The Inverse Source Problem for Wavelet Fields

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Abstract—The theory of the inverse source problem is employed to compute a class of continuously distributed and compactly supported three-dimensional (volume) sources that radiate the scalar wavelets investigated by Kaiser as well as certain electromagnetic generalizations of these scalar fields. These efforts have shown that the scalar wavelet fields can be radiated by a distributional source (generalized function) supported on a circular disk of radius \( a \) or an oblate spheroid surrounding that disk. Our main goal here is to replace this distributional source by a more conventional volume source that radiates the same wavelet field outside its support volume. The equivalent volume sources computed in this paper are supported on (three-dimensional) spherical shells whose outer radius \( a_+ > a \) and inner radius \( a_- < a \) are arbitrary. These sources are analytic functions of position within their support volumes for any finite, but arbitrarily large temporal frequency \( \omega \), and possess positive \( L^2 \) norm among all possible solutions to the inverse source problem with the given support volume constraint. Electromagnetic versions of the wavelet sources and fields are shown to be easily derived from their scalar wave counterparts.

Index Terms—Inverse source problem, pulsed beam, reliability, wavelet field.

I. THE SCALAR WAVELET FIELD

We consider the scalar wavelet field (refer to an overview and the relevant references)

\[
w(\mathbf{r}, t) = \frac{g(\tau - \hat{\tau})}{4\pi \hat{\tau}}\quad (1)
\]

where

\[
\tau = t - is, \quad \hat{\tau} = \sqrt{(\mathbf{r} - c \mathbf{a})^2} = \sqrt{r^2 - a^2 - 2iar \cos \theta} > 0\quad (2)
\]

with \((\mathbf{r}, t)\) being the space and time coordinates of a general observation point and \((\mathbf{a}, s)\) being parameters that can be selected at will so as to achieve desired behavior of the wavelet pulse. Here, \(a = |\mathbf{a}|\) and we have chosen the \(z\)-axis to be parallel to the vector \(\mathbf{a}\) so that \(\theta\) is the polar angle (see Fig. 1 for an illustration of the basic geometry under consideration). The condition \(\Re \hat{\tau} > 0\) amounts to choosing the disk

\[
D(\mathbf{a}) = \{ \mathbf{r} \in \mathbb{R}^3 : r \leq a, \mathbf{a} \cdot \mathbf{r} = 0 \}
\]

as the branch cut of the complex square root \(\hat{\tau} = g(t - is)\) is the analytic driving signal (positive-frequency part) associated with a real driving signal \(g(t)\) (in particular, see, e.g., [7, Eq. (5)], or [8, Eqs. (A1), (A2), and (A3)]) and as such it is analytic in the lower-half complex plane \((s > 0)\) and decays away from the real axis. The parameters \((\mathbf{a}, s)\) are subject to the restriction

\[
s > \frac{a}{c}\quad (3)
\]

where \(c\) is the constant propagation speed. This guarantees that \(\tau - \hat{\tau}/c\) is in the lower-half complex plane and that \(g(\tau - \hat{\tau}/c)\) is peaked around \(\theta = 0\) where the imaginary part of \(\tau - \hat{\tau}/c\) attains a (negative) maximum. The analytic signal \(g\) thus localizes the wavelet in both time and angle. The resulting pulse duration along \(\theta = 0\) is

\[
T = s - \frac{a}{c} > 0\quad (4)
\]

\(w(\mathbf{r}, t)\) is therefore a pulsed beam propagating in the direction of \(\mathbf{a}\) whose localization in time and angle are determined by the choice of the driving signal \(g\) and the parameters \(\mathbf{a}\) and \(s\).

Such “complex-source pulsed beams” have been studied by many authors; see [9], [10] for recent reviews as well as [11, Ch. 11]. Excellent localization in both time and angle is achieved by taking \(s \approx a/c\) and

\[
g(\tau - \frac{\hat{\tau}}{c}) = A \left(\frac{\tau - \hat{\tau}}{c}\right)^{-n}, \quad n \geq 4
\]

and both localizations improve with increasing \(n\). (Theoretically, the pulsed beam can be made arbitrarily short and narrow.) In addition to good time-angle localization, \(w\) is remarkable in that it has no side lobes. The scalar wavelet field can be radiated from a singular source distribution (generalized function)

Fig. 1. Illustration of the geometry.
supported spatially in the disk $D(a)$ whose time behavior is determined by the choice of $a$ and the analytic driving signal $g$ (see [5], [6] for the details). The associated real-valued scalar field $w(r,t)$ is obtained by taking the real part of $w$

$$w(r,t) = 2\Re w(r,t).$$

(4)

Our main goal in this paper is to compute a class of compactly supported three-dimensional (volume) sources $q(r,t)$ to the wave equation, i.e.,

$$\left(\nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2}\right) w(r,t) = -q(r,t)$$

(5)

that will radiate the scalar wavelet field defined in (1) everywhere outside the volume $V = \{ r \in \mathbb{R}^3 : r < a_+ \}$ bounded by a sphere of radius $a_+$ which is assumed to bound the source support volume $V_0 \subseteq V$. (Remark: Later in the presentation we will focus on the particular case of a three-dimensional spherical shell $V_0 = \{ r \in \mathbb{R}^3 : a_- \leq r \leq a_+ \}$ of inner radius $a_-$ and outer radius $a_+$. We will also compute volume sources that radiate certain electromagnetic (EM) generalizations of the wavelet field. The fields radiated by the volume sources will be identical to the original wavelet field given by (1) everywhere outside $V$ but will generally differ from the field in (1) within $V$. The radiated wavelet field and source will be related in the frequency domain via the standard equation

$$W(r,\omega) = \int_0^\infty dr'Q(r',\omega) \frac{e^{ik|r-r'|}}{4\pi|\mathbf{r}-\mathbf{r'}|}$$

