Exact and asymptotic measures of multipartite pure-state entanglement

Charles H. Bennett,[1,*] Sandu Popescu,[2,†] Daniel Rohrlich,[3,‡] John A. Smolin,[1,§] and Ashish V. Thapliyal[4,‖]
[1]*IBM Research Division, Yorktown Heights, New York 10598*
[2]*Isaac Newton Institute, Cambridge University, Cambridge, United Kingdom and BRIMS, Hewlett-Packard Laboratories, Stoke Gifford, Bristol BS12 6QZ, United Kingdom*
[3]*School of Physics and Astronomy, Tel Aviv University, Tel Aviv, Israel*
[4]*Department of Physics, University of California, Santa Barbara, California 93106*

(Received 28 December 1999; published 12 December 2000)

Hoping to simplify the classification of pure entangled states of multi (m)-partite quantum systems, we study exactly and asymptotically (in n) reversible transformations among nth tensor powers of such states (i.e., n copies of the state shared among the same m parties) under local quantum operations and classical communication (LOCC). For exact transformations, we show that two states whose marginal one-party entropies agree are either locally unitarily equivalent or else LOCC incomparable. In particular we show that two tripartite Greenberger-Horne-Zeilinger states are LOCC incomparable to three bipartite Einstein-Podolsky-Rosen (EPR) states symmetrically shared among the three parties. Asymptotic transformations yield a simpler classification than exact transformations; for example, they allow all pure bipartite states to be characterized by a single parameter—their partial entropy—which may be interpreted as the number of EPR pairs asymptotically interconvertible to the state in question by LOCC transformations. We show that m-partite pure states having an m-way Schmidt decomposition are similarly parametrizable, with the partial entropy across any nontrivial partition representing the number of standard quantum superposition or ''cat'' states $|0^{\otimes m}\rangle+|1^{\otimes m}\rangle$ asymptotically interconvertible to the state in question. For general m-partite states, partial entropies across different partitions need not be equal, and since partial entropies are conserved by asymptotically reversible LOCC operations, a multicomponent entanglement measure is needed, with each scalar component representing a different kind of entanglement, not asymptotically interconvertible to the other kinds. In particular we show that the $m=4$ cat state is not isentropic to, and therefore not asymptotically interconvertible to, any combination of bipartite and tripartite states shared among the four parties. Thus, although the $m=4$ cat state can be prepared from bipartite EPR states, the preparation process is necessarily irreversible, and remains so even asymptotically. For each number of parties m we define a minimal reversible entanglement generating set (MREGS) as a set of states of minimal cardinality sufficient to generate all m-partite pure states by asymptotically reversible LOCC transformations. Partial entropy arguments provide lower bounds on the size of the MREGS, but for $m>2$ we know no upper bounds. We briefly consider several generalizations of LOCC transformations, including transformations with some probability of failure, transformations with the catalytic assistance of states other than the states we are trying to transform, and asymptotic LOCC transformations supplemented by a negligible $[o(n)]$ amount of quantum communication.

DOI: 10.1103/PhysRevA.63.012307 PACS number(s): 03.67.Hk

I. INTRODUCTION

Entanglement, first noted by Einstein, Podolsky, and Rosen [1] and Schrödinger [2], is an essential feature of quantum mechanics. Entangled two-particle states, by their experimentally verified violations of Bell inequalities, have played an important role in establishing widespread confidence in the correctness of quantum mechanics. Three-particle entangled states, though more difficult to produce experimentally, provide even stronger tests of quantum nonlocality.

The canonical two-particle entangled state is the Einstein-Podolsky-Rosen-Bohm pair

$$|00\rangle+|11\rangle. \tag{1}$$

(We omit normalization factors when it will cause no confusion.) The canonical tripartite entangled state is the Greenberger-Horne-Zeilinger-Mermin state

$$|000\rangle+|111\rangle, \tag{2}$$

while the corresponding m-partite state

$$|0^{\otimes m}\rangle+|1^{\otimes m}\rangle \tag{3}$$

is called an m-particle cat (m-cat) state, in honor of Schrödinger's cat.

More recently it has been realized that entanglement is a useful resource for various kinds of quantum-information processing, including quantum-state teleportation [3], cryptographic key distribution [4], classical communication over quantum channels [5–7], quantum error correction [8], quantum computational speedups [9], and distributed computation

*Email address: bennetc@watson.ibm.com
†Email address: sp230@newton.cam.ac.uk
‡Email address: rohrlich@post.tau.ac.il
§Email address: smolin@watson.ibm.com
‖Email address: ash@physics.ucsb.edu

[10,11]. In view of its central role [12] in quantum information theory, it is important to have a qualitative and quantitative theory of entanglement.

Entanglement only has meaning in the context of a multipartite quantum system, whose Hilbert space can be viewed as a product of two or more tensor factors corresponding physically to subsystems of the system. We often think of subsystems as belonging to different observers, e.g. Alice has subsystem A, Bob has subsystem B, and so on.

Mathematically, an *unentangled* or *separable* state is a mixture of product states; operationally it is a state that can be made from a pure product state by local operations and classical communication (LOCC). Here local operations include unitary transformations, additions of ancillas (i.e., enlarging the Hilbert space), measurements, and throwing away parts of the system, each performed by one party on his or her subsystem. Mathematically, we represent LOCC by a multilocal superoperator, i.e. a completely positive linear map that does not increase the trace, and can be implemented locally with classical coordination among the parties.[1] Classical communication between parties allows local actions by one party to be conditioned on the outcomes of earlier measurements performed by other parties. This allows, among other things, the creation of mixed states that are classically correlated but not entangled.

Mathematically speaking, a pure state $|\Psi^{ABC\cdots}\rangle$ is separable if and only if it can be expressed as a tensor product of states belonging to different parties:

$$|\Psi^{ABC\cdots}\rangle=|\alpha^A\rangle\otimes|\beta^B\rangle\otimes|\gamma^C\rangle\otimes\cdots. \quad (4)$$

A mixed state $\rho^{ABC\cdots}$ is separable if and only if it can be expressed as a mixture of separable pure states:

$$\rho^{ABC}=\sum_i p_i|\alpha_i^A\rangle\langle\alpha_i^A|\otimes|\beta_i^B\rangle\langle\beta_i^B|\otimes|\gamma_i^C\rangle\langle\gamma_i^C|\otimes\cdots. \quad (5)$$

where the probabilities $p_i\geqslant 0$ and $\Sigma_i p_i=1$. States that are not separable are said to be *entangled* or *inseparable*.

Besides the gross distinction between entangled and unentangled states, various inequivalent *kinds* of entanglement can be distinguished, in recognition of the fact that not all entangled states can be interconverted by local operations and classical communication. For example, bipartite entangled states are further subdivided into *distillable* and *bound* entangled states, the former being states which are pure or from which some pure entanglement can be produced by LOCC, while the latter are mixed states which, though inseparable, have zero distillable entanglement.

Within a class of states having the same kind of entanglement (e.g., bipartite pure states) one can seek a scalar measure of entanglement. Five natural desiderata for such a measure (cf. Refs. [17–21]) are the following

(i) It should be zero for separable states.

(ii) It should be invariant under local unitary transformations.

(iii) Its expectation should not increase under LOCC.

(iv) It should be additive for tensor products of independent states, shared among the same set of observers [thus if Ψ^{AB} and Φ^{AB} are are bipartite states shared between Alice and Bob, and E is an entanglement measure, $E(\Psi^{AB}\otimes\Phi^{AB})$ should equal $E(\Psi^{AB})+E(\Phi^{AB})$].

(v) It should be stable [22] with respect to transfer of a subsystem from one party to another, so that in any tripartite state Ψ^{ABC}, the bipartite entanglement of AB with C should differ from that of A with BC by at most the entropy of subsystem B.

For bipartite pure states it has been shown [17,19,21] that asymptotically there is only one kind of entanglement and partial entropy is a good entanglement measure (E) for it. It is equal, both to the state's *entanglement of formation* [the number of Einstein-Podolsky-Rosen (EPR) pairs asymptotically required to prepare the state by LOCC], and the state's *distillable entanglement* (the number of EPR pairs asymptotically preparable from the state by LOCC). Here partial entropy is the von Neumann entropy $S(\rho)=\mathrm{tr}(\rho\ln_2\rho)$ of the reduced density matrix obtained by tracing out either of the two parties.

In Sec. II, we define exact and asymptotic reducibilities and equivalences under LOCC alone, and with the help of "catalysis," or asymptotically negligible amounts of quantum communication. In Sec. III we use these concepts to develop a framework for quantifying tripartite and multipartite pure-state entanglement, in terms of a canonical set of states which we call a minimal reversible entanglement generating set (MREGS). This framework leads to an additive, multicomponent entanglement measure, based on asymptotically reversible LOCC transformations among tensor powers of such states, and having a number of scalar components equal to the number of states in the MREGS, in other words the number of asymptotically inequivalent kinds of entanglement.

