [3, Section 2] For a graph G, the *graph associahedron* $\mathcal{P}G$ is a simple, convex polytope whose face poset is isomorphic to the set of tubings of G, ordered such that $U \prec U'$ if U is obtained from U' by adding tubes.

Let G be a graph with n nodes. Let M_G be the collection of maximal tubings of G, where each tubing U in M_G contains $n - 1$ compatible tubes.[1] It is important to realize that U naturally assigns a unique tube $t(v)$ to each node v of G: Let $t(v)$ be the smallest tube in U containing v; if no tube of U contains v, then $t(v)$ is all of G.

[1]Indeed, the vertices of $\mathcal{P}G$ are in bijection with the elements of M_G.

We define a map f_U from the nodes of G to the integers as follows: If $v = t(v)$, then $f_U(v) = 0$. All other nodes v of G must satisfy the recursive condition

$$\sum_{v_i \in t(v)} f_U(v_i) \;=\; 3^{|t(v)|-2}. \tag{2.1}$$

Figure 2 gives some examples of integer values of nodes associated to tubings.

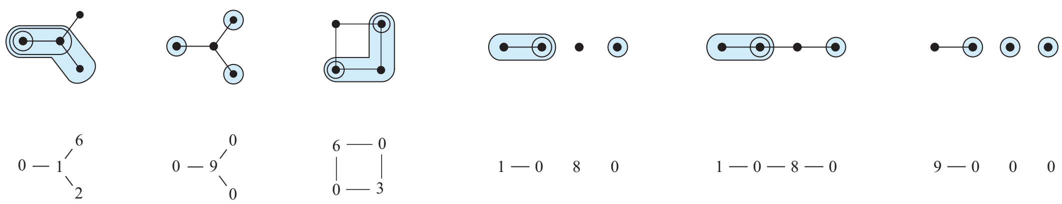

FIGURE 2. Integer values of nodes associated to tubings.

Let G be a graph with an ordering $v_1, v_2, \ldots, v_n$ of its nodes. Define the map $c : M_G \to \mathbb{R}^n$ where

$$c(U) = (f_U(v_1), f_U(v_2), \ldots, f_U(v_n)).$$

Theorem 1. *If G is a graph with n nodes, the convex hull of the points $c(M_G)$ in $\mathbb{R}^n$ yields the graph associahedron $\mathcal{P}G$.*

The proof of this theorem is given at the end of the paper. Notice the natural action of the symmetric group on the ordering of the nodes of G.

3. Examples

3.1. **Simplex.** Let G be the graph with n (disjoint) nodes. The set M_G of maximal tubings has n elements, each corresponding to choosing $n-1$ out of the n possible nodes. An element of M_G will be assigned a point in $\mathbb{R}^n$ consisting of zeros for all coordinates except one with value 3^{n-2}. Thus, $\mathcal{P}G$ is the convex hull of the n vertices in $\mathbb{R}^n$ yielding the $(n-1)$-simplex. Figure 3 shows this when $n = 3$, resulting in the 2-simplex in $\mathbb{R}^3$.

3.2. **Permutohedron.** Let G be the complete graph on n nodes. Each maximal tubing of G can be seen as a sequential nesting of all n nodes. In other words, they are in bijection with permutations on n letters. The elements of M_G will be assigned coordinate values based on all permutations of $\{0, 1, 3^1 - 3^0, \ldots, 3^{n-2} - 3^{n-3}\}$. Theorem 1 shows $\mathcal{P}G$ as the convex hull of the $n!$ vertices in $\mathbb{R}^n$, resulting in the permutohedron. Figure 4 shows this when $n = 3$, yielding the hexagon, the two-dimensional permutohedron.

