TODAY 8:00 AM

EINSTEIN'S FIRST PROOF

BY STEVEN STROGATZ

A physicist's genius turns up in his boyhood geometry.
ILLUSTRATION BY TOMI UM

On November 26, 1949, Albert Einstein published an essay in the *Saturday Review of Literature* in which he described two pivotal moments in his childhood. The first involved a compass that his father showed him when he was four or five. Einstein recalled his sense of wonderment that the needle always pointed north, even though nothing appeared to be pulling it in that direction. He came to a conclusion, then and there, about the structure of the physical world: "Something deeply hidden had to be behind things." The second moment occurred soon after he turned twelve, when he was given "a little book dealing with Euclidean plane geometry." The book's "lucidity," he wrote—the idea that a mathematical assertion could "be proved with such certainty that any doubt appeared to be out of the question"—provoked "wonder of a totally different nature." Pure thought could be just as powerful as geomagnetism.

This month, we celebrate the hundredth anniversary of Einstein's general theory of relativity, one of his many ideas that brought lucidity to the deeply hidden. With all the surrounding hoopla, it would be nice if we could fathom something of what he actually accomplished and how he did it. That turns out to be a tall order, because general relativity is tremendously complex. When Arthur Eddington—the British astrophysicist who led the team that confirmed Einstein's predictions, during a solar eclipse in 1919—was asked if it was really true that only three people in the world understood the theory, he said nothing. "Don't be so modest, Eddington!" his questioner said. "On the contrary," Eddington replied. "I'm just wondering who the third might be."

Fortunately, we can study an earlier, simpler example of Einstein's thinking. Even before he received the little geometry book, he had been introduced to the subject by his uncle Jakob, an engineer. Einstein became particularly enamored of the Pythagorean theorem and—"after much effort," he noted in the *Saturday Review*—he wrote his own mathematical proof of it. It is my intention to lead you through that proof, step by logical step. It's Einstein's first masterpiece, and certainly his most accessible one. This little gem of reasoning foreshadows the man he became, scientifically, stylistically, and temperamentally. His instinct for symmetry, his economy of means, his iconoclasm, his tenacity, his penchant for thinking in pictures—they're all here, just as they are in his theory of relativity.

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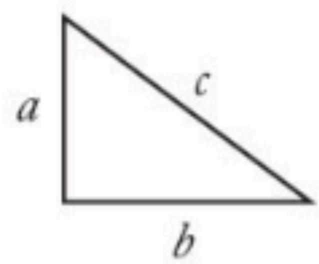
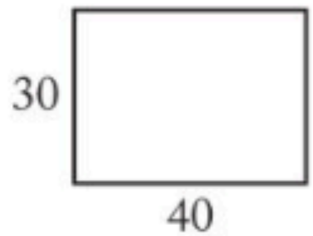
You may once have memorized the Pythagorean theorem as a series of symbols: $a^2 + b^2 = c^2$. It concerns right triangles, meaning triangles that have a right (ninety-degree) angle at one of their corners. The theorem says that if a and b are the lengths of the triangle's legs (the sides that meet at the right angle), then the length of the hypotenuse (the side opposite the right angle) is given by c, according to the formula above.



Teen-agers have this rule drummed into them by the millions, year after year, in schools around the world, but most don't give it much thought. Maybe you never did, either. Once you do, though, the questions start coming. What makes it true? How did anyone ever come up with it? Why?

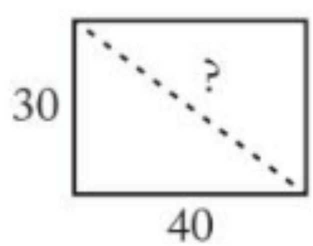
For a clue to that last question, consider the etymology of the word *geometry*. It derives from the Greek roots *gē* (meaning "earth" or "land") and *metria* ("measurement"). It's easy to imagine ancient peoples and their monarchs being concerned with the measurement of fields or plots of land. Officials needed to assess how much tax was to be paid, how much water they would need for irrigation, how much wheat, barley, and papyrus the farmers could produce.

