Observer-Based $\mathcal{H}_\infty$ Fault-Tolerant Control for Linear Systems With Sensor and Actuator Faults

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Abstract—In this paper, the design problem of $\mathcal{H}_\infty$ observer-based fault-tolerant control is considered for three faulty cases: sensor only, actuator only, and both sensor and actuator faults. The observers are designed for estimating both states and faults. To reduce the fault effects on the system, virtual observers are first introduced. Based on the virtual observer, real observers are established because the virtual observers include unmeasurable information of the system. By using the estimated information from the observers, observer-based $\mathcal{H}_\infty$ fault-tolerant controllers are designed. A numerical example is provided to demonstrate the effectiveness of the proposed method.

Index Terms—$\mathcal{H}_\infty$ control, fault-tolerant control (FTC), observer-based control, sensor and actuator fault.

I. INTRODUCTION

BY INCREASING the complexity of a control system, the reliability of the control system becomes more important. In practice, the control of a system or plant consists of measurements from sensors and control action from actuators; so the performance of the control system depends heavily on the quality of the sensor and actuator measurements. However, it is inevitable to have faults in a real system owing to external disturbances, malfunction of hardware or software components, and other unavoidable factors. When faults appear in sensor and/or actuator, the characteristics of the sensor and/or actuator may change over time; therefore affecting the performance of the designed controller, or even the stability of the overall system. Indeed, studies on fault-tolerant control (FTC) methods against sensor and actuator faults have attracted much attention from researchers and practitioners [1]–[8]. There are two main FTC approaches: passive and active. The typical passive approach designs the control law with fixed gain for both faulty-free and fault cases such that the control systems exhibit robustness for possible system fault cases by using robust control techniques. This approach is usually easy to design and implement, but it may have limited control performance and fault acceptability. In contrast, the active approach adjusts the structure or parameters of the controllers according to the online fault information. As such, active FTC schemes can achieve better robustness and fault acceptability than the passive ones. In active FTC schemes, fault estimation is a key task, and many methods are available for estimating faults, such as Kalman filter [9], neural networks [10], and adaptive methods [11]. Polycarpou [10] presented a learning method based on neural networks for estimating faults in nonlinear multi-input multoutput systems. In [11], an adaptive method was used to establish an updated law for estimating actuator fault online. Based on the estimated fault information, the proposed reliable $\mathcal{H}_\infty$ controller was updated automatically to compensate for the fault effects toward the plant. In [12], a new fault estimation method, namely, $k$-step-fault-estimation, was proposed. By using the information of online $k$-step-fault-estimation, an FTC with dynamic output feedback was designed for the Takagi–Sugeno (T–S) fuzzy systems with time-varying delays.

As stated above, it is very natural and important to consider faults in sensors and actuators. However, most FTC studies deal with either sensor or actuator faults, but not both. Only limited investigations handle both sensor and actuator faults simultaneously. In [13], a reliable event-triggered controller was designed for networked control systems with sensor and actuator faults. The faults were modeled as random variables, and the relevant probability information was captured and analyzed. In [14], multiple intermittent faults in both sensors and actuators were considered, and treated as external disturbances. By using the $\mathcal{H}_\infty$ method, reliable $\mathcal{H}_\infty$ controllers were designed for a class of discrete-time nonlinear systems, such that the fault effects on the system reduce under a given performance threshold. In [15], the problem of fault estimation and FTC for a class of T–S fuzzy Itô stochastic systems with sensor and actuator faults was considered. The faults were handled by proposing a new descriptor fuzzy sliding-mode observer method to obtain the simultaneous estimates of system states, sensor faults, and actuator faults. Owing to the limited studies in the current literature, further investigations to design robust FTC schemes for undertaking both sensor and actuator faults in the control system are required, and this is the main objective of this paper.

On the other hand, external disturbances are an undesired factor, and they are unavoidable in real-world systems. It is necessary to reduce the effects of noise or disturbances to a certain
positive definite matrix (respectively, positive semidefinite).

When a sensor fault occurs, the real measurement of the system

output can be represented by

\[ y^F(t) = Cx(t) + f_s(t) \quad (2) \]

where \( f_s(t) \in \mathbb{R}^q \) is the sensor fault.

In addition, when an actuator fault occurs, the control input can be represented by

\[ u(t) = u_F(t) + f_a(t) \quad (3) \]

where \( f_a(t) \in \mathbb{R}^m \) is the actuator fault and \( u_F(t) \) is the control input signal.

**III. MAIN RESULTS**

In this section, three faulty cases are considered, sensor fault, actuator fault, and both sensor and actuator faults. All cases proceed as follows. First, an observer for estimating the states of system \((1)\) and the respective faults is designed. Second, an observer and observer-based controllers to stabilize the system for three faulty cases. A numerical example is provided to illustrate the effectiveness of the proposed scheme.

**A. Case 1: Sensor Fault**

In this section, we design the following observer for system \((1)\) with sensor fault \((2)\):

\[
\begin{aligned}
\dot{x}(t) &= A x(t) + B u(t) + D w(t) \\
y(t) &= C x(t)
\end{aligned}
\quad (1)
\]

where \( x(t) = (x_1, x_2, \ldots, x_n)^T \in \mathbb{R}^n \) is the state vector of the system, \( y(t) = (y_1, y_2, \ldots, y_q)^T \in \mathbb{R}^q \) is the output vector of the system, \( u(t) = (u_1, u_2, \ldots, u_m)^T \in \mathbb{R}^m \) is the control input, \( w(t) = (w_1, w_2, \ldots, w_p)^T \in \mathbb{R}^p \) is the external disturbance which belongs to \( L_2(0, \infty) \), and \( A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times m}, C \in \mathbb{R}^{q \times n}, D \in \mathbb{R}^{q \times p} \) are known system matrices.

