Spin Flips and Quantum Information for Antiparallel Spins

N. Gisin[1] and S. Popescu[2,3]

[1]*Group of Applied Physics, University of Geneva, 1211 Geneva 4, Switzerland*
[2]*Isaac Newton Institute, University of Cambridge, 20 Clarkson Road, Cambridge, CB3 0EH, United Kingdom*
[3]*BRIMS Hewlett Packard Labs, Bristol, United Kingdom*
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We consider two different ways to encode quantum information, by parallel or antiparallel pairs of spins. We find that there is more information in the antiparallel ones. This purely quantum mechanical effect is due to entanglement, not of the states but occurring in the course of the measuring process. We also introduce a range of quantum information processing machines, such as spin flip and anticloning.

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Quantum information differs from classical information because it obeys the superposition principle and because it can be entangled. The huge potential of quantum information processing has renewed the interest in the foundations of two of the major scientific theories of the twentieth century: information theory and quantum mechanics [1].

Despite the very intensive recent work on quantum information, surprising effects are continuously being discovered. Here we describe yet another surprise. In a nutshell, we enquire whether quantum information is better stored by two parallel spins or two antiparallel ones?

In more detail, our paper is centered around the following problem of quantum communication. Suppose Alice wants to communicate to Bob a space direction $\vec{n}$, chosen at random. She may do that by one of the following two strategies. In the first case, Alice sends Bob two spin $1/2$ particles polarized along $\vec{n}$, i.e., $|\vec{n}, \vec{n}\rangle$. When Bob receives the spins, he performs some measurement on them and then guesses a direction $\vec{n}_g$ which has to be as close as possible to the true direction $\vec{n}$. The second strategy is almost identical to the first, with the difference that Alice sends $|\vec{n}, -\vec{n}\rangle$; i.e., the first spin is polarized along $\vec{n}$ but the second one is polarized in the *opposite* direction. The question is whether these two strategies are equally good or, if not, which is better.

To put things in a better perspective, consider first a simpler problem. Suppose Alice wants to communicate to Bob a space direction $\vec{n}$ and she may do that by one of the following two strategies. In the first case, Alice sends Bob a single spin $1/2$ particle polarized along $\vec{n}$, i.e., $|\vec{n}\rangle$. The second strategy is identical to the first, with the difference that when Alice wants to communicate to Bob the direction n she sends him a single spin $1/2$ particle polarized in the opposite direction, i.e., $|-\vec{n}\rangle$. Which of these two strategies is better?

If the particles would be classical spins then, obviously, both strategies would be equally good, as an arrow defines equally well both the direction in which it points and the opposite direction. Is the quantum situation the same?

First, we should note that in general, by sending a single spin $1/2$ particle, Alice cannot communicate to Bob the direction $\vec{n}$ with absolute precision. Nevertheless, it is still obviously true that the two strategies are equally good. Indeed, all Bob has to do in the second case is to perform exactly the same measurements as he would do in the first case, only that when his results are such that in the first case he would guess $\vec{n}_g$, in the second case he guesses $-\vec{n}_g$.

We are thus tempted to conjecture the following.

Conjecture: Similar to the classical case, for the purpose of defining an arbitrary direction $\vec{n}$, a quantum mechanical spin polarized along $\vec{n}$ is as good as a spin polarized in the opposite direction. In particular, the two two-spin states $|\vec{n}, \vec{n}\rangle$ and $|\vec{n}, -\vec{n}\rangle$ are equally good.

Surprisingly, however, as we show here, this conjecture is not true.

The main reason behind this effect is, once more, entanglement. Here entanglement does not refer to the two spins—whether parallel or antiparallel they are always in a direct product state—but to the eigenvectors of the optimal measurement. (Indeed, as Massar and Popescu demonstrated [2] the optimal measurement on parallel spins requires entanglement.)

This result has also led us to many new questions. For example, we were led to consider a *universal quantum spin-flip (UQSF) machine,* a machine that flips an unknown spin as well as possible, and an *anticloning* machine, i.e., a machine which takes as input N parallel spins, polarized in an unknown direction, and generates some supplementary spins polarized in the opposite direction. We also point out the relation between spin flip and partial transpose.