(6)

where $k = 2\pi/\lambda$ is the wavenumber, $V_0$ is the source volume and $W(r,\omega)$ and $Q(r,\omega)$ are the frequency domain wavelet field and source, respectively. The time-domain quantities are then recovered from $W$ and $Q$ via the relationships

$$u(r,t) = \frac{1}{2\pi} \int_0^\infty d\omega W(r,\omega) e^{-i\omega t}$$

$$q(r,t) = \frac{1}{2\pi} \int_0^\infty d\omega Q(r,\omega) e^{-i\omega t}.$$ 

EM wavelet fields can be constructed by taking a scalar wavelet field to be a Debye potential in the so-called Debye representation of the EM field [12]–[14]. For example, in regions outside the source volume, general electric and magnetic wavelet fields can be constructed from the prescription

$$E(r,\omega) = \nabla \times \nabla \times [r W_e(r,\omega)] + ik \nabla \times [r W_h(r,\omega)]$$

(7a)

$$H(r,\omega) = -ik \nabla \times [r W_e(r,\omega)] + \nabla \times [r W_h(r,\omega)]$$

(7b)

where $W_e$ and $W_h$ are two scalar wavelet fields generated by scalar sources $Q_e$ and $Q_h$. EM sources generating these fields can then be easily derived (Section VI) in terms of $Q_e$ and $Q_h$ from the source representation (6) of the scalar fields.

We will employ the well developed theory of the inverse source problem (ISP) (see [1]–[4] and the references therein) to construct volume sources for the wavelet fields that are supported within spherical shells. Although other source geometries are possible [15], the spherical geometry is analytically friendlier, and leads to source distributions that may be realized in actual applications. We should keep in mind throughout the development that the parameters $a$ and $s$ play key roles in the theory. As mentioned above, $a$ is the radius of the circular disk over which the (distributional) source in [5], [6] is supported and $s = a/c > 0$ is a measure of the time duration of the wavelet pulse. In particular, the distributional wavelet source is entirely contained within a spherical volume of radius $a_+ > a$ centered at the origin (see Fig. 1).

II. RADIATION PATTERN AND RADIATED ENERGY OF THE SCALAR WAVELET FIELD

The frequency domain representation of the wavelet field (1) is found by taking its temporal Fourier transform. We find [5]

$$W(r,\omega) = \int_{-\infty}^{\infty} dt \, w(r,t) e^{i\omega t} = G(\omega) \frac{e^{-\omega s} e^{ikr}}{4\pi r}$$

(8a)

where $G(\omega)$ is the Fourier transform of the (complex-valued) boundary value of $g(t \rightarrow is)$ on the real axis as $s \rightarrow 0^+$ and which vanishes at negative frequencies

$$G(\omega) = 0 \text{ for } \omega < 0$$

(8b)

so that $e^{-\omega s}$ decays exponentially with $\omega > 0$. In view of (8b), we will, from this point on, tacitly assume that the angular frequency $\omega$ and the associated wavenumber $k = \omega/c$ are strictly real and positive.

The radiation pattern of a field is defined to be the coefficient of the leading order term in the asymptotic expansion of the frequency domain representation of the field as $kr \rightarrow \infty$ along some fixed direction defined by the unit vector $\hat{r}$. For $r \gg a$, it follows from (2) that

$$\hat{r} \sim r - i\mathbf{a} \cdot \hat{r} - i\mathbf{a} \cos \theta$$

where $\theta$ is the angle between the unit vectors $\mathbf{a}$ and $\hat{r}$. It then follows from (8a) that

$$W(r,\omega) \sim f(\theta,\omega) \frac{e^{ikr}}{4\pi r}$$

(9a)

where

$$f(\theta,\omega) = G(\omega) e^{-\omega s} e^{ka \cos \theta} = G(\omega) e^{-k(c-a \cos \theta)}$$

(9b)

is the radiation pattern of the scalar wavelet field. The radiation pattern of the field radiated by a wavelet source is thus rotationally symmetric about the unit vector $\mathbf{a}$.

Under the reasonable assumption that $G(\omega)$ is bounded, the radiation pattern decays exponentially with positive $\omega$ and $k = \omega/c$. Note that the radiation pattern depends only on the selection of $G(\omega)$ and the fundamental parameters $s$ and $\mathbf{a}$ of the wavelet field. As we mentioned above, $a = |\mathbf{a}|$ represents the radius of the distributional source disk $D(\mathbf{a})$, whose normal is given by $\mathbf{a} = \mathbf{a}/a$, while $s$, along with the driving signal $g$, governs the time behavior of the source and hence that of the radiated wavelet. It can be shown from the analysis in [16] that the disk $D(\mathbf{a})$ is the minimum convex surface containing sources.
that can produce the radiation pattern in (9b), thus the distributional disk source is optimal in that it employs the minimally required spatial resources for the launching of that radiation pattern. Yet this is accomplished for the disk source at the expense of functional singularities. Our central objective in this work is to apply inverse source theory to derive spatially larger, yet functionally better behaved, volume sources radiating the wavelet.

