Research Statement

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I am a probability theorist with a focus on probabilistic and combinatorial aspects of statistical mechanics. Before describing my interests in more detail I will explain a few pieces of jargon: the notions of *critical phenomena* and *phase transitions*. The aim is convey the motivation for my research, along with some context for where this work lies within probability theory. To keep the discussion concrete the ideas will be illustrated via percolation theory, a subject that is close to everyday intuition.

Consider pouring water on top of a container filled with ground coffee. If the coffee is ground very coarsely the water will quickly pass through the grounds to the bottom of the container. Conversely, water will not pass through very finely ground coffee at all. In the mathematical idealization of this process called *percolation* there is a *critical* grind size: grinds smaller than this size do not allow water to pass through, while grinds coarser than this size allow water to pass through. This is an example of a *critical phenomenon*: a global property of a system made of microscopic constituents changes its behaviour at a precise parameter value. This is closely related to the idea of a *phase transition*; in this case a transition from a non-percolative phase to a percolative phase.

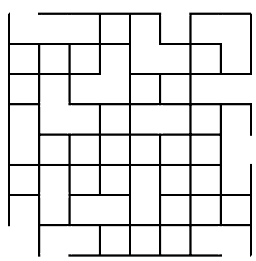


Figure 1. Percolation on a subset of $\mathbb{Z}^2$.

The mathematical idealization retains, with probability $0 \leq p \leq 1$, each edge of the lattice $\mathbb{Z}^d$. All other edges are deleted. The retained edges represent where water can flow, and they form a random subgraph $\mathsf{Perc}(p) \subset \mathbb{Z}^d$. The question of whether or not water will pass through ground coffee becomes the question of whether or not $\mathsf{Perc}(p)$ has an infinite connected component. Finely ground coffee packs more densely, so smaller values of p corresponds to finer grind sizes. Notice that $\mathsf{Perc}(0) = \emptyset$ and $\mathsf{Perc}(1) = \mathbb{Z}^d$. There is a $p_c = p_c(d)$ such that for $p < p_c$ almost surely no infinite component exists, while for $p > p_c$ almost surely an infinite component exists. The parameter p_c is called the *critical point* of the model, and p_c is non-trivial, meaning $0 < p_c < 1$, if $d \geq 2$.

The existence of an infinite component is a qualitative question, and it is interesting to quantify the dependence of $\mathsf{Perc}(p)$ on p. One quantification is given by the *two-point function* $G_p(x)$, which is the probability that the origin o and a vertex x are in the same connected component of Perc. In the prosaic language of coffee, this is the probability the grounds located at o and at x are both wet. Again the critical point p_c plays a role: $G_p(x)$ decays exponentially fast in $\|x\|$ in the non-percolative phase $p < p_c$, and $G_p(x)$ is bounded below by a positive constant in the percolative phase $p > p_c$. Exactly at $p = p_c$ the behaviour is more complex, and $G_{p_c}(x) = G_{\mathrm{crit}}(x)$ is believed to satisfy

$$G_{\mathrm{crit}}(x) \sim \|x\|^{-(d-2)-\eta_d} \qquad \text{as } \|x\| \to \infty \tag{1}$$

for some $\eta_d \in \mathbb{R}$ when $d \geq 2$; $f(x) \sim g(x)$ means $f(x)/g(x)$ tends to a positive constant as $\|x\| \to \infty$. The so-called *critical exponent* η_d gives a qualitative description of the behaviour of the model when $p = p_c$.

Equation (1) is a theorem when $d > 10$ [12], and $\eta_d = 0$ in these cases. It is believed that (1) holds and $\eta_d = 0$ when $d > 6$; this is a theorem for closely related models [16]. In the terminology of critical phenomena $d_c = 6$ is called the *upper critical dimension*, and $\eta_d = 0$ for $d > d_c$ is one manifestation of the *mean-field* behaviour of percolation above the upper critical dimension. When $\eta_d = 0$ Equation (1) is a statement that percolation has Gaussian behaviour on large scales.

The conjectural formula (1) and many other related quantitative predictions are believed to be *universal*. This means the values of critical exponents like η_d are expected to be independent of the microscopic details of the model, e.g., if $\mathbb{Z}^d$ is replaced with another d-dimensional lattice. Universality is largely an open problem [25]. As mean-field behaviour is Gaussian behaviour, the project of establishing universality above d_c is a project to establish Gaussian behaviour well beyond the hypotheses of classical central limit theorems.