I. Optimal fidelity for parallel and antiparallel spins.— In the previous section we presented our main problem as a quantum communication problem. We can present it also in a different way, which brings it closer to a well-known problem. Indeed, we can completely dispense with Alice and consider that there is a source which emits pairs of parallel (or antiparallel) spins, and Bob's task is to identify the state as well as possible.

For concreteness, let us define Bob's measure of success as the fidelity,

$$F = \int d\vec{n} \sum_g P(g \mid \vec{n}) \frac{1 + \vec{n}\vec{n}_g}{2}, \qquad (1)$$

where $\vec{n}\vec{n}_g$ is the scalar product in between the true and the guessed directions, the integral is over the different directions $\vec{n}$, and $d\vec{n}$ represents the *a priori* probability that a state associated to the direction $\vec{n}$, i.e., $|\vec{n}, \vec{n}\rangle$ or $|\vec{n}, -\vec{n}\rangle$, respectively, is emitted by the source; $P(g \mid \vec{n})$ is the probability of guessing $\vec{n}_g$ when the true direction is $\vec{n}$. In other words, for each trial Bob gets a score which is a (linear) function of the scalar product between the true and the guessed direction, and the final score is the average of the individual scores.

When the different directions $\vec{n}$ are randomly and uniformly distributed over the unit sphere, an optimal measurement for pairs of parallel spins $\psi = |\vec{n}, \vec{n}\rangle$ has been found by Massar and Popescu [2]. Bob has to measure an operator A whose eigenvectors ϕ_j, $j = 1 \ldots 4$ are

$$|\phi_j\rangle = \frac{\sqrt{3}}{2} |\vec{n}_j, \vec{n}_j\rangle + \frac{1}{2} |\psi^-\rangle, \qquad (2)$$

where $|\psi^-\rangle$ denotes the singlet state and the Bloch vectors $\vec{n}_j$ point to the four vertices of the tetrahedron:

$$\vec{n}_1 = (0, 0, 1) \qquad \vec{n}_2 = \left(\frac{\sqrt{8}}{3}, 0, -\frac{1}{3}\right)$$

$$\vec{n}_3 = \left(\frac{-\sqrt{2}}{3}, \sqrt{\frac{2}{3}}, -\frac{1}{3}\right) \qquad (3)$$

$$\vec{n}_4 = \left(\frac{-\sqrt{2}}{3}, -\sqrt{\frac{2}{3}}, -\frac{1}{3}\right).$$

(The phases used in the definition of $|\vec{n}_j\rangle$ are such that the four states ϕ_j are mutually orthogonal.) The exact values of the eigenvalues corresponding to the above eigenvectors are irrelevant; all that is important is that they are different from each other, so that each eigenvector can be unambiguously associated to a different outcome of the measurement. If the measurement results correspond to ϕ_j, then the guessed direction is $\vec{n}_j$. The corresponding optimal fidelity is $3/4$ [2].

A related case is when the directions $\vec{n}$ are *a priori* on the vertices of the tetrahedron, with equal probability $1/4$. Then the above measurement provides a fidelity of $5/6 \approx 0.833$, conjectured to be optimal.

Let us now consider pairs of antiparallel spins, $|\psi\rangle = |\vec{n}, -\vec{n}\rangle$, and the measurement whose eigenstates are

$$\theta_j = \alpha |\vec{n}_j, -\vec{n}_j\rangle - \beta \sum_{k \neq j} |\vec{n}_k, -\vec{n}_k\rangle \qquad (4)$$

with $\alpha = 13/(6\sqrt{6} - 2\sqrt{2}) \approx 1.095$ and $\beta = (5 - 2\sqrt{3})/(6\sqrt{6} - 2\sqrt{2}) \approx 0.129$. The corresponding fidelity for uniformly distributed $\vec{n}$ is $F = (5\sqrt{3} + 33)/[3(3\sqrt{3} - 1)^2] \approx 0.789$, and for $\vec{n}$ lying on the tetrahedron $F = (2\sqrt{3} + 47)/[3(3\sqrt{3} - 1)^2] \approx 0.955$. In both cases the fidelity obtained for pairs of antiparallel spins is larger than for pairs of parallel spins.