A. Radiated Energy of the Scalar Wavelet Field

Any field radiated by a compactly supported and bounded source must satisfy certain realizability conditions in the frequency domain. The first of these is that the source must radiate a finite amount of energy. The energy radiated out of any closed surface $\partial V$ that completely surrounds the source volume $V_0 \subseteq V$ is given by

$$E_q = \frac{1}{\pi} \int_0^\infty d\omega E_q(\omega)$$  \hspace{1cm} (10a)

where

$$E_q(\omega) = 2\kappa \omega^3 \int_{\partial V} dS W^*(r, \omega) \frac{\partial W(r, \omega)}{\partial n}$$  \hspace{1cm} (10b)

is the energy spectrum with $W$ being the (positive frequency part) of the radiated wavelet field and where $\kappa$ is a constant that depends on the type of scalar field (e.g., acoustic, optical, etc.). Realizability clearly requires that the total radiated energy $E_q$ be finite.

It is not difficult to show that the energy spectrum $E_q(\omega)$ is independent of the particular closed surface $\partial V$ over which the integral (10b) is computed. Taking this surface to be an infinite sphere we find that

$$E_q(\omega) = 2\pi \kappa \omega^3 |G(\omega)|^2 \int_{4\pi} d\Omega |f(\theta, \omega)|^2$$  \hspace{1cm} (11)

where $f(\theta, \omega)$ is the radiation pattern of the field and the integration is over $4\pi$ steradian. On making use of the expression (9b) for the radiation pattern of the wavelet field we conclude that

$$E_q(\omega) = 2\pi \kappa \omega^3 |G(\omega)|^2 \int_{4\pi} d\Omega e^{-2\omega s} e^{2k\alpha \cos \theta}$$

= $\frac{2\pi \kappa \omega}{a} e^{-2\omega s} |G(\omega)|^2 e^{2k\alpha a} - e^{-2k\alpha}].$

On substituting the above result into (10a) we then conclude that the total energy radiated by the wavelet source is

$$E_q = \frac{2\kappa}{a} \int_0^\infty d\omega \omega^3 |G(\omega)|^2 e^{-2\omega s} e^{2k\alpha a} - e^{-2k\alpha}$$  \hspace{1cm} (12)

which is clearly finite so long as $s > a/c$ as is assumed. It is interesting to note that realizability of the wavelet source as measured by the finite radiated energy requirement involves both the source radius $a$ as well as its time behavior as governed by the parameter $s$. Classically, realizability conditions are stated in the frequency domain and depend only on the source radius.

In the special case where $G(\omega) = 1, \omega > 0$ the integral in (12) can be performed and we find that

$$E_q = \frac{2\kappa \omega^3 s}{(c^2 s^2 - a^2)^2}.$$  \hspace{1cm} (13)

III. THE INVERSE SOURCE PROBLEM

The inverse source problem (ISP) has as its goal the computation of a volume source $Q(r, \omega)$ supported in a given volume $V_0$ that will radiate a specified field everywhere outside that source volume. It is well known [12, 17] that the radiation pattern uniquely defines the radiated field everywhere outside the source support, so that the ISP can be formulated as that of reconstructing an unknown source of support $V_0$ that generates a given radiation pattern. The radiation pattern of the scalar wavelet field has already been computed in (9b) and can be expressed in terms of the source $Q$ by making use of (6). We find that

$$W(r, \omega) \sim f(\hat{r}, \omega) e^{ikr}$$  \hspace{1cm} (14a)

where the radiation pattern $f(\hat{r}, \omega)$ is defined in terms of the scalar wavelet source via the equation

$$f(\hat{r}, \omega) = \int_{V_0} d^3 r' Q(\hat{r}, \omega) e^{-ikr'},$$  \hspace{1cm} (14b)

The ISP then reduces to solving (14b) for the source $Q(r, \omega)$ in terms of the scalar wavelet radiation pattern $f(\hat{r}, \omega)$ specified for all observation directions $\hat{r}$ and for some set of temporal frequencies $\omega$.

We will take the $z$ axis of our coordinate system to lie along the direction of the fixed unit vector $\hat{a}$ and express the plane wave $\exp(-ik\hat{r} \cdot r')$ in the multipole expansion

$$e^{-ik\hat{r} \cdot r'} = 4\pi \sum_{l=0}^{\infty} \sum_{m=-l}^{l} (-i)^l j_l(kr') Y_l^m(\theta, \phi) Y_l^{m*}(\theta', \phi')$$  \hspace{1cm} (15)

where $\theta$, $\phi$, and $\theta'$, $\phi'$ are the polar and azimuthal angles of the unit vector $\hat{r}$ and the vector $r'$, respectively, and $j_l$ and $Y_l^m$ denote spherical Bessel functions of the first kind and the spherical harmonics. On substituting (15) into (14b) we obtain the result

$$f(\hat{r}, \omega) = 4\pi \sum_{l=0}^{\infty} \sum_{m=-l}^{l} (-i)^l a_l^m(\omega) Y_l^m(\theta, \phi)$$  \hspace{1cm} (16a)

where

$$a_l^m(\omega) = \int_{V_0} d^3 r' Q(\hat{r}', \omega) j_l(kr') Y_l^{m*}(\theta', \phi').$$  \hspace{1cm} (16b)

Now since the radiation pattern of the scalar wavelet field given in (9b) depends only on the polar angle $\theta$ and not the azimuthal
angle $\phi$ we can integrate both sides of (16a) over $[0, 2\pi]$ in $\phi$ which yields the simplified form

$$f(\theta, \omega) = \frac{1}{2\pi} \int_0^{2\pi} d\phi f(\mathbf{r}, \omega)$$

$$= \sum_{l=0}^{\infty} (-1)^l (2l+1) f_l(\omega) P_l(\cos \theta)$$

(17a)

where $P_l$ denotes the ordinary Legendre polynomial of degree $l$ (associated Legendre function of degree $l$ and order zero), and where we have used the result

$$\frac{1}{2\pi} \int_0^{2\pi} d\phi Y_l^m(\theta, \phi) = \sqrt{\frac{2l+1}{4\pi}} P_l(\cos \theta) \delta_{m,0}$$

and have defined

$$f_l(\omega) = \int_{V_0} d^3 r Q(\mathbf{r}, \omega) j_l(|kr|) P_l(\cos \theta),$$