For general m-partite states, partial entropy arguments give lower bounds on the number of entanglement components as a function of m, and allow us to show that some states, e.g., the $m=4$ cat state, are not exactly, nor even asymptotically, interconvertible into any combination of EPR pairs shared among the parties.

On the other hand, we show that the subclass of multipartite pure states having an m-way Schmidt decomposition is describable by a single parameter, its partial entropy representing the number of standard cat states $|0^{\otimes m}\rangle+|1^{\otimes m}\rangle$ asymptotically interconvertible to the state in question. Section

[1]General quantum dynamics can be represented mathematically by completely positive linear maps that do not increase trace [13,14]. Such a map say $\mathcal{L}$ can be written as $\mathcal{L}(\rho)=\Sigma_i L_i\rho L_i^\dagger$, where $\Sigma_i L_i^\dagger L_i\leqslant 11$. The equality holds for trace-preserving superoperators which correspond physically to nonselective dynamics, e.g., a measurement followed by forgetting which outcome was produced. In general the superoperators may be trace decreasing and correspond to selective operations, e.g., a measurement followed by throwing away some outcomes. If $\mathcal{L}$ is a multilocally implementable superoperator, it must be a separable superoperator, i.e., a completely positive trace-preserving map of the form shown above, where the L_i's are products of local operators—$L_i=L_i^A\otimes L_i^B\ldots$. Note that not all separable superoperators are multilocally implementable [15,16].

III D treats tripartite pure-state entanglement, showing in particular that, using exact LOCC transformations, two Greenberger-Horne-Zeilinger (GHZ) states can neither be prepared from nor used to prepare the isentropic combination of three EPR pairs shared symmetrically among the three parties.

II. REDUCIBILITIES, EQUIVALENCES, AND LOCAL ENTROPIES

Reducibility formalizes the notion of a transformation of one state to another being possible under certain conditions, while equivalence formalizes the notion of this transformation being reversible—possible in both directions. While studying entanglement it is useful to discuss state transformation under LOCC. This is because a good entanglement measure should not increase under LOCC. So, if two states are equivalent under LOCC operations, they will have the same entanglement. This is the key idea we will use in Sec. III to quantify entanglement.

We start by first looking at partial entropies. Partial entropies have the nice property that for pure states their average does not increase under LOCC

Suppose the m parties holding a pure state Ψ are numbered $1,2,\ldots,m$. Let X denote a nontrivial subset of the parties and let $\overline{X}$ be the set of remaining parties. Then the reduced density matrix of subset X of the parties is defined as

$$\rho_x(\Psi)=\mathrm{tr}_{\tilde{X}}(|\Psi\rangle\langle\Psi|). \tag{6}$$

The *partial entropy* of subset X is the von Neumann entropy

$$S_X(\Psi)=-\mathrm{tr}(\rho_x(\Psi)\ln_2\rho_x(\Psi)). \tag{7}$$

When $X=\{l\}$ consists of a single party, $\rho_{\{l\}}$ is called the *marginal density matrix* of party l, and $S(\rho_{\{l\}})$ is the *marginal entropy* of party l. Two states are said to be *isentropic* if for each subset X of the parties $S_X(\Psi)=S_X(\Phi)$. Two states Ψ and Φ are said to be *marginally isentropic* if $S_{\{l\}}(\Psi)=S_{\{l\}}(\Phi)$ for each party l.

Now we are ready to show that for any subset X of parties, the partial entropy S_X is nonincreasing under LOCC. We state this as a lemma.

Lemma 1: If a multipartite system is initially in a pure state Ψ, and is subjected to a sequence of LOCC operations resulting in a set of final pure states Φ_i with probabilities p_i, then for any subset X of the parties

$$S_X(\Psi)\geqslant\sum_i p_i S_X(\Phi_i). \tag{8}$$

Proof: The result follows from the fact that average bipartite entanglement (partial entropy) of bipartite pure states cannot increase under LOCC, cf. Ref. [23].

A. Reducibilities and equivalences: Exact and stochastic

We start with a LOCC state transformation involving single copies of states. If the state transformation is exact, we say it is an exact reducibility. If the state transformation succeeds some of the time we say it is stochastic, and if the state transformation needs the presence of another state, which is is recovered after the protocol, it is called catalytic reducibility. In this section we define these more precisely. We start with exact reducibility.

We say a state Φ is *exactly reducible* to a state Ψ (written $\Phi\leqslant_{\mathrm{LOCC}}\Psi$ or just $\Phi\leqslant\Psi$) by local operations and classical communication if and only if

$$\Phi=\mathcal{L}(\Psi),\quad \exists\ \mathcal{L} \tag{9}$$

where $\mathcal{L}$ is a multilocally implementable trace preserving superoperator. Alternatively we may say that the LOCC protocol $\mathcal{P}_{\mathcal{L}}$ corresponding to the superoperator $\mathcal{L}$ transforms Ψ to Φ exactly.

Intuitively this means that the state transformation from Ψ to Φ can be done by LOCC with probability 1. (Where it will cause no confusion, for pure states we use a plain Greek letter such as Ψ to represent both the vector $|\Psi\rangle$ and the projector $|\Psi\rangle\langle\Psi|$.)

The relation of exact LOCC reducibility for bipartite pure states has been studied in Refs. [24] and [25], which give necessary and sufficient conditions for it in terms of majorization of the eigenvalues of the reduced density matrix. Nielsen [25] used notation reminiscent of a chemical reaction: where we say $\Phi\leqslant\Psi$, he says $\Psi\rightarrow\Phi$. Both notations mean that, given one copy of Ψ, we can with certainty, by local operations and classical communication, make one copy of Φ.

Chemical reactions often involve catalysts, molecules which facilitate a reaction without being used up, so it is natural to look for analogous quantum-state transformations. Jonathan and Plenio recently found an example of successful catalysis for bipartite states, where a catalyst allows a transformation to be performed with a certainty which could only be done with some chance of failure in the absence of the catalyst [26].

We say that Φ is *catalytically reducible* ($\leqslant_{\mathrm{LOCC}c}$) to Ψ if and only if there exists a state Υ such that

$$\Phi\otimes\Upsilon\leqslant_{\mathrm{LOCC}}\Psi\otimes\Upsilon. \tag{10}$$

An interesting fact about catalysis is that, because the catalyst is not consumed, one copy of it is sufficient to transform arbitrarily many copies of Ψ into Φ:

$$\Phi\Upsilon\leqslant_{\mathrm{LOCC}}\Psi\Upsilon\Rightarrow\Phi^n\Upsilon\leqslant_{\mathrm{LOCC}}\Psi^n\Upsilon,\quad\forall\ n. \tag{11}$$

Another important form of state transformation involves probabilistic outcomes, where the procedure for the reducibility may fail some of the time as in ''entanglement gambling'' [17]. We capture this idea in stochastic reducibility.

We say a state Φ is *stochastically reducible* to a state Ψ under LOCC with yield p if and only if

$$\Phi=\frac{\mathcal{L}(\Psi)}{\mathrm{tr}[\mathcal{L}(\Psi)]},\quad\exists\ \mathcal{L} \tag{12}$$

where $\mathcal{L}$ is a multilocally implementable superoperator such that $\mathrm{tr}[\mathcal{L}(\Psi)]=p$.

This means that a copy of Φ may be obtained from a copy of Ψ with probability p by LOCC operations. Exact reducibility corresponds to the case $p=1$. For any reducibility, one may define corresponding notions of equivalence and incomparability.

Two states Φ and Ψ are said to be *exactly equivalent* ($\equiv_{\rm LOCC}$ or simply $\equiv$) if $\Phi \leqslant \Psi$ and $\Psi \leqslant \Phi$. This means that the two states are exactly interconvertible by classically coordinated local operations. In chemical notation this would be $\Psi \rightleftharpoons \Phi$. Conversely, states Φ and Ψ are said to be *exactly incomparable* if neither is exactly reducible to the other. Catalytic and stochastic equivalence and incomparability may be defined analogously.[2]

In passing we note that many other reducibilities (and their corresponding equivalences) can be considered, e.g., reducibilities via local unitary (LU) operations [27] $\leqslant_{\rm LU}$, stochastic reducibility with catalysis, and reducibilities without communication or with one-way communication [28].