II. Spin flips.—As we have seen in the previous section, parallel and antiparallel spins are not equivalent. Let us try to understand why.

That there could be any difference between communicating a direction by two parallel spins or two antiparallel spins seems, at first sight, extremely surprising. After all, by simply flipping one of the spins we could change one case into the other. For example, if Bob knows that Alice indicates the direction by two antiparallel spins he has only to flip the second spin and then apply all the measurements as in the case in which Alice sends from the beginning parallel spins. Thus, apparently, the two methods are bound to be equally good.

The problem is that one cannot flip a spin of unknown polarization. Indeed, it is easy to see that the flip operator V defined as

$$V|\vec{n}\rangle = |-\vec{n}\rangle \qquad (5)$$

is not unitary but antiunitary. Thus there is no physical operation which could implement such a transformation.

But then a couple of question arise. First, why is it still the case that a single spin polarized along $\vec{n}$ defines the direction as well as a single spin polarized in the opposite direction?

What happens is that although Bob cannot implement an *active* transformation, i.e., cannot flip the spin, he can implement a *passive* transformation, that is, he can flip his measuring devices. Indeed, there is no problem for Bob in flipping all his Stern-Gerlach apparatuses, or, even simpler than that, to merely rename the outputs of each Stern-Gerlach "up" → down and "down" → up.

But given the above, why can not Bob solve the problem of two spins in the same way, just by performing a passive transformation on the apparatuses used to measure the second spin? The problem is the entanglement. Indeed, if the optimal strategy for finding the polarization direction would involve separate measurements on the two spins, then two parallel spins would be equivalent to two antiparallel spins. (This would be true even if which measurement is to be performed on the second spin depends on the result of the measurement on the first spin.) But, as shown in the previous section, the optimal measurement is *not* a measurement performed separately on the two spins but a measurement that acts on both spins simultaneously, that is, the measurement of an operator whose eigenstates are entangled states of the two spins. For such a measurement there is no way of associating different parts of the measuring device with the different spins, and thus there is no way to make a passive flip associated to the second spin. Consequently there is no way, neither active nor passive to implement an equivalence between the parallel and antiparallel spin cases.

This result illustrates once more that entanglement can produce results "classically impossible," similar to Bell inequality and to nonlocality without entanglement [2,3].

III. Spin flips and the partial transpose of bipartite density matrices.—We have claimed in the previous section that when we perform a measurement of an operator whose eigenstates are entangled states of the two spins, there is no way of making a passive flip associated with the second spin. We comment in more detail about this point.

Physically it is clear that in the case of a measuring device corresponding to an operator whose eigenstates are entangled states of the two spins, we cannot identify one part of the apparatus as acting solely on one spin and another part of the apparatus as acting on the second spin. Thus we cannot simply isolate a part of the measuring device and rename its outcome. But perhaps one could make such a passive transformation at a *mathematical* level, that is, in the mathematical description of the operator associated to the measurement and then physically construct an apparatus which corresponds to the new operator.

The optimal measurement on two parallel spins is described by a nondegenerate operator whose eigenstates $|\phi_j\rangle$ are given by (2) and (3). It is convenient to consider the projectors $P^j = |\phi_j\rangle\langle\phi_j|$ associated with the eigenstates. As is well known, any unit-trace Hermitian operator, and, in particular, any projectors, can be written as

$$P^j = \frac{1}{4}(I + \vec{\alpha}^j\vec{\sigma}^{(1)} + \vec{\beta}^j\vec{\sigma}^{(2)} + R^j_{k,l}\sigma^{(1)}_k\sigma^{(2)}_l) \quad (6)$$

with some appropriate coefficients $\vec{\alpha}^j$, $\vec{\beta}^j$, and $R^j_{k,l}$. (The upper indexes on the spin operators mean "particle 1" or "2.") Why then could we not simply make the passive spin flip by considering a measurement described by the projectors

$$\tilde{P}^j = \frac{1}{4}(I + \vec{\alpha}^j\vec{\sigma}^{(1)} - \vec{\beta}^j\vec{\sigma}^{(2)} - R^j_{k,l}\sigma^{(1)}_k\sigma^{(2)}_l) \quad (7)$$

