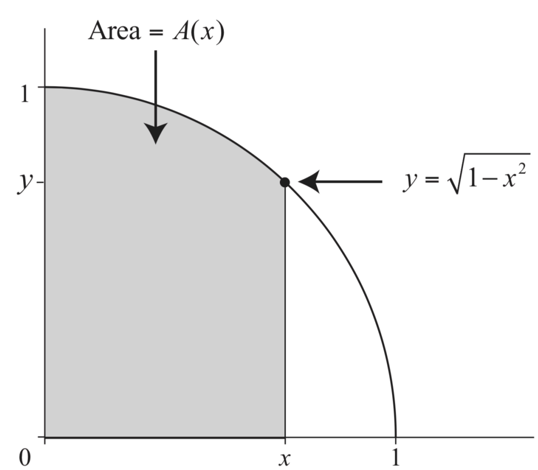
Appendix 2: Newton schools Leibniz on power series in the "Epistola Posterior"

Newton wrote two long letters to Leibniz. (Actually, they were addressed to an intermediary, Henry Oldenburg, the secretary of the Royal Society of London, for transmission to Leibniz.) Historians of mathematics call these famous letters the Epistola Prior and the Epistola Posterior. Newton's mood feels very different in them. In the first he's cagey and guarded, taking pains not to reveal too much. By turns condescending and intimidating, his tone seems designed to push Leibniz away, to show him who's boss, to convince him he's out of his depth. Newton barrages Leibniz with results so impressive, so dazzling, and so opaque, that his intent must have been to shock and awe. He doesn't want Leibniz to understand. He wants him to cower.

But in the second letter, written a few months later, Newton softens up. He tells a story from his student days, when he was just beginning to learn mathematics. Sounding almost nostalgic, he recounts how he discovered the binomial series by a process of guessing and checking. Here I'd like to show you his reasoning in some detail, because it is so charming and accessible. It reminds me of games little kids like to play: guess the next number, find the pattern, what do you think comes next?

As we discussed in Chapter 7, young Newton's inspiration for developing power series came from reading Wallis's *Arithmetica Infinitorum*. Mimicking Wallis's inductive method of finding pi, he considered the problem of finding the area of a "circular segment" of adjustable width x. This is the region under the circle $y = \sqrt{1 - x^2}$ that lies above the portion of the horizontal axis from 0 to x. Here x could be any number from 0 to 1, and 1 is the radius of the circle. When $x = 1$, the area boils down to that of a quarter of the unit circle, namely $\pi/4$. For other values of x, nothing was known.



Newton's first step was to reason by analogy. Instead of aiming directly for the area of the circular segment, he investigated the areas of analogous segments bounded by the following curves:

$y = (1 - x^2)^{0/2}$,
$y = (1 - x^2)^{1/2}$,
$y = (1 - x^2)^{2/2}$,
$y = (1 - x^2)^{3/2}$,
$y = (1 - x^2)^{4/2}$,
$y = (1 - x^2)^{5/2}$,
$y = (1 - x^2)^{6/2}$.

Notice that all of them involve half-number powers of the expression $1 - x^2$. Newton was most interested in the second curve, $y = (1 - x^2)^{1/2}$, because it is equivalent to the upper half of the perfect circle given by the equation $x^2 + y^2 = 1$. Since that circle has a radius of 1, its area is pi, as Newton well knew. So if he could find a way to determine the area under the curve $y = (1 - x^2)^{1/2}$, that might give him an unprecedented means of approximating pi. That was originally the grand plan. But along the way he found something even better: a method for replacing complicated curves by infinite sums of simpler building blocks in the form of powers of x.

Newton began with a crafty sidestep. He knew that the areas under the first, third, fifth, and seventh curves in the list (the ones with whole-number powers like 0/2 = 0 and 2/2 = 1 and 4/2 = 2) would be easy to calculate, because they simplify algebraically. For example,

$y = (1 - x^2)^{0/2} = (1 - x^2)^0 = 1.$

Similarly,

$y = (1 - x^2)^{2/2} = (1 - x^2)^1 = 1 - x^2$

and

$y = (1 - x^2)^{4/2} = (1 - x^2)^2 = 1 - 2x^2 + x^4.$

But no such simplification is available for the circle or the other curves with the half powers in them. The circle involves a nasty square root (a 1/2 power) and the others involve odd powers of square roots (3/2 and 5/2). At the time, no one knew how to find the area under any of them.