(17b)

Equation (17) form the basis for our treatment of the ISP for the scalar wavelet field. These equations are completely equivalent to (14b) for the special case of a rotationally symmetric radiation pattern about $z$ axis and their solution yields a complete solution to the ISP for this class of radiation patterns. Our goal is the computation of band limited volume sources that will radiate the wavelet fields over the finite, but arbitrary, bandwidth of the source. In the scalar wave case the error introduced by the band limited approximation can be measured by the total radiated energy $E_{\text{call}}$ produced by the difference between the ideal and band limited sources which is given by the r.h.s. of (12) but with a lower limit in the integral equal to the highest frequency $\omega_0$ of the band limited approximation. We then conclude that

$$E_{\text{call}} \leq 2K \max |G(\omega)|^2 \int_{\omega_0}^{\infty} d\omega \omega e^{-2\omega(s-\alpha/c)}$$

$$\leq \frac{K}{a} \max |G(\omega)|^2 \frac{\omega_0 e^{-2\omega(s-\alpha/c)}}{s - \alpha/c}$$

(18)

where $\max |G(\omega)|^2$ is the maximum value of $G(\omega)$ over all frequencies. It is clear from (18) that the error introduced by the band limited approximation can be made arbitrarily small by taking $\omega_0$ sufficiently large as long as $s > \alpha/c$ which is one of the conditions already imposed on the wavelet fields. The same occurs in the EM case where the error as measured by the radiated energy of the source produced by the difference between the ideal and band limited sources can be made arbitrarily small by taking the source bandwidth sufficiently large.

A. Computing the Band Limited Source

It follows from (17) that the radiation pattern determines the projection of the source $Q$ onto the set of functions

$$\Lambda_l(\mathbf{r}, \omega) = \begin{cases} j_l(kr) P_l(\cos \theta), & \mathbf{r} \in V_0 \\ 0, & \text{else} \end{cases}$$

(19)

Any solution to the ISP (for rotationally symmetric radiation patterns) must then be of the general form [2]-[4], [18], [19]

$$Q(\mathbf{r}, \omega) = \sum_{l=0}^{\infty} \frac{f_l(\omega)}{N_l(\omega)} \Lambda_l(\mathbf{r}, \omega) + \delta Q(\mathbf{r}, \omega)$$

(20a)

where

$$N_l(\omega) = \int_{V_0} d^3 r |\Lambda_l(\mathbf{r}, \omega)|^2$$

(20b)

is the square of the $L^2$ norm of $\Lambda_l$ and $\delta Q$ is a so-called non-radiating source (see [4], [20] for overviews and further references) that is supported within $V_0$ and orthogonal to the set \{\$\Lambda_l$\}. Of particular interest is the solution having minimum $L^2$ norm or functional energy, sometimes referred to in the literature as the minimum-energy (ME) source [2]-[4], [18], [19], which results from taking the non-radiating component of $Q$ equal to zero:

$$Q_0(\mathbf{r}, \omega) = \sum_{l=0}^{\infty} \frac{f_l(\omega)}{N_l(\omega)} \Lambda_l(\mathbf{r}, \omega),$$

(21)

We note in passing that the source $Q_0$, like the radiation pattern, is rotationally symmetric about the $z$ (beam) axis.

In this paper we limit our attention to the special case where the source volume $V_0$ is a spherical shell having an outer radius $a_+$ and an inner radius $a_+ < a_+$. In this case the functions $\Lambda_l$ become

$$\Lambda_l(\mathbf{r}, \omega) = \begin{cases} j_l(kr) P_l(\cos \theta), & a_- \leq r \leq a_+ \\ 0, & \text{else} \end{cases}$$

(22)

These functions are orthogonal over any spherical shell having outer radius $a_+$ and inner radius $a_- < a_+$ with $L^2$ norm square given by

$$N_l(\omega) = \frac{4\pi}{2l+1} \int_{a_-}^{a_+} r^2 dr j_l^2(kr)$$

$$= 4\pi \left[ j_l^2(a_+, \omega) - j_l^2(a_-, \omega) \right]$$

(23a)

where

$$j_l^2(b, \omega) = \frac{1}{2l+1} \int_0^b r^2 dr j_l^2(kr)$$

$$= \frac{b^3}{4l+2} - \left[ j_l(kb) - j_{l-1}(kb) j_{l+1}(kb) \right].$$

(23b)

The above manipulations are purely formal and it is necessary to check that the series (21) converges and defines a square integrable function in $V_0$. In particular, we require that the set of multipole moments $f_l(\omega)$ satisfy the so-called Picard condition [3], [19]

$$\int_{V_0} d^3 r |Q_0(\mathbf{r}, \omega)|^2 = \sum_{l=0}^{\infty} \frac{f_l(\omega)^2}{N_l(\omega)} < \infty$$

(24a)

which defines the class of multipole moments, and their respective radiated fields for $\mathbf{r} \in V_0$, that are realizable with square integrable sources supported within the source volume $V_0$. It is clear that Picard’s condition will be automatically satisfied if the
ME source $Q_0$ is bounded over its support volume $V_0$ as will be the case for the wavelet source derived in Section IV.