In this paper, we consider both the sensor and actuator faults. When a sensor fault occurs, the real measurement of the system

\[ y^F(t) = Cx(t) + f_s(t) \quad (2) \]

where \( f_s(t) \in \mathbb{R}^q \) is the sensor fault.

In addition, when an actuator fault occurs, the control input can be represented by

\[ u(t) = u_F(t) + f_a(t) \quad (3) \]

where \( f_a(t) \in \mathbb{R}^m \) is the actuator fault and \( u_F(t) \) is the control input signal.

The following assumptions and definition are needed to derive our main results.

**Assumption 1:** The pair \((A, B)\) is controllable and \((A, C)\) is observable.

**Assumption 2:** \( \dot{f}_s(t) \) belongs to \( L_2(0, \infty) \).

**Definition 1:** A system is said to be stable with \( H_\infty \) performance if the following conditions are satisfied:

1) With zero disturbance, the system is asymptotically stable.
2) With zero initial condition and for a given positive constant \( \gamma \), the following condition holds:

\[ \int_0^\infty x^T(t)x(t)dt < \gamma^2 \int_0^\infty w^T(t)w(t)dt \]

where \( x(t) \) is the state vector of the system and \( w(t) \) is disturbance in the system which belongs to \( L_2(0, \infty) \).

**Remark 1:** As stated in [7] and [12], Assumption 2 means that the fault derivatives are energy-bounded. Therefore, it is natural to consider a fault satisfying Assumption 2 in practice.
Let $F_1 = \begin{bmatrix} I_n & 0 \end{bmatrix}$ and $F_2 = \begin{bmatrix} 0_{n \times q} & I_q \end{bmatrix}$, then we can easily calculate the following:

\[
\begin{bmatrix} E_1 \\ E_2 \\ F_1 \end{bmatrix} \begin{bmatrix} F_1 & F_2 \\ F_1 & E_1, E_2 \end{bmatrix} = I_{n+q}
\]

(6)

which means $[E_1]^{-1} = [F_1 F_2]$. Multiplying $F_1$ to both sides of (5) yields that

\[
F_1 E_1 \dot{z}(t) = F_1 A_1 z(t) + F_1 B u(t) + F_1 D w(t)
\]

and by using fact (6), i.e., $F_1 E_1 + F_2 E_2 = I_{n+q}$, we have:

\[
\dot{z}(t) = F_1 A_1 z(t) + F_1 B u(t) + F_1 D w(t) + F_2 E_2 \dot{z}(t)
\]

(8)

Consider the following virtual observer:

\[
\dot{z}_F(t) = F_1 A_1 z_F(t) + F_1 B u(t) + F_2 E_2 \dot{z}(t) + L(y^F(t) - E_2 z_F(t))
\]

(9)

where $L$ is an observer gain matrix to be determined later. Define the estimation error as $e(t) = z(t) - z_F(t)$. Then, the error dynamics can be obtained as follows:

\[
\dot{e}(t) = (F_1 A_1 - LE_2) e(t) + F_1 D w(t)
\]

(10)

**Theorem 1:** For a given positive constant $\delta$, error system (10) is asymptotically stable with disturbance attenuation level $\delta$ if there exists a positive definite matrix $P \in \mathbb{R}^{(n+q) \times (n+q)}$ and any matrix $H \in \mathbb{R}^{(n+q) \times q}$ satisfying the following LMI:

\[
\Pi = \begin{bmatrix} \Delta & P F_1 D \\ \ast & -\delta^2 I \end{bmatrix} < 0
\]

(11)

where $\Delta = PF_1 A_1 + A_1^T F_1^T P - H E_2 - E_2^T H^T + I$. Observer (4) is designed with the following parameters:

\[
\mathcal{A} = F_1 A_1 - LE_2 \quad \mathcal{B} = F_1 B \quad \mathcal{C} = F_2 \quad \mathcal{L} = -F_1 A_1 - LE_2 F_2 \quad L = P^{-1} H.
\]

(12)

**Proof:** We consider the following Lyapunov functional:

\[
V_F(t) = e^T(t) P e(t).
\]

(13)

If LMI (11) holds, then the time derivative of $V_F(t)$ along the trajectories of the error dynamics (10) leads to

\[
\dot{V}_F(t) = \begin{bmatrix} e(t) \\ w(t) \end{bmatrix}^T \Pi \begin{bmatrix} e(t) \\ w(t) \end{bmatrix} - e^T(t) e(t) + \delta^2 w^T(t) w(t)
\]

\[
\leq -e^T(t) e(t) + \delta^2 w^T(t) w(t).
\]

Integrating both sides of (14) from 0 to $\infty$ satisfies

\[
\int_0^\infty \dot{V}_F(t) + e^T(t) e(t) dt < \int_0^\infty \delta^2 w^T(t) w(t) dt.
\]

Since $\dot{V}_F(0) = 0$ and $\dot{V}_F(\infty) \geq 0$, the above equation becomes

\[
\int_0^\infty e^T(t) e(t) dt < \int_0^\infty \delta^2 w^T(t) w(t) dt.
\]

(15)

Note that, this equation is the same as the second condition of Definition 1. In addition, when $w(t) = 0$, LMI (11) naturally guarantees $\dot{V}_F(t) < 0$ which means the asymptotical stability of error system (10) without $w(t)$.