Imagine a rectangular field, thirty yards by forty.

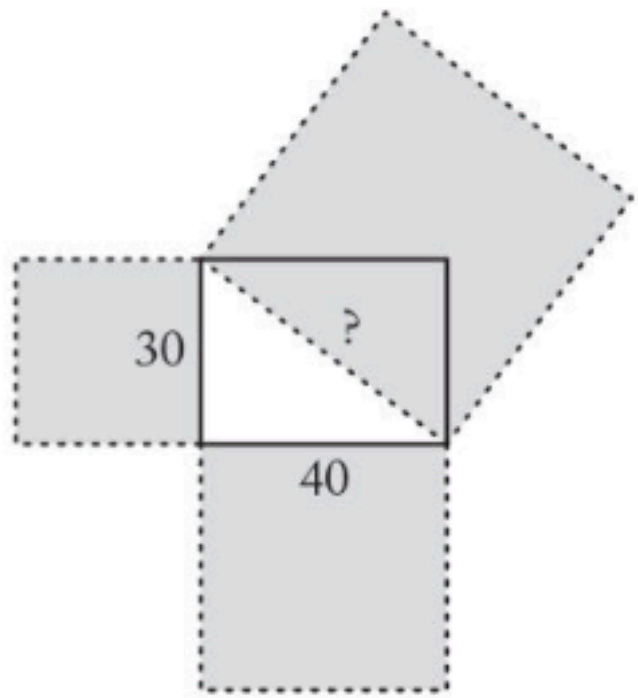


How much land is that? The meaningful measure would be the area of the field. For a thirty-by-forty lot, the area would be thirty times forty, which is twelve hundred square yards. That's the only number the tax assessor cares about. He's not interested in the precise shape of your land, just how much of it you have.

Surveyors, by contrast, do care about shapes, and angles, and distances, too. In ancient Egypt, the annual flooding of the Nile sometimes erased the boundaries between plots, necessitating the use of accurate surveying to redraw the lines. Four thousand years ago, a surveyor somewhere might have looked at a thirty-by-forty rectangular plot and wondered, How far is it from one corner to the diagonally opposite corner?

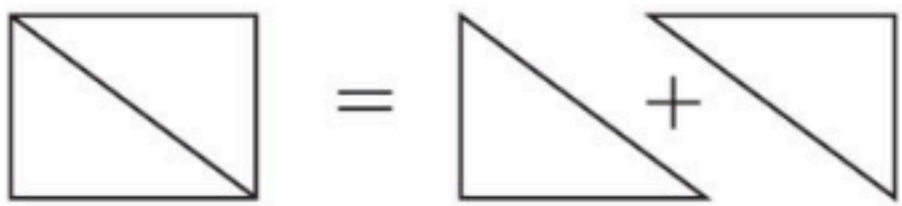


The answer to that question is far less obvious than the earlier one about area, but ancient cultures around the world—in Babylon, China, Egypt, Greece, and India—all discovered it. The rule that they came up with is now called the Pythagorean theorem, in honor of Pythagoras of Samos, a Greek mathematician, philosopher, and cult leader who lived around 550 B.C. It asks us to imagine three fictitious square plots of land—one on the short side of the rectangle, another on the long side, and a third on its diagonal.



Next, we are instructed to calculate the area of the square plots on the sides and add them together. The result, 900 + 1,600 = 2,500, is, according to the Pythagorean theorem, the same as the area of the square on the diagonal. This recipe yields the unknown length that we are trying to calculate: fifty yards, since 50 × 50 = 2,500.

