Phase-space approach to relativistic quantum mechanics. I. Coherent-state representation for massive scalar particles

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We construct a family of equivalent representations $U_\lambda (\lambda > 0)$ of the restricted Poincaré group $P_k^+$ for a massive scalar particle on spaces $K_\lambda$ of functions defined over "phase space" $P_k$. Each $P_k$ is a submanifold of the forward tube, and $K_\lambda$ consists of restrictions on holomorphic solutions of the Klein-Gordon equation to $P_k$. Each $K_\lambda$ has a resolution of the identity in terms of "coherent states" $\psi_\lambda$, $\zeta \in P_k$, which are wavepackets characterized by an invariant extremal property.

1. INTRODUCTION

This is the first in a series of papers devoted to a phase-space formulation of relativistic quantum mechanics. In this paper we construct representations of the "coherent-state" type for a free massive scalar particle. In forthcoming papers we extend the present formalism to particles with spin, supply our "phase spaces" with natural symplectic structures, and formulate a covariant phase-space quantization. The results of this paper were announced in Ref. 1.

We begin by sketching the coherent-state representation.

In addition to the well-known configuration-space and momentum-space representations of quantum mechanics for a nonrelativistic particle, there is a class of representations on spaces of functions over classical phase space, $\mathbb{R}^{2n}$, the most common of which is known as the "coherent-state" representation. The simplest such representation is constructed as follows: let $X_k$ and $P_k$ be the position and momentum operators for a particle in $\mathbb{R}^n (k=1, \ldots, n)$ and form the nonnormal operators $a_k = X_k + iP_k$. These are found to have an overcomplete set of eigenvectors $|\psi_k \rangle$, $\langle \psi_k | a_k = \pm \sqrt{2} \langle \psi_k |$, which mean that the complex conjugation, one for each $z = x - iy \in \mathbb{C}^n$, and each $\psi_k$ is a minimum-uncertainty wave packet with $\langle \psi_k | x \rangle = x_k$ and $\langle \psi_k | y \rangle = y_k$. The coherent-state representation is then the representation of wavefunctions $\psi$ by functions $f(z) \equiv \langle \psi_k | f \rangle$. These functions are entire and satisfy

$$
\langle \psi | f \rangle = \pi^{-\frac{n}{2}} \int_{\mathbb{R}^n} \hat{f}(x) \hat{g}(x) \exp\left(-|x|^2/2\right) d^n x,
$$

(1.1)

where $|x|^2 = |x_1|^2 + \cdots + |x_n|^2$, $d^n x$ is Lebesgue measure, and the left-hand side denotes the inner product of $f$ and $g$ in the given Hilbert space $\mathcal{H}$ (say, of functions over configuration space).

In spite of its usefulness and intuitive appeal, the coherent-state representation is generally regarded as something of a fluke. The formal combinations $X_k \pm iP_k$, on which it is based, cannot be justified in physical terms, and the use of non-Hermitian operators as anything other than a technical device is regarded with suspicion.

It is one of the aims of this paper to show that representations similar to the above can in fact spring from physical principles, and that the resulting formalism can, as above, be interpreted as a phase-space representation of the given quantum system. The general argument goes as follows: The positivity of the quantum Hamiltonian permits the extension of the oneparameter unitary group $\exp(-i\tau H)$ ($\tau = t - i\beta, \beta > 0$) on a holomorphic semigroup $\exp(-i\tau H)$. On a classical level, evolution in complex time (were it possible) would result in a complexification of the configuration space (hence complex space-time). This has a counterpart at the quantum level in that wavefunctions evolved in complex time, $\exp(-i\tau H)f = \exp(-i\tau H)\psi = \exp(-i\tau H)f$, may be continued analytically from $\mathbb{R}^n$ (configuration space) to a subset (possible all) of $\mathbb{C}^n$. In particular, if the given system is a free nonrelativistic particle, this continuation is even possible at the classical level and gives the complexified position $z(\tau) = x + \tau p$ ($p = (x_0 + ip_0) - ip_0$, which is a combination of the type $x-iy$. Hence the complexified space can, at every complex "instant" $t-i\beta$, be interpreted as a classical phase space. Moreover, the set of analytically continued solutions carries a representation of the quantum dynamics on functions over phase space.

In Sec. 2 we develop this idea for a free scalar nonrelativistic particle and arrive at a representation which essentially coincides with the usual coherent-state representation. An analogous construction is carried out in Sec. 3 for a relativistic free scalar particle (with positive mass). The ensuing formalism appears to be new and has the general features of the coherent-state representation. The "phase spaces" $P_k$ of Sec. 3 are products of $\mathbb{R}^n$ (configuration space) with an $n$-dimensional hyperboloid (roughly, a mass shell). It is shown that in the nonrelativistic limit ($c \to \infty$) the formalism goes over smoothly to the formalism of Sec. 2. In Sec. 4 we study the relativistic coherent states $\psi_n$, $z \in P_k = \mathbb{C}^n$. We show that $\psi_n$ is a wavepacket with $\langle \psi_n | x \rangle = x_k$ and $\langle \psi_n | y \rangle = b \nu_k$, where $b$ is a constant and $X_k$ are the position operators obtained by Newton and Wigner" by axiomatizing the notion of "localized states". These results partly justify calling $P_k$ a "phase space." The $\psi_n$ are shown to be characterized by an extremal property which, we suggest, is a covariant substitute for minimal uncertainty.

