A SAMPLING THEOREM FOR SIGNALS IN THE
JOINT TIME-FREQUENCY DOMAIN

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ABSTRACT

Certain analog systems, such as the ear, have outputs which are
time-dependent spectra. Such a system can be modeled as the Fourier
transform of a windowed version of the input signal, sometimes called a
short-term Fourier transform. It is shown that the input signal can be
fully recovered by sampling the output at time-intervals $T$ and
frequency-intervals $F$, where $TF \leq 1$. 
A short-term Fourier transform of a signal \( x(t) \) may be defined as follows: For each time, \( s \), choose a window \( h_s(t) \) which specifies how much emphasis is given to the signal at time \( t \) in computing the output at time \( s \). Then the short-term Fourier transform of \( x(t) \) is defined as

\[
X(f,s) = \int_{-\infty}^{\infty} e^{-j2\pi ft} h_s(t)x(t)dt.
\]  

(1)

It is usually assumed that the system is time-invariant, i.e. \( h_s(t) = h_0(t-s) \), but this is not necessary for our purposes. We do assume that the system is causal \( (h_s(t) = 0 \text{ for } t > s) \) and has a finite memory \( \tau \) independent of \( s \) \( (h_s(t) = 0 \text{ for } t \leq s-\tau) \). Choose a time-interval \( \Delta s = \) and a frequency-interval \( \Delta f = F \), and consider the samples

\[
X_{kn} = X(kF,nT)
\]

(2)

of \( X(f,s) \) for integers \( k \) and \( n \). We wish to know under what conditions \( x(t) \) can be recovered from these samples. Suppose \( F < \frac{1}{T} \). Then the windowed signal \( h_nT(t)x(t) \) vanishes outside the interval \( nT - \frac{1}{F} < t < nT \) hence can be expanded in a Fourier series

\[
h_nT(t)x(t) = \sum_k c_{kn} e^{j2\pi kt}
\]

(3)
To recover \( x(t) \), we must now assume that the function
\[
    g(t) = \left[ \sum_{n} \left| h_n(t) \right|^2 \right]^{-1}
\]

\[
    \sum_{n} h_n(t)^2 \tau(t) = F \sum_{n} X_k h_n(t) e^{j2\pi ft}
\]

\[
    \lim_{T \to \infty} \tau(t) = F h_n(t) \sum_{n} X_k h_n(t) e^{j2\pi ft}
\]

Thus
\[
    \int \tau(t) x(t) dt = F \int h_n(t) \sum_{n} X_k h_n(t) e^{j2\pi ft} dt
\]

To make the expansion (3) valid for all \( t \) (i.e., eliminate the periodic repetitions on the right-hand side), multiply both sides by \( h_n(t) \):

\[
    c_n = \int_{-\infty}^{\infty} e^{-j2\pi ft} h_n(t) x(t) dt
\]
is bounded for all \( t \). This implies that \( T \leq \tau \) (for otherwise

\[
\sum_n |h_{nT}(t)|^2 = 0 \quad \text{for} \quad nT < t \leq nT + T - \tau.
\]

If \( g(t) \) is bounded, we may recover \( x(t) \):

\[
x(t) = \mathcal{F} g(t) \sum_n \sum_k x_{kn} h_{nT}(t) e^{j2\pi knT}
\]

To summarize, the conditions necessary for recovery are

\[
T \leq \tau \leq \frac{1}{F},
\]

from which it follows that \( FT \leq 1 \), i.e., we have at least one sample per unit area in the time-frequency plane. We wish to note the following:

1. All sums over \( n \) are finite, since \( h_{nT}(t) = 0 \) unless \( nT - \tau < t \leq nT \).

2. In principle it is unnecessary to assume that \( x(t) \) is band-limited, as in the Nyquist sampling theorem. In fact, if the windows \( h_{nT}(t) \) are reasonable (e.g., piecewise continuous), it is only required that \( x(t) \) be square-integrable on each interval \( nT - \tau \leq t \leq nT \). Thus \( x(t) \) need only have finite average power.

3. In practice, the formula (8) cannot be used as it stands because it would require an infinite number of samples \( x_{kn}(k=0, \pm1, \pm2, \ldots) \) at each time \( nT \). But a good approximation can be expected by truncating the sum at finite \( k \). In fact, this gives something akin
to a band-limited interpolation of \( x(t) \), since the Fourier transform of (8) gives

\[
X(f) = \sum_n \sum_k X_{kn} L_n(f-kF)
\]  

(10)

where \( L_n(f) \) is the Fourier transform of \( Fg(t) h_n(t) \). Thus if \( h_n(t) \) is real and not too rough, \( L_n(f) \) behaves as an approximate low-pass filter and is negligible for \(|f| \gg 1/\tau\). If the sum over \( k \) is replaced by a finite sum with \(|kF| \leq B\), then (8) requires

\[
\frac{2B}{\bar{T}} \geq 2B
\]

samples per second. In the extreme case (\( FT = 1 \)), this coincides with the Nyquist rate.

4. If in (6) we sum only over \( n_1 \leq n \leq n_2 \), then \( x(t) \) may still be recovered, but only in the interval \( n_1 T - \tau < t < n_2 T \), since

\[
\sum_{n=n_1}^{n_2} |h_n(t)|^2 = 0
\]

otherwise. The formula (8) is still valid in this interval, where \( n \) in both \( g(t) \) and (8) runs from \( n_1 \) to \( n_2 \). (However, \( g(t) \) may become unbounded as \( t \to n_2 T \) and \( t \to n_1 T - \tau \).) This appears to be an advantage over the Nyquist theorem, where truncation of the time-samples is not so trivial.

5. When applying the Nyquist sampling theorem to a signal which is not band-limited, it is in practice necessary first to pass the signal
through a low-pass filter to avoid aliasing errors. This is unnecessary when using (8).

6. Eq. (8) has an obvious generalization to several independent variables, hence can also be applied to optical signals.

Finally, eq. (8) has the following suggestive interpretation: The signal

\[ F g(t) * h_n(t) = e^{j2\pi kFt} \]

may be regarded as a musical note of frequency \( kF \) (\( k \)-th harmonic of the fundamental frequency \( F \)) and duration \( \tau \), played at the time \( nT - \tau \). Then the collection of samples \( X_{kn} \) may be viewed as a score for \( x(t) \) in the key \( F \) and tempo \( T \), and (8) states that \( x(t) \) may be synthesized by "playing" its score!

REFERENCES