1. INTRODUCTION

The coherent-state representation and its variants [1-3] have found many applications in quantum physics, in particular as a tool for the study of the classical limit [4-6]. For finite degrees of freedom, such representations are usually confined to non-relativistic systems. The purpose of this paper is to construct similar representations which are applicable to relativistic particles. In section 2 we develop a family of representations for the dynamics of a free non-relativistic particle which is closely related to the coherent-state representation. This family is extended in section 3 to include relativistic particles. In section 4 we summarize some properties of the new wave packets.

2. NON-RELATIVISTIC PARTICLE

The wave function $f(x,t)$ for a non-relativistic free particle in $\mathbb{R}^n$ evolves under the Schrödinger equation

$$i \frac{\partial f}{\partial t} = H f, \quad H = -\frac{1}{2m} \Delta.$$  \hspace{1cm} (2.1)

The solutions are given by
\[ f(\hat{x}, t) = (e^{-i\hat{x} \cdot \hat{p}})(\hat{x}) = (2\pi)^{-n/2} \int e^{-i\hat{x} \cdot \hat{p} + i\hat{\tau} \cdot \hat{p}} d^n\hat{p} \quad (2.2) \]

where \( \hat{f}(\hat{p}) \) is the Fourier transform of the initial function \( f(\hat{x}, 0) \). Now let \( \hat{x} = \hat{x} - i\hat{\tau} \in \mathbb{C}^n \) and let \( \tau = \tau - i\beta \) be in the lower-half plane \( \mathbb{C}^- \) (i.e., \( \beta > 0 \)). Then \( \exp(-i\hat{x} \cdot \hat{p} + i\hat{\tau} \cdot \hat{p}) \) decays rapidly as \( |\hat{\tau}| \to +\infty \) and eq. \( (2.2) \) defines a function \( f(\hat{x}, \tau) \) holomorphic in \( \mathbb{P} = \mathbb{C}^n \times \mathbb{C}^- \). Let \( G = \{ f(\hat{x}, \tau) : \hat{x} \in L^2(\mathbb{R}^n) \} \) be the vector space of all such functions. Then for each \( \beta > 0 \) the function \( f_\beta(\hat{x}, t) = f(\hat{x} - i\hat{\tau}, t - i\beta) \) satisfies \( (2.1) \) in \( \hat{x} \) and \( t \). Let \( G_\beta \) be the space of all such functions \( f_\beta(\hat{x}, t) \). On \( G_\beta \) define the map \( (e^{-i\hat{\tau} \cdot \hat{p}}(\hat{x}, s) = f_\beta(\hat{x}, s + t). \) We are going to make \( G_\beta \) into a Hilbert space such that \( e^{-i\hat{\tau} \cdot \hat{p}} \) is unitary for every real \( t \), giving us a unitary representation of dynamics on \( G_\beta \) for every \( \beta > 0 \).

Although these representations are all unitarily equivalent, the spaces \( G_\beta \) have some interesting properties, as we shall see.

Thus let \( \beta > 0 \) and \( \hat{x} = \hat{x} - i\hat{\tau} \in \mathbb{C}^n \). Then

\[ f_\beta(\hat{x}, 0) = (e^{-i\beta \cdot \hat{p}})(\hat{x}) = (2\pi)^{-n/2} \int e^{-i\hat{x} \cdot \hat{p} - i\beta \cdot \hat{p}} d^n\hat{p} \quad (2.3) \]

where

\[ (e^{\beta \cdot \hat{p}})(\hat{x}) = (2\pi)^{-n/2} \exp(-i\hat{x} \cdot \hat{p}) \quad (2.4) \]

with Fourier transform

\[ (e^{\beta \cdot \hat{\tau}})(\hat{x}) = (2\pi)^{-n/2} \exp(-i\hat{x} \cdot \hat{\tau} / \beta) \quad (2.5) \]