Conjectures similar to (1), and in some cases results, hold for a great variety of statistical mechanical models. Two important examples are spin systems, which model magnetism, and interacting random walks, which model linear polymers. I describe work I have done about critical behaviour for models of these types when $d > d_c$ in Section 1. There are also many interesting questions about non-critical models, e.g., establishing the existence of distinct phases, or describing typical properties within a phase. In Section 2 I describe my work on random spatial permutations, which are related to Bose-Einstein condensates. Lastly, in Section 3 I discuss projects involving systems of particles subjected to excluded volume constraints. Each section also highlights questions and directions I intend to pursue in the future.

1. High-dimensional critical phenomena

The *lace expansion* is a perturbative method originally developed by Brydges and Spencer to study self-avoiding walk on $\mathbb{Z}^d$ for $d \geq 5$ [4]. Since its introduction it has been generalized to apply to a variety of probability models in high dimensions. What constitutes high dimensions depends on the model being considered, but the importance of the lace expansion is clear: for the highly non-Markovian models to which it is applied, it is typically the only known technique for establishing mean-field behaviour at the critical point. An excellent overview of the method is [33], although this does not cover several important recent developments [20, 30, 31, 38].

Despite these successes the lace expansion is currently rather far from being a fully developed tool. This section describes my research that has used lace expansion ideas. While each work discussed concerns a different model, there is a common thread: the models all suffer from a lack of repulsion. Repulsion, defined in the next section, is traditionally crucial for lace expansion analyses. An ongoing theme in my research is the development of lace expansion methods for models that lack repulsion; there are many interesting models of this type. The long-term goal of this research is to make the lace expansion a generically applicable tool for proving mean-field behaviour in sufficiently high dimensions.

1.1. **Self-attracting self-avoiding walk.**

Let $\mathcal{W}$ be a non-negative weight on walks in $\mathbb{Z}^d$. This defines a probability measure $\mathbb{P}$ on n-step walks ω that begin at the origin by declaring $\mathbb{P}[\omega]$ to be proportional to $\mathcal{W}(\omega)$. Models of random walks defined by a weight $\mathcal{W}$ are called *repulsive* if

$$\mathcal{W}(\omega_1 \circ \omega_2) \leq \mathcal{W}(\omega_1)\mathcal{W}(\omega_2), \tag{2}$$

when $\omega_1 \circ \omega_2$ is a walk made by concatenating two walks ω_1 and ω_2. The inequality (2), called the *repulsion inequality*, says that forgetting the past of a walk can only increase the weight of the future. It holds, for example, if $\mathcal{W}$ is the weight of *self-avoiding walk*: $\mathcal{W}(\omega) = 1$ if ω is self-avoiding, and $\mathcal{W}(\omega) = 0$ otherwise. Equation (2) implies that the number $c_n = c_n(0)$ of n-step self-avoiding walks is *submultiplicative*, i.e., $c_{n+m} \leq c_n c_m$. Submultiplicativity in turn implies that $c_n^{1/n}$ has a limiting value $\mu(0)$, and more generally submultiplicativity plays an important role in the study of self-avoiding walk [25]. The repulsion inequality has use far beyond submultiplicativity, however, and is a crucial ingredient in the lace expansion analysis of self-avoiding walk.

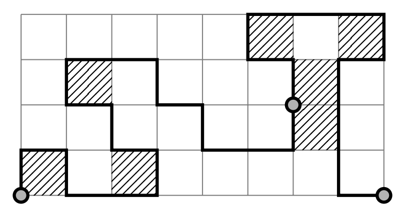


Figure 2. A self-avoiding walk ω. Shaded plaquettes, i.e., unit squares in $\mathbb{Z}^d$, indicate pairs of adjacent edges of ω; the walk has weight $(1+\kappa)^7$.