Physically, reducibility via local unitary operations and that via local unitary operations along with a change in the local support (corresponding to the increase or decrease in the local Hilbert space dimensions) are the same because we could think of the extra dimensions as being present from the start and extend the local unitary operation to the larger space. Thus, from now on, when we say local unitary operations we mean local unitary operations along with a possible change in the local support, i.e., isometric transformations.[3]

We now look at some conditions for two states to be exactly equivalent. From lemma 1 it is clear that if two states are equivalent they must be isentropic, but not all isentropic states are equivalent. We are now in a position to demonstrate some important facts about exact LOCC reducibility.[4]

Theorem 1: If Ψ and Φ are two marginally isentropic pure states, then they are either locally unitarily (LU) equivalent or else LOCC incomparable.

Corollary 1: Two states are LOCC equivalent if and only if they are LU equivalent:

$$\Psi \equiv_{\rm LOCC} \Phi \Leftrightarrow \Psi \equiv_{\rm LU} \Phi, \quad \forall \quad \Psi, \Phi \tag{13}$$

Corollary 2: States that are marginally but not fully isentropic are necessarily LOCC incomparable.

Proof: To prove this it suffices to show that for marginally isentropic states Ψ and Φ, if $\Phi \leqslant \Psi$ then they must be local unitarily equivalent. In light of the nonincrease of partial entropy under LOCC (cf. lemma 1) and the fact that these two states are marginally isentropic, a LOCC protocol that converts one state to the other must conserve the marginal entropies at each step. Suppose the LOCC protocol $\mathcal{P}$ transforms Ψ to Φ exactly. In general such a protocol consists of a sequence of local transformations each done by one party followed by communication of (some of) the information gained to other parties. Without loss of generality assume that Alice performs the first operation of such a protocol converting Ψ to Φ, which gives the resulting ensemble $\mathcal{E} = \{p_i, \psi_i\}$. Since Alice's operation cannot change the density matrix $\rho^{BC\cdots}$ "seen" by the remaining parties,

$$\rho^{BC\cdots} = \sum_i p_i {\rm tr}_A(|\psi_i\rangle\langle\psi_i|). \tag{14}$$

As argued earlier, the average entropy must not change, i.e.,

$$S_{BC\ldots}(\Psi) = S_A(\Psi) = \sum_i p_i S_{BC\ldots}(\psi_i). \tag{15}$$

By the strict concavity of the von Neumann entropy [29] each of the resultant states ψ_i must have the same reduced density matrix, from the viewpoint of all the other parties besides Alice, as the original state Ψ did:

$$ {\rm tr}_A(|\psi_i\rangle\langle\psi_i|) = {\rm tr}_A(|\Psi\rangle\langle\Psi|), \quad \forall \quad i \tag{16}$$

Therefore, the states ψ_i must be related by isometries acting on Alice's Hilbert space alone:

$$|\psi_i\rangle = U_i^A \otimes I^{BC\cdots}|\Psi\rangle, \tag{17}$$

where U_i^A are unitary transformations acting on Alice's Hilbert space, which may have more dimensions than the support of $|\Psi\rangle$ in Alice's space (this would correspond to Alice having unilaterally chosen to enlarge her Hilbert space, which she is always free to do). Thus Alice's measurement process, which appears on its face to be a stochastic process not entirely under her control, could in fact be faithfully simulated by having her simply toss a coin to choose a "measurement result" i with probability p_i, then perform the deterministic operation U_i on her portion of the joint state, and then finally report the result i to all the other parties. In the next step of the protocol, another party performs similar operations and sends classical information as to which unitary it performed and so on for each step. Thus the entire protocol consists of local unitary transformations, enlargement of Hilbert space and classical communication, maintaining at each step the overall state to be pure. The protocol ends when the state Φ has been obtained. Since this is an exact reducibility of one pure state to another, for each possible sequence of local unitaries, the result must be Φ. Thus we can define a new protocol $\mathcal{P}'$ that consists of choosing just one such sequence of local unitaries and it will take

[2]Very recently Dürr, Vidal, and Cirac (LANL eprint, quant-ph/0005115) found a tripartite pure state of three qubits which is stochastically incomparable with the GHZ state. They also showed that if two pure states are chosen randomly in the tensor product Hilbert space of four or more parties, then, with probability 1, they are stochastically incomparable: neither state can be produced from the other by LOCC with any chance of success.

[3]Unitary operations are characterized by $U^\dagger U = \mathbb{1} = UU^\dagger$. However, if we are want general transformations that preserve the norm of vectors, all we need is $U^\dagger U = \mathbb{1}$, where the U's could be rectangular matrices. Such U are called isometric [29].

[4]These results strengthen Vidal's result [21] that LU equivalence$\Leftrightarrow$LOCC equivalence for bipartite pure states, and Kempe's result [30] that if two multipartite pure states have isospectral marginal density matrices, then they are either LU equivalent or LOCC incomparable.

Ψ to Φ, showing that the two states are local unitarily equivalent. The first corollary follows from the fact that if two states are LOCC equivalent, they must be isentropic and therefore marginally isentropic. The second follows from the fact if that the two states were LU equivalent, they would be fully isentropic, not merely marginally so.

B. Asymptotic reducibilities and equivalence, and their relation to partial entropies

Before we discuss asymptotic reducibilities and equivalences, let us define a quantitative measure of similarity of two states. One such measure, the *fidelity* [31,32] of a mixed state ρ relative to a pure state ψ, is given by $F(\rho,\psi)=\langle\psi|\rho|\psi\rangle$. It is the probability that ρ will pass a test for being ψ, conducted by an observer who knows the state ψ. For mixed states ρ and σ it is given by the more symmetric expression $F(\rho,\sigma)=[\mathrm{tr}(\sqrt{\sigma}\rho\sqrt{\sigma})^{1/2}]^2$.

Exact reducibility is too weak a reducibility to give a *simple* classification of entanglement—even for bipartite pure states, there are infinitely many incomparable $\leqslant_{\mathrm{LU}}$ equivalence classes, which would lead to infinitely many distinct kinds of bipartite entanglement. Linden and Popescu [27] have explored the orbits of multipartite states under local unitary operations, and shown that the number of LU invariants increases exponentially with the number of parties and with the number of qubits possessed by each party.

One natural way to strengthen the notion of reducibility is to make it asymptotic. We first consider ''asymptotic LOCC reducibility'' [17,28] which expresses the ability to convert n copies of one pure state into a good approximation of n copies of another, in the limit of large n. A possibly stronger reducibility, which we will call ''asymptotic LOCCq reducibility,'' expresses the ability to do the state transformation with the help of a limited $[o(n)]$ amount of quantum communication, in addition to the unlimited classical communication and local operations allowed in ordinary LOCC reducibility. Another natural way of strengthening asymptotic reducibility is to allow catalysis; defining ''catalytic asymptotic LOCC reducibility'' (LOCCc), in direct analogy with the exact case. We show that asymptotic LOCCc reducibility is at least as strong as (i.e., can simulate) LOCCq reducibility.

Ordinary asymptotic LOCC reducibility is enough to simplify the classification of all bipartite pure states and some classes of m-partite states, so that, for any given m, a finite repertoire of standard states (EPR, GHZ, etc.), which we will later call a minimal reversible entanglement generating set or MREGS, can be combined to prepare any member of class in an asymptotically reversible fashion, regardless of the size of the Hilbert spaces of the parties. Whether this classification can be extended to cover general m-partite states for $m>2$ while maintaining a finite repertoire size is an open question. Let us start by defining ordinary asymptotic LOCC reducibility.

State Φ is *asymptotically reducible* ($\leqslant_{\mathrm{LOCC}}$ or simply $\leqslant$) to state Ψ by local operations and classical communication if and only if

$$|(n/n')-1|<\delta,$$
$$F(\mathcal{L}(\Psi^{\otimes n'}),\Phi^{\otimes n})\geqslant 1-\epsilon,\ \ \forall\ \ \delta>0,\ \ \epsilon>0,\ \ \ \exists\ n,n',\mathcal{L}. \tag{18}$$

Here $\mathcal{L}$ is a multilocally implementable superoperator that converts n' copies of Ψ into a high fidelity approximation to n copies of Φ. In chemical notation we can write this as $\Psi\hookrightarrow\Phi$.

A natural extension of asymptotic LOCC reducibility occurs if we allow catalysis. Thus we define asymptotic LOCCc reducibility as follows. We say Φ is *asymptotically LOCCc reducible* ($\leqslant_{\mathrm{LOCC}c}$) to Ψ if there exists some state Υ such that

$$\Phi\Upsilon\leqslant\Psi\Upsilon, \tag{19}$$

where we say the state Υ is a catalyst for this reducibility. As with exact catalysis [Eq. (11)], asymptotic catalysis allows an arbitrarily large ratio of reactant to catalyst:

$$\Phi\Upsilon\leqslant\Psi\Upsilon\Rightarrow\Phi^n\Upsilon\leqslant\Psi^n\Upsilon,\ \ \forall\ \ n. \tag{20}$$

Another way of extending asymptotic LOCC reducibility is to allow a sublinear amount of quantum communication during the transformation process.