Since we will tacitly assume that the source is strictly band limited it then follows that the Picard condition (24a) automatically guarantees that the source is square integrable over both space and time; i.e., that

$$\int_{-\infty}^{\infty} dt \int_{V_0} d^3r \vert q_0(r, t) \vert^2 = \frac{1}{2\pi} \int_0^\infty d\omega \int_{V_0} d^3r \vert Q_0(r, \omega) \vert^2 = \frac{1}{2\pi} \int_0^\infty d\omega \sum_{l=0}^\infty \frac{\vert f_l(\omega) \vert^2}{N_l(\omega)} < \infty \tag{24b}$$

where $\omega_0$ is the largest positive frequency of the source and $q_0(r, t)$ is the time-domain minimum $L^2$ norm source:

$$q_0(r, t) = \frac{1}{\pi} \int_0^{\omega_0} d\omega Q_0(r, \omega)e^{-i\omega t} = \frac{1}{2\pi} \int_0^{\omega_0} d\omega Q_0(r, \omega)e^{-i\omega t} \tag{25}$$

where $Q_0$ has been extended to negative frequencies by the reality condition

$$Q_0(r, -\omega) \equiv Q_0(r, \omega)^*, \quad \omega > 0.$$ 

It follows from the above results that the time dependent wavelet field can be radiated by the time dependent ME source given by

$$q_0(r, t) = \frac{1}{\pi} \int_0^{\omega_0} d\omega \sum_{l=0}^\infty f_l(\omega)N_l(r, \omega)e^{-i\omega t} = \sum_{l=0}^\infty q_l(r, t)P_l(\cos \theta) \tag{26a}$$

where

$$q_l(r, t) = \frac{1}{\pi} \int_0^{\omega_0} d\omega \frac{f_l(\omega)}{N_l(\omega)}j_l(\nu r)e^{-i\omega t}, \quad a_- \leq r \leq a_+ \tag{26b}$$

and the multipole moments $f_l$ are determined from the radiation pattern via (17b) and must satisfy the condition (24b).

IV. SOLUTION OF THE ISP FOR THE SCALAR WAVELET FIELD

In order to solve the ISP for the wavelet field (via (21)) it is necessary to compute the multipole moments $f_l(\omega)$ from the wavelet field’s radiation pattern in (9b). If we make use of the expansion

$$e^{i\kappa \cos \theta} = \sum_{l=0}^\infty (i)^l(2l+1)j_l(\kappa r)P_l(\cos \theta) \tag{27}$$

we find that the radiation pattern of the wavelet field (9b) can be expanded as

$$f(\theta, \omega) = \mathcal{F}(\omega)e^{-i\omega \delta} \sum_{l=0}^\infty (i)^l(2l+1)j_l(\kappa r)P_l(\cos \theta), \quad \omega > 0 \tag{28}$$

from which it follows that the multipole moments of the wavelet field (as defined in (17)) are given by

$$f_l(\omega) = \mathcal{F}(\omega)e^{-i\omega \delta}j_l(\kappa r), \quad \omega > 0. \tag{29}$$

and are thus proportional to the spherical Bessel functions with imaginary argument $ika$. The ME solution of the ISP for the wavelet field is then obtained in the frequency domain by substituting the expression (29) into (21), with $N_l(\omega)$ given by (23a).

We obtain the result

$$Q_0(r, \omega) = \mathcal{F}(\omega)e^{-i\omega \delta} \sum_{l=0}^\infty j_l(\kappa a)N_l(r, \omega) \tag{30}$$

A. Boundedness and Analyticity of the Source

In this section we prove that the scalar wavelet source is bounded and analytic over the source region if $a_+ > a$, as we are requiring. It of course follows from this that the source is square integrable so that Picard’s condition (24a) is automatically satisfied if $a_+ > a$. We first prove boundedness which follows from the fact that the infinite sum in the wavelet source expression (30) is bounded above by the function

$$S(r, \omega) = \sum_{l=0}^\infty \left| \frac{j_l(\kappa a)L_l(r)}{N_l(\omega)} \right| \tag{31}$$

where $N_l(\omega)$ is previously defined in (23) (in arriving at this result we have used the property $\vert PL(\cos \theta) \vert \leq 1$ for real $\theta$), so that for the pertinent positive frequencies the ME source $Q_0$ in (30) is bounded for bounded $G(\omega)$. It is clear that the individual terms in the above series [31] are bounded for any finite $l$ so that we only have to prove that the series (31) converges to show that the wavelet source is bounded. For large index $l$, the spherical Bessel functions $j_l(z)$ have the asymptotic form [22]

$$j_l(z) \sim \sqrt{\frac{\pi}{2z}} \left( \frac{z}{2} \right)^{l+1/2}, \quad l \gg |z|, \quad z \in \mathbb{C}. \tag{32}$$

Hence

$$\left| j_l(\kappa a) \right| \sim \left( \frac{i^l}{2^{l+1}} \right) \frac{(ka)^l}{(l+\frac{3}{2})}, \quad l \gg ka \tag{33a}$$

and

$$\left| j_l(\kappa a)j_l(\kappa r) \right| \sim \left( \frac{i^{2l}}{2^{2l+1}} \right) \frac{(ka)^l(r^2)}{(l+\frac{3}{2})}, \quad l \gg ka. \tag{33b}$$

From the same results, the quantities $j_l^2(a_+, \omega)$ defined in (23b) are found to have the asymptotic form

$$j_l^2(a_+, \omega) = \frac{1}{2^l+1} \int_0^{a_+} j_l^2(\kappa r)^2 dr \sim \frac{\pi a_+^2(ka_+)^2}{2^{l+2}(2l+1)(2l+3)^2}, \quad l \gg ka_+ \tag{33c}$$
so that by using also (23a) one finds that

$$\frac{|j_0(ika)| |j_0(kr)|}{N_\ell(\omega)} \sim \frac{L^2}{\pi a^2} \left( \frac{ar}{ak} \right)^I, \quad l \gg ka. \quad (34)$$

Applying the ratio test we then conclude that the series (31) will converge and the source will be bounded so long as

$$\frac{ar}{ak} < 1 \rightarrow r < \frac{a_+}{a}$$

which, because of our requirement that $a_+ > a$, will certainly hold as long as $r \leq a_+$, as is the case for all points in the spherical shell of outer radius $a_+$.