The self-avoiding walk model was introduced as a model of a linear polymer in a good solvent, and it is also of interest to study models of linear polymers in poor solvents [8]. Mathematically this is done by adding a self-attraction to the model, i.e., to study walks with weight

$$\mathcal{W}_\kappa(\omega) = (1+\kappa)^{c(\omega)} \mathbb{1}_{\{\omega \text{ is self-avoiding}\}}, \qquad \kappa \geq 0$$

where $c(\omega)$ is function chosen to quantify the number of self-contacts the walk ω makes. A natural choice for $c(\omega)$ is the number of times two edges of ω occupy disjoint edges of a plaquette, see Figure 2. The model resulting from this choice of $c(\omega)$ is called *κ-ASAW*, abbreviating κ-attractive self-avoiding walk. When $\kappa > 0$ the repulsion inequality does not hold: consider splitting the walk depicted in Figure 2 at the indicated vertices. Further, it is not generally true that $c_{n+m}(\kappa) \leq c_n(\kappa)c_m(\kappa)$, where $c_n(\kappa)$ is the total $\mathcal{W}_\kappa$-weight of n-step walks starting at o. These facts make the analysis of κ-ASAW difficult, as repulsion is one of the only *a priori* tools available for analyzing self-avoiding walk.

In joint work with Alan Hammond we have shown how the loss of submultiplicativity can be overcome [15]. A first result is that the connective constant $\mu(\kappa) = \lim_{n\to\infty} c_n^{1/n}(\kappa)$ exists on $\mathbb{Z}^d$ when $\kappa > 0$ is sufficiently small. Moreover, we show that it is possible to carry out a lace expansion analysis, leading to the following theorem.

result: *existence of connective constant for κ-ASAW*

Theorem ([15])**.** *For $d \geq 5$, $\kappa > 0$ sufficiently small, and sufficiently spread-out step distributions the critical κ-ASAW two-point function $G_{\mathrm{crit}}(x)$ satisfies* (1)*.*

result: *Gaussian decay of critical κ-ASAW 2-point function*

The function $G_{\mathrm{crit}}(x)$, which has not been defined, plays a similar role for κ-ASAW as the two-point function does in the context of percolation theory. The theorem says that the self-attraction does not alter the behaviour of the model – it

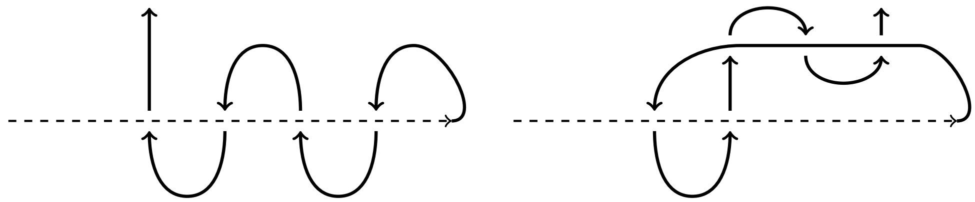


Figure 3. Consider the number of loops erased by loop erasure for (i) the concatenation of the dashed and solid walks and (ii) the walks as being separate. On the left (i) has weight λ^4 while (ii) has weight 1. On the right (i) has weight λ while (ii) has weight λ^3. This shows λ-LWW is not repulsive for $\lambda \neq 0, 1$.

behaves like simple random walk when $d \geq 5$. The hypothesis of a "sufficiently spread out" step distribution means walks ω are permitted to take steps in a sufficiently large origin-symmetric subset of $\mathbb{Z}^d$.

The central idea in studying κ-ASAW is that each energetic reward $(1+\kappa)$ is associated with an entropic penalty due to the self-avoidance constraint. When κ is small enough this can be exploited to show that forgetting the past is beneficial *on average*, and it is possible to exploit this gain in a manner that is compatible with a lace expansion analysis. The proof of the existence of the connective constant uses this idea as well, but in the context of a classical result for self-avoiding walk called the Hammersley-Welsh argument.

A similar result was previously obtained under stronger restrictions on the step distribution [36]; in particular the hypotheses of [36] require the step distribution to have infinite range. The removal of the infinite range assumption solves an open problem of den Hollander [8, Chapter 4.8 (5)].

1.2. **Loop-weighted Walk.** *Loop-weighted walk* is a one-parameter family of non-Markovian random walks that are punished or rewarded for containing loops. The definition relies on the notion of *loop erasure*: for a finite walk ω, $\mathsf{LE}(\omega)$ is the self-avoiding walk that results from chronologically removing loops from ω. If ω is not already self-avoiding this means ω is traced until the first time a vertex v is visited twice, and when this occurs, the loop that begins and ends at v is removed. This process is repeated until the result is a self-avoiding walk.