Therefore, according to Definition 1, error system (10) is asymptotically stable with $H_\infty$ performance under disturbance attenuation level $\delta$.

However, virtual observer (9) may have poor estimation accuracy because of the term $F_2 E_2 \dot{z}(t)$. So, by defining $\eta(t) = z_F(t) - F_2 E_2 z(t)$, (9) can be rewritten as

\[
\dot{\eta}(t) = F_1 A_1 \eta(t) + F_1 B u(t) + L(y^F(t) - E_2 z_F(t))
\]

\[
= (F_1 A_1 - LE_2) \eta(t) + F_1 B u(t) + L y^F(t)
\]

\[
= (F_1 A_1 - LE_2) \eta(t) + F_1 B u(t) + L y^F(t)
\]

(16)

which is the same to the designed observer (4) with parameters (12). Finally, we can obtain $z_F(t) = \eta(t) + F_2 E_2 \dot{z}(t)$. As a result, we derive a real observer based on the virtual observer by eliminating the time derivative term. A similar approach was employed in [24] to design a delay compensator for compensating the effect of communication delay in discrete-time networked-control systems. In addition, in [24, Remark 3], a practical technique has been introduced to obtain a feasible solution of the proposed theorem when the system parameters do not satisfy certain conditions, and the technique works in our proposed method as well.

Until now, we have designed an $H_\infty$ observer for system (1), and have $z_F(t)$ as the estimated value for $x(t)$ and $f_2(t)$. So, now we are ready to derive the observer-based controller.

We consider that the observer-based control input has the form of $u(t) = K E_1 z_F(t)$, where $K$ is the control gain to be determined later. Then, system (1) becomes

\[
\dot{x}(t) = A x(t) + B K E_1 z_F(t) + D w(t)
\]

\[
= (A + BK) x(t) - B K E_1 e(t) + D w(t)
\]

\[
= (A + BK) x(t) - B d(t) + D w(t)
\]

(17)

where $d(t) = K E_1 e(t)$.

**Theorem 2:** Under Theorem 1, for a given positive constant $\alpha$ and designed observer (4) with parameters (12) satisfying LMI (11), system (1) with sensor fault (2) is asymptotically stable with disturbance attenuation level $\gamma$ if there exists a positive matrix $R \in \mathbb{R}^{n \times n}$ and any matrix $G \in \mathbb{R}^{m \times n}$ satisfying the following LMI:

\[
\begin{bmatrix} \bar{\Delta} & R & -B & D \\ -I & 0 & 0 & 0 \\ * & -\alpha I & 0 & 0 \\ * & * & -\delta^2 I & 0 \end{bmatrix} < 0
\]

(18)

where $\bar{\Delta} = AR + RA^T + BG + G^T B^T$ and $\delta$ is obtained in Theorem 1. In addition, the control gain can be calculated by
$K = GR^{-1}$ and the disturbance attenuation level is given as
$$\gamma = \sqrt{(\alpha \lambda_{\text{max}}(K^T K) + 1)\delta^2}.$$

Proof: We consider the following Lyapunov functional:
$$V(t) = x^T(t)Qx(t)$$
where $Q \in \mathbb{R}^{n \times n}$ is a positive definite matrix. Then, its time derivative is
$$\dot{V}(t) = \zeta(t)^T \Phi \zeta(t) - x^T(t)\dot{x}(t) + \alpha d^T(t)d(t) + \kappa^2 w^T(t)w(t)$$
where $\zeta(t) = [x^T(t) \ d^T(t) \ w^T(t)]$ and
$$\Phi = \begin{bmatrix} \hat{\Delta} & -QB & QD \\ * & -\alpha I & 0 \\ * & * & -\delta^2 I \end{bmatrix}$$
where $\hat{\Delta} = QA + A^TQ + QBK + K^TBTQ + I$.

Let $R = Q^{-1}$ and pre and postmultiplying $\text{diag}\{R, I, I\}$ with $\Phi$, we have
$$\Gamma = \begin{bmatrix} \hat{\Delta} & -B & D \\ * & -\alpha I & 0 \\ * & * & -\delta^2 I \end{bmatrix}$$
where $\hat{\Delta} = AR + RA^T + BKR + K^TBT^2 + RR$.