1) Analyticity of the Source: The functions $A_I(\mathbf{r}, \omega)$ are analytic functions of $r$, $\theta$ throughout any closed region within the source volume $a_- < r < a_+$. Since the series (30) converges absolutely and uniformly throughout such regions it then follows that it represents an analytic function of $r$, $\theta$ throughout such regions. The absolute and uniform convergence of this series follows after manipulations analogous to those given above to show boundedness of the series (30) by invoking the Weierstrass M test where the comparison series has terms $M_I = |j_0(ika)| |j_0(kr)|/N_\ell(\omega)$. Here we wish to comment that Picard’s condition (24a) for wavelet fields can also be established after some manipulations from (32) and the ratio test. The reader is referred to [3] for further discussion of the connection between Picard’s condition and the absolute and uniform convergence of minimum energy sources for more general fields.

V. COMPUTER SIMULATION

We performed a simulation in the frequency domain where we computed the scalar wavelet source for the case where the inner radius $a_-$ was set equal to zero and the outer radius $a_+$ was set equal to $a + \epsilon > a$ (for some small $\epsilon$). We selected the wavelet parameter $a$ to be $a = 10\lambda$, with $\lambda$ equal to unity. Due to practical reasons (in particular, the truncation of the minimum energy series, as explained next), in the numerical simulation, the strict inequality requirement $a_+ > a$ was not necessary, and we used $a_+ = a$ with successful results. In fact, it is easily verified that the terms in this series decay exponentially fast with $l$ for $l > [ka/2]$ where the bracket $[\cdot]$ stands for the next highest integer. This is illustrated in Fig. 2 which shows a plot of the coefficients $|j_0(\omega)|^2 N_\ell(\omega)$ as a function of the index $I$ for $ka_+ = ka = 20\pi \approx 64$. For a source with outer radius $a = 10\lambda$ this yields an effective cutoff index of $I_{\text{cutoff}} \approx [ka/2] \approx 32$. We show a plot of $|j_0(\omega)|^2 N_\ell(\omega)$ as a function of index $I$ in the top of Fig. 2 and plots of the ideal and regularized radiation patterns in the lower part of the figure where the regularized pattern was computed using $I_{\text{cutoff}} = 32$. In the simulation we used a value of $s = a/c$ so that the radiation pattern is normalized to have a maximum value of unity at $\theta = 0$ (on axis). It is clear from the figure that the regularized source computed using $I_{\text{cutoff}} \approx [ka/2]$ will generate the wavelet radiation pattern to great accuracy. However, it must be realized that this source is regularized and will not radiate the wavelet field in the very near field of the source (within a wavelength of the source).

The regularized source was computed using (30) with a maximum $l$ value equal to $I_{\text{cutoff}} = [ka/2] = 32$. This source is rotationally symmetric about the beam ($z$) axis. We show in Fig. 3 mesh plots of the real part (top) and magnitude (bottom) of this source distribution over any plane containing the beam axis. It should be clear from this figure that the imaginary part of the source is basically simply a phase shifted version of the real part. To check on the accuracy of the source reconstruction we computed the radiation pattern directly from the source using (17a) and (17b). We obtained a radiation pattern virtually identical to the regularized pattern shown in Fig. 2.

VI. EM WAVELET SOURCES

The preceding scalar field results are important in their own right, e.g., for the physical realization of well-behaved volume
sources of compact support launching acoustic wavelet fields
[8]. We conclude the paper by showing that the same results
are also of interest in the physical realization of EM wavelet
sources and their fields. The proposed construction procedure
for EM wavelet fields is based on the Debye representation given
in (7) where \( W_e \) and \( W_h \) are two scalar wavelet fields defined
by parameters \( \alpha_e, \beta_e \) and \( \alpha_h, \beta_h \), respectively.

A. Radiation Pattern of the EM Wavelets

The frequency domain representation of the scalar wavelet field
was obtained in (8a) and is given by

\[
W(\mathbf{r}, \omega) = G(\omega)e^{-i\omega t - \frac{ik\mathbf{r}}{\omega}} \tag{35}
\]

where \( \mathbf{r} \) is defined in (2). Let us take this scalar wavelet to be the
electric Debye potential so that the electric and magnetic fields are
given by

\[
\begin{align*}
\mathbf{E}(\mathbf{r}, \omega) &= \nabla \times \nabla \times [\frac{1}{\epsilon} W_e(\mathbf{r}, \omega)] \\
\mathbf{H}(\mathbf{r}, \omega) &= -\frac{\lambda}{\mu} \nabla \times [\frac{1}{\epsilon} W_e(\mathbf{r}, \omega)].
\end{align*} \tag{36}
\]

If we select the \( z \) axis to lie along the vector \( \mathbf{a} \) then we have that

\[
\nabla \times \mathbf{r} = \mathbf{r} \times \nabla = \phi \frac{\partial}{\partial \theta} + \theta \frac{1}{\sin \theta} \frac{\partial}{\partial \phi}.
\]