obtained by the flip of the operators associated second spin, $\vec{\sigma}^{(2)} \rightarrow -\vec{\sigma}^{(2)}$? The reason is that the transformed operators $\tilde{P}^j$ are no longer projectors. Indeed, each projection operator P^j could also be viewed as a density matrix $\rho^j = P^j = |\phi_j\rangle\langle\phi_j|$. The passive spin flip (6) → (7) is nothing more that the partial transpose of the density matrices ρ^j with respect to the second spin. But each density matrix ρ^j is *nonseparable* (because they describe the entangled state $|\Phi\rangle$). But according to the well-known result of the Horodeckis [4,5] the partial transpose of a nonseparable density matrix of two spin $1/2$ particles has a negative eigenvalue and thus it cannot represent a projector anymore. Obviously, however, if the optimal measurement would have consisted of independent measurements on the two spins, each projector would have been a direct product density matrix and the spin flip would have transformed them into new projectors and thus led to a valid new measurement.

IV. Spin flips, entropy, and the global structure of the set of states.—There is yet another surprise in the fact that antiparallel spins can be better distinguished than parallel ones. Consider the two sets of states, that of parallel spins and that of antiparallel spins. The distance in between any two states in the first set is equal to the distance in between the corresponding pair of states in the second set. That is,

$$|\langle\vec{n},\vec{n}\,|\,\vec{m},\vec{m}\rangle|^2 = |\langle\vec{n},-\vec{n}\,|\,\vec{m},-\vec{m}\rangle|^2. \quad (8)$$

Nevertheless, *as a whole,* the antiparallel spin states are farer apart than the parallel ones. Indeed, the antiparallel spin states span the entire 4-dimensional Hilbert space of the two spin $\frac{1}{2}$, while the parallel spin states span only the 3-dimensional subspace of symmetric states. This is similar to a three spin example discovered by Jozsa and Schlienz [6].

Furthermore, the ensembles of parallel and of antiparallel spins in random directions have quantum entropies of $\log(3)$ and $\log(12)$, respectively; that is, the ensemble of antiparallel pairs has a larger entropy. One should note, however, that in general there is no relation between the entropy of an ensemble and the fidelity with which the states can be recognized. Indeed a larger entropy does not necessarily imply higher fidelity, nor does higher fidelity imply larger entropy.

V. The universal quantum spin-flip and anticloning machines.—As we have already noted in Sec. II, a perfect universal quantum spin-flip machine, i.e., a machine that would reverse any spin $\frac{1}{2}$ state $|\vec{n}\rangle \rightarrow |-\vec{n}\rangle$ is impossible—it would require an antiunitary transformation. However, following the lesson of the cloning machine [7], let us ask how well one could approximate such a machine.

Analog to an optimal universal cloning machine, let us define an optimal *universal quantum spin-flip machine.* By definition, a UQSF is a machine which acting on a spin $1/2$ particle implements the transformation

$$|\vec{n}\rangle \rightarrow \rho(\vec{n}) \quad (9)$$

such that $\rho(\vec{n})$ is as close as possible to $|-\vec{n}\rangle$. For concreteness, we define "as close as possible" to mean "according to the usual fidelity" $F = \int d\vec{n}\langle-\vec{n}|\rho(\vec{n})|-\vec{n}\rangle$. Furthermore, to be "universal" we require that the fidelity is independent of the initial polarization of the spin, that is, that all states are flipped equally well. [Obviously, in order to be able to implement the transformation (9) which is nonunitary, the UQSF machine is allowed to entangle the spin with an ancilla.]

Following the technique of [8], developed for optimal eavesdropping in the six state protocol of quantum cryptography, one finds that the fidelity of the optimal quantum spin-flip machine is of $2/3$ (which appears as the maximal disturbance Eve can introduce in the quantum channel).

One simple way to implement this optimal spin flip consists in first measuring the spin in an arbitrary direction, then producing a spin pointing in the direction opposite to the measurement result.