The Pythagorean theorem is true for rectangles of any proportion—skinny, blocky, or anything in between. The squares on the two sides always add up to the square on the diagonal. (More precisely, the *areas* of the squares, not the squares themselves, add up. But this simpler phrasing is less of a mouthful, so I'll continue to speak of squares adding up when I really mean their areas.) The same rule applies to right triangles, the shape you get when you slice a rectangle in half along its diagonal.



The rule now sounds more like the one you learned in school: $a^2 + b^2 = c^2$. In pictorial terms, the squares on the sides of a right triangle add up to the square on its hypotenuse.

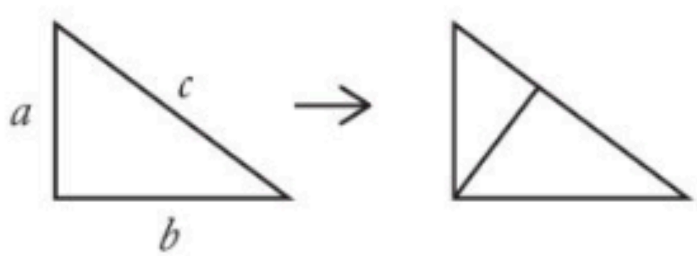
But why is the theorem true? What's the logic behind it? Actually, hundreds of proofs are known today. There's a marvelously simple one attributed to the Pythagoreans and, independently, to the ancient Chinese. There's an intricate one given in Euclid's Elements, which schoolchildren have struggled with for the past twenty-three hundred years, and which induced in the philosopher Arthur Schopenhauer "the same uncomfortable feeling that we experience after a juggling trick." There's even a proof by President James A. Garfield, which involves the cunning use of a trapezoid.

Einstein, unfortunately, left no such record of his childhood proof. In his *Saturday Review* essay, he described it in general terms, mentioning only that it relied on "the similarity of triangles." The consensus among Einstein's biographers is that he probably discovered, on his own, a standard textbook proof in which similar triangles (meaning triangles that are like photographic reductions or enlargements of one another) do indeed play a starring role. Walter Isaacson, Jeremy Bernstein, and Banesh Hoffman all come to this deflating conclusion, and each of them describes the steps that Einstein would have followed as he unwittingly reinvented a well-known proof.

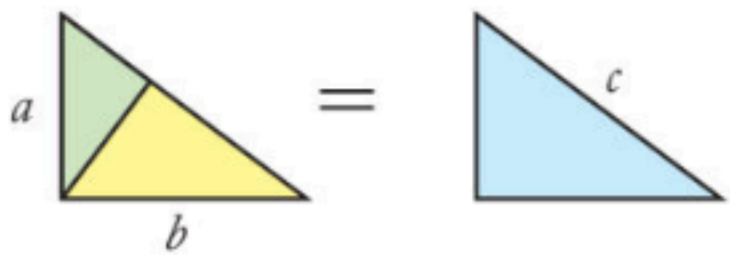
Twenty-four years ago, however, an alternative contender for the lost proof emerged. In his book "Fractals, Chaos, Power Laws," the physicist Manfred Schroeder presented a breathtakingly simple proof of the Pythagorean theorem whose provenance he traced to Einstein. Schroeder wrote that the proof had been shown to him by a friend of his, the chemical physicist Shneior Lifson, of the Weizmann Institute, in Rehovot, Israel, who heard it from the physicist Ernst Straus, one of Einstein's former assistants, who heard it from Einstein himself. Though we cannot be sure the following proof is Einstein's, anyone who knows his work will recognize the lion by his claw.

It helps to run through the proof quickly at first, to get a feel for its over-all structure.

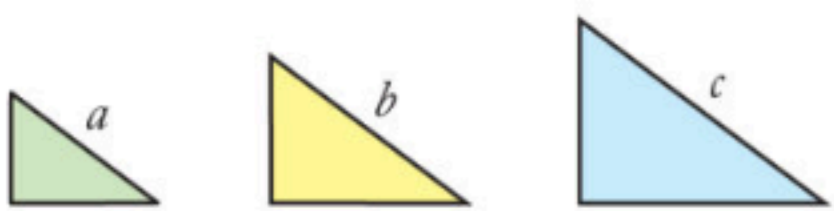
Step 1: Draw a perpendicular line from the hypotenuse to the right angle. This partitions the original right triangle into two smaller right triangles.



