Measurements, errors, and negative kinetic energy

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(Received 5 May 1993)

An analaysis of errors in measurement yields insight into the penetration of quantum particles into classically forbidden regions. In addition to physical values, realistic measurements yield "unphysical" values which, we show, can form a consistent pattern. An experiment to isolate a particle in a classically forbidden region obtains negative values for its kinetic energy. These values realize the concept of a *weak value*, discussed in previous works.

PACS number(s): 03.65.Bz, 03.65.Ca

I. INTRODUCTION

When the word "quantum" first entered the language of physics, it meant a restriction on possible values of energy. Although the quantum theory that later emerged has many other aspects, it is still axiomatic that the only observable values of a physical quantity are the eigenvalues of a corresponding quantized operator. The more precise our measurements, the more clearly this restriction stands out; thus when we obtain values that are not eigenvalues, we interpret them as errors. Still, measurements are uncertain in practice, and can even yield classically forbidden, "unphysical" values. We have uncovered remarkable regularities in the way that unphysical values can appear in sequences of measurements, suggesting that these values may not be unphysical at all. In quantum theory it seems that, not only are physical quantities not restricted, they can even take values outside the classically allowed range. Here we discuss this effect and show how it arises in the context of barrier penetration by quantum particles.

The phenomenon of barrier penetration, such as tunneling through a potential barrier, is an outstanding example of quantum behavior. Quantum particles can be found in regions where a classical particle could never go, since it would have negative kinetic energy. But in quantum theory, too, the eigenvalues of kinetic energy cannot be negative. How, then, can a quantum particle "tunnel"? The apparent paradox is resolved by noting that the wave function of a tunneling particle only partly overlaps the forbidden region, while a particle found within the forbidden region may have taken enough energy from the measuring probe to offset any energy deficit. There is no wave function that represents a particle restricted to a region where its potential energy is larger than its total energy.

Nevertheless, we will show that actual measurements of kinetic energy can yield negative values and that, un-der proper conditions, a remarkable consistency appears in these apparent errors. In a model experiment, we measure the kinetic energy of a bound particle to any desired precision. We then attempt to localize the particle within the classically forbidden region. The attempt rarely succeeds, but whenever it does, we find that the kinetic-energy measurements gave an unphysical negative result; moreover, these results cluster around the appropriate value, the difference between the total and the potential energy. This consistency, which seems to come from nowhere—a background of errors—suggests strongly that the notion of a quantum observable is richer than the one generally accepted. Previous papers suggesting this conclusion analyze a measurement of spin [1] and a quantum time machine [2].

II. ANALYSIS OF ERRORS IN MEASUREMENT

We begin by reviewing the standard von Neumann [3] theory of measurement in nonrelativistic quantum mechanics. Suppose we wish to measure a dynamical quantity C. We choose a measuring device with an interaction Hamiltonian

$$H_{\text{int}} = g(t)PC , \tag{1}$$

where P is a canonical momentum of the measuring device; the conjugate position Q corresponds to the position of a pointer on the device. The time-dependent coupling constant $g(t)$ is nonzero only for a short time interval corresponding to the measurement and is normalized so that

$$\int g(t)dt = 1 . \tag{2}$$

When the time interval is very short, we call the measurement impulsive. For an impulsive measurement, H_{int} dominates the Hamiltonian of the measured system and the measuring device. Then, since $\dot{Q} = (i/\hbar)[H_{\text{int}}, Q] = g(t)C$, we obtain (in the Heisenberg representation) the

result

$$Q_{\text{fin}} - Q_{\text{in}} = C , \tag{3}$$

where Q_{fin} and Q_{in} denote the final and initial settings of the pointer.