The \( e^{\beta \cdot \hat{x}} \) are minimum-uncertainty spherical wave packets with

\( (X_k) = x_k = \text{Re}(t_k), \quad (P_k) = (m/\beta)y_k \) and diameter \( \Delta X_k = \sqrt{\beta / 2m} \)

\( (k = 1, 2, \ldots, n) \). For \( f_\beta \) in \( G_\beta \) define
\[ \|f\|_\beta^2 = \int_{C^n} |f_{\beta}(z,0)|^2 \, du_\beta(z), \quad (2.6) \]

where
\[ du_\beta(z) = \left( \frac{m}{\pi\hbar^2} \right)^{n/2} \exp(-\frac{m\gamma^2}{\hbar^2}) \, d^n x \, d^n y. \quad (2.7) \]

**Theorem 1.** Let \( t \in \mathbb{R}, \beta > 0, f \in L^2(\mathbb{R}^n) \) and \( f_\beta = e^{-\beta H} f \).

Then
\[ \|f\|_\beta = \|f\|. \quad (2.8) \]

In particular,

(a) \( \| \cdot \|_\beta \) is a norm on \( G_\beta \) under which \( G_\beta \) is a Hilbert space,

(b) The map \( e^{-\beta H} \) is unitary from \( L^2(\mathbb{R}^n) \) onto \( G_\beta \),

(c) The map \( e^{-\beta H} \) is unitary on \( G_\beta \).

**Remarks.** 1. (2.8) can of course be polarized to give a resolution of the identity: for \( f, g \) in \( L^2(\mathbb{R}^n) \),
\[ \langle f | g \rangle_\beta \equiv \int_{C^n} \langle f | e^{\frac{B}{2}} \rangle_\beta \langle e^{\frac{-B}{2}} g \rangle_\beta \, du_\beta(z) = \langle f | g \rangle. \quad (2.9) \]

2. \( e^{-\beta H} \) intertwines \( [7] \) the dynamics on \( L^2(\mathbb{R}^n) \) with the dynamics on \( G_\beta \).

**Proof.** Let \( f \in S(\mathbb{R}^n) \). By (2.3), \( f_\beta(x-i\hbar \xi,0) = \xi \cdot g_\beta(x) \) where
\[ g_\beta(x) = \exp(-\beta p^2/2m + \gamma \cdot p) \, \tilde{f}(\xi), \]
and \( \xi \) denotes the inverse Fourier transform of \( g \). Thus by Plancherel's theorem (and Fubini's),
\[ \|f\|_\beta^2 = \left( \frac{m}{\pi\hbar^2} \right)^{n/2} \int_{\mathbb{R}^n} e^{-m\gamma^2/\hbar^2} \, d^n x \, d^n p \, e^{-\beta p^2/2m + \gamma \cdot p} |\tilde{f}(\xi)|^2 \]
\[ = \int_{\mathbb{R}^n} |\tilde{f}(\xi)|^2 \, d^n p = \|f\|^2, \]
which proves (2.8) for \( f \) in \( S(\mathbb{R}^n) \), hence also in \( L^2(\mathbb{R}^n) \) by
continuity. (a)-(c) are obvious.

3. RELATIVISTIC PARTICLE

We sketch a generalization of the results of section 2 to relativistic particles. We confine ourselves to $n=3$.

The evolution of a free scalar relativistic particle of mass $m > 0$ is given by the Klein-Gordon equation

$$
(- \frac{1}{c^2} \frac{\partial^2}{\partial t^2} + \Delta - m^2) \phi(x,t) = 0.
$$

(3.1)