Surprisingly enough, although the original goal was to flip a single spin (the input spin) the optimal UQSF machine can produce additional flipped spins at no extra cost. This follows from the fact that the optimal UQSF provides *classical* information which then can be used to prepare as many flipped spins as we want. This result is surprising because one is tempted to imagine that if we want only to flip a single spin we could do it with much better fidelity if we do not attempt to extract classical information from it. At least this is the lesson of many other quantum information processing procedures, such as cloning, teleportation, data compression, quantum computation, etc. In all these cases quantum information can be processed with much better results if we keep it all the time in quantum form rather than extracting some classical information from it and processing this classical information. The deep reason why spin flipping is essentially a classical operation is an interesting but yet open question.

One can also consider other interesting machines, for example, machines that take as input two parallel spins $|\vec{n}, \vec{n}\rangle$ and the output is as close as possible either to $|\vec{n}, -\vec{n}\rangle$, or to $|-\vec{n}\rangle$, or to $|\vec{n}, \vec{n}, -\vec{n}\rangle$. Another interesting open question is which operation can be produced with higher fidelity: from two parallel spins to antiparallel ones, or vice versa.

VI. Spin flips and quantum optics.—Entanglement is closely connected to the mathematics of partial transpose $\rho_{ij,kl}^{T} = \rho_{il,kj}$: a 2-spin $\frac{1}{2}$ (mixed) state is separable if and only if its partial transpose has non-negative eigenvalues [4,5]. Interestingly, partial transposes can be seen as a representation of spin flips. Indeed, the partial transpose of a product operator reads

$$
\begin{aligned}
(a_0 + \vec{a}\vec{\sigma}) \otimes (b_0 + \vec{b}\vec{\sigma}) \;\rightarrow\; & (a_0 + \vec{a}\vec{\sigma}) \\
& \otimes (b_0 + \vec{b}\vec{\sigma} - 2b_z\sigma_z)
\end{aligned}
\tag{10}
$$

(where the σ_k are the usual Pauli matrices); hence, a partial transpose is a reflection of the second spin through the x-z plane (the plane depends on the basis, as partial transpose is basis dependent). Note that this is a practical way of representing the polarization of a photon reflected by a mirror: the upper and lower hemispheres of the Poincaré sphere are exchanged, corresponding to the change of right handed and left handed elliptic polarization states. To complete the connection with spin flips, add after the reflection a π rotation around the axes orthogonal to the reflection plane (like the Faraday rotator in Faraday mirrors [9]), this flips the second spin. Now, the proof that perfect UQSF machines do not exist can be reformulated: a perfect UQSF machine would turn entangled states into states with negative eigenvalues. Physically, the use of a mirror acting only on the second photon is of course still possible, but one must note that mirrors change right handed reference frames into left handed ones. This is acceptable as long as one can describe the two photons separately, but leads to erroneous predictions if applied to entangled photons.

VII. Conclusion.—We have proved that there is more information about a random space direction $\vec{n}$ in a pair of antiparallel spins $|\vec{n}, -\vec{n}\rangle$ than in a pair of parallel spins $|\vec{n}, \vec{n}\rangle$. This demonstrates again the role played by entanglement in quantum information processing: not a source of paradoxes, but a means of performing tasks which are impossible classically. It also draws attention to the global structure of the state space of combined systems. Related questions concern the optimal quantum spin-flip machine and the optimal quantum machine that turns parallel to antiparallel pairs of spins.

[1] See, e.g., Phys. World **11**, No. 3 (1998).
[2] S. Massar and S. Popescu, Phys. Rev. Lett. **74**, 1259 (1995).
[3] C. H. Bennett, D. P. DiVincenzo, C. A. Fuchs, T. Mor, E. Rains, P. W. Shor, J. A. Smolin, and W. K. Wootters, quant-ph/9804053.
[4] A. Peres, Phys. Rev. Lett. **76**, 1413 (1996).
[5] M. Horodecki, R. Horodecki, and P. Horodecki, Phys. Lett. A **223**, 1 (1996).
[6] R. Jozsa and J. Schlienz (private communication).
[7] V. Bužek and M. Hillery, Phys. Rev. A **54**, 1844 (1996).
[8] H. Bechmann and N. Gisin, quant-ph/9807041.
[9] M. Martinelli, J. Mod. Opt. **39**, 451 (1992).
[10] V. Buzek, M. Hillery, and R. Werner, quant-ph/9901053.