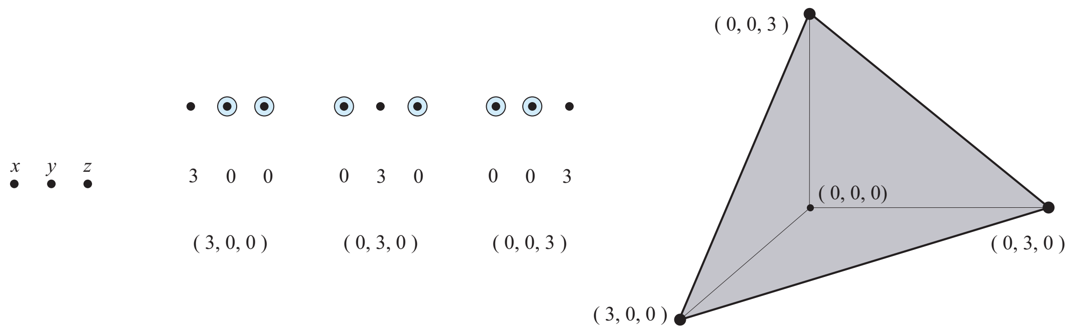

FIGURE 3. Maximal tubings of G and its convex hull, resulting in the simplex.

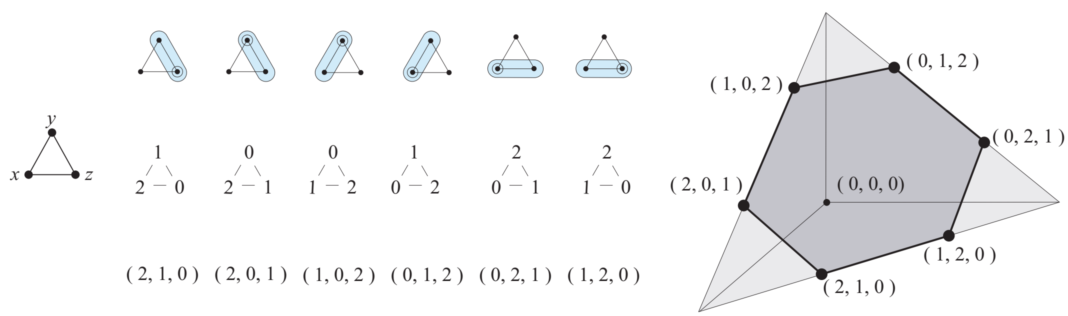

FIGURE 4. Maximal tubings of G and its convex hull, resulting in the permutohedron.

3.3. **Associahedron.** Let G be an n-path. The number of such maximal tubings is in bijection with the Catalan number $\frac{1}{n+1}\binom{2n}{n}$. Due to Theorem 1, the convex hull of these vertices in $\mathbb{R}^n$ yields the $(n-1)$-dimensional associahedron. Stasheff originally defined the associahedron for use in homotopy theory in connection with associativity properties of H-spaces [12]. Figure 5 displays the $n = 3$ case, resulting in the pentagon, the two-dimensional associahedron.

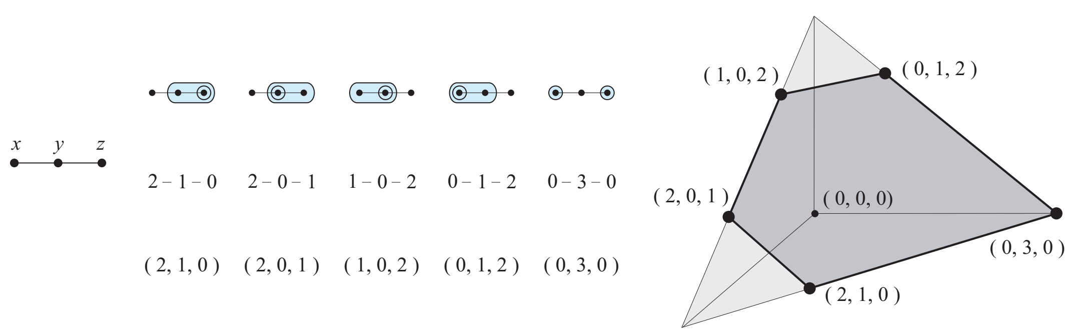

FIGURE 5. Maximal tubings of G and its convex hull, resulting in the associahedron.

3.4. **Cyclohedron.** Let G be an n-cycle. In this case, the number of maximal tubings is the type B Catalan number $\binom{2n-2}{n-1}$. Theorem 1 shows $\mathcal{P}G$ as the cyclohedron, a polytope originally manifested in the work of Bott and Taubes in relation to knot and link invariants [2]. Figure 4 shows this when $n = 3$, since the permutohedron and cyclohedron are identical in dimension two.