We consider only positive-energy solutions. These are given by

$$
\phi(x,t;\mu) = (e^{-iH\mu} \phi)(x) = (2\pi)^{-3/2} \int_{\mathbb{R}^3} e^{-i\mathbf{p} \cdot \mathbf{x}} \hat{\phi}(\mathbf{p}) d\Omega(\mathbf{p}),
$$

where $\hat{\phi}$ is the ordinary Fourier transform on $\mathbb{R}^3$. For every $\hat{\phi}$ in $L^2(\mathbb{R}^3)$ the solution $\phi(x,t;\mu)$ is the boundary-value of a function $\phi(x,t;\mu_0) = \phi(x)$ holomorphic in the forward tube $\mathcal{T}$,

$$
\mathcal{T} = \{ x-iy \in \mathbb{C}^4 : x \in \mathbb{R}^4, y \in \mathbb{V}_+ \},
$$

where

$$
\mathbb{V}_+ = \{ y \in \mathbb{R}^4 : y_0 > |\mathbf{y}| \}
$$

is the open forward light cone. This is so because

$$
|e^{-i\mathbf{p} \cdot \mathbf{x}}| = e^{-\gamma p} < \exp(-\gamma_0 |\mathbf{y}| |\mathbf{p}|)
$$

decays rapidly as $|\mathbf{p}| \to \infty$ for fixed $x$ in $\mathcal{T}$. $\mathcal{T}$ will replace $\mathcal{D} = \mathbb{C}^3 \times \mathbb{C}^-$ of section 2 and is strictly contained in $\mathcal{D}$. The analogue of $\mathcal{G}$ is the space $K = \{ \phi(x,t;\mu) : \phi \in L^2(\mathbb{R}) \}$. To obtain
counterparts of the $G_\beta$ we need a phase space. In section 2 that
was the set $\{(\tilde{z}, \tau) \in \mathcal{P} : \tau = -i\beta\} \cong \mathbb{C}^3$. This will not do since it
is not contained in $T$. Thus we deform it: let

$$P_\lambda = \{z = x-iy \in \mathbb{C}^4 : z_0 = -i \sqrt{\lambda^2 + y^2}\}, \quad \lambda \geq 0.$$ 

The functions

$$f_\lambda(\tilde{z}, x_0) = f(\tilde{z}-i\gamma, x_0 - i \sqrt{\lambda^2 + y^2})$$

satisfy (3.1) in $\tilde{z}$ and $x_0 = ct$. Let $K_\lambda = (f_\lambda(\tilde{z}, x_0) : f \in L^2(\mathcal{H}))$
and denote the map $\hat{f}(\hat{p}) \to f_\lambda(\tilde{z}, x_0)$ by $U_\lambda$. Define dynamics on $K_\lambda$
by

$$(e^{-iH\lambda} \hat{f}_\lambda)(\tilde{z}, x_0) = f_\lambda(\tilde{z}, x_0 + x')/$$

For $\lambda > 0$,

$$\hat{f}_\lambda(\tilde{z}, 0) = (2\pi)^{-3/2} \int \exp\left(-\sqrt{\lambda^2+y^2}\omega+iz \cdot \vec{p}\right)\hat{f}(\vec{p})\,d\omega(\vec{p})$$

$$= (e_\lambda^\lambda | \hat{f} ) \tag{3.3}$$

where

$$(e_\lambda^\lambda | \hat{p} ) = (2\pi)^{-3/2} \exp\left(-\sqrt{\lambda^2+y^2}\omega+iz \cdot \vec{p}\right) \tag{3.4}$$

and all inner products are in $L^2(\mathcal{H})$ until further notice. The $e_\lambda^\lambda$ are in $L^2(\mathcal{H})$ for $z = x-iy$ in $P_\lambda$ and $z' = x'-iy'$ in $P_\lambda$, (where $\lambda, \lambda' > 0$),

$$\langle e_\lambda^\lambda | e_\lambda^{\lambda'} \rangle = (2\pi)^{-3} \int \exp\left(-(y_0+y_0')\omega+iz \cdot \vec{p}\right)\hat{f}(\vec{p})\,d\omega(\vec{p})$$