On using this in the expression for the \( \mathbf{H} \) field we find that

\[
\mathbf{H}(\mathbf{r}, \omega) = -i\frac{\lambda}{\mu} \nabla \times [\frac{1}{\epsilon} W_e(\mathbf{r}, \omega)] = \frac{\lambda}{\mu} \phi \frac{\partial}{\partial \theta} W_e(\mathbf{r}, \omega)
\]

since \( W_e \) does not depend on \( \phi \). On substituting from (35) we then find that

\[
\mathbf{H}(\mathbf{r}, \omega) = \frac{\lambda}{\mu} \phi \left\{ i k G(\omega) e^{-i\omega t} \frac{\partial}{\partial \theta} \right\} e^{i\frac{k\mathbf{r}}{\omega}} \tag{37}
\]

The partial derivatives in (37) are readily performed and we find that

\[
\mathbf{H}(\mathbf{r}, \omega) = \frac{\lambda}{\mu} \phi \left\{ i k G(\omega) e^{-i\omega t} \left[ \frac{k \sin \theta}{r} + \frac{i a r \sin \theta}{r^2} \right] \right\} e^{i\frac{k\mathbf{r}}{\omega}} \tag{38}
\]

which behaves asymptotically as

\[
\mathbf{H}(\mathbf{r}, \omega) \sim \frac{\lambda}{\mu} \phi \left\{ i k^2 a G(\omega) e^{-i\omega t} \sin \theta e^{i\frac{k\mathbf{r}}{\omega}} \right\} \tag{39a}
\]

A parallel calculation yields the following far field result for the
electric field component:

\[
\mathbf{E}(\mathbf{r}, \omega) \sim \frac{\lambda}{\mu} \phi \left\{ i k^2 a G(\omega) e^{-i\omega t} \sin \theta e^{i\frac{k\mathbf{r}}{\omega}} \right\} \tag{39b}
\]

As in the scalar wave case we define the radiation pattern of the
EM field to be the coefficient of the leading order term in the
asymptotic expansion of the field as \( k\mathbf{r} \to \infty \) along some fixed
direction defined by the unit vector \( \mathbf{\hat{r}} \). It then follows from (39)
that the radiation patterns of the \( E \) and \( H \) fields are given by

\[
\begin{align*}
f_{\mathbf{e}}(\theta, \omega) &= f_{\mathbf{h}}(\theta, \omega) = \frac{1}{\frac{\lambda}{\mu} \phi} e^{i\theta} \int d^3e Q_e(\mathbf{r}', \omega) e^{ik|\mathbf{r}' - \mathbf{r}|} \tag{40a} \\
f_{\mathbf{h}}(\theta, \omega) &= f_{\mathbf{h}}(\theta, \omega) = \frac{1}{\frac{\lambda}{\mu} \phi} e^{i\theta} \int d^3h Q_h(\mathbf{r}', \omega) e^{ik|\mathbf{r}' - \mathbf{r}|} \tag{40b}
\end{align*}
\]

where \( f_{\mathbf{e}} \) and \( f_{\mathbf{h}} \) are the magnetic and electric field radiation patterns
and \( f \) is the radiation pattern of the scalar wavelet field
defined in (9b). We see that the radiation patterns of the EM
wavelet field are directly proportional to the scalar wavelet
radiation pattern. The factor \( \sin \theta \) means, of course, that the EM
radiation patterns have a “hole” in the center of the pattern (at
\( \theta = 0 \)) but also possess the desirable narrow beam pattern of
the scalar wavelet field. The above calculation used the electric
Debye potential but a completely parallel computation leads to
an analogous result for the magnetic Debye potential.

B. EM Wavelet Charge-Current Distributions

If we express the wavelet fields in (7) via the source representation
(36) we obtain the results

\[
\begin{align*}
\mathbf{E}(\mathbf{r}, \omega) &= \nabla \times \nabla \times \left[ \frac{r}{\epsilon} \int_{V_0} d^3r' Q_e(\mathbf{r}', \omega) e^{i\frac{k|\mathbf{r}' - \mathbf{r}|}{\omega}} \right] \\
&\quad + i k \nabla \times \left[ \frac{r}{\epsilon} \int_{V_0} d^3r' Q_h(\mathbf{r}', \omega) e^{i\frac{k|\mathbf{r}' - \mathbf{r}|}{\omega}} \right] \\
\mathbf{H}(\mathbf{r}, \omega) &= - \frac{r}{\mu} \nabla \times \left[ \frac{r}{\epsilon} \int_{V_0} d^3r' Q_e(\mathbf{r}', \omega) e^{i\frac{k|\mathbf{r}' - \mathbf{r}|}{\omega}} \right] \\
&\quad + \nabla \times \nabla \times \left[ \frac{r}{\epsilon} \int_{V_0} d^3r' Q_h(\mathbf{r}', \omega) e^{i\frac{k|\mathbf{r}' - \mathbf{r}|}{\omega}} \right] \tag{41a}
\end{align*}
\]

where \( Q_e \) and \( Q_h \) are the scalar volume sources computed in
the preceding section. It is important to note that the above
representations only apply to the divergence free components
of the electric and magnetic field vectors. However, since the magnetic
field is everywhere divergence free it is valid over all of space
and, hence, will form the basis for the computation of the
volume current and charge distributions.\(^1\)

We can compute the curl of the volume current directly from
the magnetic field vector through the equation (in Gaussian
units)

\[
\nabla \times \mathbf{J}(\mathbf{r}, \omega) = -\frac{4\pi}{c} \nabla \times \mathbf{J}(\mathbf{r}, \omega). \tag{42}
\]