4. Constructing the Graph Associahedron

For a graph G with n nodes, let Δ be the $(n-1)$-simplex in which each facet (codimension 1 face) corresponds to a particular node of G. Thus, each proper subset of nodes of G corresponds to a unique face of Δ, defined by the intersection of the faces associated to those nodes. The following construction of the graph associahedron is based on truncations of a simplex.

Theorem 2. [3, Section 2] *For a given graph G, truncating faces of Δ which correspond to tubes of G in increasing order of dimension results in a realization of $\mathcal{P}G$.*

Figure 6 shows a tetrahedron truncated according to a graph, resulting in $\mathcal{P}G$. The truncations along two-dimensional faces are omitted from this picture since they do not change the combinatorial structure of the polytope. Note that the facets of $\mathcal{P}G$ are labeled with 1-tubings. One can verify that the edges correspond to all possible 2-tubings and the vertices to 3-tubings.

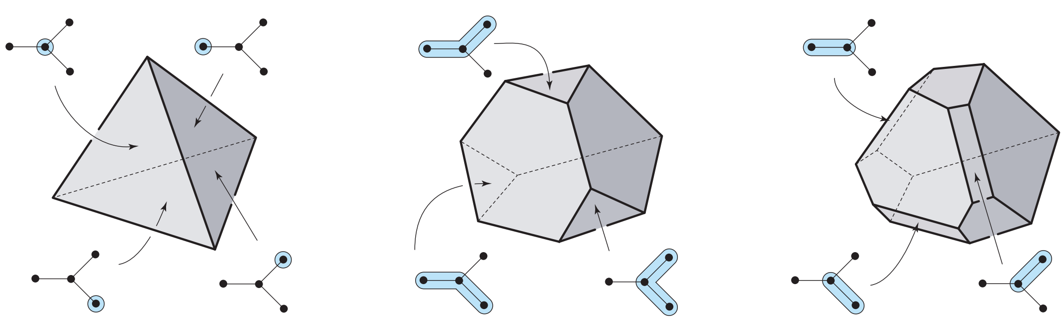

Figure 6. Iterated truncations of the 3-simplex based on an underlying graph.

Our goal is to make Theorem 2 more precise by explicitly constructing a simplex Δ along with a set of truncating hyperplanes resulting in the realization given in Theorem 1. The approach taken is influenced by the works of Loday [8] and Stasheff and Shnider [13, Appendix B].

Proof. Consider the affine hyperplane H of $\mathbb{R}^n$ defined by

$$\sum x_i = 3^{n-2}. \tag{4.1}$$

The intersection of the quadrant $\{(x_1, \ldots, x_n) \mid x_i \geq 0\}$ with H yields a standard $(n-1)$-simplex Δ. Let u be a tube of G containing k nodes; this corresponds to an $(n-1-k)$-dimensional face of Δ, seen as the hyperplane

$$\sum_{v_i \in u} x_i = 0$$

of $\mathbb{R}^n$ restricted to Δ. Associate to u the half-space h_u defined as[2]

$$\sum_{v_i \in u} x_i \geq 3^{k-2}. \tag{4.2}$$

We claim the intersection of these half-spaces with Δ, one for each tube of G, results in $\mathcal{P}G$. Even though by Theorem 2 appropriate faces of Δ have been truncated, the validity of this construction still needs to be shown. This is provided by the proof of Theorem 2 in [3, Section 2.4] which demands two conditions be met:

Let p be a vertex of $\mathcal{P}G$ viewed as a maximal tubing U of G.

(1) If tube $u \in U$, then p must lie in the intersection of Δ and the supporting hyperplane of h_u.
(2) If tube $u \notin U$, then p must lie in the intersection of Δ and the interior of the half-space h_u.

The first condition is satisfied simply by the construction of the coordinates of the vertices of $\mathcal{P}G$ as given by Eq. (2.1). To demonstrate the second condition, we must show that if $u \notin U$, then

$$\sum_{v_i \in u} f_U(v_i) > 3^{k-2}.$$

To each node v in G, recall that U assigns the smallest tube $t(v)$ in U containing v. We claim there exists a node v_* in u such that $u \subset t(v_*)$. Assume otherwise. Then for any node v_1 of u, since $u \not\subset t(v_1)$, we can choose a node v_2 in $u \setminus t(v_1)$ adjacent to $t(v_1)$.[3] The adjacency of v_2, along with U being a tubing implies $t(v_1) \subset t(v_2)$. We can continue to choose nodes v_{i+1} in $u \setminus t(v_i)$ adjacent to $t(v_i)$ resulting in a nested sequence $t(v_1) \subset t(v_2) \subset \cdots \subset t(v_{i+1})$. Since this process will exhaust all nodes in u, there must exist a node v_* in u such that $u \subset t(v_*)$.