If LMI (18) holds, then it guarantees that $\Gamma < 0$ and $\Phi < 0$ by Schur complement. Therefore, (20) can be defined as follows:
$$\dot{V}(t) + x^T(t)\dot{x}(t) < \alpha d^T(t)d(t) + \kappa^2 w^T(t)w(t)$$
and by integrating both side of above equation from 0 to $\infty$ and using (15) and facts $V(0) = 0$, $V(\infty) > 0$, and $e^T(t)e(t) = e^T(t)E_1^T E_1 e(t) + e^T(t)E_2^T E_2 e(t)$ where $E_3 = [I_q \ 0_m]$, we have
$$\int_0^\infty x^T(t)x(t)dt < \int_0^\infty \alpha d^T(t)d(t) + \kappa^2 w^T(t)w(t)dt$$
and using (15) and facts $V(0) = 0$, $V(\infty) > 0$, and $e^T(t)e(t) = e^T(t)E_1^T E_1 e(t) + e^T(t)E_2^T E_2 e(t)$ where $E_3 = [I_q \ 0_m]$, we have
$$\int_0^\infty x^T(t)x(t)dt < \int_0^\infty \alpha d^T(t)d(t) + \kappa^2 w^T(t)w(t)dt$$
and using (15) and facts $V(0) = 0$, $V(\infty) > 0$, and $e^T(t)e(t) = e^T(t)E_1^T E_1 e(t) + e^T(t)E_2^T E_2 e(t)$ where $E_3 = [I_q \ 0_m]$, we have
$$\int_0^\infty x^T(t)x(t)dt < \int_0^\infty \alpha d^T(t)d(t) + \kappa^2 w^T(t)w(t)dt$$
and using (15) and facts $V(0) = 0$, $V(\infty) > 0$, and $e^T(t)e(t) = e^T(t)E_1^T E_1 e(t) + e^T(t)E_2^T E_2 e(t)$ where $E_3 = [I_q \ 0_m]$, we have
$$\int_0^\infty x^T(t)x(t)dt < \int_0^\infty \alpha d^T(t)d(t) + \kappa^2 w^T(t)w(t)dt$$
where $\gamma$ is defined in Theorem 2. According to the same procedure for Theorem 1, system (1) is asymptotically stable with $H_\infty$ performance under disturbance attenuation level $\gamma$ by Definition 1. This completes the proof.

B. Case 2: Actuator fault

For estimating the state of system (1), $x(t)$, and actuator fault, $f_m(t)$, we design the following observer:
$$\begin{cases} \hat{n}(t) = \hat{\Lambda}\hat{n}(t) + \bar{B}u_F(t) + \bar{L}y(t) \\ \bar{z}_F(t) = \hat{n}(t) + \bar{C}y(t) \\ \dot{\hat{n}}(t) = A\hat{n}(t) + B\bar{u}_F(t) + \bar{L}\bar{y}(t) \end{cases}$$
where $\bar{n}(t)$ is an auxiliary variable, matrices $\hat{\Lambda}, \bar{B}, \bar{C}$, and $\bar{L}$ are the observer parameters to be determined later, and $\bar{z}_F(t)$ is the estimation of $x(t)$ and $f_m(t)$.

When an actuator fault occurs, system (1) can be reformulated as
$$\begin{cases} \bar{E}_1\dot{\bar{z}}(t) = A_2\bar{z}(t) + B_uF(t) + Dw(t) \\ y(t) = \bar{E}_3\bar{z}(t) \end{cases}$$
where $\bar{z}(t) = [x(t) \ f_m(t)]$, $A_2 = [A \ B]$, $\bar{E}_1 = [I_n \ 0_{n \times m}]$, and $\bar{E}_3 = [C \ \bar{O}_{q \times m}]$.

We define $\bar{E}_2 = [B_1 C \ \bar{I}_m]$ where $B_1 \in \mathbb{R}^{m \times q}$ is a full column rank (if $m > q$) or row rank (if $m < q$) matrix, then it is clear that $\text{rank}(\bar{E}_2) = n + m$ which means it is full rank and its inverse exists.

Let $\bar{F}_1 = [I_q \ B_2 ]$, $\bar{F}_2 = [0_m \ \bar{I}_m]$, then we can easily calculate the following:
$$\begin{bmatrix} \bar{E}_1 \\ \bar{E}_2 \end{bmatrix} = \begin{bmatrix} \bar{F}_1 & \bar{F}_2 \end{bmatrix} = \bar{I}_{n+m}$$
$$\begin{bmatrix} \bar{F}_1 & \bar{F}_2 \end{bmatrix} = \begin{bmatrix} \bar{E}_1 \\ \bar{E}_2 \end{bmatrix} = \bar{I}_{n+m}$$
which means $[\bar{F}_1 \ \bar{F}_2]^{-1} = [\bar{F}_1 \ \bar{F}_2]$.

By using the same procedure in Section 1, multiplying $\bar{F}_1$ to both sides of (22) and using (23) yields
$$\dot{\bar{z}}(t) = \bar{F}_1\bar{A}_2\bar{z}(t) + \bar{F}_1B_uF(t) + \bar{F}_1Dw(t) + \bar{F}_2\bar{E}_3\dot{\bar{z}}(t)$$
$$+ \bar{L}(y(t) - \bar{E}_3\bar{z}(t))$$
where $\bar{L}$ is an observer gain matrix to be determined later.

We define the estimation error as $\bar{e}(t) = \bar{z}(t) - \bar{z}_F(t)$, then the error system can be obtained as follows:
$$\dot{\bar{e}}(t) = (\bar{F}_1\bar{A}_2 - \bar{L}\bar{E}_3)\bar{e}(t) + \bar{F}_D\bar{w}(t)$$
where $\bar{F}_D = [\bar{F}_1 D \ \bar{F}_2]$ and $\bar{w}(t) = [\bar{w}(t)]$.

Theorem 3: For a given positive constant $\delta$, error system (26) is asymptotically stable with disturbance attenuation level $\delta$ if there exists a positive definite matrix $\bar{P} \in \mathbb{R}^{(n + m) \times (n + m)}$ and any matrix $\bar{H} \in \mathbb{R}^{(n + m) \times q}$ satisfying the following LMI:
$$\bar{P} = \begin{bmatrix} \bar{F}_1 & \bar{F}_2 \end{bmatrix}$$
$$\begin{bmatrix} \bar{F}_1 & \bar{F}_2 \end{bmatrix} = \begin{bmatrix} \bar{E}_1 \\ \bar{E}_2 \end{bmatrix} = \bar{I}_{n+m}$$
where $\bar{P} = \bar{F}_1 A_2 - \bar{L}\bar{E}_3$ and $\bar{w}(t) = [\bar{w}(t)]$.