2. NONRELATIVISTIC PARTICLE

The wave function of a free, spinless nonrelativistic particle in $\mathbb{R}^n$ evolves under the Schrödinger equation

$$
\frac{\partial \psi}{\partial t} = iH \psi, \quad H = -\frac{\Delta}{2m},
$$

(2.1)
The solutions are given by

\[ f(x, t) = \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} \exp(-itp^2/2m + ix \cdot p) \hat{f}(p) \, dp. \]  

(2.2)

where \( \hat{f}(p) \) is the Fourier transform of the initial function \( f(x, 0) \in L^2(\mathbb{R}^3) \). Now let \( x = x - iy \in C^* \) and let \( \tau = t - i\beta \) in the lower half-plane \( C^- (\beta > 0) \). Then \( \exp(-itp^2/2m + iz \cdot p) \) decays rapidly as \( 1/p_1 \to -\infty \), and Eq. (2.2) defines a function \( f(z, \tau) = \{ \exp(-i\tau H) \} \hat{f}(z) \), holomorphic in \( \tau \in C^- \times C^* \). Let \( H = \{ f(z, \tau) \} \hat{f}(p) \) \( \in L^2(\mathbb{R}^3) \) be the vector space of all such functions. Then, for each \( \beta > 0 \), the function \( f_\beta(z) = f(z, -i\beta) = \{ \exp(-i\beta H) \} \hat{f}(z) \) is entire in \( C^* \). Let \( H_\beta \) be the space of all such functions \( f_\beta(z) \). On \( H_\beta \) define the map \( \exp(-i\beta H) \) by

\[ \exp(-i\beta H)\{ \exp(-i\beta H) \} \hat{f}(z) = \exp(-\beta H) \{ \exp(-i\beta H) \} \hat{f}(z), \]  

(3.3)

We shall make \( H_\beta \) into a Hilbert space such that \( \{ \exp(-i\beta H) \} \) is a unitary representation of the dynamics on \( H_\beta \).

Thus, let \( \beta > 0 \) and \( z = x - iy \in C^* \). Then

\[ f_\beta(z) = \{ \exp(-i\beta H) \} \hat{f}(z) \]

(2.4)

\[ = (2\pi)^{-3/2} \int_{\mathbb{R}^3} \exp(-i\beta p^2/2m + iz \cdot p) \hat{f}(p) \, dp, \]

(2.5)

with Fourier transform

\[ \langle \phi^\beta | x \rangle = (2\pi)^{-3/2} \exp(-i\beta p^2/2m + iz \cdot p). \]

(2.6)

The \( \phi^\beta \) are minimum-uncertainty spherical wavepackets with \( \langle X \rangle = m/\beta \), \( \langle X \rangle^2 = (m/\beta)^2 \), \( \Delta X = \sqrt{\beta}/2m \), and \( \Delta P = \sqrt{m/2\beta} \). They are eigenvectors of \( A_\beta = X + i(\beta/m)P \) with eigenvalue \( \lambda_\beta \).

For \( f_\beta \in H_\beta \) define

\[ ||f||^2 = \int_{\mathbb{R}^3} |f_\beta(z)|^2 \, d\mu_\beta(z), \]

(2.7)

where

\[ d\mu_\beta(z) = (m/\beta)^{-3/2} \exp(-i\beta z^2/2m) \, dx \, dy. \]

(2.8)

Theorem 1: Let \( \beta > 0 \) and \( f(p) \in L^2(\mathbb{R}^3) \). Then

\[ ||f||^2 = ||\hat{f}||^2. \]  

(2.9)

In particular,

(a) \( ||f||^2 \) is a norm on \( H_\beta \) under which \( H_\beta \) is a Hilbert space.

(b) The map \( \exp(-i\beta H) \) is unitary on \( H_\beta \).

(c) The map \( \exp(-i\beta H) \) is unitary from \( L^2(\mathbb{R}^3) \) onto \( H_\beta \) and intertwines the dynamics on \( L^2(\mathbb{R}^3) \) with the dynamics on \( H_\beta \).

Remark: Equation (2.9) can be polarized to give a resolution of the identity: For \( f, \hat{g} \), in \( L^2(\mathbb{R}^3) \),

\[ \langle \hat{g} | \langle f | x \rangle \rangle = \int_{\mathbb{R}^3} \phi^\beta(z) \langle \phi^\beta | \hat{g} \rangle \, d\mu_\beta(z) \]

(2.10)

\[ = \langle \hat{g} | \langle f \rangle \rangle_{L^2(\mathbb{R}^3)}. \]

Hence \( \langle \phi^\beta | f \rangle \) is a "representation" of \( f \) by an entire function. The connection with the coherent-state representation is as follows: Set \( m = \beta = 1 \) and let \( f(z) = \pi^{3/4} \exp(z^2/4) \phi(z) \). Then (2.10) becomes

\[ \pi^{3/2} \int_{\mathbb{R}^3} \langle \phi | \hat{g} \rangle \exp(-i\beta x^2/2) \, dx \, dy = \langle \hat{g} | \langle f \rangle \rangle_{L^2(\mathbb{R}^3)}, \]

(2.11)

so that \( \hat{f}(z) \) is (essentially) the ordinary coherent-state representation [in most of the literature, \( z = (x - iy)/\sqrt{2} \); the weight function is then \( \exp(-|z|^2) \)].

Proof: Let \( f \in L^2(\mathbb{R}^3) \). By (2.4), \( f_\beta(x - iy) = \phi^\beta \hat{f}(x) \) where \( \phi^\beta(p) = \exp(-i\beta p^2/2 + ip \cdot y) \hat{f}(p) \) and \( \hat{f}(p) \) denotes the inverse Fourier transform of \( f \). Thus, by Plancherel's theorem (and Fubini's),

\[ ||f||^2 = (m/\beta)^{3/2} \int \exp(-i\beta x^2/2m + 2iy \cdot p) \hat{f}(p) \, dp \]

(2.12)

\[ = \int \| \hat{f}(p) \|^2 \, dp = ||\hat{f}||^2, \]

which proves (2.9) for \( f \in L^2(\mathbb{R}^3) \), hence also for \( f \in L^2(\mathbb{R}^3) \) by continuity. (a)–(c) are obvious.