In an ideal measurement the initial position of the pointer is precisely defined, say $Q_{\text{in}} = 0$, and so from its final position we read the precise value of C. But in practice, measurements involve uncertainty. To model a source of uncertainty, we can take the initial state of the pointer to be

$$\Phi_{\text{in}}(Q) = (\epsilon^2 \pi)^{-1/4} e^{-Q^2/2\epsilon^2} . \tag{4}$$

The uncertainty in the initial position of the pointer produces errors of order ϵ in the determination of C; when $\epsilon \rightarrow 0$ we recover the ideal measurement. Suppose that the system under study is initially in an eigenstate of C with eigenvalue c_i. Ideal measurements can yield only the result c_i. But when the pointer itself introduces uncertainty, other results are possible, indeed a scatter of results, with a spread of about ϵ, and peaked at the eigenvalue c_i. If the measuring device works as described, then *any* measured value is possible, although large errors are exponentially suppressed. There is no mystery in the appearance of such errors; they are expected, given the uncertainty associated with the measuring device. Measurements of a positive definite operator such as p^2 could even yield negative values. Of course, the dial of the measuring device might have a pin preventing negative readings, but let us assume that it does not. Even if the negative values themselves are unphysical, they are part of a distribution representing the measurement of a physical quantity. They should not be thrown out, since they give information about the distribution and contribute to the best estimate of the peak value.

The standard theory of measurement not only allows errors, it also prescribes their interpretation: they constitute scatter around a true physical value which can only be one of the eigenvalues of the operator measured. Of course, the system under study may not be in an eigenstate of the measured operator. Then results of measurements will be distributed according to quantum probabilities, folded with errors due to the measuring device. Since these errors originate in the measuring device and not in the system under study, it seems that they cannot depend on any property of the system. However, closer analysis of these errors in the context of *sequences* of measurements reveals a pattern which, far from being random, clearly reflect properties of the system under study. The pattern emerges only after selection of a particular final state of the system. In the next section, we take a particular example and analyze it in detail to show how and where the surprise appears.

III. NEGATIVE KINETIC ENERGY

Our example may be summarized as follows: we prepare a sufficiently large ensemble of particles bound in a potential well, in an eigenstate of energy, and measure the kinetic energy of each particle to a given precision.

The results of these measurements are predictably scattered, and even include some negative values, although the kinetic-energy spectrum is positive. Then we measure the position of each particle and select only those cases where the particle is found within some region "far enough" from the well—with "far enough" depending on how precisely the kinetic energy was measured. In almost all such cases, we find that the measured kinetic energy was *negative*. Not only are the measured values negative, they also cluster around a particular negative value appropriate to particles in the classically forbidden region. Also, the spread of the clustering is the characteristic spread for kinetic-energy measurements with this device.

We begin with a particle trapped in a potential well. The Hamiltonian is [4]

$$H = \frac{p^2}{2m} + V(x) , \tag{5}$$

with $V(x) = -V_0$ for $|x| < a$ and $V(x) = 0$ for $|x| > a$. We prepare an ensemble of particles in the ground state of this Hamiltonian, with energy $E_0 < 0$:

$$|\Psi_{\text{in}}\rangle = |E_0\rangle . \tag{6}$$

Following von Neumann, we model a measurement of kinetic energy with an interaction Hamiltonian

$$H_{\text{int}} = g(t)P\frac{p^2}{2m} , \tag{7}$$

where P is a canonical momentum conjugate to the position Q of a pointer on the measuring device. As in Eq. (2), we assume that the coupling between the particle and device is turned on so briefly that the Hamiltonian reduces to H_{int}, and we obtain for the operator Q

$$Q_{\text{fin}} - Q_{\text{in}} = \frac{p^2}{2m} . \tag{8}$$

The initial state of the pointer is given by Eq. (4). The uncertainty in the initial state of the pointer leads to errors of order ϵ in the measurement of kinetic energy.

Initially, the particle and device are in a product state $\Psi_{\text{in}}(x)\Phi_{\text{in}}(Q)$; after the interaction is complete, the state is

$$e^{-(i/\hbar)Pp^2/2m}\Psi_{\text{in}}(x)\Phi_{\text{in}}(Q) , \tag{9}$$

in which the particle and the device are correlated. Now we consider kinetic-energy measurements followed by a final measurement of position, with the particle found far outside the potential well. For the final state we choose a Gaussian wave packet with its center far from the potential well,

$$\Psi_{\text{fin}}(x) = (\delta^2 \pi)^{-1/4} e^{-(x-x_0)^2/2\delta^2} , \tag{10}$$

and we require $\delta > \alpha\hbar^2/m\epsilon$. We can now be more definite about what it means for the particle to be far enough from the potential well; the condition on x_0 is

$$\alpha x_0 \gg (\alpha^2\hbar^2/2m\epsilon)^2 . \tag{11}$$

Since $\alpha^2\hbar^2/2m = |E_0|$, the expression in parentheses is

the ratio of the magnitude of the effect $|E_0|$ to the precision of the measurement ϵ. This condition is derived in the Appendix. Note that for more precise measurements of kinetic energy ($\epsilon \to 0$), the final state is selected at increasing distances from the potential well ($x_0 \to \infty$).

