An idiots guide to the Boltzmann, Lotka and Volterra model, in the context of a dynamic retail model.

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Formulating the problem

Consider a spatial region with residential areas (i) and shopping centers (j). Define:

$\{X_i\}$	The total money spent by shoppers from residential area i.
$\{Z_j\}$	The floor space of shopping center j.
$\{Y_{ij}\}$	The money spent in shopping center j by residents of area i.
$\{c_{ij}\}$	The impedance or *cost* of a resident in i spending in center j.
$\{b_j\}$	The benefit of a resident in i spending in center j.
$\{D_j\}$	The total money taken by shopping center j .
κ	The running costs per unit floor space .

Here, the subscript details which zone is being refereed to. For example, Z_1 could be the retail floor space of Bluewater and Z_2 the floor space of Lakeside.

The *attractiveness* of shopping center j to a shopper may be thought of as

$$\text{benefit} - \text{cost} = \bar{a}b_j - \bar{b}c_{ij}, \tag{1}$$

weighted by some constants $\bar{a}$ and $\bar{b}$. Notice the benefit is determined only by the features of j, while cost takes into account where the shopper comes from.

Assume that the benefit of j is the natural log of its floor space:

$$b_j = \log Z_j \tag{2}$$

A center with a large floor space is beneficial, while if a center has a tiny floor space (below some critical threshold) it becomes a disadvantage for the consumer to shop there.

Consider the information we have about the system in the form of a hierarchy.

- At the top level - the macro state - we have information about the total money spent in each center Z_j and from area X_i.
- Below this, a more detailed picture would be given by how much money from each i is spent in each j. This second (and lower) macro state is the set of spending flows Y_{ij}.
- Below this in the micro-state, a complete description to the entire problem would be given by the spending pattern of each individual in i; where they shopped and how much they spent.

Assuming information is given about the top level macro-state, X_i, Z_j, and nothing else is known, this method aims to find the most probable set of spending flows Y_{ij}. (Later, a dynamic model for Z is outlined). Also known are the following:

(I)	$\sum_j Y_{ij} = X_i$	The money spent by shoppers from i in all centers j must equal the total.
(II)	$\sum_{ij} Y_{ij} c_{ij} = C$	The total cost has some value C, analogous in some sense to system energy.
(III)	$\sum_{ij} Y_{ij} \log Z_j = B$	There is some total benefit or capacity in the system.

Modelling the spending flow Yij

If Y is the total amount of money available in the system, we may write down the number of possible (distinguishable) micro-states which could give rise to a specific set of spending flows Y_{ij} as in Wilson (1970):

$$W(\{Y_{ij}\}) = \frac{Y!}{\prod_{ij} Y_{ij}!}. \tag{3}$$

If all micro-states are equally probable then $W(\{Y_{ij}\})$ will be proportional to the likelyhood that a given Y_{ij} will occur. The aim of this method is find the *most likely* set of spending flows of all possible and this may be done by maximising (3). Doing so may be shown to be equivalent to maximising the entropy of a system:

First, taking the log of (3):

$$\log W(\{Y_{ij}\}) = \log Y! - \sum_{ij} \log Y_{ij}!, \tag{4}$$

and applying Stirling's approximation for the log of a factorial

$$\log W(\{Y_{ij}\}) = \log Y! - \sum_{ij} Y_{ij} \log Y_{ij} - Y_{ij} \tag{5}$$

or, better

$$\log W(\{Y_{ij}\}) = \log Y! - \sum_{ij} Y_{ij} \log Y_{ij} + \sum_{i} \left(\sum_{j} Y_{ij} \right) \tag{6}$$

From (I), the last term is just $\sum_i X_i$. Since Y and X_i are both fixed, maximising (6) is equivalent to maximising

$$S = -\sum_{ij} Y_{ij} \log Y_{ij} \tag{7}$$

which is identical to the model for probabilistic entropy given in Shannon (1929)

Jaynes (1957) notes:

"In making inferences on the basis of partial information we must use that probability distribution [in our case the set of spending flows Y_{ij}] *which has maximum entropy subject to whatever is known. This is the only unbiased assumption we can make; to use any other would amount to arbitrary assumption of information which by hypothesis we do not have"*

The task then is to maximise (7), subject to the constraints (I), (II) and (III).