3. RELATIVISTIC PARTICLE

In the last section we obtained unitary maps from \( L^2(\mathbb{R}^3) \) onto Hilbert spaces \( H_\beta \) where the role of \( \beta \) functions is played by spherical wavepackets \( \phi^\beta \) in \( L^2(\mathbb{R}^3) \) [\( H_\beta \) is continuously imbedded in \( L^2(\mathbb{R}^3) \) by restricting \( f_\beta(z) \) to \( \mathbb{R}^3 \)]. This formalism is nonrelativistic since the inner product for \( L^2(\mathbb{R}^3) \) is not Lorentz-invariant. In this section we define covariant counterparts of the \( \phi^\beta \) and prove the analog of Theorem 1. We begin with the relativistic version of the free-particle Schrödinger equation, namely the Klein–Gordon equation (for a free scalar particle of mass \( m > 0 \) in \( n + 1 \) space–time dimensions):

\[ \left( -\frac{1}{c^2} \frac{\partial^2}{\partial t^2} + \Delta - m^2 \right) f(x, t) = 0. \]

(3.1)

The positivity of the energy played an essential role in Sec. 2, and will do so again here. Hence we confine ourselves to positive-energy solutions. These are given by

\[ f(x) = f(x, x_0) = [\exp(-ix_0H)] f(x) \]

(3.2)

\[ = (2\pi)^{-n/2} \int_{\mathbb{R}^3} \exp(-ix_0p) \hat{f}(p) \, dp \]

where

\[ x_0 = ct, \]

\[ H = (m^2c^2 - \Delta)^{1/2}, \]

\[ x_0 = x - \omega \times x \cdot p, \]

\[ \omega = (m^2c^2 + p^2)^{1/2}, \]

\[ d\Omega(p) = dp / \omega \]

is the Lorentz-invariant measure on the mass shell, and \( \hat{f}(p)/\omega \) is the ordinary Fourier transform of the initial function \( f(x, 0) \) (considered, say, as a tempered distribution in \( \mathbb{R}^3 \)). For every \( \hat{f}(p) \in L^2(\Omega) \), i.e., with

\[ ||\hat{f}||^2 = \int_{\mathbb{R}^3} ||\hat{f}(p)||^2 \, d\Omega(p) < \infty, \]

(3.3)

the corresponding solution \( f(x) \) is the boundary value of a function \( f(z) \) holomorphic in the forward tube.
where $V_\ast = \{ y = (y_0, y) \in R^{n+1} \mid y_0 > |y| \}$ is the open forward light cone in $R^{n+1}$. This is because $\exp(-izp) = \exp(-y_0 + y \cdot p) = \exp(-y_0 - iy \cdot p)$, hence $\exp(-izp)$ decays rapidly as $|p| \to \infty$ for fixed $z = x - iy \in \mathbb{T}$. $\mathbb{T}$ will replace the domain $\mathcal{D} = C^0 \times C$ of Sec. 2, and is strictly contained in $\mathbb{T}$. Thus, for $z \in \mathbb{T}$,

$$
\langle f(z) | f \rangle = \int_{\mathbb{T}} \exp(-izp) \psi(p) d\Omega(p)
$$

(3.4)

where

$$
\langle f(z) | f \rangle = \int_{\mathbb{T}} \exp(-izp) \psi(p) d\Omega(p)
$$

(3.5)

and $\langle f(z) | f \rangle$ denotes the inner product in $L^2(\Omega)$. The vector $f(z)$ is in $L^2(\Omega)$, since for $z \in \mathbb{T}$,

$$
\langle f(z) | f \rangle = \int_{\mathbb{T}} \exp(-izw) f(p) d\Omega(p)
$$

(3.6)

where $\Delta$ is the familiar two-point function for the free scalar field of mass $m$, which is defined by

$$
\eta = \frac{1}{2\pi} \log r + \frac{1}{2} \ln r^2 + \frac{1}{2} \ln r^2 + \frac{1}{2} \ln r^2 + \frac{1}{2} \ln r^2 + \frac{1}{2} \ln r^2 + \frac{1}{2} \ln r^2 + \frac{1}{2} \ln r^2 + \frac{1}{2} \ln r^2 + \frac{1}{2} \ln r^2 + \frac{1}{2} \ln r^2 + \frac{1}{2} \ln r^2 + \frac{1}{2} \ln r^2 + \frac{1}{2} \ln r^2 + \frac{1}{2} \ln r^2 + \frac{1}{2} \ln r^2 + \frac{1}{2} \ln r^2 + \frac{1}{2} \ln r^2
$$

(3.7)

with $\lambda = c \to 0$. We will show that $P_\lambda$ is a suitable phase space. For $\lambda = 0$ Eq. (3.7) defines $P_\lambda$ as a subset of the boundary of $\mathbb{T}$. The sets $P_\lambda$ are clearly not invariant under Lorentz transformations. To make the formalism manifestly Poincaré-covariant, we will also need the sets

$$
P_\lambda = \{ x \in R^n \} \quad \text{for} \quad x = x_\lambda + i \eta \in C_0^\prime,
$$