Loop-weighted walk with parameter λ (λ-LWW) assigns a weight $\mathcal{W}(\omega) = \lambda^k$ if loop erasure removes k loops from ω. See Figure 3. This induces a probability measure on walks of length n when $\lambda \geq 0$; if $\lambda = 1$ the model is simple random walk, while if $\lambda = 0$ it is self-avoiding walk. For $0 < \lambda < 1$ the weight discourages walks from containing loops and interpolates between self-avoiding and simple random walk, while for $\lambda > 1$ the weight encourages walks to contain loops.

result: *random walk representation for correlations*

I introduced loop-weighted walk in [19]. Aside from its interest as an interpolation between self-avoiding and simple random walk, λ-LWW also provides a discrete-time random walk representation of correlation functions related to the $\mathsf{O(N)}$ models of statistical mechanics. This random walk representation is distinct from the representations mentioned in Section 1.3; its derivation is related to combinatorial approaches to the Ising model [17, 21] and relies on Viennot's heaps of pieces [40].

Loop-weighted walk does not satisfy the repulsion inequality (2). See Figure 3. As described earlier, this makes the application of the lace expansion difficult, but I have shown that λ-LWW's lack of repulsion can be circumvented [19]. The strategy relies on proving that a partial resummation of λ-LWW yields a model of self-interacting and self-avoiding walks that *is* repulsive. This enables a lace expansion analysis of λ-LWW by (i) performing the partial resummation, (ii) performing a lace expansion on the resulting self-interacting and self-avoiding walk, and (iii) undoing the resummation to express the result in terms of λ-LWW. One particular consequence of the lace expansion analysis is that λ-LWW scales diffusively.

result: *diffusive behaviour of λ-LWW in high dimensions*

Theorem ([19]). *If $\langle \cdot \rangle_n^\lambda$ denotes expectation for the n-step λ-LWW measure in $\mathbb{Z}^d$, and d is sufficiently large then*

$$\langle |\omega_n|^2 \rangle_n^\lambda \sim n, \qquad \text{as } n \to \infty.$$

1.3. **Continuous-time lace expansion.** The *n-component φ^4 model* is a model of ferromagnetism of central importance in mathematical physics. For a finite volume $\Lambda \subset \mathbb{Z}^d$ there is a so-called *spin* $\varphi_x \in \mathbb{R}^n$ at each vertex $x \in \Lambda$, and these spins interact with one another in a manner similar to how spins interact in the Ising model. Namely, there is a tendency for nearby spins to align, but this tendency must compete against entropic fluctuations. The precise definition of the model, which is not important here, depends on two parameters $g > 0$ and $\nu \in \mathbb{R}$. A quantity of significant interest is the *spin-spin correlation* $G_{g,\nu}(x,y) = \langle \varphi_x \cdot \varphi_y \rangle_{g,\nu}$, where the angled brackets denote expectation with respect to the φ^4 measure. This is particularly interesting when the parameter ν is chosen to be $\nu_c(g,d)$, i.e., at the critical point of the model, in which case we write G_{crit}.

In joint work in preparation with David Brydges and Mark Holmes we introduce for the first time a lace expansion in continuous time [3]. Extending the lace expansion to continuous-time models of walks is not just intrinsically interesting, it also provides a way to apply the lace expansion to the φ^4 model. This is because of the existence of continuous-time random walk representations for spin systems [2, 11, 34]. Our expansion applies to several models; in particular we are able to analyze the n-component φ^4 model for $n = 1, 2$, with the result being that the critical two-point function G_{crit} obeys (1) with $\eta = 0$ when $d \geq 5$ and $g > 0$ is sufficiently small. This is the first time such a result has been proven for the 2-component φ^4 model.

result: *lace expansion for continuous-time weakly self-avoiding walk and the 2-component φ^4 model*

A crucial aspect of our analysis of φ^4 models is that the continuous-time random walk representation leads to a lace expansion that is *not* what one gets when using a more traditional inclusion-exclusion type argument. This alternate expansion is essential in our argument, as it enables us to use correlation inequalities that are otherwise not available.