The state of the measuring device after the measurement, and after the particle is found in the state $\Psi_{\text{fin}}(x)$, is obtained by projecting the correlated state of the particle and measuring device onto the final state of the particle $\Psi_{\text{fin}}(x)$. Apart from normalization, the final state of the measuring device is

$$\Phi_{\text{fin}}(Q) = \langle \Psi_{\text{fin}} | e^{-(i/\hbar)Pp^2/2m} | \Psi_{\text{in}} \rangle \Phi_{\text{in}}(Q) , \tag{12}$$

where $\Psi_{\text{in}}(x) = |E_0\rangle$. For simplicity, we take $V(x)$ in Eq. (5) to be a δ-function potential ($a \to 0$). Then $\Psi_{\text{in}}(x)$ is $\sqrt{\alpha}\exp(-\alpha|x|)$.

The exponent in Eq. (12) contains the operators P and p. It is convenient to express $\Psi_{\text{in}}(x)$ via its Fourier transform

$$\Psi_{\text{in}}(x) = \frac{\hbar\alpha^{3/2}}{\pi} \int dp \frac{e^{-ipx/\hbar}}{\alpha^2\hbar^2 + p^2} \tag{13}$$

and replace the operator p with its eigenvalue. The exponential of $-iPp^2/2m\hbar$ effects a translation of Q in $\Phi_{\text{in}}(Q)$ and we obtain (up to a normalizing factor)

$$\Phi_{\text{fin}}(Q) = \frac{\hbar\alpha}{\pi} e^{\alpha x_0 - \alpha^2\delta^2/2}$$
$$\times \int dp \frac{e^{-p^2\delta^2/2\hbar^2 - ipx_0/\hbar}}{\alpha^2\hbar^2 + p^2} \Phi_{\text{in}}(Q - p^2/2m) . \tag{14}$$

This integral has poles at $p = \pm i\alpha\hbar$; we may evaluate it by integration on a contour including a line of p with imaginary part $-ip_0$ for any $p_0 > \hbar\alpha$. The integral in Eq. (14) then reduces to two terms: a pole term

$$\Phi_{\text{in}}(Q + \alpha^2\hbar^2/2m) \tag{15}$$

and the integral Eq. (14) with p replaced by $p - ip_0$. The pole term represents the measuring device with its pointer shifted to the negative value $-\alpha^2\hbar^2/2m$. If the final state included only this term, measurements would yield $-\alpha^2\hbar^2/2m$ for the kinetic energy, up to a scatter ϵ characteristic of the measuring device.

The correction to the pole term is the integral in $p - ip_0$,

$$\frac{\hbar\alpha}{\pi} e^{\alpha x_0 - \alpha^2\delta^2/2} \int_{-\infty}^{\infty} dp \frac{e^{-(p-ip_0)^2\delta^2/2\hbar^2 - i(p-ip_0)x_0/\hbar}}{\alpha^2\hbar^2 + (p-ip_0)^2}$$
$$\times \Phi_{\text{in}}\left[Q - (p-ip_0)^2/2m\right] . \tag{16}$$

We can bound the magnitude of the correction by replacing the integrand with its absolute value. The integral over the absolute value converges (see Appendix). Since we replaced the integrand with its absolute value, the only dependence on x_0 that remains is the exponential $e^{(\alpha - p_0/\hbar)x_0}$. Since $\alpha < p_0/\hbar$, the correction to

$\Phi_{\text{in}}(Q + \alpha^2\hbar^2/2m)$ can be made arbitrarily small by taking x_0 large, as in Eq. (11). In this limit, the final state of the measuring device shows the unphysical result $-\alpha^2\hbar^2/2m$ for the kinetic energy.