(3.8)

where $\eta = \frac{1}{2\pi} \log r + \frac{1}{2} \ln r^2 + \frac{1}{2} \ln r^2 + \frac{1}{2} \ln r^2 + \frac{1}{2} \ln r^2 + \frac{1}{2} \ln r^2 + \frac{1}{2} \ln r^2 + \frac{1}{2} \ln r^2 + \frac{1}{2} \ln r^2 + \frac{1}{2} \ln r^2 + \frac{1}{2} \ln r^2 + \frac{1}{2} \ln r^2 + \frac{1}{2} \ln r^2 + \frac{1}{2} \ln r^2 + \frac{1}{2} \ln r^2 + \frac{1}{2} \ln r^2 + \frac{1}{2} \ln r^2 + \frac{1}{2} \ln r^2
$$

(3.9)

where

$$
f(z) = f_0(\Delta - iy) = (1 + \frac{1}{\omega} \exp(-iy \cdot p) \psi(p))^{-1}(z)
$$

(3.10)

where

$$
\| f \| = \int_{\mathbb{T}} \frac{1}{\psi(p)} d\mu_\lambda(z),
$$

(3.11)

and

$$
C_\lambda = \frac{1}{2\pi} \frac{\lambda}{\mu c} \frac{f(\lambda^{-1} z - a)}{\lambda^{-1} z - a}
$$

(3.12)

These facts, and others needed later, follow from certain properties of the $K_{\nu}^{\prime}$ which we summarize in Appendix A. We may regard $\mu_\lambda$, either as a measure on $P_\lambda$ or as a measure on $C_0$. In the latter interpretation (which will also be useful) we write (3.10) as

$$
\| f \| = \int_{\mathbb{T}} f(z) \mu_\lambda(z),
$$

(3.13)

where $f_\lambda(z) = f_0(\Delta - iy \cdot p)^{-1/2}$ is the restriction of $f \in K_{\nu}$ to $P_\lambda$. Let $K_\lambda = \{ f(z) \mid f \in K_{\nu} \}$ be the space of all such restrictions (boundary values, if $\lambda = 0$) and the map $f \mapsto f_\lambda$ from $L^2(\Omega)$ onto $K_\lambda$ by $D_\lambda$. Similarly let $K_\lambda'$ be the space of restrictions $f(x, y) = f(x - iy, x_\lambda - iy \cdot p)^{-1/2}$ to $P_\lambda'$ and denote the corresponding map by $D_\lambda'$. Since each $f(x, y) \in K_\lambda'$ satisfies (3.1) in $x \in R^{1+1}$, $K_\lambda'$ is simply the space of solutions with initial values in $K_\lambda$. Notice that (3.14) is defined for $f_\lambda \in K_\lambda$, as well as for $f \in K_\lambda'.

Now $L^2(\Omega)$ carries a unitary, irreducible representation of the restricted Poincaré group $P_\lambda'$ given by

$$
(U(a, \lambda)) f(p) = \exp(iap) f(\lambda^{-1} p),
$$

(3.14)

where

$$
(U(a, \lambda)) f(p) = \exp(iap) f(\lambda^{-1} p)
$$

(3.15)

and

$$
\lambda \in R^{1+1},
$$

(3.16)

In (3.15) $p = (p_0, p)$ denotes a point on the mass shell (a homogeneous space for the Lorentz group) rather than the corresponding momentum vector $p$. The representation (3.15) defines a corresponding representation on $K_{\nu}$ given by

$$
(U(a, \lambda)) f(z) = f(\lambda^{-1} (z - a))
$$

(3.17)

(where we have extended the action of $P_\lambda'$ to $T$ by linearity). Now $P_\lambda'$ is a homogeneous space of $P_\lambda$ [in fact, $P_\lambda' = SO(n)$ since the stability subgroup at, say, $(0, -i \lambda) = SO(n)$]. Hence (3.17) gives a representation $U_{\lambda'}$ on $K_{\nu}$ by restriction (taking boundary values, if $\lambda = 0$). Since extension sets up a one—one correspondence between $K_\lambda$ and $K_\lambda'$, we also have a representation $U_{\lambda'}$ on $K_{\nu}$, but this one is less direct since $P_\lambda$ is not invariant under $P_\lambda'$.

The next theorem, which is our first main result, shows that $U_{\lambda}, U_{\lambda'}$, and $U_{\lambda'}$ are unitarily equivalent.

Theorem 2: Let $\lambda \to 0$ and $f \in L^2(\Omega)$. Then

$$
\| f \| = \| f \|
$$

(3.18)

In particular,

(a) $\| f \|$ is a norm on $K_{\nu}$, under which $K_{\nu}$ is Hilbert space.

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We can now make precise the sense in which a function \( f \in \mathcal{K} \) takes on its boundary value on \( P_\alpha \).

**Corollary 2.** (a) Each \( \mathcal{K}_\alpha \) is a closed subspace of \( L^2(\mathbb{C}^*) \). (b) Let \( 0 < \lambda < \lambda' \) and \( \hat{f} \in L^2(\mathbb{C}) \). Then

\[
\|f - \hat{f}\|_{L^2(\mathbb{C}^*)} = \int dy |f(p)|^2 \omega^2 \lesssim \omega^2 \lesssim \omega.
\]

**Proof:** (a) follows from (3.18), and (b) follows essentially from the proof of Theorem 2:

\[
\|f - \hat{f}\|_{L^2(\mathbb{C}^*)} = \int dy |f(p)|^2 \omega^2 \lesssim \omega^2 \lesssim \omega.
\]

But \( \omega/C_\lambda < J < \omega/C_\lambda \); hence

\[
\|f - \hat{f}\|_{L^2(\mathbb{C}^*)} < (C_\lambda^2 - C_\lambda^2) \|\hat{f}\|_2^2,
\]

which implies (b).

