Revisiting Bell’s theorem for a class of down-conversion experiments

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A certain class of parametric down-conversion Bell-type experiments has the following features. In the idealized perfect situation it is in only 50% of the cases that each observer receives a photon; in the other 50% one observer receives both photons of a pair while the other observer receives none. The standard approach is to discard the events of the second type. Only the remaining ones are used as the data input to some Bell inequalities. This raises justified doubts about whether such experiments could ever be genuine tests of local realism. We propose to take into account these “unfavorable” cases and to analyze the entire pattern of polarization and localization correlations. This departure from the standard reasoning enables one to show that the experiments are indeed a true test of local realism.

The first generation of Bell tests opened with the experiment of Freedman and Clauser [1]. The tremendous progress in experimental quantum optics during recent years, and especially the advent of parametric down-conversion techniques have led a new generation of experiments proposing to test Bell’s inequalities [2]. Of these the experiments, those of Alley and Shih [3], Ou and Mandel [4], and Kiss, Shih, Sergienko, and Alley [5] are some of the most famous examples. They form a distinctive class: polarization correlations are measured and, to produce the required phenomena, use is made of a technique essentially involving two linear optical devices, namely a wave plate and a single beam splitter (Fig. 1). Although these experiments have successfully produced certain long-distance quantum-mechanical correlations, it is generally believed [6] that they could never, not even their idealized versions, be considered as true tests of local realism. In the present paper we prove that this general belief is wrong and that the above experiments are in fact much better than they were previously thought to be: in principle, they could be tests of the most general premises of local realism, and the only obstacles they face are purely technical (such as the present-day low efficiency of the photon detectors, misalignments, etc.). The tools we use to prove this thesis are a generalization of some of the ideas introduced by Clauser and Horne [7].

The claim that the above experiments can never be true tests of local realism is usually based on the following argument. In the original setting, proposed by Bell and then widely analyzed by many others [8], experiments are performed on an ensemble of pairs of particles prepared in such

FIG. 1. Example of an experimental setup for testing Bell’s inequalities. M, mirror; WP, wave plate; BS, beam splitter; D, detectors; PDC, parametric down-conversion crystal.
a way that one particle in each pair is directed towards an observer while the other particle is directed toward another observer situated far from the first one. However, in the parametric down-conversion experiments we are interested in [3–5] the pairs of particles (photons) are prepared in a different way—only in 50% of the cases does each observer receive a photon; in the other 50% of cases one observer receives both photons of a pair while the other observer receives none. The usual superficial method to deal with this situation is to discard all the unfavorable cases in which both photons end at the same observer and to retain only the cases in which each observer receives a photon. Therefore, it can be justifiably claimed that one runs directly into the well-known problem of subensemble postselection [6–9]. If one restricts the analysis to a small enough subensemble of the original ensemble of pairs, one can never rule out all possible local hidden-variable models.

However, a more careful look at the argument formulated in the last two sentences shows its weakness. First of all, the idea that the “unfavorable” cases should be discarded appeared probably because of the desire to forcefully map the problem under investigation into the one originally formulated by Bell. But instead of doing this one could actually make use of these “unfavorable” cases by taking them also into account and analyzing the entire pattern of correlations. (This is the approach we shall take in the present paper.) Second, while it is widely believed that subensemble selection (when the selected subensemble is small enough) always prevents one from observing violations of local realism, this is actually not true. Indeed, there are cases in which postselection is a serious drawback (e.g., the “detector efficiency” problem [7]), but there are also other cases in which subensemble selection raises no problems [10,11]. Consequently one should not dismiss at the outset the possibility of observing violations of local realism just because subensemble selection has been performed, but each situation should be investigated carefully.

Consider first the experimental setting used in [3–5] and illustrated in Fig. 1 [12]. A type-I parametric down-conversion source is used to generate pairs of photons in which both photons have the same energy and linear polarization (say $\hat{x}$) but propagate in two different directions. One of the photons passes through a 90° polarization rotator, the wave plate $WP$, emerging polarized along $\hat{y}$. The two photons are then directed by two mirrors $M$ onto the two sides of a (polarization-independent) “50-50” beam splitter BS, which, for simplicity, we consider to be symmetric. Each observer is equipped with a polarizing beam splitter, orientated along an arbitrary axis, randomly chosen just before the photons are supposed to arrive. Each polarizing beam splitter is followed by two detectors, $D^+_1$, $D^-_1$ and $D^+_2$, $D^-_2$, respectively, where the lower index indicates the corresponding observer and the upper index the two exit ports of the polarized beam splitter (“+” meaning parallel to the polarization axis of the beam splitter and “−” meaning orthogonal to that axis). All optical paths are assumed to be equal.