Proof: We consider the Lyapunov functional as $\dot{V}_F(t) = e^T(t)\bar{P}e(t)$, then LMI (27) yields
$$\dot{V}_F(t) = \begin{bmatrix} \bar{e}(t) & \bar{w}(t) \end{bmatrix}^T \begin{bmatrix} \bar{P} & \bar{F}_D \\ \bar{F}_D & \delta^2 \bar{I} \end{bmatrix} \bar{e}(t) + \delta^2 \bar{w}(t) \bar{w}(t)$$
$$\leq -\bar{e}^T(t)\bar{e}(t) + \delta^2 \bar{w}(t) \bar{w}(t)$$
Integrating both sides of (29) from 0 to \( \infty \) and using \( \dot{V}_F(0) = 0 \)
and \( \dot{V}_F(\infty) \geq 0 \), we have
\[
\int_0^\infty \dot{V}_F(t) \tilde{e}(t) dt < \int_0^\infty \tilde{\sigma}^2 \tilde{w}^T(t) \tilde{w}(t) dt.
\]
(30)

Therefore, by Definition 1, if LMI (27) holds, then the error system (26) is asymptotically stable with \( \mathcal{H}_\infty \) performance under disturbance attenuation level \( \delta \).

Furthermore, to eliminate the term \( F_2 B_1 \bar{E}_3 \tilde{z}(t) \) in the virtual observer (25), we define \( \bar{\eta}(t) = \bar{z}_F(t) - F_2 B_1 \bar{E}_3 z(t) \), then (25) can be rewritten as
\[
\dot{\bar{\eta}}(t) = \bar{F}_1 A_2 \bar{z}_F(t) + \bar{F}_1 B u_F(t) + \bar{L}(y(t) - \bar{E}_3 \bar{z}_F(t))
= (\bar{F}_1 A_2 - \bar{L} \bar{E}_3) \bar{z}_F(t) + \bar{F}_1 B u_F(t) + \bar{L} y(t)
= (\bar{F}_1 A_2 - \bar{L} \bar{E}_3) \bar{\eta}(t) + \bar{F}_1 B u_F(t)
+ (\bar{L} - \bar{F}_1 A_2 - \bar{L} \bar{E}_3) \bar{F}_2 B_1 y(t)
\]
(31)

which is the same to the designed observer of (21) with parameters (28). Finally, we obtain \( \bar{z}_F(t) = \bar{\eta}(t) + F_2 B_1 \bar{E}_3 z(t) = \bar{\eta}(t) + F_2 B_1 y(t) \). This completes the proof.

Now we are ready to derive the observer-based controller using \( \bar{z}_F(t) \), which is obtained from the designed \( \mathcal{H}_\infty \) observer (21) with parameters (28) satisfying LMI (27) and disturbance attenuation level \( \delta \).

We consider the observer-based control input in the form of
\[
u_F(t) = (\bar{K} \bar{E}_1 - \bar{E}_4) \bar{z}_F(t)
\]
(32)
where \( \bar{E}_1 = [0_{m \times n} \ I_m] \) and \( \bar{K} \) is the control gain to be determined later. Then, system (1) with actuator fault (3) becomes
\[
\dot{x}(t) = A x(t) + B ((\bar{K} \bar{E}_1 - \bar{E}_4) \bar{z}_F(t) + f_a(t)) + D w(t)
= (A + \bar{B} \bar{K}) x(t) - B (\bar{K} \bar{E}_1 + \bar{E}_4) \bar{e}(t) + D w(t)
= (A + \bar{B} \bar{K}) x(t) - B \bar{d}_1(t) + B \bar{d}_2(t) + \bar{D} w(t)
\]
(33)
where \( \bar{D} = [D \ 0] \), \( \bar{d}_1(t) = \bar{K} \bar{E}_1 \bar{e}(t) \), and \( \bar{d}_2(t) = \bar{E}_4 \bar{e}(t) \).

**Theorem 4:** Under Theorem 3, for given positive constants \( \bar{\alpha} \) and \( \beta \) and designed observer (21) with parameters (28) satisfying LMI (27), system (1) with actuator fault (3) is asymptotically stable with disturbance attenuation level \( \tilde{\gamma} \) if there exists a positive matrix \( \bar{R} \in \mathbb{R}^{n \times n} \) and any matrix \( \bar{G} \in \mathbb{R}^{n \times n} \) satisfying the following LMI:
\[
\begin{bmatrix}
\bar{S} & \bar{R} & -B & B & \bar{D} \\
* & -\bar{I} & 0 & 0 & 0 \\
* & * & -\bar{\alpha} \bar{I} & 0 & 0 \\
* & * & * & -\beta \bar{I} & 0 \\
* & * & * & * & -\tilde{\gamma}^2 \bar{I}
\end{bmatrix} < 0
\]
(34)

where \( \bar{S} = A \bar{R} + \bar{R} A^T + B \bar{G} + \bar{G}^T B^T \) and \( \tilde{\gamma} \) is obtained in Theorem 3. In addition, the control gain can be calculated by \( \bar{K} = \bar{G} \bar{R}^{-1} \) and the disturbance attenuation level is given as
\[
\tilde{\gamma} = \sqrt{(\bar{\alpha} \lambda_{\text{max}}(K^T K) + 1 + \bar{\beta}) \bar{\delta}^2}.
\]

**Proof:** We consider the following Lyapunov functional:
\[
\dot{V}(t) = x^T(t) \bar{Q} x(t)
\]
(35)
where \( \bar{Q} \in \mathbb{R}^{n \times n} \) is a positive definite matrix.