We conclude this section by showing that the \( \varepsilon_\beta \)-representation on \( \mathcal{K}_\alpha \) is indeed a relativistic version of the \( \varepsilon_\beta \)-representation on \( \mathcal{H}_\alpha \). For given \( \beta > 0 \), define

\[
\hat{f}^\beta_{\varepsilon_\alpha}(x - iy) = \exp(-\beta p^2/2m - y \cdot p)f(p) \exp(-\beta \hat{p}^2/2m + x \cdot p) \exp(-\beta \hat{p}^2/2m - x \cdot p) \exp(-\beta p^2/2m + y \cdot p)
\]

Theorem 3: Let \( \beta > 0 \) and \( \hat{f}(p) \in L^2(\mathbb{R}^*) \). Then \( \hat{f}^\beta_{\varepsilon_\alpha}(x) \in L^2(\mathbb{C}^*) \) and

\[
J(c) = \|mc \exp(\beta mc^2\hat{f}(p) - \beta \hat{p}^2/2m)\|_{L^2(\mathbb{C}^*)}^2 = \sum_{k=2}^\infty \frac{P_k(2m \omega)}{2m} \|\hat{f}\|_{L^2(\mathbb{C}^*)}^2 = 0 \quad \text{as} \quad c \to \infty,
\]

where \( f_{\beta \varepsilon_\alpha} \) is the function in \( \mathcal{K}_{\beta \varepsilon_\alpha} \) corresponding to \( \hat{f} \in L^2(\mathbb{R}^*) \). The proof is given as Appendix B.

**4. THE WAVEPACKETS \( \varepsilon_\beta \)**

In this section we study the "relativistic coherent states" \( \varepsilon_\beta \). We show that they are centered about \( \mathbf{x} = \text{Re}(\mathbf{z}) \), travel with average momentum proportional to \( \mathbf{y} \), and are characterized by a property which is a covariant analog of minimal uncertainty.

To compute the position of the center of \( \varepsilon_\beta \) we need position operators. It was shown by Newton and Wigner,11 that certain group-theoretical postulates about (idealized) "localized states"—e.g., that any space translate of a localized state be "orthogonal" to the state?—uniquely determine a set of self-adjoint operators [here given on \( L^2(\mathbb{C}^*) \)]

\[
X_{\mathbf{k}} = i \left( \frac{\partial}{\partial \mathbf{p}} - \frac{P_k}{2m} \right), \quad k = 2, \ldots, n,
\]

(4.1)

whose (generalized) eigenvectors are the localized states. (The notion of being localized in this sense, however, depends on the frame of reference.) In a later paper, dealing with quantization, we shall show that the
operators (4.1) can also be obtained naturally from the formalism of Sec. 3. For the purpose of this section, we simply adopt (4.1) as the definition of position operators.

We begin by computing the expectation of $x_5$ in $e_5$:

$$
\langle e_5 | X e_5 \rangle = \int \frac{dp}{\omega} \langle e_5 | p \rangle \left( \frac{\partial}{\partial p} - \frac{p}{2\omega} \right) (\phi | e_5 \rangle
$$

$$
= \int \frac{dp}{\omega} \langle e_5 | p \rangle \frac{\partial}{\partial p} \left( \frac{\phi}{\omega} \right)
= \text{Re} \int \frac{dp}{\omega} \langle e_5 | p \rangle \frac{\partial}{\partial p} \left( \frac{\phi}{\omega} \right) \left( \omega^{-1/2} \exp(i\hat{p} p) \right)
= x_5 \langle e_5 | e_5 \rangle.
$$

Thus
$$
\langle X_5 \rangle = x_5 = \text{Re} \langle e_5 | e_5 \rangle.
$$

To find the expectation of $P_\mu$, let

$$
G(m, y) = \int_s e^{i2yp} d\Omega(p) = 2(2\pi)^{3/2} K_{2\lambda m}(2\pi m)
$$

$$
= a(m)\psi^{-1}K_{2\lambda m}(\varphi) = b(\varphi)\psi^{-1}K_{2\lambda m}(\varphi),
$$

where $a(m) = 2(2\pi)^{3/2}$, $b(\varphi) = 2(2\pi)^{3/2} \varphi$, $\varphi = 2\lambda m$, and $\lambda = \lambda(m) = (\lambda y_0)^{1/2}$ with all the $y_0$ considered as independent variables. $G(m, y)$ will be a "partition function" (as in statistical mechanics) for generating expectations. Thus, using (A2),

$$
\int \frac{dp}{\omega} \exp(-2yp) d\Omega(p) = -\frac{1}{2} \frac{\partial}{\partial \varphi} K_{2\lambda m}(\varphi)
$$

$$
= -2m^2 y_0 a(m) \frac{\partial}{\partial \varphi} (\varphi^{-1} K_{2\lambda m}(\varphi))
= 2m^2 y_0 a(m) \varphi^{-1} K_{2\lambda m}(\varphi),
$$

hence

$$
\langle P_\mu \rangle = -\frac{1}{2} \frac{\partial}{\partial \varphi} K_{2\lambda m}(2\lambda m) \frac{m}{\lambda} y_0
$$

$$
\langle x_5 \rangle = \text{Re} \langle e_5 | e_5 \rangle.
$$

(4.2)