The quantum state of the two photons just before entering the detection stations is

$$|\Psi\rangle = \frac{1}{2}(|1x\rangle_1|1\hat{y}\rangle_2 - |1\hat{y}\rangle_1|1\hat{x}\rangle_2 + i|1\hat{x},1\hat{y}\rangle_1|0\rangle_2$$

(1)

where the subscripts 1 or 2 on the ket vectors represent the two regions of space where the photons arrive, i.e., near observer 1 and near observer 2, the notation 0 inside the ket vectors denotes vacuum, and $1\hat{x}$ and $1\hat{y}$ represent one photon polarized along either the $\hat{x}$ or $\hat{y}$ directions, respectively. The first two terms in Eq. (1) correspond to cases in which each observer will register a single photon, while the last two terms correspond to cases in which one of the observers will register two photons [13] while the other observer will register none.

Quite often the discussion starts by just simply chopping off the last two terms in Eq. (1). This approach, since it is effectively a postselection procedure, raises serious, justified doubts [6], as to whether the experiments indeed are tests of local realism. But in fact there is no reason why one should reject the other cases.

Let us denote by $P(i, \hat{\xi}; j, \hat{\eta})$ the joint probability for the outcome $i$ to be registered by observer 1 when his polarizing beam splitter is oriented along the direction $\hat{\xi}$ and the outcome $j$ to be registered by observer 2 when her polarizing beam splitter is oriented along $\hat{\eta}$. Here $i,j = 1–6$ and have the following meaning:

1. One photon in $D^-$, no photon in $D^+$,
2. One photon in $D^+$, no photon in $D^-$,
3. No photons,
4. One photon in $D^+$ and one photon in $D^-$,
5. Two photons in $D^+$, no photon in $D^-$,
6. Two photons in $D^-$, no photons in $D^+$.

We have not included the possibility of more than one pair of photons being emitted. The probability of this happening, under the usual experimental conditions, is very small, and thus will lead to negligible effects.

A Clauser-Horne-Shimony-Holt (CHSH) type inequality, which is obeyed by all local hidden-variable models but is violated by quantum mechanics for the full state (1), can be obtained in the following way. Let us associate with each outcome registered by observer 1 or 2 a corresponding value $a_i^\hat{x}$ or $b_j^\hat{y}$, respectively, where $a_i^\hat{x} = b_j^\hat{y} = -1$, while all the other values are equal to 1, and let us denote by $E(a_i^\hat{x} b_j^\hat{y})$ the expectation value of their product

$$E(a_i^\hat{x} b_j^\hat{y}) = \sum_{i,j} a_i^\hat{x} b_j^\hat{y} P(i, \hat{\xi}; j, \hat{\eta}).$$

(2)

Now, in a local hidden-variable model

$$P(i, \hat{\xi}; j, \hat{\eta}) = \int d\lambda \rho(\lambda) P_1(i, \hat{\xi}; \lambda) P_2(j, \hat{\eta}; \lambda),$$

(3)

where $\lambda$ is the local hidden variable with $\rho$ its distribution function [$\int d\lambda \rho(\lambda) = 1$], and $P_1(i, \hat{\xi}; \lambda)$ and $P_2(j, \hat{\eta}; \lambda)$ the local probabilities [$\Sigma_{P_1}(i, \hat{\xi}; \lambda) = 1 = \Sigma_{P_2}(j, \hat{\eta}; \lambda)$]. It is straightforward to see [8] that according to any local hidden-variable model the CHSH inequality holds, i.e.,
\[ |E(a^\hat{x} b^\hat{y}) + E(a^\hat{x} b^\hat{y}') + E(a^\hat{x}' b^\hat{y}) - E(a^\hat{x}' b^\hat{y}')| \leq 2, \quad (4) \]

for any directions \( \hat{x}, \hat{x}', \hat{y}, \text{ and } \hat{y}' \).

On the other hand, according to quantum mechanics

\[ E(a^\hat{x} b^\hat{y}) = \langle \Psi | A^\hat{x} B^\hat{y} | \Psi \rangle, \quad (5) \]