1.4. **Future work relating to the lace expansion.** I would like to comment on two models of random walks that are similar to self-avoiding walk to which I would like to apply lace expansion ideas. The first is self-attracting *weakly* self-avoiding walk. This means that self-intersections are not strictly forbidden, but they are energetically discouraged. The second is *cyclic-time random walk*, introduced by Angel to study a random spatial permutation model called the *random stirring model* [1]. The behaviour of cyclic time random walk is related to the existence of infinite cycles in the random stirring model.

goal: *Gaussian behaviour of more general attractive self-avoiding walks*

goal: *existence of infinite cycles in high dimensional random spatial permutations*

Both of these models are similar to the κ-ASAW model of Section 1.1, in that they feature a self-attraction. There is an additional difficulty: the entropic gain of forgetting the past does not necessarily offset the energetic loss. However, this situation only occurs for particularly entropically unfavourable pasts, so what is needed is the development of lace expansion techniques that exploit this.

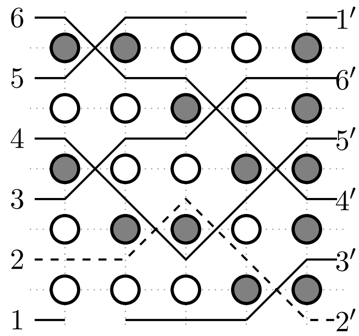


Figure 4. A directed spatial permutation with $C = 1$. The torus has been cut open for illustrative purposes. The top and bottom edges are to be identified with one another, and the left and right edges are to be identified with one another, with j being glued to j'.

goal: *extend the lace expansion to environments without translation invariance*

Another direction that I plan to pursue is the development of the lace expansion for graphs other than $\mathbb{Z}^d$. One interesting class of models are self-avoiding walks in random environments, which have been studied in the physics literature for some time but are only beginning to be studied mathematically. See [24] and references therein. The main challenge is that current lace expansion methods rely on translation invariance, which ceases to hold in the context of random environments.

2. Spatial Permutations

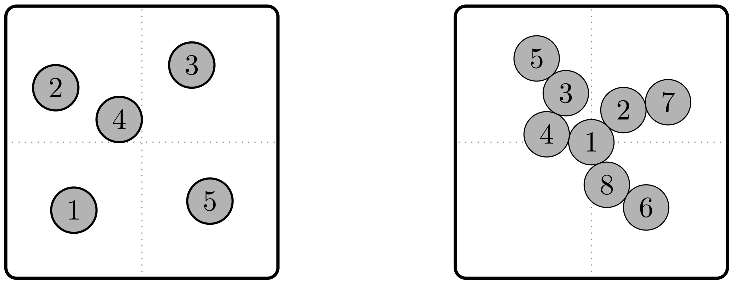
Let σ be a permutation on n letters, and let c_i be the length of the i^{th} largest cycle in σ. Let $\mathcal{X}(\sigma)$ denote the cycle structure of σ, i.e., the vector $n^{-1}(c_1, c_2, \dots, c_k)$ if σ contains k cycles. It is a classical result that if σ is a uniformly chosen permutation then as $n \to \infty$ the cycle structure $\mathcal{X}(\sigma)$ converges weakly to the *Poisson-Dirichlet distribution* with parameter 1 [23, 39]. This distribution, denoted $\mathsf{PD}(1)$, is a probability distribution on the unit simplex $\{(t_1, t_2, \dots) \in [0,1]^{\mathbb{N}} \mid t_1 \geq t_2 \geq \dots; \sum t_i = 1\}$.

A natural way to sample a random permutation is by Markov Chain Monte Carlo methods, and this requires a chain whose stationary distribution is uniform on permutations. One such chain is called *random transpositions*: each step of the chain picks two elements $i \neq j$ and transposes them. After enough steps of the chain, which turns out to be $\frac{1}{2} n \log n$ steps, the resulting permutation σ_t is close to being uniformly chosen, and as a result $\mathcal{X}(\sigma_t)$ is nearly distributed as $\mathsf{PD}(1)$ [9].