The integral is difficult to evaluate, and we merely derive an upper bound in the rest frame. Setting $y = 0$ and $y_0 = \lambda$,

$$
\langle (x_5 - x_5)^2 \rangle \leq \lambda G^{-1} \int \left( 1 + \frac{1}{2\gamma y_0} \right)^2 \exp(-2\lambda \omega) d\Omega(p)
$$

$$
\leq \lambda G^{-1} \int \left( 1 + \frac{1}{2\gamma y_0} \right)^2 \exp(-2\lambda \omega) d\Omega(p)
$$

$$
= \lambda^2 + \lambda G^{-1} \int \left( 1 + \frac{1}{2\gamma y_0} \right)^2 \exp(-2\lambda \omega) d\Omega(p).
$$

Now

$$
-\frac{1}{2\lambda m} \frac{3G}{3m} \int \left( 1 + \frac{1}{2\gamma y_0} \right)^2 \exp(-2\lambda \omega) d\Omega(p),
$$

hence

$$
\langle (x_5 - x_5)^2 \rangle \leq \lambda^2 + \lambda \frac{3G}{2mG} \int \left( 1 + \frac{1}{2\gamma y_0} \right)^2 \exp(-2\lambda \omega) d\Omega(p),
$$

$$
\langle (x_5 - x_5)^2 \rangle \leq \lambda^2 + \lambda \frac{3G}{2mG} \left( 1 + \frac{1}{2\gamma y_0} \right)^2 \exp(-2\lambda \omega)
$$

$$
\leq \lambda^2 + \lambda m \frac{3G}{2mG} \exp(-2\lambda \omega).
$$

The position uncertainty therefore satisfies

$$
\langle (x_5 - x_5)^2 \rangle \leq \lambda^2 + \lambda m \frac{3G}{2mG} \exp(-2\lambda \omega),
$$

hence

$$
\langle (x_5 - x_5)^2 \rangle \leq \lambda^2 + \lambda m \frac{3G}{2mG} \exp(-2\lambda \omega),
$$

(4.12)

For $\nu = (n - 1)/2 = 1$ (which is in fact the physical case), (4.13) must be replaced with

$$
\langle (x_5 - x_5)^2 \rangle \leq \lambda^2 + \lambda m \frac{3G}{2mG} \exp(-2\lambda \omega).
$$

(4.13')
Thus $\Delta X_0 \rightarrow 0$ when $\lambda \rightarrow 0$.

We can now draw consequences from the above computations. Equations (4.2) and (4.4) confirm that $e_{x_0}$ is a wavepacket centered about $x$ with expected energy-momentum proportional to $(v, y)$. Note that

$$\langle P_\theta (x) \rangle^{1/2} = m \frac{K_{n+1}(2\lambda m)}{K_n(2\lambda m)} \equiv m_0 > m.$$  \(4.14\)

We shall call $m_0$ the “effective mass” for the particle in $P_\lambda$. The factor $K_{n+1}/K_n$ represents a kind of renormalization which takes into effect the fluctuations in energy-momentum. $m_0$ has the asymptotic behavior

$$v/\lambda^2 m_0 \rightarrow m.$$  \(4.15\)

Equations (4.7)–(4.10), (4.13), and (4.15) show the following pattern: When $\lambda m \rightarrow 0$, the expectations and uncertainties of physical observables in the state $e_x$ become independent of the mass $m$. Thus, roughly, when $\lambda m = 0$, approaches the boundary of $T$, analyticity fails and fluctuations take over. On the other hand, we have seen that $\lambda m = 0$ gives a smooth transition to the nonrelativistic formalism (Theorem 3). Thus we expect $P_\lambda \rightarrow m_0 / \lambda = m_{0}/\lambda$, $C_{\lambda m} \rightarrow m / \lambda$, and $\langle N_0 - x_0 \rangle^{1/2} \rightarrow \beta / 2m = \lambda / 2m$. The first two are born out by (4.7) and (4.10). Equation (4.13'), though consistent with this expectation, shows that in obtaining the estimate (4.12) we gave up too much ground.

The nonrelativistic wavepackets $e_{x_0}$ have the attractive feature of being minimum-uncertainty states. So far we have not shown that the $e_x$ have a similar property. Now uncertainty operators do not seem to be a natural measure of the optimality of relativistic states. The position operators $X_0$ are not covariant, and furthermore it is not obvious how to define an invariant counterpart to the uncertainty product. We conclude by proving that the $e_x$ are characterized by a simple, invariant property which we propose as an adequate substitute for minimal uncertainty. For $z \in T$ let

$$\tilde{e}_z (w) = \langle e_z | e_{x_0} \rangle / \| e_{x_0} \|, \quad w \in T.$$

**Theorem 4:** Let $z \in T$. Then $\tilde{e}_z$ is the unique (up to a constant phase factor) solution to the following problem: Find $f \in K$ such that $\| f \| = 1$ and $|f(z)|$ is a maximum.

**Proof:** We have

$$|\langle e_z | f \rangle| \leq \| e_{x_0} \| / \| f \|,$$

and equality holds if and only if $f$ is a constant multiple of $e_z$.

**Remark:** Theorem 4 can be restated as a variational principle\(^{13}\): $\tilde{e}_z (w) = \langle e_z | e_{x_0} \rangle / \| e_{x_0} \|$ is the unique function $f \in K$ such that $f(z) = 1$ and $\| f \|$ is a minimum. The above form seems to be more appropriate for quantum mechanics. See also Ref. 20.

5. CONCLUSION

We have developed a formalism analogous to that of the coherent-state representation. By this analogy we have called $P_\lambda$ a “phase space.” We then showed that, at least so far as the $e_x$ are concerned, $P_\lambda$ is indeed a space parametrized by coordinates and momenta. Now in the classical notion of phase space, a central role is played by Poisson brackets and canonical transformations, i.e., by symplectic structure.\(^{21, 22}\) These geometrical aspects will be dealt with in a later paper, where the present formalism will be given a geometrical foundation and made manifestly covariant.