where \( A^\hat{x} \) and \( B^\hat{y} \) are the corresponding quantum observables, defined as follows: \( A^\hat{x} \) is an operator that has a non-degenerate eigenvalue \( A^\hat{x} = -1 \) corresponding to the eigenstate \( |1\hat{\xi}\rangle \) (which represents one photon polarized orthogonal to \( \hat{\xi} \)) and which corresponds to observer 1 obtaining the outcome \( i = 1 \) and a multiple degenerate eigenvalue \( A^\hat{x} = 1 \) corresponding to the rest of the Hilbert space (the space spanned by \( |1\hat{\xi}\rangle, |0\rangle, |1\hat{\xi},1\hat{\xi}\rangle, |2\hat{\xi}\rangle, \text{ and } |2\hat{\xi}\rangle_2 \), which correspond to observer 1 obtaining outcomes \( i = 2 - 6 \); \( B^\hat{y} \) is defined in a similar way. In other words, the operators \( A \) and \( B \) are equal to the usual polarization operators on the subspace of “favorable” cases (yielding \( +1 \) if the photon’s polarization is parallel to that of the polarization analyzer and \(-1 \) if it is perpendicular) and equal to the identity operator on the subspace of “unfavorable” cases. Let us also define \( |\Psi_1^\uparrow\rangle \) and \( |\Psi_2^\uparrow\rangle \) as the normalized projections of \( |\Psi\rangle \) on the subspaces of “unfavorable” and “favorable” cases respectively, i.e.,

\[ |\Psi_1^\uparrow\rangle = \frac{i}{\sqrt{2}}(|1\hat{\xi},1\hat{\gamma}\rangle_1|0\rangle_2 + |0\rangle_1 |1\hat{\xi},1\hat{\gamma}\rangle_2) \quad (6) \]

and

\[ |\Psi_2^\uparrow\rangle = \frac{1}{\sqrt{2}}(|1\hat{\xi},1\hat{\gamma}\rangle_1|1\hat{\gamma}\rangle_2 - |1\hat{\gamma}\rangle_1 |1\hat{\xi}\rangle_2). \quad (7) \]

Then, since the local operators \( A \) and \( B \) do not mix the local vacuum and local two-photon states with the local one photon states, it follows that

\[ I_{CHSH} = \langle \Psi | A^\hat{x} B^\hat{y} + A^\hat{x} B^\hat{y} + A^\hat{x}' B^\hat{y} - A^\hat{x}' B^\hat{y} | \Psi \rangle \]

\[ = \frac{1}{2} \langle \Psi_1^\uparrow | A^\hat{x} B^\hat{y} + A^\hat{x} B^\hat{y} + A^\hat{x}' B^\hat{y} - A^\hat{x}' B^\hat{y} | \Psi_1^\uparrow \rangle \]

\[ + \frac{1}{2} \langle \Psi_2^\uparrow | A^\hat{x} B^\hat{y} + A^\hat{x} B^\hat{y} + A^\hat{x}' B^\hat{y} - A^\hat{x}' B^\hat{y} | \Psi_2^\uparrow \rangle \]

\[ = \frac{1}{2} \times 2 + \frac{1}{2} \times \langle \Psi_2^\uparrow | A^\hat{x} B^\hat{y} + A^\hat{x} B^\hat{y} + A^\hat{x}' B^\hat{y} - A^\hat{x}' B^\hat{y} | \Psi_2^\uparrow \rangle \]

\[ = A^\hat{x}' B^\hat{y} | \Psi_2^\uparrow \rangle. \quad (8) \]

The expectation value in the last term in Eq. (8) is nothing other than the usual quantum CHSH expression computed in the \( |\Psi_2^\uparrow\rangle \) state, which for suitable chosen directions \( \hat{x}, \hat{x}', \hat{\gamma}, \text{ and } \hat{\gamma}' \) can yield \( 2\sqrt{2} \). Choosing such directions in Eq. (8) it follows that, for the idealized perfect experiment

\[ I_{CHSH} = 1 + \sqrt{2} > 2, \quad (9) \]

which is in contradiction to the limit imposed by local hidden-variable models [14].

In calculating the probabilities above we have assumed that the total number of events is equal to the total number of pairs detected. This is equivalent to assuming an event-ready configuration in which the source clicks (or gives off some other appropriate signal) when the photons are emitted \([8]\). However, the experiments \([3-5]\) were not event ready, since there was no way to know that a pair of photons had been emitted (event-ready configurations have only been suggested \([15]\)).

Thus, it could be possible, for example, that whether or not two photons are detected at one end (whether we are in an \( i = 4,5,6 \) or an \( i = 3 \) case) depends on the setting of the polarizing beam splitter. Polarizer settings, by biasing the ensemble considered, seemingly introduce the possibility of a loophole.

To solve this problem, first of all, we must decide what we are going to regard as an event \([7]\). The experiment runs for a certain time \( T \). This time can be divided up into short intervals of duration \( \tau \). At the beginning of each time interval the polarizing beam splitters are set in a new, randomly chosen direction. The time interval \( \tau \) must be chosen to be smaller than \( L/c \), where \( L \) is the distance between the detector stations, so that there is no possibility of relativistic causal signals being transmitted during this time interval. Also, to avoid extra complications, it should be chosen such that there is a very small probability of more than one pair being emitted during \( \tau \); and it should be bigger than the time resolution of the detectors, so that two photons from the same pair would almost certainly be detected during the same time interval. A practical choice would probably be \( \tau = 10 \) ns.