State Φ is said to be *asymptotically* LOCCq *reducible* ($\leqslant_{\mathrm{LOCC}q}$) to state Ψ if

$$(k/n)<\delta,$$
$$F(\mathcal{L}(\Gamma^{\otimes k}\otimes\Psi^{\otimes n}),\Phi^{\otimes n})\geqslant 1-\epsilon,\ \ \forall\ \ \delta>0,$$
$$\epsilon>0,\ \ \ \exists\ \ n,k,\mathcal{L}. \tag{21}$$

where Γ denotes the m-cat state $|0^{\otimes m}\rangle+|1^{\otimes m}\rangle$.

The m-cat states used here are a convenient way of allowing a sublinear amount $o(n)$ of quantum communication, since they can be used as described in Sec. III D to generate EPR pairs between any two parties which in turn can be used to teleport quantum data between the parties. The $o(n)$ quantum communication allows the definition to be simpler in one respect: a single tensor power n can be used for the input state Ψ and output state Φ, rather than the separate powers n and n' used in the definition of ordinary asymptotic LOCC reducibility without quantum communication, because any $o(n)$ shortfall in number of copies of the output state can be made up by using cat states to synthesize the extra output states *de novo*. This definition is more natural than that for ordinary asymptotic LOCC reducibility in that the input and output states are allowed to differ in any way that can be repaired by an $o(n)$ expenditure of quantum communication, rather than only in the specific way of being n versus n' copies of the desired state where $n-n'$ is $o(n)$.

Clearly $\leqslant_{\mathrm{LOCC}}$ implies $\leqslant_{\mathrm{LOCC}q}$ and $\leqslant_{\mathrm{LOCC}c}$, because ordinary asymptotic reducibility is a special case of the two other kinds of asymptotic reducibility. We can also show that asymptotic LOCCq reducibility implies asymptotic LOCCc reducibility, because any $\leqslant_{\mathrm{LOCC}q}$ protocol can be simulated by a $\leqslant_{\mathrm{LOCC}c}$ protocol with the m-cat state Γ as a catalyst, only a sublinear (and therefore asymptotically negligible) amount of which is consumed. In more detail, if $\Phi\leqslant_{\mathrm{LOCC}q}\Psi$, then from Eq. (21), for each ϵ and δ, there exist n

and k such that $\Psi^{\otimes n}$ can be converted to a $1-\epsilon$ faithful approximation to $\Phi^{\otimes n}$ with the help of $k<n\delta$ cat states' worth of quantum communication. This implies that n copies of Ψ and k copies of Γ can be converted into a $1-\epsilon$ faithful approximation to n copies of Φ without any quantum communication. By supplying extra $n-k$, nonparticipatory copies of Γ, which are present both before and after the transformation, and discarding k of the copies of Φ which the transformation has produced (even if the copies are entangled, this cannot decrease the fidelity), we obtain that a $1-\epsilon$ faithful approximation to $(\Phi\otimes\Gamma)^{\otimes(n-k)}$ can be prepared from $(\Psi\otimes\Gamma)^{\otimes n}$. This satisfies the conditions [Eq. (18)] for asymptotic reducibility,

$$\Phi\otimes\Gamma\leqslant_{\rm LOCC}\Psi\otimes\Gamma, \tag{22}$$

or, invoking the definition [Eq. (9)] of asymptotic catalytic reducibility,

$$\Phi\leqslant_{{\rm LOCC}c}\Psi, \tag{23}$$

which was to be demonstrated. While the converse (i.e., that asymptotic catalytic reducibility can be simulated by LOCCq transformations) seems plausible, we have not been able to prove it except in special cases.

Asymptotic reducibilities and equivalences can have noninteger yields. This can be expressed using tensor exponents that take on any non-negative real value, so that $\Phi^{\otimes x}\leqslant\Psi^{\otimes y}$ denotes

$$|(n/n')-x/y|<\delta,\quad F(\mathcal{L}(\Psi^{\otimes n'}),\Phi^{\otimes n})\geqslant 1-\epsilon,$$
$$\forall\,\delta>0,\quad \exists\ \ n,n'. \tag{24}$$

In this case we say x/y is the asymptotic efficiency or yield with which Φ can be obtained from Ψ. In chemical notation this could be expressed by $\Psi\hookrightarrow(x/y)\Phi$, keeping in mind that the coefficient x represents an asymptotic yield or number of copies of the state Φ, not a scalar factor multiplying the state vector. Clearly, if a stochastic state transformation with yield p is possible from Ψ to Φ then $\Psi\hookrightarrow p\Phi$ because of the law of large numbers and the central limit theorem.

We are now in a position to define the most important tool in quantifying entanglement, namely, asymptotic equivalence. We say that $\Psi^{\otimes x}$ and $\Phi^{\otimes y}$, with $x,y\geqslant 0$, are *asymptotically equivalent* ($\Psi^{\otimes x}\approx\Phi^{\otimes y}$) if and only if $\Phi^{\otimes y}$ is asymptotically reducible to $\Psi^{\otimes x}$, and vice versa. Two states are said to be *asymptotically incomparable* if neither is asymptotically reducible to the other.

Although we will mainly be concerned with asymptotic equivalence ($\approx$), two possibly stronger reducibilities mentioned earlier—asymptotic LOCC reducibility with a catalyst ($\leqslant_{{\rm LOCC}c}$) and asymptotic LOCC reducibility with a small amount of quantum communication ($\leqslant_{{\rm LOCC}q}$)—give rise to their own corresponding versions of equivalence and incomparability. Since $\leqslant_{{\rm LOCC}c}$ transformations can simulate both $\leqslant_{{\rm LOCC}q}$ and $\leqslant_{\rm LOCC}$, the $\approx_{{\rm LOCC}c}$ reducibility can be expected to give rise to the simplest (coarsest) classification of states into equivalence classes, and the simplest (fewest independent components) entanglement measures for multipartite states. It was very recently shown [40] that even $\leqslant_{{\rm LOCC}c}$ is not coarse enough to connect every isentropic pair of states. [The converse—that asymptotically LOCCc-equivalent states must be isentropic—follows from the nonincrease of pure states' partial entropies under LOCC: if Ψ can be efficiently converted into Φ, even asymptotically and even with the help of a catalyst, then for each subset X of the parties, $S_X(\Phi)$ cannot exceed $S_X(\Psi)$; otherwise an increase of partial entropy could be made to occur in violation of lemma 1.] We collect the relations we have proved in this section from the definitions of the various reducibilities, using Lemma 1 and Theorem 1, and express them as follows.

Theorem 2: The following implications hold among the reducibilities, equivalences, and partial entropies of a pair of multipartite pure states.

For reducibilities and entropy inequalities (omitting mention of the states Ψ and Φ where it will create no confusion), we have

$$\begin{aligned}&(\Phi\equiv_{LU}\Psi)\\&\quad\Rightarrow<_{\rm LOCC}\Rightarrow\leqslant_{\rm LOCC}\Rightarrow\leqslant_{{\rm LOCC}q}\\&\quad\Rightarrow\leqslant_{{\rm LOCC}c}\Rightarrow S_X(\Phi)\leqslant S_X(\Psi),\ \ \forall X.\end{aligned} \tag{25}$$

For equivalences and entropy equalities we have

$$\begin{gathered}(\Phi\equiv_{\rm LU}\Psi)\Leftrightarrow\equiv_{\rm LOCC}\Rightarrow\approx_{\rm LOCC}\Rightarrow\approx_{{\rm LOCC}q}\\ \Rightarrow\approx_{{\rm LOCC}c}\Rightarrow S_X(\Phi)=S_X(\Psi),\\ \forall X,\ (\text{i.e.,}\Phi\ \text{ and }\ \Psi\ \text{ are isentropic})\Rightarrow\\ \Phi\ \text{ and }\Psi\ \text{ are marginally isentropic}\Rightarrow\\ (\Phi\equiv_{\rm LU}\Psi)\ \text{or}\ \Psi\ \text{ and }\ \Psi\ \text{are LOCC incomparable.}\end{gathered} \tag{26}$$

Figure 1 illustrates several of these relations.

C. Bipartite entanglement: a reinterpretation

As an example of the usefulness of these concepts let us reexpress the bipartite pure state entanglement result [17] in terms of asymptotic equivalence. In this new language, any bipartite pure state Ψ^{AB} is asymptotically equivalent to $S_A(\Psi^{AB})$ EPR pairs: this is the number of EPR pairs that, asymptotically, can be obtained from and are required to prepare Ψ^{AB} by classically coordinated local operations.