Then, its time derivative is
\[
\dot{V}(t) = \bar{z}_F(t) \dot{\bar{\eta}}(t) - x^T(t) x(t) + \bar{\alpha} \bar{d}_1^T(t) \bar{d}_1(t)
+ \bar{\beta} \bar{d}_2^T(t) \bar{d}_2(t) + \tilde{\sigma}^2 \bar{w}^T(t) \bar{w}(t)
\]
(36)

where \( \bar{\eta}(t) = \bar{z}_F(t) - F_2 B_1 \bar{E}_3 z(t) \) and
\[
\bar{\Phi} = \begin{bmatrix}
\bar{S} & -\bar{Q} \bar{B} & \bar{Q} \bar{B} & \bar{Q} \bar{D} \\
* & -\bar{\alpha} \bar{I} & 0 & 0 \\
* & * & -\beta \bar{I} & 0 \\
* & * & * & -\tilde{\gamma}^2 \bar{I}
\end{bmatrix}
\]

where \( \bar{S} = \bar{A} \bar{R} + \bar{R} \bar{A}^T + B \bar{G} + \bar{G}^T B^T \) and \( \tilde{\gamma} \) is obtained in Theorem 4. According to the same procedure in Theorem 1, system (1) is asymptotically stable with \( \mathcal{H}_\infty \) performance under disturbance attenuation level \( \tilde{\gamma} \) by Definition 1. This completes the proof.

**C. Case 3: Sensor and Actuator Faults**

In this section, both sensor and actuator faults are considered. To this end, we first consider the following observer:
\[
\begin{align*}
\dot{\hat{\eta}}(t) &= \hat{A} \hat{\eta}(t) + \hat{B} u_F(t) + \hat{L} \hat{y}(t) \\
\hat{z}_F(t) &= \hat{Q} \hat{z}(t) + \hat{C} \hat{y}(t)
\end{align*}
\]
(37)

where \( \hat{\eta}(t) \) is an auxiliary variable, matrices \( \hat{A}, \hat{B}, \hat{C}, \) and \( \hat{L} \) are the observer parameters to be determined later, and \( \hat{z}_F(t) \) is the estimation of \( x(t), f_s(t), \) and \( f_a(t) \).

According to the same procedure of the previous sections, system (1) with sensor fault (2) and actuator fault (3) can be
denoted as follows:

\[
\begin{cases}
\dot{z}(t) = \hat{F}_1 A_3 z(t) + \hat{F}_1 B u_F(t) + \hat{F}_1 D w(t) \\
y^F(t) = \hat{E}_2 \hat{z}(t)
\end{cases}
\]

(38)

where \( \hat{z} = [x^T(t) f_1^T(t) f_2^T(t)]^T, A_3 = [A \ 0_{n \times q} \ B], \) and

\[
\hat{E} = \begin{bmatrix}
\hat{E}_1 \\
\hat{E}_2 \\
\hat{E}_3
\end{bmatrix} = \begin{bmatrix}
I_n & 0_{n \times q} & 0_{n \times m} \\
-C & I_q & 0_{q \times m} \\
0_{m \times n} & -B_1 & I_m
\end{bmatrix}
\]

(39)

Then it is clear that \( \hat{E} \) is a full rank matrix and its inverse matrix can be obtained as \( \hat{E}^{-1} = \hat{F}, \) i.e., \( \hat{F}_1 \hat{E}_1 + \hat{F}_2 \hat{E}_2 + \hat{F}_3 \hat{E}_3 = I. \)

Let us define the following virtual observer:

\[
\begin{align*}
\dot{\hat{z}}_F(t) &= \hat{F}_1 A_3 \hat{z}_F(t) + \hat{F}_1 B u_F(t) + (\hat{F}_2 + \hat{F}_3 B_1) \hat{E}_2 \hat{z}(t) \\
&\quad + \hat{L}(y^F(t) - \hat{E}_2 \hat{z}_F(t))
\end{align*}
\]

(40)

where \( \hat{L} \) is an observer gain matrix to be determined later.

Defining the estimation error as \( \hat{e}(t) = \hat{z}(t) - \hat{z}_F(t) \) leads to the following error dynamics:

\[
\dot{\hat{e}}(t) = (\hat{F}_1 A_3 - \hat{L} \hat{E}_2) \hat{e}(t) + \hat{D} \hat{w}(t)
\]

(41)

where \( \hat{D} = [D \ 0] \) and \( \hat{w}(t) \) is defined in (26).

**Theorem 5:** For a given positive constant \( \delta, \) error system (40) is asymptotically stable with disturbance attenuation level \( \delta \), if there exists a positive definite matrix \( \hat{P} \in \mathbb{R}^{(n+m+q) \times (n+m+q)} \) and any matrix \( \hat{H} \in \mathbb{R}^{(n+m+q) \times n} \) satisfying the following LMI:

\[
\hat{P} \hat{D} \hat{P}^T + \begin{bmatrix}
\hat{F}_1 A_3 & \hat{E}_2 \hat{L} & \hat{E}_2 \hat{L} & 0
\end{bmatrix} \\
\begin{bmatrix}
0_{n \times q} & 0_{q \times m} & 0_{m \times n} & 0_{m \times m}
\end{bmatrix} < 0
\]

(42)

**Proof:** When we consider the Lyapunov functional as \( \hat{V}_F(t) = \hat{e}^T(t) \hat{P} \hat{e}(t), \) according to the similar procedure of the proofs for Theorems 1 and 3, Theorem 5 can be easily proven. So, the proof is omitted here.