We thus obtain a correlation between position measurements and prior kinetic-energy measurements: nearly all particles found far outside the potential well yielded negative values of kinetic energy. On the other hand, we could look at the entire set of data differently. We could consider all particles that produced negative values of kinetic energy and ask about their final position. We would find nearly all these particles *inside* the well. The correlation works one way only. Prior kinetic-energy measurements on particles found far from the well cluster around a negative value, but position measurements on particles yielding negative values of kinetic energy cluster around zero. How do we interpret this one-way correlation?

IV. INTERPRETATION

Our example suggests that particles in a classically forbidden region have negative kinetic energy. But the conventional interpretation of quantum mechanics has no place for negative kinetic energy. Measurements correspond to eigenvalues or to expectation values only. These must be positive in the case of kinetic energy, so negative measured values of kinetic energy must be errors.

However, the conventional interpretation involves an assumption about how measurements are made. The conventional interpretation considers measurements on ensembles of systems prepared in an initial state, without any conditions on the final state of the systems. Such an ensemble, defined by initial conditions only, may be termed a *preselected* ensemble. By contrast, we consider measurements made on *preselected* and *postselected* ensembles, defined by both initial and final conditions. The experiment of the preceding section is an example of a measurement on a pre- and postselected ensemble. It is natural to introduce pre- and postselected ensembles in quantum theory: in the quantum world, unlike the classical world, complete specification of the initial state does not determine the final state.

A measurement on a pre- and postselected ensemble involves a preselection, a measurement, and a postselection. Aharonov, Bergmann, and Lebowitz [5] (ABL) gave a formula for the result of the intermediate measurement. Let an operator C be measured at time t between a preselected state $|a\rangle$ at time t_1 and a postselected state $|b\rangle$ at time t_2. If C has eigenvalues c_j, then the probability $\mathcal{P}(c_j)$ that the intermediate measurement of C yields c_j is [5,6]

$$\mathcal{P}(c_j) = \frac{|\langle b|U(t_2,t)|c_j\rangle\langle c_j|U(t,t_1)|a\rangle|^2}{\sum_k |\langle b|U(t_2,t)|c_k\rangle\langle c_k|U(t,t_1)|a\rangle|^2} . \tag{17}$$

Still, the ABL formula applies to *ideal* intermediate measurements. Equation (17) presupposes that the measurement of C yields one of its eigenvalues c_j. Real measurements, on the other hand, are subject to error. At the same time, the disturbance they make is bounded. These

two aspects of real measurements go together. Suppose our measuring device interacts very weakly with the systems in the ensemble. We pay a price in precision. On the other hand, the measurements hardly disturb the ensemble, and therefore they characterize the ensemble during the whole intermediate time. Even noncommuting operators can be measured at the same time if the measurements are imprecise. When such measurements are made on pre- and postselected ensembles, they yield surprising results. An operator yields *weak* values that need not be eigenvalues, or even classically allowed [1,7]. The negative kinetic energy of Sec. III is an example of a weak value.

Let us briefly review how weak values arise. The initial wave function of the measuring device is $\Phi_{\text{in}}(Q)$. After an impulsive measurement of an operator C and projection onto a final state, the final state of the measuring device is

$$\langle b | e^{-iPC/\hbar} | a \rangle \Phi_{\text{in}}(Q) = \sum_i \langle b | c_i \rangle \langle c_i | a \rangle \Phi_{\text{in}}(Q - c_i) .$$

$$(18)$$

If $\Phi_{\text{in}}(Q)$ is sharply peaked, then the various terms $\Phi_{\text{in}}(Q - c_i)$ will be practically orthogonal, and the probability of obtaining c_i as an outcome follows the ABL formula, Eq. (17). But suppose $\Phi(Q)$ has a width ϵ. Its Fourier transform has a width in P of $\hbar/\epsilon$. Small $|P|$ corresponds to a measuring device that is coupled weakly to the measured system. If ϵ is large, then $|P|$ is small, and we can expand the exponential in Eq. (18) to first order in P to obtain

$$\langle b | e^{-iPC/\hbar} | a \rangle \Phi(Q) \approx \langle b | 1 - iPC/\hbar | a \rangle \Phi(Q)$$

$$\approx \langle b | a \rangle e^{-iPC_w/\hbar} \Phi(Q) . \quad (19)$$

Here

$$C_w \equiv \frac{\langle b | C | a \rangle}{\langle b | a \rangle} \quad (20)$$

is the *weak value* of the operator C for the pre- and postselected ensemble defined by $\langle b |$ and $| a \rangle$.