Fortunately, the areas under the curves with whole-number powers were straightforward. Take the curve $y = (1 - x^2)^{4/2} = (1 - x^2)^2 = 1 - 2x^2 + x^4$. Newton knew that the area of the corresponding segment is $x - 2x^3/3 + x^5/5$. This followed from repeated application of a well-known rule for power functions: For any whole-number power $n \geq 0$, the area under the curve $y = x^n$ over the interval from 0 to x is given by $x^{n+1}/(n+1)$. Wallis had guessed this rule with his inductive method, and Fermat had proven it by the brilliant method we discussed in Appendix 1. Other mathematicians were aware of this rule as well. So it was an absolutely standard piece of knowledge for Newton as a student.

It allowed him to find the area of the first, third, fifth, and seventh curves in the list above. Let's write A_n for the area under the curve $y = (1 - x^2)^{n/2}$, where $n = 0, 1, 2, \ldots$. Then

$$
\begin{aligned}
A_0 &= x \\
A_1 &= ? \\
A_2 &= x - \tfrac{1}{3}x^3 \\
A_3 &= ? \\
A_4 &= x - \tfrac{2}{3}x^3 + \tfrac{1}{5}x^5 \\
A_5 &= ? \\
A_6 &= x - \tfrac{3}{3}x^3 + \tfrac{3}{5}x^5 - \tfrac{1}{7}x^7
\end{aligned}
$$

and so on. Newton's idea was to fill in the gaps. His ultimate goal was to guess A_1 based on what he could see in the other series. One thing was immediately clear: all the A_n began simply with x. That suggested amending the table like so:

$$
\begin{aligned}
A_0 &= x \\
A_1 &= x - ? \\
A_2 &= x - \tfrac{1}{3}x^3 \\
A_3 &= x - ? \\
A_4 &= x - \tfrac{2}{3}x^3 + \tfrac{1}{5}x^5 \\
A_5 &= x - ? \\
A_6 &= x - \tfrac{3}{3}x^3 + \tfrac{3}{5}x^5 - \tfrac{1}{7}x^7.
\end{aligned}
$$

Then, to replace the next batch of question marks, Newton looked at the x^3 terms. As he pointed out to Leibniz, he observed "that the second terms $\frac{0}{3}x^3,\ \frac{1}{3}x^3,\ \frac{2}{3}x^3,\ \frac{3}{3}x^3$, etc., were in arithmetical progression" (he was referring to the 0, 1, 2, 3 in the numerators of the x^3 terms) "and hence that the first two terms of the series to be intercalated" (the unknown $A_1,\ A_3$, and A_5) "ought to be $x - \frac{1}{3}\left(\frac{1}{2}x^3\right), x - \frac{1}{3}\left(\frac{3}{2}x^3\right), x - \frac{1}{3}\left(\frac{5}{2}x^3\right)$, etc."

Thus, at this stage the patterns suggested to Newton that A_1 should begin as

$$A_1 = x - \tfrac{1}{3}\left(\tfrac{1}{2}x^3\right) + \cdots .$$

As his roving eye hunted for other patterns, Newton noticed that the denominators in the A_n always contained odd numbers in increasing order. For instance, look at A_6. It has 1, 3, 5, and 7 in its denominators. That same pattern worked for A_4 and A_2. Simple enough. That pattern took care of all the denominators.