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**APPENDIX A**

We collect here some properties of the modified Bessel functions $K_\nu$ and evaluate some integrals needed in Secs. 3 and 4.

The functions $K_\nu (\xi)$ are defined\(^{14}\) for $Re \nu > -\frac{1}{2}$ and $Re \xi > 0$ by

$$K_\nu (\xi) = \sqrt{\pi} (\xi / 2)^\nu \int_0^\infty \exp (-\xi \cosh t) \sinh^n t \, dt.$$  \(A1\)

They satisfy

$$\left( -\frac{1}{\xi} \frac{d}{d\xi} \right)^m (\xi K_\nu (\xi)) = \xi^{-m} K_{\nu - m} (\xi),$$

$$\left( -\frac{1}{\xi} \frac{d}{d\xi} \right)^m (\xi^{-\nu} K_\nu (\xi)) = \xi^{\nu - m} K_{\nu - m} (\xi),$$

for $m = 1, 2, \ldots$ and

$$K_\nu (\xi) = \frac{1}{\Gamma (\nu) (\xi / 2)^\nu} \xi^{-\nu}, \quad \xi > 0 \ (\nu \neq 0),$$

$$K_\nu (\xi) \sim -\ln (\xi / 2), \quad \xi \rightarrow 0,$$

$$K_\nu (\xi) \sim \sqrt{\pi / 2 \xi} \xi^{\nu / 2}, \quad \xi \rightarrow \infty.$$  \(A3\)

In Sec. 4 we use

$$\frac{\Gamma (\nu + k)}{\Gamma (\nu)} \left( \frac{\xi}{2} \right)^{1 + \frac{k}{2}} \frac{K_{\nu + k} (\xi)}{K_\nu (\xi)} = 1 + \frac{\xi^2 + 2k}{2\xi},$$

$$\frac{2n + 1}{\xi^2} \frac{K_{n+1} (\xi)}{K_n (\xi)} \left( \frac{K_{n+1} (\xi)}{K_n (\xi)} \right)^2 = 1 + \frac{n}{\xi^2}.$$  \(A4\)

To evaluate

$$I (v, y) = \int_{-\infty}^\infty \frac{d^p \rho}{(1 + p^2)^{7/2}} \exp \left[ -2y \rho_0 (1 + p^0)^{1/2} + 2y \cdot \rho \right] \rho,$$

note that $I$ is Lorentz-invariant; hence

$$I (\lambda, 0) = I (\lambda, 0) = \int_{(1 + p^0)^{1/2}}^\infty \frac{d^p \rho}{(1 + p^2)^{7/2}} \exp \left[ -2 (1 + p^0)^{1/2} \right]$$

$$= \frac{8n + 1}{\pi} \int_0^\infty \frac{r^{n-1} dr}{(1 + r)^{7/2}} \exp \left[ -2 (1 + r)^{1/2} \right]$$

$$= \frac{8n + 1}{\pi} \int_0^\infty \sinh r dr \exp \left[ -2 \lambda \cosh r \right]$$

$$= \frac{2n + 1}{\lambda} K_n (\lambda v), \quad v = \frac{n - 1}{2}.$$  \(A5\)

Consequently, using (A2),
\[
\int_{\mathbb{R}^n} \frac{dp}{(1 + p^2)^{1/2}} \rho_{a} \exp[-2y_0 (1 + p^2)^{1/2} + 2y \cdot p] \\
= -\frac{1}{2} \frac{\partial}{\partial y} \int dy \phi(y_0, y) \\
= 4 \phi_{a} \left( -\frac{1}{2 \lambda} \frac{\partial}{\partial \lambda} \right) \left( \frac{\pi}{\lambda} \right)^{n/2} K_\nu(2\lambda) \\
= 2 \frac{\partial}{\partial a} \left( \frac{\pi}{\lambda} \right)^{n/2} K_{\nu+1}(2\lambda),
\]
where \( \rho_{a} = (1 + p^2)^{1/2} \).

**APPENDIX B. PROOF OF THEOREM 3**

We can set \( m = \beta = 1 \) without loss. Note

\[
f_{f_{x-y}}(x-y) = \exp(-y^2/2) \langle c_{x-y} \rangle_L^2 \langle a \rangle.
\]

Hence by (2.8) and (2.9),

\[
\|f_{f_{x-y}}\|_{L^2} \leq \frac{\pi^{n/2}}{\Gamma(n/2)} \sum_{0} = 0.5
\]

Note also

\[
\|c_{f_{x-y}}\|_{L^2} \leq \frac{\pi^{n/2}}{\Gamma(n/2)} \sum_{0} = 0.5
\]

Now

\[
J_x = \int dx \, dy \, \left| \left( \frac{c_{x-y}}{c_{x-y}} \exp(c - yp) - \exp(-\frac{1}{2}(p - y)^2) \right) \right|^2 \\
= \int dp \int dy \left( \frac{c_{x-y}}{c_{x-y}} \exp(c - yp) - \exp(-\frac{1}{2}(p - y)^2) \right)^2.
\]