The definition of a weak value provides us with an intuitive language for describing quantum processes. In our example, the operators of total energy E, kinetic energy K, and potential energy V do not commute. Therefore, the classical formula $E = K + V$ does not apply to the quantum operators E, K, and V, but only to their expectation values; and the expectation value of K in any state is positive. However, the formula applies to weak values, as follows immediately from the definition, Eq. (20):

$$E_w = K_w + V_w , \quad (21)$$

and the weak value of K is *not* necessarily positive. We can compute it as $K_w = E_w - V_w$. We know $E_w = E_0 = -\alpha^2 \hbar^2 / 2m$, since the preselected state is an energy eigenstate, and V_w vanishes since the postselected state is far from the potential well. Then we have $K_w = -\alpha^2 \hbar^2 / 2m$, the "unphysical" value obtained above in our example. Weak values do not appear in the con-

ventional formulation of quantum mechanics, but they appear in measurements.

Equation (19) shows how weak values emerge from an imprecise measurement (ϵ large). But the weak value emerged from a *precise* measurement of kinetic energy in our example. Instead of the condition on the initial state of the measuring device (ϵ large), we had a condition on the final state of the particle (x_0 large and $\delta > \alpha \hbar^2 / m \epsilon$). What do these measurements have in common? Equation (19) assumes a weak measurement interaction which disturbs the measured system within limits. When $|P|$ is small, the measurement hardly intrudes between the pre- and postselected states of the system. The pre- and postselected states define the measured system during the intervening time. But when $|P|$ is not small, we can still control the effect of a measurement. In our example, we preselect a state with negative total energy and postselect a state where the potential vanishes. It is not enough to postselect particles outside the well. The kinetic-energy measurement disturbs the particles, and they may not remain bound. We must somehow postselect particles so far from the well that measurements of kinetic energy could not have kicked them there. Then both negative total energy and vanishing potential will characterize the particles throughout the measurement.

To see what to postselect, let us write Eq. (12) as an integral over x instead of over p:

$$\Phi_{\text{fin}}(Q) = \int_{-\infty}^{\infty} dx \, e^{-(x - x_0)^2 / 2\delta^2} e^{-(i/\hbar)Pp^2/2m}$$

$$\times e^{-\alpha |x|} \Phi_{\text{in}}(Q) , \quad (22)$$

up to normalization. If we could ignore the part of the integral near $x = 0$, we could replace p^2 with $-\alpha^2 \hbar^2$ in Eq. (22), and the final state of the measuring device would be $\Phi_{\text{fin}}(Q) = \Phi_{\text{in}}(Q + \alpha^2 \hbar^2 / 2m)$. Although we cannot ignore this part of the integral, we can choose $\Psi_{\text{fin}}(x)$ to suppress it. $\Psi_{\text{fin}}(x)$ will suppress the integral near $x = 0$ if Eq. (11) holds and $\delta > \alpha \hbar^2 / m \epsilon$. We have already derived these conditions (see the Appendix). Now we show intuitively, using time symmetry, how they keep particles away from the well.

Defining an ensemble via an initial state breaks time symmetry. To preserve time symmetry, we may select both an initial and a final state, thus defining a pre- and postselected ensemble. Both the ABL formula, Eq. (17), and the definition of a weak value, Eq. (20), manifest time symmetry. We may think of quantum states propagating forwards and backwards in time [7]. The initial state evolves forwards in time, and by time symmetry the final state evolves backwards in time; *both* states influence an intermediate measurement. Indeed, the adjoint of Eq. (22) represents reversed time evolution with $\Psi_{\text{fin}}(x)$ as the preselected state and $\Psi_{\text{in}}(x)$ as the postselected state. If we reverse the time evolution, the weak value remains the same, as well as the condition on $\Psi_{\text{fin}}(x)$. Applying the time evolution operator to $\Psi_{\text{fin}}(x)$,

$$e^{(i/\hbar)Pp^2/2m} e^{-(x - x_0)^2 / 2\delta^2}$$

$$= (1 - i\hbar P / m\delta^2)^{-1/2} e^{-(x - x_0)^2 / 2(\delta^2 - i\hbar P/m)} , \quad (23)$$

we see that the effect of the measurement is to broaden $\Psi_{\text{fin}}(x)$. While time evolution of $\Psi_{\text{in}}(x)$ can kick particles out of the well, time evolution of $\Psi_{\text{fin}}(x)$ can bring particles *to* the well. *Either* forward *or* backward time evolution of $\Psi_{\text{fin}}(x)$ can bring particles to the well, although forward time evolution is more familiar.