This dynamical picture of random permutations has a natural generalization. View the n letters as being the vertices of the complete graph, so each transposition is a transposition across an edge of the graph. What happens when the graph has more structure, i.e., what does a random *spatial* permutation look like? Aside from its intrinsic interest, this question turns out to be closely connected to the behaviour of quantum spin systems [37]. It is believed, but largely open to prove, that in a great variety of situations the limiting cycle structure is $\mathsf{PD}(1)$.

2.1. Directed spatial permutations on asymmetric tori.

In joint work in preparation with Alan Hammond we have introduced a model of random spatial permutations and established that its limiting cycle structure is $\mathsf{PD}(1)$ [14]. The remainder of this section describes the model and main strategy. The strategy is relatively



A hard sphere gas configuration in $\mathbb{R}^2$. Each sphere carries a copy of Lebesgue measure.

A branched polymer in $\mathbb{R}^2$. Each edge carries a copy of suface measure on the unit sphere.

Figure 5. $Z_\Lambda^{\mathsf{BP}}(z)$ and $Z_\Lambda^{\mathsf{HCG}}(z)$ are given by $\sum \frac{z^k}{k!}\mathrm{Vol}_k$, where Vol_k is the volume of k-particle configurations in $\Lambda \subset \mathbb{R}^d$. The sum starts at $k=0$ for the hard sphere gas and $k=1$ for branched polymers.

general and should be useful in other contexts, albeit with additional technical complications.

Fix a square $[0,n]^2 \subset \mathbb{Z}^2$ and identify the top vertices $\{i,n\}$ and bottom vertices $\{i,0\}$ with one another to form a cylinder. Choose $C \in \mathbb{N}$ and identify vertices $\{n,i\}$ with vertices $\{0, i+C \mod n\}$. This corresponds to twisting the cylinder before attaching its ends to form a torus. See Figure 4. To obtain a model of random spatial permutations assign a variable $\sigma_{x,y} \in \{-1,0,1\}$ to each site (x,y) of the torus, and think of $\sigma_{x,y}$ as an arrow taking (x,y) to $(x+1, y+\sigma_{x,y})$. The collection of arrows defines a random map $\bar{\Phi}$ from the torus to itself: each vertex is mapped to where its arrow points. Let Φ denote $\bar{\Phi}$ conditioned to be a bijection. $\bar{\Phi}$ is a random spatial permutation, as it permutes the vertices of the torus.

Theorem ([14]). *For C a large enough multiple of $\lfloor\sqrt{n\log n}\rfloor$ the limiting cycle structure $\mathcal{X}(\Phi)$ converges weakly to* $\mathsf{PD}(1)$ *as $n\to\infty$.*

result: *limiting cycle distribution is* $\mathsf{PD}(1)$

To study the cycle structure we run a Markov chain whose equilibrium distribution converges, as $n\to\infty$, to the law of Φ. The effect at each step of the dynamics is to either split a cycle into two cycles or to merge two cycles into a single cycle; in Figure 4 the shaded circles are the locations at which the dynamics can alter the adjacent arrows. When C is large there is enough control of the local structure of cycles to show that each cycle is composed of many pieces, and these individual pieces are spatially well mixed. This implies that under the dynamics of the Markov chain the probability of two cycles merging is proportional to the product of their lengths. Given this we can adapt an argument of Schramm in the context of random transpositions on the complete graph to establish convergence to $\mathsf{PD}(1)$ [32].

2.2. **Future directions for directed spatial permutations.** Schramm showed that the distribution of cycle lengths of random transpositions on the complete graph not only converges to $\mathsf{PD}(1)$, but does so in $\frac{1}{2}n$ steps – this is significantly faster than the permutation becomes a uniform permutation [32]. It would be interesting to discover if this phenomenon persists for the directed spatial permutation model.

goal: *establish mixing time of cycle structure*

3. Excluded volume phenomena and expansion methods

3.1. **Generalized dimensional reduction for branched polymers.** In the context of statistical mechanics *dimensional reduction* refers to relationships between models in different dimensions. An interesting theorem of this type was established by Brydges and Imbrie, who related the *hard sphere gas* model to the *continuum branched polymer* model [6]. Eschewing precise definitions in favour of figures, see Figure 5, their theorem states:

$$\lim_{\Lambda\uparrow\mathbb{R}^d} \frac{1}{|\Lambda|} \log Z_\Lambda^{\mathsf{HCG}}(z) = -2\pi Z^{\mathsf{BP}}_{\mathbb{R}^{d+2}}(-\frac{z}{2\pi}) \tag{3}$$

whenever the right-hand side converges; the limit is omitted when $d = 0$. Note the left-hand side is a quantity associated to $\mathbb{R}^d$, while the right-hand side is associated to $\mathbb{R}^{d+2}$. This formula contains a wealth of information about branched polymers in $d = 2, 3$ since the left-hand side is computable for $d = 0, 1$.