On substituting the expression (41b) for the \( \mathbf{H} \) field in the l.h.s.
of the above equation we find that

\[
\begin{align*}
\nabla^2 + \frac{k^2}{\omega^2} \mathbf{H}(\mathbf{r}, \omega) &= \frac{4\pi}{c} \nabla \times \mathbf{J}(\mathbf{r}, \omega) \\
&= \frac{4\pi}{c} \left[ \nabla \times \left( \frac{r}{\epsilon} \int_{V_0} d^3r' Q_e(\mathbf{r}', \omega) \left\{ \nabla^2 + \frac{k^2}{\omega^2} \right\} e^{i\frac{k|\mathbf{r}' - \mathbf{r}|}{\omega}} \right) \right] \\
&\quad + \nabla \times \nabla \times \left[ \frac{r}{\epsilon} \int_{V_0} d^3r' Q_h(\mathbf{r}', \omega) \left\{ \nabla^2 + \frac{k^2}{\omega^2} \right\} e^{i\frac{k|\mathbf{r}' - \mathbf{r}|}{\omega}} \right] \\
&= i k \nabla \times \nabla \times \mathbf{J}(\mathbf{r}, \omega) - \nabla \times \nabla \times \mathbf{Q_h}(\mathbf{r}, \omega)
\end{align*}
\]

where we have used the fact that the D’Alembertian operator
commutes with the vector operators \( \nabla \times \mathbf{r} \) and \( \nabla \times \nabla \times \mathbf{r} \). On
making use of (42) we then find that

\[
\nabla \times \mathbf{J}(\mathbf{r}, \omega) = \frac{-c}{4\pi} \left( i k \nabla \times \nabla \times \mathbf{Q_e}(\mathbf{r}, \omega) - \nabla \times \nabla \times \mathbf{Q_h}(\mathbf{r}, \omega) \right). \tag{43}
\]

\(^1\)These distributions will not be unique and, in particular, will be indeterminant
up to arbitrary non-radiating distributions localized within \( V_0 \).
As mentioned above the wavelet current is only be determined up to a divergence free component and the charge distribution is arbitrary so long as it satisfies the charge conservation equation. One choice of charge-current distribution is obtained by simply taking
\[
J(r, \omega) = -\frac{c}{4\pi} \nabla \times r Q_h(r, \omega), \quad \rho(r, \omega) = 0, \quad (44)
\]
It is clear that this volume current will satisfy (43) and thus radiate the EM wavelet field
\[
E(r, \omega) = ik \nabla \times [r W_h(r, \omega)]
\]
\[
H(r, \omega) = \nabla \times \nabla \times [r W_h(r, \omega)] \quad (45)
\]
where \(W_h\) is the scalar wavelet field radiated by \(Q_h\).

VII. CONCLUSION

This research has employed the well developed theory for the scalar wave inverse source problem to derive a class of volume source distributions of compact support whose generated fields outside their region of localization coincide with the scalar wavelet fields. Such fields have localization as well as representational compactness characteristics that make them of interest in a number of applications [5], [8]–[10]. Although alternative approaches for realizability of the wavelet fields have been investigated before [5]–[8], the approach investigated in the present paper fits more naturally within the standard theory of antenna and source synthesis from the radiation pattern of the field (see, e.g., [23] and the references therein for an overview of the use of inverse source theoretic ideas such as the ones developed in the present paper in the companion problem of antenna synthesis). The relevant theory borrows from standard linear inverse theory and multipole expansions of the radiated field, and also provides a mathematical framework for addressing the question of realizability of given fields (the wavelet fields) under realistic source and field constraints such as the bounding of physical energies and of measures of source excitation such as the familiar \(L^2\) norm. Particular emphasis was given in the paper to sources confined within spherical shells, but the derived general approach can be extended to more general configurations.

To facilitate analytical characterization of both the physical field energy as well as the \(L^2\) norm or functional energy of the proposed wavelet sources, particular attention was given in the paper to temporally band limited signals. The sources derived in the paper possess minimum \(L^2\) norm among all sources that are confined within a given support volume and generate the wavelet field outside that source volume. The derived sources are analytic functions of position within that support and, in fact, they are, themselves, wave fields, truncated within the given support, as expected from the general inverse source theory [2]–[4]. A construction procedure for EM wavelet fields that is based on the Debye representation was also presented. It was found that the EM wavelet fields have radiation patterns that are directly proportional to the scalar wavelet fields and can be radiated by charge-current distributions that are simply expressible in terms of the volume sources computed for the scalar wavelet field.

Interesting avenues for continuation of the work reported here include the formulation of the inverse problem for wavelet fields directly in the time domain [2] and in the spheroidal geometry [15]. Another avenue for inquiry is the comparative study of the wavelet fields and sources of the present paper with more conventional well-collimated short-pulse sources studied in the literature [24], [25], which like the wavelet fields also enable high time-angle localization. Also, from a fundamental physical point of view, the question of what one considers a physically realizable source or field remains open. In the present treatment, we have only partly addressed the matter of realizability of wavelet sources and their fields while conveniently borrowing from the well-established mathematical theory of the inverse problem for square integrable sources wherein the source \(L^2\) norm plays a key role. But future work should incorporate other solution constraints, particularly those more directly connected to the source-field interaction integral characterizing the physical source energy dynamics [23]. Yet, while the present approach is not the last word on the realizability of wavelet fields, it does represent an advancement, based on a solid formal framework, within which we expect to be able to formulate the right questions and their answers in the future. We plan to report further developments on these and other related topics elsewhere.

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REFERENCES

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