Next, we design the observer-based controller using the designed observer of (37) with parameters (42) satisfying LMI (41).

We consider the observer-based control input in the form of

\[
u_F(t) = (\hat{K} \hat{E}_1 - \hat{E}_2) \hat{z}_F(t) \]

(43)

where \( \hat{E}_2 = [0_{m \times n} \ 0_{m \times q} \ I_m] \) and \( \hat{K} \) is the control gain to be determined later.

Then, system (1) with both sensor fault (2) and actuator fault (3) becomes

\[
\begin{align*}
\dot{x}(t) &= Ax(t) + B((\hat{K} \hat{E}_1 - \hat{E}_2) \hat{z}_F(t) + f_s(t)) + Dw(t) \\
&= (A + B \hat{K})x(t) - B(\hat{K} \hat{E}_1 + \hat{E}_2) \hat{e}(t) + Dw(t) \\
&= (A + B \hat{K})x(t) - B \hat{d}_1(t) + B \hat{d}_2(t) + D \hat{w}(t)
\end{align*}
\]

(44)

where \( \hat{d}_1(t) = \hat{K} \hat{E}_1 \hat{e}(t), \hat{d}_2(t) = \hat{E}_2 \hat{e}(t), \) and \( D \) is defined in (33).

**Theorem 6:** Under Theorem 5, for the given positive constants \( \alpha \) and \( \beta \) and designed observer (37) with parameters (42) satisfying LMI (41), system (1) with sensor fault (2) and actuator fault (3) is asymptotically stable with disturbance attenuation level \( \hat{\delta} \), if there exists a positive matrix \( \hat{R} \in \mathbb{R}^{n \times n} \) and any matrix \( \hat{G} \in \mathbb{R}^{m \times n} \) satisfying the following LMI:

\[
\begin{bmatrix}
\hat{\Xi} & \hat{R} & -B & B & \hat{D} \\
* & -I & 0 & 0 & 0 \\
* & * & -\alpha I & 0 & 0 \\
* & * & * & -\beta I & 0 \\
* & * & * & * & -\delta^2 I
\end{bmatrix} < 0
\]

(45)

where \( \hat{\Xi} = A \hat{R} + \hat{R} A^T + B \hat{G} + \hat{G}^T \hat{B}^T \hat{B} \) and \( \hat{\delta} \) are obtained in Theorem 5. In addition, the control gain can be calculated by \( \hat{K} = \hat{G} \hat{R}^{-1} \) and the disturbance attenuation level is given as \( \hat{\delta} = \sqrt{(\alpha \lambda_{\max}(\hat{K}^T \hat{K}) + \beta + 1) \hat{\delta}^2}. \)

**Proof:** When we consider the Lyapunov functional as \( \hat{V}(t) = \hat{e}^T(t) \hat{Q} \hat{x}(t), \) and following the similar procedure in Theorem 4, Theorem 6 can be easily obtained. So, the proof is omitted here.

**Remark 3:** Very recently, remarkable works on handling sensor and actuator faults were reported [25], [26]. In [25], a linear continuous-time switched system with simultaneous disturbances, sensor and actuator faults was considered and both descriptor type reduced-order observer and sliding mode observer were designed to estimate states of the system, faults, and disturbance. The paper [26] was concerned with the FTC problem of a nonlinear Markovian jump systems with output disturbances, actuator, and sensor faults simultaneously. First, a descriptor type sliding mode observer was designed to monitor states of the system, faults, and disturbance, and then a feedback controller was designed to stabilize the system. These works have taken into account simultaneous sensor and actuator faults the same as this paper and need to know the upper bound value of sensor and actuator faults and disturbance, but this paper does not need them. We utilize the \( \mathcal{H}_\infty \) control method to improve the estimation accuracy against the unknown sensor and actuator fault and disturbance.

**IV. NUMERICAL EXAMPLE**

A system with two masses and two springs modeled in the form of (1) with the following parameters [27] is used for
illustrating the effectiveness of the proposed method:

\[
A = \begin{bmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
-2 & 1 & -0.5 & 0 \\
2 & -2 & 0 & -1
\end{bmatrix},
B = \begin{bmatrix}
0 \\
0 \\
1 \\
0
\end{bmatrix},
\]

\[
C = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0
\end{bmatrix},
D = \begin{bmatrix}
0.1 \\
0.1 \\
0.1
\end{bmatrix}.
\]

Given the following parameters: \( \delta = \delta = \delta = 0.4, \alpha = 0.3, \alpha_1 = 0.1, \beta = 9, \alpha_2 = 0.03, \) and \( \beta = 16, \) we can obtain the following values for each case by using Theorems 1–6:

**Case 1:**

\[
\begin{align*}
\gamma &= 3.0785 \\
K &= \begin{bmatrix}
-11.6418 & 0.1060 & -6.8401 & -3.4324 \\
257.0666 & 246.0590 \\
342.8792 & 331.9765 \\
-69.5056 & -69.6612 \\
-1.4743 & -13.6080 \\
-252.9797 & -242.8634 \\
-337.3556 & -326.1748
\end{bmatrix},
\end{align*}
\]

\[
L = \begin{bmatrix}
36.4010 & -26.7597 \\
-10.1884 & 9.9278 \\
295.4673 & -230.6275 \\
-3.1969 & 4.1860 \\
872.3001 & -685.6052
\end{bmatrix}.
\]