$$= -2i\delta_\lambda(z-\tilde{z'}) \cdot \hat{p}$$

$$= \frac{mc}{4\pi^2\hbar} \chi_1(2\nu c), \tag{3.5}$$

where $y_0$ denotes $\sqrt{\lambda^2+y^2}$, $\chi_1$ is the two-point function for the
free scalar field of mass $m$ \cite{8} and $2\eta = \left[-(z-\bar{z})^2\right]^{1/2}$ is defined by analytic continuation from $\left[-(z-\bar{z})^2\right]^{1/2} = [4y^2]^{1/2} = 2\lambda$ for $z = z' = x+iy$ in $P_\lambda$. $K_n$ ($n = 0,1,2,\ldots$) denotes a modified Bessel function. For $\lambda=0$, (3.3) still gives $f_0(\bar{z},0)$ and the functions $e_{2\lambda}^0$ are still defined, but are no longer in $L^2(\Omega)$, as (3.5) shows.

For $f_{\lambda} \in K_\lambda$ ($\lambda \geq 0$) define

$$\|\hat{f}\|_{\lambda}^2 = \int_{C^3} |\hat{f}_{\lambda}(\bar{z},0)|^2 du_{\lambda}(\bar{z})$$

(3.6)

where

$$du_{\lambda}(\bar{z}) = C_\lambda d^3xd^3y$$

(3.7)

with $C_\lambda = [2\pi(\lambda mc)^2K_0(2\lambda mc)]^{-1}$ for $\lambda > 0$ and $C_0 = (mc)^4/\pi$. Then our main result is the following

**Theorem 2.** Let $\lambda \geq 0$ and $\hat{f} \in L^2(\Omega)$. Then

$$\|\hat{f}\|_{\lambda} = \|\hat{f}\|_2$$

(3.8)

In particular,

(a) $\|\cdot\|_{\lambda}$ is a Lorentz-invariant norm on $K_\lambda$ under which $K_\lambda$ is a Hilbert space.

(b) The map $U_{\lambda}$ is unitary from $L^2(\Omega)$ onto $K_\lambda$.

(c) $e_{\lambda}^0$ is unitary on $K_\lambda$.

The remarks following Theorem 1 apply here as well.

Comparing the measures (2.7) and (3.7), note that $du_{\lambda}$ has no weight function. This is a consequence of the curvature of the phase space $P_\lambda$. The "weight" has been absorbed into the functions $f_{\lambda}$ themselves, which are consequently bounded:

$$|f_{\lambda}(\bar{z},0)|^2 = |(a^\lambda_\pm f_\pm)|^2 \leq \|e_{\lambda}^0\|_{2}^2 \|\hat{f}\|_{\lambda}^2 = \frac{me}{4\pi^2\lambda} K_1(2\lambda mc)\|\hat{f}\|_2^2$$

(3.9)

Finally note that Theorem 2 gives us a unitary, irreducible
representation of the restricted Poincaré group on $K_{\lambda}$. Define
the action on $K$ by $(U(g)f)(z) = f(g^{-1}z)$, $g \in \mathfrak{p}^+$. This induces an
action on $K_{\lambda}$ with the desired properties.

4. CONCLUSION

The $e_{\frac{\lambda}{2}}$ have other interesting properties which we can only
mention here for lack of space. In the state $e_{\frac{\lambda}{2}}$, the particle
appears as a wave packet centered about $\bar{x} = \text{Re}(\bar{z})$ with expected
momentum proportional to $\bar{y} = -\text{Im}(\bar{z})$. The wave packet, which is
spherical in the rest frame, shows contraction in the direction
of motion and has minimal uncertainties in a natural sense. Its
diameter increases from zero (when $\lambda mc \rightarrow 0$) to $\sim \sqrt{\lambda mc} (\text{when}
\lambda mc \rightarrow \infty)$. Thus $e_{\frac{\lambda}{2}}$ describes an extended, relativistic particle.

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