In proving this result, the concepts of entanglement concentration and dilution [17] are central. The process of asymptotically reducing a given bipartite pure state to EPR singlet form is *entanglement dilution*, and that of reducing EPR singlets to an arbitrary bipartite pure state is *entanglement concentration*. Then the above result means that entanglement concentration and dilution are reversible in the sense of asymptotic equivalence, i.e., they approach unit efficiency and fidelity in the limit of large number of copies n. The crucial requirement for these methods to work is the existence of the Schmidt biorthogonal (normal or polar) form

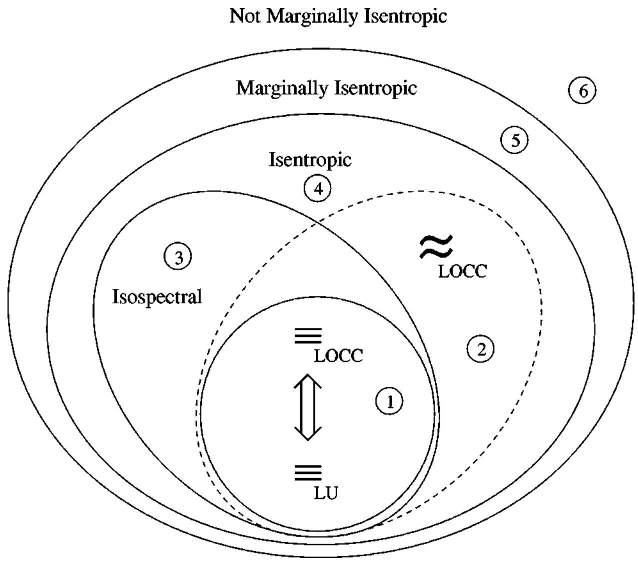

FIG. 1. Relation of exact and asymptotic equivalences to equality of local entropies. Two states are exactly equivalent under local operations and classical communication (LOCC) if and only if they are equivalent under local unitary (LU) operations alone. An example (circled 1) is the LU interconvertibility of the two Bell states $|00\rangle+|11\rangle$ and $|00\rangle-|11\rangle$. An exact equivalence of course implies asymptotic equivalence (dotted region) including (circled 2) the asymptotic equivalence between an EPR pair and an isentropic but not isospectral two-trit state of the form $\alpha|00\rangle+\beta|11\rangle+\gamma|22\rangle$. Asymptotically equivalent states are necessarily isentropic, but not the converse. For example (circled 3), the isentropic—and indeed isospectral—tripartite 2-GHZ and 3-EPR states (see Sec. III D) were very recently shown [40] to be incomparable with respect asymptotic LOCC reducibility. This example also illustrates the fact (cf. Ref. [30]) that isospectral states of three or more parties need not be LU equivalent. A tensor product of circled 2 type states with circled 3 type states yields isentropic states (circled 4) that are neither asymptotically LOCC equivalent nor isospectral. States that are marginally but not fully isentropic (circled 5) must be incomparable with respect to exact LOCC reducibility. Finally, at the periphery (circled 6) are states that are not even marginally isentropic. These include incomparable pairs such as AB-EPR vs BC-EPR, and properly reducible pairs such as GHZ vs EPR, but no cases of exact or even asymptotic equivalence.

for bipartite pure states [37], that is, the fact that any bipartite pure state $|\Psi^{AB}\rangle$ can be written in a biorthogonal form,

$$\Psi_{AB}=\sum_i \lambda_i|i^A\rangle\otimes|i^B\rangle, \qquad (27)$$

where $|i^A\rangle$ and $|i^B\rangle$ form orthonormal bases in Alice's and Bob's Hilbert space, respectively, where by choice of phases of local bases the coefficients λ_i can be made real and nonnegative.

III. TRIPARTITE AND MULTIPARTITE PURE-STATE ENTANGLEMENT

In this section we use the tools we developed earlier to propose a framework for quantifying multipartite pure-state entanglement. Discussions in Sec. II were valid for pure as well as mixed states. However from now on we will restrict out attention to pure states.

In Sec. III A we consider the natural generalization of the bipartite states, namely, the m-party states with an m-way Schmidt decomposition which we call m-orthogonal states. We show that for each m such states can be characterized by a scalar entanglement measure, which may be interpreted as the number of m-cat states asymptotically equivalent to the state in question: a single-parameter case. In Sec. III B we introduce the concepts of entanglement span, entanglement coefficients and minimal entanglement generating sets, as elements of a general framework for quantifying multipartite pure-state entanglement. In Sec. III C we derive lower bounds on the cardinality of MREGS's. In Sec. III D where we study the question of interconversion between m-cat and EPR states. In Sec. III E we show the uniqueness of the entanglement coefficients for natural MREGS possibilities for tripartite states.

A. Schmidt-decomposable or m-orthogonal states

We consider Alice, Bob, Claire, . . . , Matt as m observers who have one subsystem each of an m-part system in a joint m-partite pure state. Some m-partite pure states, but not all, can be written in an *m-orthogonal form* analogous to the Schmidt biorthogonal form. We call such states m-orthogonal or Schmidt decomposable. Thus an m-partite pure state $|\Psi^{ABC\cdots}\rangle$ is *Schmidt decomposable* or *m orthogonal* if and only if it can be written in a form

$$|\Psi^{ABC\ldots M}\rangle=\sum_i \lambda_i|i^A\rangle\otimes|i^B\langle\otimes|i^C\rangle\cdots\otimes|i^M\rangle, \qquad (28)$$

where $|i^A\rangle,|i^B\rangle,|i^C\rangle,\ldots,|i^M\rangle$ are orthonormal bases for the corresponding party. Note that by change of phases of local bases, each of the Schmidt coefficients λ_i can be made real and non-negative. In any m-orthogonal state, the reduced entropy seen by any observer, indeed by any nontrivial subset of observers, is the same, being given by the Shannon entropy of the squares of the Schmidt coefficients. Already this makes it obvious that not all tripartite and higher states are Schmidt decomposable since, for any $m>2$, it is clear that there are pure m-partite states having unequal partial entropies for the different observers. Peres [33] gave necessary and sufficient conditions for a multipartite pure state to be Schmidt decomposable. Thapliyal [34] recently gave another characterization, showing that an m-partite pure state is Schmidt decomposable if and only if each of the $m-1$ partite mixed states obtained by tracing out one party is separable.

For such Schmidt decomposable states, the notions of entanglement concentration and dilution, developed for bipartite states, generalize in a straightforward manner, so that for an m-partite state $\Psi^{ABC\cdots}$ the local entropy, as seen by any party, or indeed any nontrivial subset of the parties, gives the asymptotic number of m-partite cat states into which it can be asymptotically interconverted. That is, if $\Psi^{ABC\ldots M}$ is a Schmidt decomposable multipartite state, then

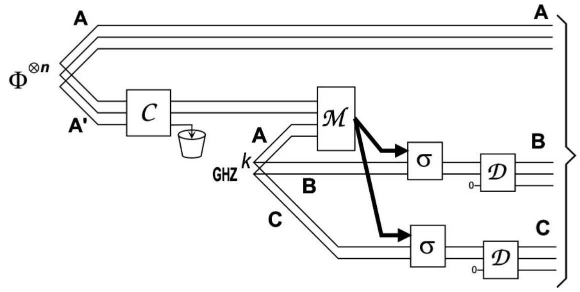

FIG. 2. Entanglement dilution for Schmidt decomposable tripartite states. Alice prepares a local supply of n bipartite states $\Phi^{AA'}$ isospectral to the Schmidt decomposable tripartite state $\Psi^{A,B,C}$ she wishes to share, and Schumacher compresses their A' halves $(\mathcal{C})$ to $k \approx nS(\rho_A)$ qubits. Then, using k previously shared GHZ states, she teleports the compressed qubits to Bob and Charlie simultaneously (Here $\mathcal{M}$ denotes a Bell measurement, the thick lines a $2k$-bit classical message Alice broadcasts to both Bob and Claire, and σ the conditional Pauli rotation which completes the teleportation process). Finally, Bob and Claire Schumacher decompress $(\mathcal{D})$ their k qubits to cover n qubits each, in a state closely approximating n copies of the diluted Schmidt decomposable tripartite state Ψ^{ABC} they wished to share.

$$|\Psi^{ABC\ldots M}\rangle \approx \text{cat}^{ABC\ldots M}\rangle^{\otimes S_A(\Psi^{ABC\ldots M})}. \tag{29}$$

Entanglement concentration on an m-orthogonal state $\Psi^{ABC\ldots M}$, like its bipartite counterpart, can be done by parallel local actions of the observers, without any communication. Starting with a number n of copies of the state to be concentrated, each party makes an incomplete von Neumann measurement, collapsing the system onto a uniform superposition over an eigenspace of one eigenvalue in the product Schmidt basis. After enough such states have been accumulated to span a Hilbert space of dimension slightly more than some power k of 2^m, another measurement suffices, with high probability, to collapse the state onto a maximally entangled m-partite state in a Hilbert space of dimension 2^{mk}, which can then be transformed by local operations into a tensor product of k m-partite cat states.