Equation (23) is awkward because P is an operator. For a given value of $|P|$, the semiclassical probability for the measurement to bring a particle to the well is the absolute square of Eq. (23) for $x=0$. Thus, for any state $\Phi_{\text{in}}(Q)$ with $|P|$ strictly bounded, such as $(\sqrt{\epsilon}/\sqrt{\pi}Q)\sin(Q/\epsilon)$, a sufficient condition on x_0 is

$$x_0 \gg 2\alpha(\delta^2 + \hbar^2 P^2/m^2\delta^2) \tag{24}$$

for all P. However, the Gaussian state $\Phi_{\text{in}}(Q)$ of Eq. (4) includes Fourier modes with arbitrary $|P|$. Large $|P|$ are suppressed, but for no x_0 are they suppressed altogether. In the state $\Phi_{\text{in}}(Q)$, the probability of a given P is

$$\frac{\epsilon}{\hbar\sqrt{\pi}}e^{-P^2\epsilon^2/\hbar^2} . \tag{25}$$

Folding this probability with the absolute square of Eq. (23), we obtain

$$\frac{\epsilon}{\hbar\sqrt{\pi}}\int_{-\infty}^{\infty}dP\frac{e^{-x_0^2/(\delta^2+\hbar^2P^2/m^2\delta^2)}}{(1+\hbar^2P^2/m^2\delta^4)^{1/2}}e^{-P^2\epsilon^2/\hbar^2} \tag{26}$$

as the probability for the measurement to bring particles to the well. For large x_0 the integral is dominated by large $|P|$; we may replace $\delta^2 + \hbar^2P^2/m^2\delta^2$ by $\hbar^2P^2/m^2\delta^2$ and neglect the denominator to get an upper bound

$$e^{-2x_0 m\delta\epsilon/\hbar^2} . \tag{27}$$

If we *pre*select $\Psi_{\text{fin}}(x)$, Eq. (27) represents the fraction of the preselected ensemble that we would expect to find at the well. But the probability to *post*select Ψ_{in} is suppressed by a factor $e^{-2\alpha x_0}$ for large x_0. We want a pre- and postselected ensemble dominated by particles outside the well, and so we require the latter probability to be much larger than the former, that is,

$$\delta > \alpha\hbar^2/m\epsilon , \tag{28}$$

with x_0 large. These are the conditions imposed on $\Psi_{\text{fin}}(x)$ in Sec. III. We need *both* conditions to restrict particles to the classically forbidden region. When these conditions hold, V_w vanishes, and a kinetic-energy measurement yields K_w, even though the measurement is precise.

This is an important lesson: the right pre- or postselection allows us to increase the precision of the intermediate measurement. The price is that we must wait for increasingly rare events. As measurements of kinetic energy become more precise ($\epsilon \to 0$), they disturb the particle more. To get negative kinetic energies, we must postselect particles further from the potential well ($x_0 \to \infty$). As the precision of the measurement increases, negative kinetic energies become less and less frequent; in the limit of ideal measurements, the probability vanishes, and so ideal measurements of kinetic energy never yield negative

values. It is easy to see that if ϵ approaches 0 while x_0 is held fixed, so does the chance to measure negative kinetic energies. Taking the limit $\epsilon \to 0$ in Eqs. (4) and (14) turns $\Phi_{\text{in}}(Q-p^2/2m)$ into a δ function, and the final state of the measuring device becomes (up to normalization)

$$\frac{\cos(\sqrt{2mQ}\,x_0/\hbar)}{\sqrt{2mQ}\,(\alpha^2\hbar^2+2mQ)} \tag{29}$$

for positive Q, and zero for negative Q. The ABL formula predicts exactly this distribution of kinetic energies.

V. CONCLUSIONS

We have seen that measurements of the kinetic energy of a particle in a potential well can yield negative values consistently. These measurements involve selecting a final state of the particle far from the well. The negative values represent the *weak* value of the kinetic-energy operator.