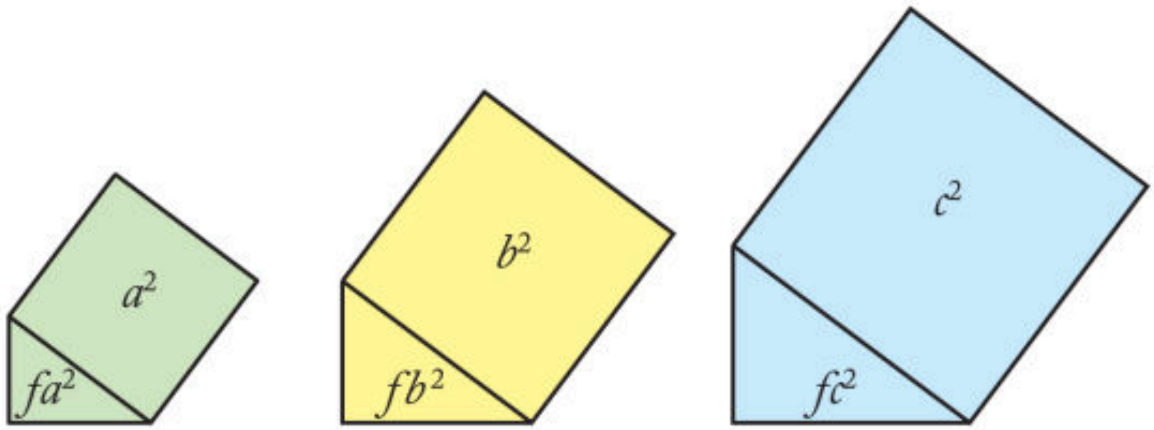
Step 2: Note that the area of the little triangle plus the area of the medium triangle equals the area of the big triangle.



Step 3: The big, medium, and little triangles are similar in the technical sense: their corresponding angles are equal and their corresponding sides are in proportion. Their similarity becomes clear if you imagine picking them up, rotating them, and arranging them like so, with their hypotenuses on the top and their right angles on the lower left:



Step 4: Because the triangles are similar, each occupies the same fraction f of the area of the square on its hypotenuse. Restated symbolically, this observation says that the triangles have areas fa^2, fb^2, and fc^2, as indicated in the diagram.



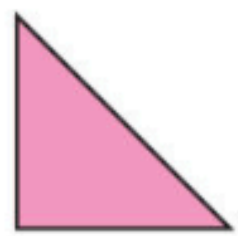
(Don't worry if this step provokes a bit of head-scratching. I'll have more to say about it below, after which I hope it'll seem obvious.)

Step 5: Remember, from Step 2, that the little and medium triangles add up to the original big one. Hence, from Step 4, $fa^2 + fb^2 = fc^2$.

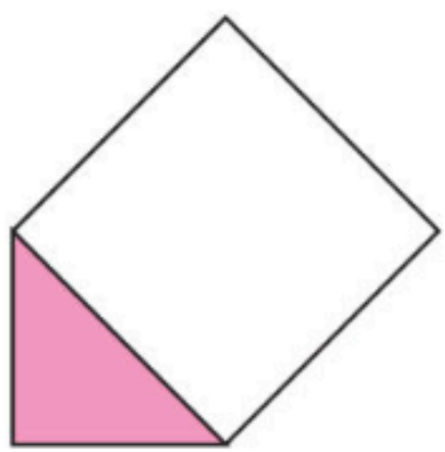
Step 6: Divide both sides of the equation above by f. You will obtain $a^2 + b^2 = c^2$, which says that the areas of the squares add up. That's the Pythagorean theorem.