result: *constructive proof of dimensional reduction formula*

result: *generalized dimensional reduction formulas*

I have given a constructive proof of a generalized dimensional reduction formula that contains (3) as a special case [18]. My results are generalizations in two senses. First, I establish dimensional reduction formulas for some non-spherical bodies. Second, I show such formulas hold far beyond the context of branched polymers associated to trees: there is a dimensional reduction formula associated to any central essential hyperplane arrangement. The case of branched polymers corresponds to the braid arrangement.

Aside from yielding generalizations, my proof is quite explicit and provides a rather different perspective from the Brydges-Imbrie proof of (3), which relied on a nonconstructive supersymmetric localization lemma [6].

Specializing the constructive proof of [18] to prove (3) has three main ingredients. The first is the *Mayer expansion*, a classical theorem of statistical mechanics which gives a convergent power series representation for the left-hand side of (3). The second is the collection of *invariance lemmas* due to [22], which state that the volume of planar branched polymers and related objects are independent of the radii of the constituent disks. The final ingredient is *Archimedes's theorem*, which decomposes surface measure on a sphere in $\mathbb{R}^{d+2}$ into a product of Lebesgue measure on the ball in $\mathbb{R}^d$ and surface measure on a circle. The generalized dimensional formulas associated to central hyperplane arrangements make use of invariance lemmas due to [26] for generalized planar polymer models.

result: *dimensional reduction for finite volume corrections*

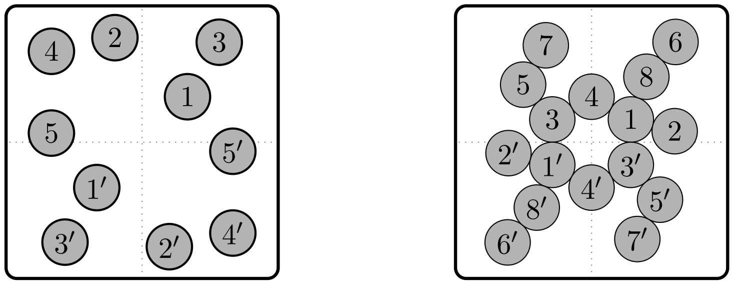
One explicit consequence of these generalized dimensional reduction formulas is an identification of the lowest order finite-volume corrections for the symmetrized hard sphere gas:

$$\lim_{\Lambda\uparrow\mathbb{R}^d} \frac{Z_\Lambda^{\mathsf{SHG}}(z)}{Z_\Lambda^{\mathsf{Bulk}}(z)} = Z^{\mathsf{SBP}}_{\mathbb{R}^{d+2}}(-\frac{z}{2\pi}).$$

See Figure 6 for a depiction of these models. The term $Z^{\mathsf{SBP}}_{\mathbb{R}^{d+2}}$ is the partition function for symmetric branched polymers. $Z_\Lambda^{\mathsf{Bulk}}$ is the bulk contribution to $Z_\Lambda^{\mathsf{SHG}}(z)$, the symmetric hard core gas partition function. The bulk term has an explicit power series representation and no symmetry constraints.

3.2. **Expansions for continuum multibody interactions.** The *Mayer expansion*, mentioned above in Section 3.1, is an important tool for the study of statistical mechanics models in their low density regimes [5, 13]. It provides a convergent power series expansion for many quantities of interest and establishes the existence and properties of the low density phase. When applicable it provides a very powerful analytical tool.

For models in which the interaction between particles is a *two-body* interaction, e.g., the hard sphere gas of Section 3.1, there is a well-developed theory of the Mayer expansion in both discrete and continuous settings. For *multibody* interactions



A symmetric hard core gas configuration in $\mathbb{R}^2$.