**Case 2:**

\[
\begin{align*}
\bar{\gamma} &= 3.1520 \\
\bar{K} &= \begin{bmatrix}
-19.1722 & 0.4201 & -10.8645 & -5.9299 \\
36.4010 & -26.7597 \\
-10.1884 & 9.9278 \\
295.4673 & -230.6275 \\
-3.1969 & 4.1860 \\
872.3001 & -685.6052
\end{bmatrix},
\end{align*}
\]

\[
\bar{L} = \begin{bmatrix}
36.4010 & -26.7597 \\
-10.1884 & 9.9278 \\
295.4673 & -230.6275 \\
-3.1969 & 4.1860 \\
872.3001 & -685.6052
\end{bmatrix}.
\]

**Case 3:**

\[
\begin{align*}
\tilde{\gamma} &= 3.5789 \\
\tilde{K} &= \begin{bmatrix}
157.1947 & -94.1093 \\
-4.2314 & 10.7003 \\
404.6966 & -260.5303 \\
52.4496 & -30.4361 \\
-117.9508 & 69.3506 \\
1.2455 & -4.6398 \\
656.3364 & -424.8111
\end{bmatrix},
\end{align*}
\]

\[
\tilde{L} = \begin{bmatrix}
36.4010 & -26.7597 \\
-10.1884 & 9.9278 \\
295.4673 & -230.6275 \\
-3.1969 & 4.1860 \\
872.3001 & -685.6052
\end{bmatrix}.
\]

For the simulation, we set the initial values, external disturbance, and sensor and actuator faults as follows:

\[
x(0) = (0.5, 0, 0.2, 0.1)\quad z_F(0) = (0, 0, 0, 0, 0, 0) \\
z_F(0) = (0, 0, 0, 0, 0)\quad \hat{z}_F(0) = (0, 0, 0, 0, 0, 0) \\
w(t) = (\sin \pi t + 1.5) e^\frac{-1.5}{t} \\
f_a(t) = (\sin \pi t) e^{-0.5t} \\
f_s(t) = \begin{cases} \\
0.5 u_F(1) \\
(0.3 \sin 0.1 t - 0.4) u_F(2) \\
\end{cases}
\]

With the above parameter setting, the simulation results are presented in Figs. 1–9. First, we define the estimation of \( x(t) \) for each case as follows: \( x_F(t) = \hat{E}_1 x(t) \) for Case 1, \( x_F(t) = \hat{E}_2 x(t) \) for Case 2, and \( \hat{x}_F(t) = \hat{E}_3 x(t) \) for Case 3. Then, the estimation errors of \( x(t) \) can be also defined as \( e_x(t) = x(t) - x_F(t) \), for Case 1, \( e_x(t) = x(t) - x_F(t) \), for Case 2, and \( e_x(t) = x(t) - \hat{x}_F(t) \), for Case 3. Figs. 1, 4, and 7 show \( e_x(t) \), \( e_z(t) \), and \( e_F(t) \), respectively, and Figs. 2, 5, and 8 depict the sensor and/or actuator faults and the respective estimation results for each case. As seen in these figures, the designed observer (4), (21), and (37) estimate the states of system (1) and unknown sensor and/or actuator faults. Figs. 3, 6, and 9 display the controlled states of system (1) in each case, which demonstrate that our designed observer-based controllers work well in making system (1) stable.

**Remark 4:** As seen in simulation results, the disturbance attenuation level for three cases, \( \gamma, \tilde{\gamma}, \hat{\gamma} \), are over 3. By definition, these factors, \( \gamma, \tilde{\gamma}, \hat{\gamma} \), are associated with LMI variables, i.e., \( \bar{K} \) and \( \alpha \) in Theorem 2, \( \tilde{K}, \tilde{\alpha}, \) and \( \hat{\beta} \) in Theorem 4, and \( \tilde{K}, \tilde{\alpha}, \) and \( \hat{\beta} \)
Fig. 3. Controlled states of Case 1.

Fig. 4. Estimation error $\bar{e}_x(t)$ of Case 2.

Fig. 5. Actuator fault and its estimation result of Case 2.

Fig. 6. Controlled states of Case 2.

Fig. 7. Estimation error $\hat{e}_x(t)$ of Case 3.

Fig. 8. Sensor and actuator faults and estimation results of Case 3.
in Theorem 6. So, we can just check how much the disturbance attenuation levels does each theorem guarantees after getting feasible solution of LMIs (18), (34), and (45). It is obvious that a small value of the disturbance attenuation level ensures better system performance because it reduce the effect of disturbance to the system under the value. Therefore, the development of an algorithm minimizing the disturbance attenuation levels is a notable issue and will be our future work.

V. CONCLUSION

In this paper, the observer-based FTC problem has been investigated for a class of linear systems with sensor and/or actuator faults. Three faulty cases have been considered, i.e., sensor fault only, actuator fault only, and both sensor and actuator faults. First, observers are designed for estimating both states of the system and faults. Virtual observers are first used to reduce the fault effects on the system. Then, real observers are obtained from the virtual observers. Based on the designed observers, new criteria for designing observer-based $\mathcal{H}_\infty$ FTC models have been established. The effectiveness of the proposed method has been demonstrated by using a numerical example.

REFERENCES


Fig. 9. Controlled states of Case 3.
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