Entanglement dilution (cf. Fig. 2) proceeds in the same way as for bipartite states, except that Alice locally prepares a supply of bipartite pure states $\Phi^{A,A'}$ having the same Schmidt spectrum as the multipartite Schmidt decomposable state $\Psi^{ABC\ldots M}$ which she wishes to share with the other parties. Here the superscript A,A' signifies that both parts of this state are in Alice's laboratory, whereas her goal is to end up with states shared among all the parties. As in bipartite entanglement dilution, Alice then Schumacher compresses the A' part of a tensor product of n copies of $\Phi^{AA'}$, resulting in approximately k compressed qubits, where k/n asymptotically approaches $S_A(\Phi)=S_A(\Psi)$, the local entropy of the Schmidt decomposable state she wishes to share. She then teleports these k compressed qubits to the other parties—Bob, Claire, etc. The teleportation is performed not with k EPR pairs, as in ordinary teleportation, but with k m-partite cat states, which she has shared beforehand with the other parties. For each of the compressed qubits, Alice performs a Bell measurement on that qubit and one leg of an m-partite cat state, and broadcasts the two-bit classical result to all the other parties, who then each apply the corresponding Pauli rotation to their leg of the shared cat state. Finally all the other parties besides Alice apply Schumacher decompression to their legs of the rotated cat states, leaving the m parties in a high-fidelity approximation to the m-partite state $(\Psi^{ABC\ldots M})^{\otimes n}$ which they wished to share.

This entanglement dilution protocol requires $2k/n$ bits of classical information per copy (of the target state) to be communicated from Alice to the other two parties. Lo and Popescu [39] showed a bipartite entanglement dilution protocol which requires $O(1/\sqrt{n})$ bits of communication per copy; thus, asymptotically, the classical communication cost per copy goes to zero for their protocol. The question then is whether a similar protocol can be found for the dilution of m-cat states into m-orthogonal states. It is easy to see that replacing teleportation through EPR states with teleportation through the m-partite cat states in their protocol gives us a protocol for entanglement dilution of the m-cat states into m-orthogonal states. This protocol again uses only $O(1/\sqrt{n})$ classical communication per copy, an asymptotically vanishing amount.

B. Framework for quantifying entanglement of multipartite pure states

Now we apply concepts of reducibilities and equivalences II in attempting to quantifying entanglement. For general m-partite states, there will be several inequivalent kinds of entanglement under asymptotically reversible LOCC (or LOCCq or LOCCc) transformations—at least as many the number of independently variable partial entropies for such states—and perhaps more. However, a good entanglement measure ought to be defined so as to assign equal entanglement (in the case of a multicomponent measure, equal in all components) to asymptotically equivalent states. This forms the basis of our framework for quantifying entanglement.

We start by looking at the concept of the *entanglement span* of a set of states. Given the set of states $\mathcal{G}=\{\psi_1,\psi_2,\ldots,\psi_k\}$, their entanglement span $[\mathcal{S}(\mathcal{G})]$ is defined as the set of states that they can reversibly generate under asymptotic LOCC. That is,

$$\mathcal{S}(\mathcal{G})=\left\{\Psi|\Psi\approx\bigotimes_{i=1}^{k}|\psi_i\rangle^{\otimes x_i}, \quad \text{with} \quad x_i\geqslant 0\right\}. \tag{30}$$

Note that the x_i give a quantitative amount of entanglement in terms of the spanning states. They are called the *entanglement coefficients*. In general these coefficients may be nonunique, for example, if two states in the set are locally unitarily related. Loosely speaking these coefficients may be nonunique if the ''kinds of entanglement'' they correspond to are not ''independent.''

Let us look at some examples. The entanglement span under LOCC of any bipartite state is the set of all bipartite states. Another example is provided by the set of m-orthogonal states. Any such state in general, and in particular the m-cat state, spans the set of all the m-orthogonal states.

Let us now introduce the concept of reversible entanglement generating sets (REGS's), which is dual to the concept of entanglement span. A set $\mathcal{G}=\{\psi_1,\psi_2,\ldots,\psi_n\}$ of states is said to be a *reversible entanglement generating set* for a class of states $\mathcal{C}$ if and only if $\mathcal{C}\subseteq\mathcal{S}(\mathcal{G})$.

Clearly, every REGS for the class of $m+1$ partite states is a REGS for each of its m-partite subsystems. In particular any REGS for the full class of m-partite states must be capable of generating an EPR pair between any two of the parties. One might suspect that the set of all $m(m-1)/2$ EPR pairs would be a sufficient REGS for generating all m-partite states, but as we will see in Sec. III C, that is not the case for $m\geqslant 4$.

To quantify entanglement, one would like to know the fewest kinds of entanglement needed to make all states in a given class. To this end we define a MREGS as a REGS of minimal cardinality. Again the set $\mathcal{G}_2=\{\mathrm{EPR}\}$ is an example of a MREGS for bipartite entanglement which induces the entanglement measure given by the partial entropy in bits.

Thus we have reduced the problem of quantifying entanglement to the problem of finding the MREGS and the corresponding entanglement coefficients. The entanglement coefficients give us the entanglement measure in terms of how many of the states in the MREGS are required to reversibly make the state by asymptotic LOCC.

If we drop the requirement of reversibility, we get the notion of an *entanglement generating set* (EGS), a set of states which can generate every state in $\mathcal{C}$ under exact or asymptotic LOCC. An EGS state needs only one member, since the m-partite cat state by itself is sufficient to generate all m-partite entangled states, though not reversibly. This can be seen because the m-cat state can give an EPR pair between any two parties by exact LOCC. So Alice can make the desired multipartite state in her lab and then teleport it using these EPR pairs, thus generating an arbitrary multipartite state exactly by LOCC, starting from the appropriate number of m-cat states. To see that the transformation is irreversible, note that an m-partite cat state can be used to prepare at most one EPR state, say between Alice and Bob, but $m-1$ EPR states, say, connecting Alice to every other party, are needed to prepare the cat state again. Thapliyal [35] showed that a pure m-partite state Ψ is an EGS state (can be transformed into a cat state by LOCC) if and only if its partial entropies S_X are positive across all nontrivial partitions X.

Section III C exhibits some simple lower bounds on the cardinality of the MREGS for tripartite and higher entangled pure states. Unfortunately we do not know any corresponding upper bounds. We cannot exclude the possibility that for tripartite and higher states an infinite number of asympotitically inequivalent kinds of entanglement might exist.

C. Lower bounds on the size of MREGS based on local entropies

It is easy to see that the Alice-Bob EPR state EPR^{AB} (regarded as a special case of an m-partite state in which all the parties besides Alice and Bob are unentangled bystanders in a standard $|0\rangle$ state) is a MREGS for the class containing all and only those states which have AB entanglement but no other entanglement, more precisely states for which S_X is zero if X includes both A and B or neither A and B, and has a constant nonzero value for all other X. Therefore, in order to generate all possible bipartite EPR pairs, the MREGS's for general m-partite pure states must have at least $m(m-1)/2$ members, which can be taken without loss of generality to be the $m(m-1)/2$ bipartite EPR states themselves.

TABLE I. Entropy ratio r_{21} for some multipartite entangled pure states.

Parties	State	r_{21}
3	Cat (GHZ)	1
	3-EPR*s* state	1
4	Cat	1
	6-EPR*s* state	4/3
5	Cat	1
	10-EPR*s* state	3/2
	Five-qubit codeword	2
6	Cat	1
	15-EPR*s* state	8/5

However, for all $m>3$ the partial entropy argument requires the MREGS's to include other states as well. Without pursuing this exhaustively [36], we will sketch how local entropy arguments can be used to derive other lower bounds on the size of the MREGS's for general m-partite states.

Let us restrict our attention to m-partite pure states Υ in which the partial entropy $S(\mathrm{tr}_x(|\Upsilon\rangle\langle\Upsilon|))$ of a subset X depends only on the number of members of X, not on which parties are members of X. Two examples of such as state are the m-way cat state, and a tensor product of $m(m-1)/2$ EPR pairs, one shared between each pair of parties. We shall call the latter an EPR*s* state. Let $r_{21}(\Upsilon)=S_{AB}(\Upsilon)/S_A(\Upsilon)$ be the ratio of two-party to one-party partial entropy in state Υ. It is easy to see that $r_{21}=1$ for cat states, independent of m, but $r_{21}=2(m-2)/(m-1)$ for EPR*s* states, the numerator of the latter expression being the number of edges, in an m-partite complete graph, joining a two-vertex subset X to its complement, while the denominator is the number of edges incident on any single vertex. Thus cat and EPR*s* states have equal r_{21} for $m=3$, but for EPRs states with larger m, the ratio exceeds 1, as shown in Table I. Therefore the 4-cat state, unlike the 4-EPR*s* state, cannot be asymptotically equivalent to any combination of the six EPR pairs, and the MREGS's for $m=4$ must have at least seven members.