The proof relies on two insights. The first is that a right triangle can be decomposed into two smaller copies of itself (Steps 1 and 3). That's a peculiarity of right triangles. If you try instead, for example, to decompose an equilateral triangle into two smaller equilateral triangles, you'll find that you can't. So Einstein's proof reveals why the Pythagorean theorem applies only to right triangles: they're the only kind made up of smaller copies of themselves. The second insight is about additivity. Why do the squares add up (Step 6)? It's because the triangles add up (Step 2), and the squares are proportional to the triangles (Step 4).

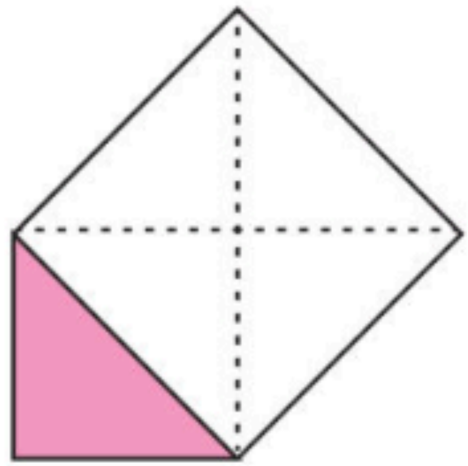
The logical link between the squares and triangles comes via the confusing Step 4. Here's a way to make peace with it. Try it out for the easiest kind of right triangle, an isosceles right triangle, also known as a 45-45-90 triangle, which is formed by cutting a square in half along its diagonal.



As before, erect a square on its hypotenuse.

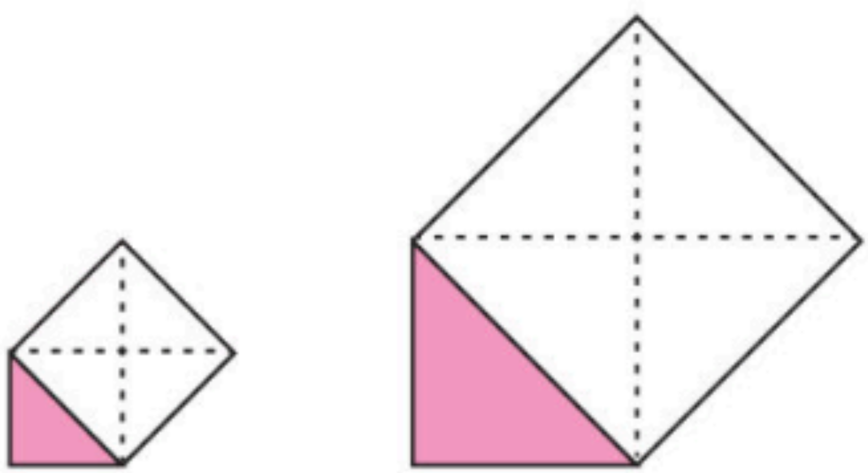


If we draw dashed lines on the diagonals of that newly built square, the picture looks like the folding instructions for an envelope.



As you can see, four copies of the triangle fit neatly inside the square. Or, said the other way around, the triangle occupies exactly a quarter of the square. That means that $f = 1/4$, in the notation above.

Now for the cruncher. We never said how big the square and the isosceles right triangle were. The ratio of their areas is *always* one to four, for any such envelope. It's a property of the envelope's shape, not its size.



That's the thrust of Step 4. It's obvious when you think of it like that, no?

The same thing works for any right triangle of any shape. It doesn't have to be isosceles. The triangle always occupies a certain fraction, f, of the square on its hypotenuse, and that fraction stays the same no matter how big or small they both are. To be sure, the numerical value of f depends on the proportions of the triangle; if it's a long, flat sliver, the square on its hypotenuse will have a lot more than four times its area, and so f will be a lot less than 1/4. But that numerical value is irrelevant. Einstein's proof shows that f disappears in the end anyway. It enters stage right, in Step 4, and promptly exits stage left, in Step 6.