Choose \( \alpha, \gamma \) such that \( \frac{1}{2} < \gamma < \alpha < 1 \). Then

\[
J_x \geq \int \left| \frac{c_{x-y}}{c_{x-y}} \right|^2 \exp[-\frac{1}{2}(p - y)^2 - 0 \text{ as } c \to -\infty; \text{ hence}
\]

\[
J_x = \int dp \int dy \left| \frac{c_{x-y}}{c_{x-y}} \right|^2 \exp[-\frac{1}{2}(p - y)^2 - 0 \text{ as } c \to -\infty; \text{ hence}
\]

\[
\times \left( \frac{c_{x-y}}{c_{x-y}} \exp(c - yp) - \exp(-\frac{1}{2}(p - y)^2) \right)^2
\]

\[
= 4 \pi^{n/2} \sum_{0} = 0.5 \exp(-\frac{1}{2}(p - y)^2 - 0 \text{ as } c \to -\infty; \text{ hence}
\]

\[
\left| \frac{c_{x-y}}{c_{x-y}} \right|^2 \exp(-\frac{1}{2}(p - y)^2 - 0 \text{ as } c \to -\infty; \text{ hence}
\]

\[
G_c(p) = \int \sum_{0} = 0.5 \exp(2c - 2yp)
\]

\[
\leq \frac{2 \pi^{n/2}}{\Gamma(n/2)} \int \sum_{0} = 0.5 \exp(-c \theta - \phi)^4 d\theta
\]

\[
\leq \frac{2 \pi^{n/2}}{\Gamma(n/2)} \int \sum_{0} = 0.5 \exp(-c \theta - \phi)^4 d\theta
\]

\[
\leq \frac{2 \pi^{n/2}}{\Gamma(n/2)} \int \sum_{0} = 0.5 \exp(-c \theta - \phi)^4 d\theta
\]

Let \( a = \sinh^{-1}(c^\gamma) \). Then, for \( |p| < c^{-\delta} \),

\[
c(a - \phi) - n/2c \geq c [\sinh^{-1}(c^\gamma) - \sinh^{-1}(c^\delta)] - n/2c
\]

\[
geq g(c).
\]

\( g(c) \) is independent of \( p \) and \( g(c) \sim c^{1-\beta} \), \( c \to \infty \). Also, \( |p| < c^{-\delta} \)

\[
J_x = \sum_{0} = 0.5 \exp(2c - 2yp)
\]

\[
\leq \frac{2 \pi^{n/2}}{\Gamma(n/2)} \sum_{0} = 0.5 \exp(-\frac{1}{2}(p - y)^2) \exp(-\frac{1}{2}(p - y)^2 - 0 \text{ as } c \to -\infty; \text{ hence}
\]

\[
\times \left| \frac{c_{x-y}}{c_{x-y}} \right|^2 \exp(-\frac{1}{2}(p - y)^2 - 0 \text{ as } c \to -\infty; \text{ hence}
\]

\[
2c^2 - 2yp = y^2 + p^2 - 2yp = (y - p)^2
\]

\[
= (y^2 - \phi)^2 - (y - p)^2
\]

\[
\geq - (y - p)^2.
\]

Hence

\[
J_x = \int dp \int \left| \frac{c_{x-y}}{c_{x-y}} \right|^2 \sum_{0} = 0.5 \exp[-\frac{1}{2}(p - y)^2] \sum_{0} = 0.5 \exp[-\frac{1}{2}(p - y)^2]
\]

\[
= \int dp \int \left| \frac{c_{x-y}}{c_{x-y}} \right|^2 \sum_{0} = 0.5 \exp[-\frac{1}{2}(p - y)^2] \sum_{0} = 0.5 \exp[-\frac{1}{2}(p - y)^2]
\]

\[
= \sum_{0} = 0.5 \exp[-\frac{1}{2}(p - y)^2] \sum_{0} = 0.5 \exp[-\frac{1}{2}(p - y)^2]
\]

\[
\times \left| \frac{c_{x-y}}{c_{x-y}} \right|^2 \exp(-\frac{1}{2}(p - y)^2 - 0 \text{ as } c \to -\infty; \text{ hence}
\]

\[
\left| \frac{c_{x-y}}{c_{x-y}} \right|^2 \exp(-\frac{1}{2}(p - y)^2 - 0 \text{ as } c \to -\infty; \text{ hence}
\]

We have used the estimate

\[
\left| (1 + v^2)^{1/2} - (1 + v^2)^{1/2} \right|
\]

\[
= \left| \int_{0}^{v} x dx \right| \leq \left| \int_{0}^{v} x dx \right| = \frac{1}{2} v^2 - 0.
\]

Hence for sufficiently large \( c \) and \( |p| < c^{-\delta} \),

\[
[(c/\omega) \exp(c^{2}\delta^2/2) - 1]^2
\]

\[
\leq \exp(c^2\delta^2) + 1 - (2c/\omega) \exp(c^{2}\delta^2/2)
\]

\[
\leq \exp(c^2\delta^2) + 1 - (2c/\omega) \exp(c^2\delta^2/2)
\]

\[
< 2(1 - \exp(c^2\delta^2/2)) + 2c^2\delta^2 + c^{-2}\exp(c^2\delta^2/2)
\]

\[
< 2c^2\delta^2 + c^{-2} + (1 + c^2\delta^2)
\]

\[
= h(c) - 0 \text{ as } c \to -\infty.
\]

Thus
\[ J = \hbar^{-c} \int_{\mathbb{R}^3} \left( \int_{\mathbb{R}^3} |\phi(y)|^2 \right)^2 \int_{\mathbb{R}^3} |\phi(p)|^2 \exp\left[-i p \cdot y \hbar \right] dp \, dy \]

which proves that \( J - 0 \) as \( c \to \infty \).

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*This work is part of the author's Ph.D. thesis (submitted to the University of Toronto, 1977).
8For other representations of the "coherent-state" type, see Refs. 9, 16.
13The definiteness of the energy is necessary in order that our representation of \( \beta^* \) be irreducible; choosing it to be positive is also in the spirit of quantum field theory, where solutions of (3.1) enter as one-particle test functions for the field. (See Ref. 14.)