For $m=5$, the table also includes an entry for the maximally entangled state of five qubits, (e.g., a codeword in the well-known five-qubit error-correcting code [38,23]) which has maximal entropy across any partition X. Since this state has an r_{21} even greater than the EPR*s* state, the MREGS's for $m=5$ must have at least 12 states. Similarly, the MREGS's for $m=6$ must have at least 31 members, without considering other entropy ratios besides r_{21} or other states besides the EPR, 4-cat, and 6-cat states.

D. Exact reducibilities between GHZ and EPR

At this point it is natural to ask whether three EPR pairs (shared symmetrically among Alice, Bob, and Claire) can be reversibly interconverted to two GHZ states. Partial entropy arguments do not resolve the question because, for both the 3-EPR state and the 2-GHZ state, the partial entropy of any nontrivial subset of the parties is two bits. Nevertheless, the impossibility of performing this conversion follows from the fact that two states are LOCC equivalent if and only if they are equivalent under local unitary operations.

To see that 2-GHZ and 3-EPR states are LOCC incomparable, first observe that, since the two states are isentropic, they must, by theorem 1, either be LOCC incomparable or LU equivalent. To see that they are not LU equivalent, observe that the mixed state obtained by tracing out Alice from the 2-GHZ state, namely $\rho_{BC}(2\text{-GHZ})$, a maximally mixed, separable state of the two parties Bob and Claire, while the corresponding mixed state obtained from the 3-EPR states, $\rho_{BC}(3\text{-EPR})$ is a distillable entangled state, consisting of the tensor product of an intact *BC* EPR pair with another random qubit held by each party. But if the 3-EPR and 2-GHZ states were LU equivalent, Bob and Claire, by performing their own local unitary transformations without reference to Alice, could make $\rho^{BC}(3\text{-EPR})$ from $\rho^{BC}(2\text{-GHZ})$. Since they cannot do this (otherwise they would be generating entanglement by LOCC), the 3-EPR and 2-GHZ states cannot be LU equivalent; therefore, by corollary 1 they must be LOCC incomparable.

Figure 3 shows the exact reducibilities that hold among EPR and GHZ states. The protocols for these reducibilities follow: To obtain an EPR pair say between Bob and Claire, Alice performs a measurement in the Hadamard basis, namely, $\{|0+1\rangle,|0-1\rangle\}$ and informs Bob and Claire about the outcome. Using this information, Bob and Claire can perform conditioned rotations that give them an EPR pair. Clearly, this LOCC protocol can be generalized to many parties, to transform an *m*-cat state into an EPR pair between any two parties, by having the remaining $m-2$ parties measure in the Hadamard basis, and communicate the result to the two parties, who then perform appropriate conditioned local unitary operations.

To obtain a GHZ state from two EPR pairs say $|\text{EPR}^{AB}\rangle$ and $|\text{EPR}^{AB}\rangle$, Alice makes a GHZ state in her lab and then uses the EPR pairs to teleport Bob's and Charlie's parts to them. Clearly, this protocol can be generalized to make a *m*-cat state from a set of *m* EPR pairs, shared by one party with all the rest.

In passing we note that any set of EPR pairs that describe a connected graph, the nodes representing parties and the edges representing the shared EPR pairs, is an EGS. This is easy to prove using teleportation, as done above.

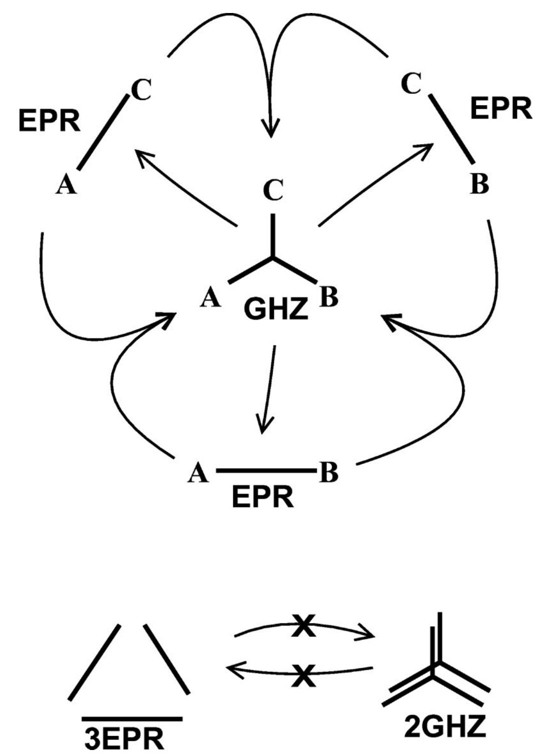

FIG. 3. Top: Two EPR pairs, together involving three parties, can be exactly transformed to one GHZ state. A GHZ state can be transformed into any one of the three EPR pairs. These transformations are exact and irreversible, involving a loss of entanglement across some bipartite boundary. Bottom: The transformations between the symmetric 3-EPR state and the 2-GHZ state, marked by an *X*, cannot be done exactly, even though the partial entropies agree, by the arguments of this section. Very recently it was shown [40] that these transformation cannot even be done asymptotically.

E. Uniqueness of entanglement coefficients

One key question about this framework for quantifying entanglement is whether entanglement coefficients are unique. Surely this is to be desired if we are to interpret the values of the coefficients as representing the amounts of different kinds of entanglement present in the given state. We do not know how to show uniqueness in general, but we can show this for some cases of interest.

For concreteness let us consider the case of three parties, say Alice, Bob, and Claire. We noted earlier that all EPR pairs shared between two parties must be in the MREGS's, so EPR^{AB}, EPR^{BC}, and EPR^{CA} must be in the MREGS's. Let us consider the entanglement span of these three EPR pairs. Assume that there exists a state Ψ in this span such that the entanglement coefficients are not unique, say (x,y,z) and (a,b,c), where *x*, *y*, and *z* (*a*, *b*, and *c*) denote the amounts of EPR^{AB}, EPR^{BC}, and EPR^{CA} in the two decompositions. Then using the fact that asymptotically LOCC-equivalent states must be isentropic, we have

$$x+y=a+b, \quad y+z=b+c, \quad z+x=c+a. \tag{31}$$

This implies that $(x,y,z)=(a,b,c)$, and thus proves uniqueness. Clearly such an argument works for the entanglement span of EPR pairs of more parties, because there are at most $m(m-1)/2$ EPR pairs shared by different parties and the isentropic condition gives the same number of independent constraints.

Now we look at the entanglement span of the above three EPR pairs and the GHZ state. If we assume the GHZ state belongs to the span of the EPR*s* state then uniqueness has already been proved. Thus let us assume that the GHZ state is asymptotically not equivalent to the EPR*s* state. Let the nonunique entanglement coefficients be (x,y,z,w) and $(x-\delta_x, y-\delta_y, z-\delta_z, w+\delta_w)$, with the first three coefficients representing the amount of the EPR*s* state, and the last representing the amount of the GHZ state. Without loss of generality we can assume $\delta_w=2\delta>0$. Again using the fact that asymptotically LOCC equivalent states must be isentropic, we have

$$\delta_w-\delta_x-\delta_y=\delta_w-\delta_y-\delta_z=\delta_w-\delta_z-\delta_x=0. \tag{32}$$

Solving these equations, we find that

$$\delta_x=\delta_y=\delta_z=\delta_w/2=\delta. \tag{33}$$

This implies that

$$\mathrm{EPP}^{AB}\otimes\mathrm{EPR}^{BC}\otimes\mathrm{EPR}^{CA}\approx_{\mathrm{LOCCc}}\mathrm{GHZ}^{\otimes 2}. \tag{34}$$

For more complicated sets $\mathcal{S}$ of states, the requirement that entanglement coefficients be positive may lead to nonuniqueness. Because of positivity, all extremal points of $\mathcal{S}$ must be in the MREGS's, and for some $\mathcal{S}$ the number of extremal points may considerably exceed the dimensionality of $\mathcal{S}$ (for example, for $n\geqslant 3$, each interior point of a regular n-gon can be expressed in multiple ways as a convex combination of vertices).

Note that there may be many MREGS's, for example any bipartite state is as MREGS's for bipartite entanglement. So how do we decide upon a *canonical MREGS*? Possible criteria include requiring the states in the MREGS to be of as low Hilbert space dimension as possible, and as high in partial entropy within that Hilbert space as possible. Thus for the bipartite case the EPR state is the canonical MREGS, up to local unitary operations.