What we're seeing here is a quintessential use of a symmetry argument. In science and math, we say that something is symmetrical if some aspect of it stays the same despite a change. A sphere, for instance, has rotational symmetry; rotate it about its center and its appearance stays the same. A Rorschach inkblot has reflectional symmetry: its mirror image matches the original. In Step 4 of his proof, Einstein exploited a symmetry known as scaling. Take a right triangle with a square on its hypotenuse and rescale both of them by the same amount, as if on a photocopier. That rescaling changes some of their features (their areas and side lengths) while leaving others intact (their angles, proportions, and area ratio). It's the constancy of the area ratio that undergirds Step 4.

Throughout his career, Einstein would continue to deploy symmetry arguments like a scalpel, getting to the hidden heart of things. He opened his revolutionary 1905 paper on the special theory of relativity by noting an asymmetry in the existing theories of electricity and magnetism: "It is known that Maxwell's electrodynamics—as usually understood at the present time—when applied to moving bodies, leads to asymmetries which do not appear to be inherent in the phenomena." Those asymmetries, Einstein sensed, must be a clue to something rotten at the core of physics as it was then formulated. In his mind, everything else—space, time, matter, energy—was up for grabs, but not symmetry. Think of the courage it required to reformulate nearly all of physics from the ground up, even if it meant revising Newton and Maxwell along the way.

Both special and general relativity are also profoundly geometrical theories. They conceive of the universe as having a dimension beyond the usual three; that fourth dimension is time. Rather than considering the distance between two points (a measure of space), the special-relativistic counterpart of the Pythagorean theorem considers the interval between two events (a measure of space-time). In general relativity, where space-time itself becomes warped and curved by the matter and energy within it, the Pythagorean theorem still has a part to play; it morphs into a quantity called the metric, which measures the space-time separation between infinitesimally close events, for which curvature can temporarily be overlooked. In a sense, Einstein continued his love affair with the Pythagorean theorem all his life.

The style of his Pythagorean proof, elegant and seemingly effortless, also portends something of the later scientist. Einstein draws a single line in Step 1, after which the Pythagorean theorem falls out like a ripe avocado. The same spirit of minimalism characterizes all of Einstein's adult work. Incredibly, in the part of his special-relativity paper where he revolutionized our notions of space and time, he used no math beyond high-school algebra and geometry.

Finally, although the young Einstein made his proof of the Pythagorean theorem look easy, it surely wasn't. Remember that, in his *Saturday Review* essay, he says that it required "much effort." Later in life, this tenacity—what Einstein referred to as his stubbornness—would serve him well. It took him years to come up with general relativity, and he often felt overwhelmed by the abstract mathematics that the theory required. Although he was mathematically powerful, he was not among the world's best. ("Every boy in the streets of Göttingen understands more about four-dimensional geometry than Einstein," one of his contemporaries, the mathematician David Hilbert, remarked.)

Many years after his Pythagorean proof, Einstein shared this lesson with another twelve-year-old who was wrestling with mathematics. On January 3, 1943, a junior-high-school student named Barbara Lee Wilson wrote to him for advice. "Most of the girls in my room have heroes which they write fan mail to," she began. "You + my uncle who is in the Coast Guard are my heroes." Wilson told Einstein that she was anxious about her performance in math class: "I have to work longer in it than most of my friends. I worry (perhaps too much)." Four days later, Einstein sent her a reply. "Until now I never dreamed to be something like a hero," he wrote. "But since you have given me the nomination I feel that I am one." As for Wilson's academic concerns? "Do not worry about your difficulties in mathematics," he told her. "I can assure you that mine are still greater."

Steven Strogatz is a professor of mathematics at Cornell. He is the author, most recently, of "The Joy of *x*."