The total number of events is \( N \), where \( T = N \tau \), and \( T \) is chosen such that \( N \) is integer. The \( n \)th event happens during the interval \((n-1)\tau \) to \( n\tau \). During this interval we record the type of event that has occurred at each end of the apparatus \((i,j = 1 \text{ to } 6) \). In this way we can form probabilities in the usual way. Typical counting rates are about \( 10^4/s \), which is much smaller than \( 1/\tau \). This means that almost all events will be of the type where no photons are detected at either end \((i = j = 3) \). For CHSH inequalities formulated in the usual way this would be a big problem since these no-photon events would drown out the interesting events and hence the inequalities would not be violated. However, as we will see, the way in which the correlation function has been defined in equation Eq. (2) solves this problem.

To include explicitly the vacuum term one can describe the state by

\[ |\Psi'\rangle = a|0\rangle_1|0\rangle_2 + \beta |\Psi\rangle, \quad (10) \]

where \( |\Psi\rangle \) is the state Eq. (1). The vacuum term is now explicitly included. Either a total of two photons or no photons will be detected. Let \( N_0 \) be the number of events for which no photons are detected. Then

\[ P(3,\hat{x},3,\hat{\gamma}) = \frac{N_0}{N}. \quad (11) \]
Since $a_j^\dagger = b_j^\dagger = 1$, we have from Eq. (2) that
\[
E(a_j^\dagger b_j) = \frac{N_0}{N} + \frac{N-N_0}{N} \langle \Psi | A_j^\dagger B_j | \Psi \rangle.
\]
(12)
Hence, now taking into account the vacuum cases, we have that
\[
I_{CHSH} = \frac{2N_0}{N} + \frac{N-N_0}{N} (1 + \sqrt{2}) > 2,
\]
(13)
where the inequality follows since $N_0 < N$. Hence, the CHSH inequalities are still violated.

The magnitude of the violation is not as great but this need not bother us. Any experiment that violates the inequalities when the vacuum events are included will also violate the experiment when they are not, and vice versa. Simply, an experiment where every event is a vacuum event would give $I_{CHSH} = 2$.

Let us comment on the general case of non-event-ready experiments. In their review [8] Clauser and Shimony make the point that the Clauser-Horne inequalities have a 0 as their upper bound, and hence they are insensitive to the overall normalization of probabilities making them suitable for non-event-ready experiments, and since the CHSH inequalities have nonzero bounds they do rely on knowing how to normalize probabilities and thus are not suitable for non-event-ready experiments. However, by employing the two tricks, (i) considering short intervals as events and (ii) putting $a_j^\dagger = b_j^\dagger = 1$ so that the CHSH inequality is saturated by an ensemble of vacuum events, we make it possible to employ the CHSH inequalities in non-event-ready–type experiments.

To summarize, if one wants to discuss the experiments [3–5] as tests of local realism, one should not discard any “unfavorable” cases but rather one has to analyze them, and there is plenty of information we can obtain apart from what are the polarization correlations. What we have just shown is that the entire pattern of polarization and localization correlations in the experiments [3–5] cannot be explained by any local hidden variable model.

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[12] We stress here that the detector stations in Fig. 1 differ from those in the actually performed experiments. We shall also demand that the detectors at these stations be capable of distinguishing between one and two photons (see below).
[13] Even if one has at his disposal no detectors that can distinguish between single counts and double counts, one can actually build such a detector out of ordinary detectors (which cannot distinguish one- and two-photon events) and beam splitters. This can be realized by splitting the incoming beam into $n$ beams by use of standard (nonpolarizing) beam splitters and placing an ordinary detector in each of these $n$ beams. When the incoming beam contains two photons, the probability that both photons end in just one of the $n$ beams rapidly goes to zero as $n$ increases. Thus for sufficiently large $n$, almost always a two-photon incoming state will result in the firing of two of the detectors, while a one-photon incoming state will fire just a single detector. This may not be the case, if the detectors have a low efficiency, but also in the standard Bell-type experiments one has very high efficiency requirements; therefore this particular problem is not a specific feature of the studied case.
[14] Instead of the CHSH inequality as used above, one can equally well use the Clauser-Horne inequality. Obviously, in this case too, information about the “unfavorable” cases has to be used.
[16] P. G. Kwiat, Phys. Rev. A 52, 3380 (1995). The author proposes a different state preparation method than the one of Refs [2,3]. He also quotes our present idea (his footnote [9]).