IV. DISCUSSION AND OPEN PROBLEMS

For bipartite pure states, the unique asymptotic measure of entanglement is known [17–19]. The present paper identifies elements of any exact or asymptotic measure of *multipartite* entanglement. For bipartite states, entanglement is a scalar: the measure of entanglement of a state reduces to a single number. For multipartite states, entanglement is a vector, i.e., there are inequivalent classes of entanglement. The inequivalence leads to the concept of a MREGS, and the requirement that any m-partite entangled state be expressible as a linear combination of the states in the m-partite MREGS. Within a class of states with equivalent entanglement, we seek a scalar measure of entanglement. Five desiderata for a scalar measure of entanglement are listed in Sec. I, and Sec. III A derives such a measure for the states we call m-orthogonal states. In this paper, however, we focus on inequivalent classes of entanglement, leaving many questions unanswered.

Very recently [40] Linden, *et al.*, using a relative entropy argument, strengthened the result of Sec. III D by showing that asymptotically reversible transformations are insufficient to interconvert 2-GHZ and 3-EPR (indeed the states remain asymptotically incomparable even with the help of a catalyst). Therefore, the MREGS for $m=3$ must contain at least four states (without loss of generality the GHZ state and the three bipartite EPR states). Of course we would like to know whether these resources are sufficient to prepare *all* tripartite pure states in an asymptotically reversible fashion.

A more fundamental problem is that although we have lower bounds on the number of inequivalent kinds of entanglement under asymptotically reversible LOCC transformations, we know of no nontrivial upper bounds. As noted earlier, even for tripartite states we do not know that the number is finite. One possible approach to this problem, which we do not explore in detail here, would be to further generalize the notion of state by allowing tensor factors to appear with negative as well as nonintegral exponents. A generalized state such as $(\mathrm{EPR}^{AB})^{\otimes 2}\otimes(\mathrm{GHZ})^{\otimes -0.3}$ (in chemical notation, $2\text{-}\mathrm{EPR}^{AB}-0.3\text{-}\mathrm{GHZ}$) would thus represent a quantum ''contract'' comprising a license, asymptotically, to consume two Alice-Bob EPR pairs along with an obligation to produce 0.3-GHZ states. Allowing negative entanglement coefficients would also solve the problem of nonuniqueness of entanglement coefficients, allowing any state to be described as a unique, but not necessarily positive, linear combination of states in a smaller MREGS.

The most powerful result we could hope for from approaches of this kind would be to show that under some appropriately strengthened (but still natural) notion of asymptotic reducibility, all isentropic states are asymptotically equivalent. A less ambitious result would be to show that for simple asymptotic reducibility, or some strengthened version of it, all isentropic states are either equivalent or incomparable, in analogy with the fact that all isentropic states must be either equivalent or incomparable under *exact* LOCC reducibility (corollary 1).

ACKNOWLEDGMENTS

We thank David DiVincenzo, Julia Kempe, Noah Linden, Barbara Terhal, Armin Uhlmann, and Bill Wootters for helpful discussions. C.H.B., J.A.S., and A.V.T. acknowledge support from the U.S. Army Research Office, Grant No. DAAG55-98-C-0041, and A.V.T. also under Grant No. DAAG55-98-1-0366. D.R. acknowledges support from the Giladi program.

[1] A. Einstein, B. Podolsky, and N. Rosen, Phys. Rev. **47**, 777 (1935).
[2] E. Schrödinger, Naturwissenschaften **23**, 807 (1935); **23**, 823 (1935); **23**, 844 (1935) Translation: Proc. APS, **124**, 323 (1980).
[3] C. H. Bennett, G. Brassard, C. Crépeau, R. Jozsa, A. Peres, and W. K. Wootters, Phys. Rev. Lett. **70**, 1895 (1993).
[4] C. H. Bennett and G. Brassard (unpublished); D. Deutsch, A.

Ekert, R. Jozsa, C. Macchiavello, S. Popescu, and A. Sanpera, Phys. Rev. Lett. **77** 2818, (1996), **80**, 2022 (1998), H.-K. Lo, in *Introduction to Quantum Computation and Information*, edited by H.-K. Lo, S. Popescu and T. Spiller (World Scientific, Singapore, 1998), pp. 76–119; H. Zbinden *ibid.* pp. 120–142.

[5] C. H. Bennett and S. J. Wiesner, Phys. Rev. Lett. **69**, 2881 (1992).

[6] C. H. Bennett, C. A. Fuchs, and J. A. Smolin, in *Quantum Communication, Computing and Measurement*, edited by O. Hirota, A. S. Holevo, and C. M. Caves (Plenum, New York, 1997).

[7] C. H. Bennett, P. W. Shor, J. A. Smolin, and A. V. Thapliyal, Phys. Rev. Lett. **83**, 3081 (1999).

[8] P. W. Shor, Phys. Rev. A **52**, R2493 (1995); D. Gottesman, Ph.D. Thesis, California Institute of Technology, 1997; LANL e-print, quant-ph/9705052.

[9] D. Deutsch, Proc. R. Soc. London, Ser. A **400**, 97 (1985); D. Deutsch *ibid.*, **425**, 73 (1989).

[10] L. K. Grover, LANL e-print, quant-ph/9704012.

[11] R. Cleve and H. Buhrman, LANL e-print quant-ph/9704026.

[12] Authors, Phys World, **11** (1998).

[13] K.-E. Hellwig and K. Kraus, Commun. Math. Phys. **11**, 214 (1969); **16**, 142 (1970); K. Kraus, *States, Effects, and Operations: Fundamental Notions of Quantum Theory* (Springer, Berlin, 1983).

[14] C. M. Caves (unpublished).

[15] C. H. Bennett, D. P. DiVincenzo, C. A. Fuchs, T. Mor, E. Rains, P. W. Shor, J. A. Smolin, and W. K. Wootters, Phys. Rev. A **59**, 1070 (1999).

[16] E. H. Rains, Phys. Rev. A **60**, 173 (1999).

[17] C. H. Bennett, H. J. Bernstein, S. Popescu, and B. Schumacher, Phys. Rev. A **53**, 2046 (1996).

[18] C. H. Bennett, G. Brassard, S. Popescu, B. Schumacher, J. A. Smolin, and W. K. Wootters, Phys. Rev. Lett. **76**, 722 (1996).

[19] S. Popescu and D. Rohrlich, Phys. Rev. A **56**, R3319 (1997).

[20] V. Vedral, M. B. Plenio, M. A. Rippin, and P. L. Knight, Phys. Rev. Lett. **78**, 2275 (1997).

[21] G. Vidal, J. Mod. Opt. **47**, 355 (2000).

[22] A. Kitaev (private communication).

[23] C. H. Bennett, D. P. DiVincenzo, J. A. Smolin, and W. K. Wootters, Phys. Rev. A **54**, 3824 (1996).

[24] H.-K. Lo and S. Popescu, LANL e-print quant-ph/9707038.

[25] M. A. Nielsen, Phys. Rev. Lett. **83**, 436 (1999).

[26] D. Jonathan and M. B. Plenio, Phys. Rev. Lett. **83**, 3566 (1999).

[27] N. Linden and S. Popescu, Fortschr. Phys. **46**, 567 (1998) N. Linden, S. Popescu, and A. Sudbery, Phys. Rev. Lett. **83**, 243 (1999).

[28] C. H. Bennett, in *Quantum Communication, Computing, and Measurement*, Proceedings of the Third International Conference on Quantum Communication and Measurement, 1996, Shizuoka, Japan, edited by O. Hirota, A. S. Holevo, and C. M. Caves (Plenum, New York, 1997), pp. 25–40.

[29] A. Wehrl, Rev. Mod. Phys. **50**, 221 (1978).

[30] J. Kempe, Phys. Rev. A **60**, 910 (1999).

[31] A. Uhlmann, Rep. Math. Phys. **9**, 273 (1976).

[32] R. Jozsa, J. Mod. Opt. **41**, 2315 (1994).

[33] A. Peres, Phys. Lett. A **202**, 16 (1995).

[34] A. V. Thapliyal, Phys. Rev. A **59**, 3336 (1999).

[35] A. V. Thapliyal (unpublished).

[36] N. Linden, S. Popescu, B. Schumacher, and A. Thapliyal (unpublished) obtained many other results on the partial entropy classification of multipartite states.

[37] L. P. Hughston, R. Jozsa, and W. K. Wootters, Phys. Lett. A **183**, 14 (1993).

[38] R. Laflamme, C. Miquel, J. P. Paz, and W. H. Zurek, Phys. Rev. Lett. **77**, 198 (1996).

[39] H.-K. Lo and Sandu Popescu, Phys. Rev. Lett. **83**, 1459 (2000).

[40] N. Linden, S. Popescu, B. Schumacher, and M. Westmoreland, LANL eprint, quant-ph/9912039.