From the point of view of standard quantum theory, all that we have produced is a game of errors of measurement. Ideal measurements of kinetic energy can yield only positive values, since all eigenvalues of the kinetic-energy operator are positive. But in practice, measurements are not exact, and even if their precision is very good, sometimes—rarely—they yield negative values. We have seen that if particles are subsequently found far from the potential well, the measured kinetic energy of these particles comes out negative. Consistently, large measurement "errors" did occur, producing a distribution peaked at the "unphysical" negative value E_0. Mathematically, this peak arises from an unusual interference. The measuring procedure pairs each particle eigenstate of kinetic energy K with a Gaussian wave packet $\exp[-(Q-K)^2/2\epsilon^2]$ of the pointer. But after projection onto the postselected particle state, these Gaussian wave packets interfere, destructively for positive K and constructively for negative K. The pointer is left in a Gaussian state centered on the negative value E_0, with a spread characteristic of the measuring device.

What special properties of nonideal measurements led to this result? First, these measurements involve only bounded disturbances of particle position. Second, since their precision is limited, they can supply, "by error," the necessary negative values. These two properties are intimately connected: any measurement of kinetic energy causing only bounded changes of position must occasionally yield negative values for the kinetic energy. The von Neumann formalism states that the change of x due to the measurement is

$$\dot{x} = \frac{i}{\hbar}g(t)[x, Pp^2/2m] . \tag{30}$$

P and p are unchanged during the measurement, so the normalization condition, Eq. (2), implies [8]

$$x_{\text{fin}} - x_{\text{in}} = Pp/m . \tag{31}$$

From here it follows that the change of x is bounded only if the pointer is in an initial state with P bounded, i.e., if the Fourier transform of $\Phi_{\text{in}}(Q)$ has compact support.

But then the support of $\Phi_{in}(Q)$ is unbounded [9], which immediately implies a nonzero probability for the pointer to indicate negative values ($Q < 0$). Indeed, the "game of errors" displays a remarkable consistency, and this consistency allows negative kinetic energies to enter physics in a natural way.

The concept of a *weak* value of a quantum operator gives precise meaning to the statement that the kinetic energy of a particle in a classically forbidden region is negative, namely the weak value of the kinetic energy is negative. Weak values are defined on pre- and postselected ensembles. The interpretation of this concept raises subtle questions about time. Our example involves preselection of particles in a bound state, measurement of their kinetic energy, and postselection of the particles far from the potential well. We associate negative values of kinetic energy with the particles. However, instead of postselecting particles far from the well, we could measure the kinetic energy again with greater precision. We would then find that almost every time the first measurement yielded a negative value, the second measurement yields a positive value, and we would interpret the negative value as an error of the measuring device. The final measurement, whether of position or of kinetic energy, is made *after* a kinetic-energy measurement has yielded negative values. Nevertheless, the interpretation of these negative values depends on the final measurement. If we measure position, we attribute them to the particle, while if we measure kinetic energy, we attribute them to the device. The effect seems to precede the cause.

The example of a particle in a potential well is a limiting case of quantum tunneling, when the barrier becomes very broad. Negative kinetic energies arise in finite barriers, too; but precise measurements of kinetic energy require postselected particles deep in the classically forbidden region, so negative kinetic energies may be difficult to observe in narrow barriers. Finally, we note a surprising extension to our result. By assuming an impulsive measurement of kinetic energy, we could neglect the Hamiltonians of the system and measuring device and consider just their interaction. It follows that we can observe particles with negative kinetic energy even if there is no binding potential at all. What matters is only the shape of the preselected particle wave function. Here, too, negative energies are consistent with other physical processes (scattering) [10].