What remained was to find a pattern in the numerators. Newton examined A_2, A_4, and A_6 again and spotted something he recognized immediately. In $A_2 = x - \frac{1}{3}x^3$ he saw a 1 multiplying the x and another 1 in the term $\frac{1}{3}x^3$ (he ignored its negative sign for the time being). In $A_4 = x - \frac{2}{3}x^3 + \frac{1}{5}x^5$, he saw numerators of 1, 2, 1. And in $A_6 = x - \frac{3}{3}x^3 + \frac{3}{5}x^5 - \frac{1}{7}x^7$, he saw numerators 1, 3, 3, 1. These numbers are the familiar rows of Pascal's triangle! But Newton didn't refer to Pascal. Instead he referred to "powers of the number 11." For example, $11^2 = 121,$ and $11^3 = 1331$. Nowadays these numbers are called binomial coefficients. They arise when expanding a binomial like $(a+b)^3 = 1a^3 + 3a^2b + 3ab^2 + 1b^3$.

Next, to extrapolate his results to half-powers, Newton needed to extend Pascal's triangle to a fantastic new regime, *halfway in between* the rows. To perform the extrapolation, he derived a general formula for the binomial coefficients in row m of Pascal's triangle and then audaciously plugged in $m = 1/2$. That gave him the numerators in the series he was seeking for A_1.

Here, in Newton's own words, is his summary to Leibniz of the patterns he noticed inductively, up to this stage in the argument:

"I began to reflect that the denominators 1, 3, 5, 7, etc. were in arithmetical progression, so that the numerical coefficients of the numerators only were still in need of investigation. But in the alternately given areas these were the figures of powers of the number 11, namely of these, that is, first 1; then 1, 1; thirdly, 1, 2, 1; fourthly 1, 3, 3, 1; fifthly 1, 4, 6, 4, 1, etc. and so I began to inquire how the remaining figures in the series could be derived from the first two given figures, and I found that on putting m for the second figure, the rest would be produced by continual multiplication of the terms of this series,

$$\frac{m-0}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} \times \frac{m-4}{5}\text{ , etc.}$$

[...] Accordingly I applied this rule for interposing series among series, and since, for the circle, the second term was $\frac{1}{3}\left(\frac{1}{2}x^3\right)$, I put $m = 1/2$, and the terms arising were

$$\frac{1}{2} \times \frac{\frac{1}{2}-1}{2} \text{ or } -\frac{1}{8}, \qquad -\frac{1}{8} \times \frac{\frac{1}{2}-2}{3} \text{ or } +\frac{1}{16}, \qquad \frac{1}{16} \times \frac{\frac{1}{2}-3}{4} \text{ or } -\frac{5}{128},$$

so to infinity. Whence I came to understand that the area of the circular segment which I wanted was

$$x - \frac{\frac{1}{2}x^3}{3} - \frac{\frac{1}{8}x^5}{5} - \frac{\frac{1}{16}x^7}{7} - \frac{\frac{5}{128}x^9}{9} \text{ etc.''}$$

Then he quickly realized that the circle $y = (1 - x^2)^{1/2}$ itself (not merely the area underneath it) could also be represented by a power series. All he needed to do was "omit the denominators" and reduce the powers by 1. Thus he was led to guess that

$$(1 - x^2)^{1/2} = 1 - \frac{1}{2}x^2 - \frac{1}{8}x^4 - \frac{1}{16}x^6 - \frac{5}{128}x^8 - \cdots.$$

To test whether this made sense, Newton checked his formula by multiplying it by itself "and it became $1 - x^2$, the remaining terms vanishing by the continuation of the series to infinity."

Stepping back a bit from the details, we see several lessons here about problem solving. If a problem is too hard, change it. If it seems too specific, generalize it. Newton used both of those tactics in the analysis above. They gave him results more important and more powerful than those he originally sought.

In particular, Newton didn't stubbornly fixate on a quarter of a circle. He looked at a much more general shape, any circular segment of width x. Rather than sticking to $x = 1$, he allowed x to run freely from 0 to 1. That revealed the binomial character of the coefficients in his series – the unexpected appearance of numbers in Pascal's triangle and their generalizations – which let Newton see patterns that Wallis and others had missed. Seeing those patterns then gave Newton the insights he needed to develop the theory of power series much more widely and generally.

The moral is: Changing a problem is not cheating. It's creative. And it may be the key to something greater.