A symmetric branched polymer configuration in $\mathbb{R}^2$.

Figure 6. Symmetrized variants of the hard sphere gas and branched polymer models.

there is largely only a discrete theory [5, 27], and existing approaches to multibody interactions in the continuum pass through discretizations [29, 28].

In ongoing work with David Brydges we are removing the need to discretize continuum multibody systems. This is interesting for at least two reasons. First, it clarifies the assumptions needed on multibody potentials, yielding a more easily applicable tool. Second, it has connections with geometric combinatorics. This latter point is similar to the interplay between the Mayer expansion for two-body potentials, the simplicial complex of connected graphs, and spanning trees. Our approach is based on developing a generalization of spanning trees that is well-suited for a multibody Mayer expansion.

in progress: *convergent cluster expansion for continuum multibody models*

3.3. **Nematic phase of hard rods on $\mathbb{Z}^2$.** The planar *nematic rod* model is a lattice model for a liquid crystal. Fix a finite box Λ in $\mathbb{Z}^2$ and a parameter $k \in \mathbb{N}$. A *rod* is a set of k colinear vertices, and a *configuration* R in Λ is a collection of vertex disjoint rods, all contained in Λ. See Figure 7. Given an *activity* $z \in [0, \infty)$, the *partition function* is

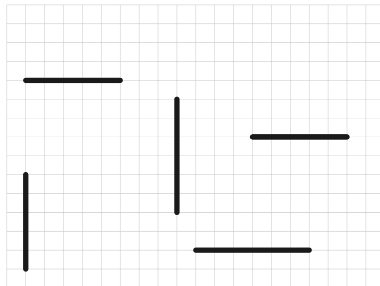
$$Z_\Lambda(z) = \sum_R z^{|R|}, \tag{4}$$

where the sum runs over all rod configurations in Λ. The probability of observing a configuration R is proportional to its contribution to (4).

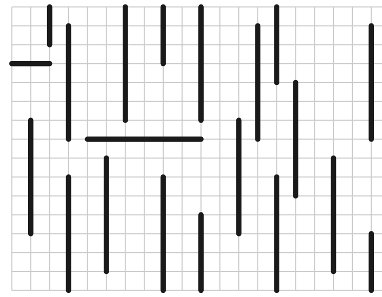
It has been proven that when $zk \ll 1$ and $zk^2 \gg 1$ there is a *nematic* phase [10]. Formally, this means two distinct translation invariant Gibbs measures μ_v and μ_h exist, and that these measure are *not* invariant under rotation by $\pi/2$. Informally this means that a typical configuration under μ_v consists of primarily vertical rods, while a typical configuration under μ_h consists of primarily horizontal rods, and the locations of these rods are very weakly correlated. See Figure 7. The key idea in the proof is that if the rod model is coarse-grained into $\frac{k}{2} \times \frac{k}{2}$ blocks the resulting model is similar to a ferromagnetic two-state Ising model. This is because having a block with vertical rods adjacent to a block with horizontal rods is entropically disadvantageous.

In [10] the measures μ_v and μ_h are constructed by taking advantage of special boundary conditions, analogous to plus and minus boundary conditions for the Ising model. In joint work in progress with Margherita Disertori we are developing a better understanding of the nematic phase by establishing that μ_v and μ_h are the only extremal Gibbs measures. This will remove the possibility that the nematic

in progress: *classify extremal Gibbs measures for nematic rods*



A configuration in the low density phase.



A configuration in the nematic phase.

Figure 7. Schematic depictions of the low density and nematic phases of the planar hard rod model; only a portion of the box Λ is depicted.

measures arise only as a result of the special boundary conditions used [10]. Our proof strategy is similar to one used for the two-dimensional Ising model [7]; due to a lack of correlation inequalities we must use cluster expansion techniques.

goal: *phase transition for $O(2)$ symmetric nematic rods*

An interesting aspect of the rod model is that the existence of the nematic phase is due solely to entropy. It would be very interesting to establish an entropically driven phase transition when the rods are able to be continuously oriented instead of having only two alignments. The transition is believed to be a Kosterlitz-Thouless transition, a phenomena which has recently attracted a great deal of attention [35]. This means the transition is marked by a change from exponential to polynomially decaying orientational correlations.

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