To do so, the method of Lagrangian multipliers is used, ie solving $\nabla\Lambda = 0$, where

$$\Lambda(Y_{ij}, \alpha, \beta, \gamma) = -\sum_{ij} Y_{ij} \log Y_{ij} + \alpha \left(\sum_{ij} Y_{ij} \log Z_j - B \right) - \beta \left(\sum_{ij} Y_{ij} c_{ij} - C \right) \cdots$$
$$- \sum_{i} \gamma_i \left(\sum_{j} Y_{ij} - X_i \right), \tag{8}$$

for Lagrangian multipliers α, β γ_i. A full description of the method may be seen here.

With this approach $\frac{\partial \Lambda}{\partial \alpha} = 0$, $\frac{\partial \Lambda}{\partial \beta} = 0$ and $\frac{\partial \Lambda}{\partial \gamma_i} = 0$ fix the constraints outlined above, and:

$$\frac{\partial \Lambda}{\partial Y_{ij}} = -\log Y_{ij} - 1 + \alpha \log Z_j - \beta c_{ij} - \gamma_i = 0 \tag{9}$$

which leads directly to the result

$$Y_{ij} = \frac{Z_j^\alpha \exp\left(-\beta c_{ij}\right)}{\exp\left(1 + \gamma_i\right)}. \tag{10}$$

Further, (I) implies

$$X_i = \sum_k \frac{Z_k^\alpha \exp\left(-\beta c_{ik}\right)}{\exp\left(1 + \gamma_i\right)}, \tag{11}$$

so, setting a normalisation term:

$$A_i = \frac{1}{\sum_k Z_k^\alpha \exp\left(-\beta c_{ik}\right)} \tag{12}$$

allows us to eliminate the Lagrangian multiplier γ_i, and to rewrite (10) as in Wilson (2007):

$$Y_{ij} = A_i X_i Z_j^\alpha \exp\left(-\beta c_{ij}\right). \tag{13}$$

The solution to this and (12) will yield the most probable set of spending flows.

Noting that Z_j^α may be written $\exp(\alpha \log Z_j$, another form of (13) is

$$Y_{ij} = X_i \frac{\exp\left(\alpha \log Z_j - \beta c_{ij}\right)}{\sum_k \exp\left(\alpha \log Z_k - \beta c_{ik}\right)} \tag{14}$$

by returning to the notion of attractiveness in (1), this form of Y_{ij} now makes intuitive sense. $\exp\left(\alpha \log Z_j - \beta c_{ij}\right)$ speaks of the total advantage or pulling power of center j to area i, weighted by the constants α and β. The denominator acts as a *competition* term, comparing the attractiveness of center j to all other centers.

So, the flow of money into a given shopping center j is its relative attractiveness to a person in residential area i times by the total spending power of i, X_i.

A dynamic model for Z_j

If Y_{ij} is known for a given time step, then D_j, the money taken by shopping center j, may be found from

$$D_j = \sum_i Y_{ij} \tag{15}$$

since the total flow of money into j must be equivalent to the sum of the flows from each residential area i. One would expect that the retail floor space of a given shopping center Z_j would be a direct result of the money it takes. Based on this idea, the following assumption is made by Harris and Wilson (1978):

- If $D_j > Z_j$, the center should grow
- If $D_j < Z_j$ the center should decline

This forms the basis of the dynamical model, where the given value of Z_j at the current time step is changed by

$$\Delta Z_j = \epsilon \left(D_j - Z_j \right) Z_j. \tag{16}$$

at the next time step, for some growth parameter ϵ. The Z_j term at the end governs the dynamics of growth near $Z_j = 0$; small centers grow slower than large ones. The process may then be repeated: finding Y_{ij} from (13) based on the original X_i and the latest Z_j, and using (15) to find the new D_j and again update Z_j.

Considering the equilibrium state $D_j = Z_j$ for all j, where no centers grow or shrink, we may use (14) in (15) to derive an explicit set of equations in Z_j and combining the Boltzmann and Lotka ,Voltera models:

$$\sum_i \left[\frac{X_i Z_j^\alpha exp(-\beta c_{ij})}{\sum_k Z_k^\alpha exp(-\beta c_{ik})} \right] = Z_j \tag{17}$$