Since $u \notin U$, the containment $u \subset t(v_*)$ is proper, implying $|t(v_*)| \geq k+1$. Therefore,

$$\begin{aligned}\sum_{v_i \in u} f_U(v_i) \quad &\geq \quad f_U(v_*) = \sum_{v_i \in t(v_*)} f_U(v_i) - \sum_{v_i \in t(v_*)-v_*} f_U(v_i) \\ &\geq \quad 3^{|t(v_*)|-2} - 3^{|t(v_*)|-3} = 2 \cdot 3^{|t(v_*)|-3} \geq 2 \cdot 3^{(k+1)-3} > 3^{k-2}.\end{aligned}$$

This satisfies the second condition, ensuring validity of the construction. □

[2] For ease of notation, define 3^{-1} to be 0 throughout the proof.

[3] This adjacency is possible because u is a tube of G.

5. Some Remarks

The realization above is determined by a function ϕ based on the tubes u of G. As seen in Eq. (2.1), the function chosen in this paper is

$$\phi(u) = 3^{|u|-2}. \tag{5.1}$$

As seen in the proof of Theorem 1, ϕ satisfies the inequality

$$\phi(u) > \phi(u_1) + \phi(u_2), \tag{5.2}$$

where tubes u_1 and u_2 are proper subsets of tube u. A geometric interpretation of property (5.2) is the avoidance of "deep cuts" during the truncation process: Consider Figure 7 as an example. Part (a) shows a 3-simplex with two vertices marked for truncation; part (b) shows appropriate truncations of the vertices, with (c) and (d) showing inappropriate cuts which are too deep. By ensuring property (5.2), two separate truncations of the corresponding

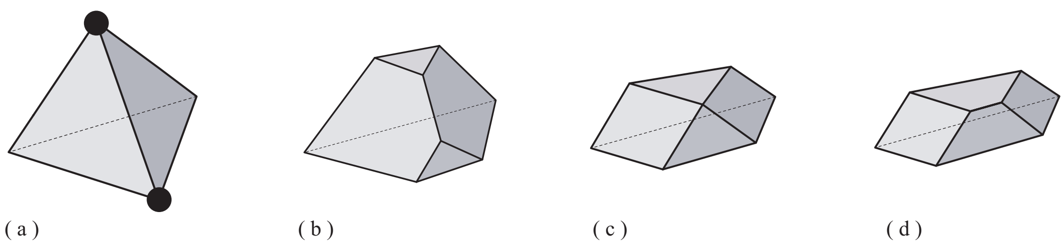

Figure 7. (a) Marked vertices of Δ along with (b) appropriate and (c)-(d) deep cuts.

cells of u_1 and u_2 will not meet and cut too deeply into a previous truncation of u.

We note that in order to recover Loday's elegant construction [8] of the classical permutohedron inside the associahedron, one can simply use

$$\phi(u) = \binom{|u|+1}{2}.$$

Although this works for the associahedron, it fails for the cyclohedron, and for graph associahedra in general. The reason is that this $\phi(u)$ does not comply with property (5.2), resulting in truncations with deep cuts.

Recently, Postnikov [10] constructs realizations of generalized permutohedra, a large family of polytopes encompassing graph associahedra. His approach is to use Minkowski sums of simplices, acquiring coordinates for vertices based on B-trees. Recasting the results in [10, Proposition 7.9] using the current terminology, his method uses a function $\phi(u)$ based on the number of *tubes* in u, whereas in contrast our method uses a $\phi(u)$ based on the number of *nodes* in u. Although Postnikov's method succeeds in a more general context, computing the exact value for his $\phi(u)$ is not an elementary notion. Indeed, the recent work of Postnikov et al. [11] is partly devoted to addressing such combinatorial issues in a much broader context.

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