ACKNOWLEDGMENTS

APPENDIX

We wish to obtain an upper bound for the magnitude of the correction term, Eq. (16), in the final state of the measuring device. The absolute value of the denominator is at least $p_0^2 - \alpha^2 \hbar^2$, so a bound is

$$\frac{\hbar \alpha}{\pi} \frac{e^{(\alpha - p_0/\hbar)x_0 - \alpha^2 \delta^2/2}}{p_0^2 - \alpha^2 \hbar^2}$$

$$\times \int dp \, e^{-(p^2 - p_0^2)\delta^2/2\hbar^2} |\Phi_{in}[Q - (p - ip_0)^2/2m]| \quad (A1)$$

and $|\Phi_{in}(Q - (p - ip_0)^2/2m)|$ is

$$(\epsilon^2 \pi)^{-1/4} e^{-(Q + p_0^2/2m)^2/2\epsilon}$$

$$\times e^{-p^4/8m^2\epsilon^2 + p^2(Q/2m\epsilon^2 + 3p_0^2/4m^2\epsilon^2)} . \quad (A2)$$

Using [11]

$$\int_{-\infty}^{\infty} dp \, e^{-\mu p^4 \pm |a| p^2}$$

$$= \frac{\pi |a|}{4\mu} e^{a^2/8\mu} \left[I_{-1/4} \left[\frac{a^2}{8\mu} \right] \pm I_{1/4} \left[\frac{a^2}{8\mu} \right] \right] \quad (A3)$$

and

$$I_\nu(x) = \frac{e^x}{(2\pi x)^{1/2}} \left[1 + O \left[\frac{1}{x} \right] \right] \quad (A4)$$

for large x, we find that Eq. (A1) leads to an exponential in

$$(\alpha - p_0/\hbar)x_0 - \delta^2 \left[\frac{p_0^2}{\hbar^2} + \frac{\alpha^2}{2} \right] + Q \left[\frac{p_0^2}{m\epsilon^2} - \frac{\delta^2 m}{\hbar^2} \right]$$

$$+ \frac{p_0^4}{m^2\epsilon^2} + \frac{\delta^4 m^2 \epsilon^2}{2\hbar^4} . \quad (A5)$$

The upper bound on the correction, Eq. (16), will be exponentially suppressed if this sum of terms is sufficiently negative. The parameter p_0 is arbitrary, aside from the constraint $p_0 > \alpha \hbar$. Since $\delta > \alpha \hbar^2/m\epsilon$ is a condition on $\Psi_{fin}(x)$, we can eliminate the dependence on Q by choosing $p_0 = \delta m \epsilon/\hbar$. Then for large enough x_0, the exponent is negative. Setting $\delta = n \alpha \hbar^2/m\epsilon$ for $n > 1$, we obtain for the exponent

$$-\alpha(n-1)x_0 + 2(n^4 - n^2)\frac{\alpha^4 \hbar^4}{4m^2\epsilon^2} , \quad (A6)$$

so that the upper bound on the correction term is exponentially suppressed if

$$\alpha x_0 \gg \left[\frac{|E_0|}{\epsilon} \right]^2 , \quad (A7)$$

as in Eq. (11).

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[4] An example involving a smooth potential $V(x)$ $= -\alpha^2 \hbar^2/m \cosh^2(\alpha x)$ is presented in Y. Aharonov, S. Popescu, D. Rohrlich, and L. Vaidman, in *Proceedings of the Fourth International Symposium on the Foundations of Quantum Mechanics, Tokyo, 1992*, edited by Masaru Tsukada, Shun-ichi Kobayashi, Susumu Kurihara, and Sadao Nomura, Jpn. J. Appl. Phys. Ser. 9 (Japanese Journal of Applied Physics, Tokyo, 1993), pp. 251–253.

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[6] If C has degenerate eigenvalues, the projectors $|c_k\rangle\langle c_k|$ appearing in Eq. (17) must be replaced by projectors onto the degenerate eigenspaces. See Y. Aharonov and L. Vaidman, J. Phys. A **24**, 2315 (1991).

[7] Y. Aharonov and L. Vaidman, Phys. Rev. A **41**, 11 (1990); Y. Aharonov and D. Rohrlich, in *Quantum Coherence*, Proceedings of the Conference on Fundamental Aspects of Quantum Theory, Columbia, SC, 1989, edited by J. S. Anandan (World-Scientific, Singapore, 1990).

[8] The uncertainty relations imply that x and p cannot both be bounded; however, Eq. (30) tells us qualitatively how the measurement interaction affects the final state of the particle and whether x_0 is sufficiently large, in the sense discussed above.

[9] If the Fourier transform of $\Phi_{in}(Q)$ has compact support, then $\Phi_{in}(Q)$ is analytic. The two derivations of our result, via contour integration and via Taylor expansion of the exponential in Eq. (19), both require $\Phi_{in}(Q)$ to be analytic.

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