Formulating the problem

Consider a spatial region with residential areas \((i)\) and shopping centers \((j)\). Define:

- \(\{X_i\}\) The total money spent by shoppers from residential area \(i\).
- \(\{Z_j\}\) The floor space of shopping center \(j\).
- \(\{Y_{ij}\}\) The money spent in shopping center \(j\) by residents of area \(i\).
- \(\{c_{ij}\}\) The impedance or cost of a resident in \(i\) spending in center \(j\).
- \(\{b_j\}\) The benefit of a resident in \(i\) spending in center \(j\).
- \(\{D_j\}\) The total money taken by shopping center \(j\).
- \(\kappa\) The running costs per unit floor space.

Here, the subscript details which zone is being referred to. For example, \(Z_1\) could be the retail floor space of Bluewater and \(Z_2\) the floor space of Lakeside.

The attractiveness of shopping center \(j\) to a shopper may be thought of as

\[
\text{benefit} - \text{cost} = \bar{a}b_j - \bar{b}c_{ij},
\]

weighted by some constants \(\bar{a}\) and \(\bar{b}\). Notice the benefit is determined only by the features of \(j\), while cost takes into account where the shopper comes from.
Assume that the benefit of $j$ is the natural log of its floor space:

$$b_j = \log Z_j$$  \hspace{1cm} (2)

A center with a large floor space is beneficial, while if a center has a tiny floor space (below some critical threshold) it becomes a disadvantage for the consumer to shop there.

Consider the information we have about the system in the form of a hierarchy.

- At the top level - the macro state - we have information about the total money spent in each center $Z_j$ and from area $X_i$.
- Below this, a more detailed picture would be given by how much money from each $i$ is spent in each $j$. This second (and lower) macro state is the set of spending flows $Y_{ij}$.
- Below this in the micro-state, a complete description to the entire problem would be given by the spending pattern of each individual in $i$; where they shopped and how much they spent.

Assuming information is given about the top level macro-state, $X_i$, $Z_j$, and nothing else is known, this method aims to find the most probable set of spending flows $Y_{ij}$. (Later, a dynamic model for $Z$ is outlined). Also known are the following:

(I) $\sum_j Y_{ij} = X_i$ \hspace{1cm} The money spent by shoppers from $i$ in all centers $j$ must equal the total.

(II) $\sum_{ij} Y_{ij} c_{ij} = C$ \hspace{1cm} The total cost has some value $C$, analogous in some sense to system energy.

(III) $\sum_{ij} Y_{ij} \log Z_j = B$ \hspace{1cm} There is some total benefit or capacity in the system.

**Modelling the spending flow $Y_{ij}$**

If $Y$ is the total amount of money available in the system, we may write down the number of possible (distinguishable) micro-states which could give rise to a specific set of spending flows $Y_{ij}$ as in Wilson (1970):

$$W(\{Y_{ij}\}) = \frac{Y!}{\prod_{ij} Y_{ij}!}. \hspace{1cm} (3)$$

If all micro-states are equally probable then $W(\{Y_{ij}\})$ will be proportional to the likelihood that a given $Y_{ij}$ will occur. The aim of this method is find the most likely set of spending flows of all possible and this may be done by maximising (3). Doing so may be shown to be equivalent to maximising the entropy.
of a system:

First, taking the log of (3):

\[
\log W(\{Y_{ij}\}) = \log Y! - \sum_{ij} \log Y_{ij}!,
\]

(4)

and applying Stirling’s approximation for the log of a factorial

\[
\log W(\{Y_{ij}\}) = \log Y! - \sum_{ij} Y_{ij} \log Y_{ij} - Y_{ij}
\]

(5)

or, better

\[
\log W(\{Y_{ij}\}) = \log Y! - \sum_{ij} Y_{ij} \log Y_{ij} + \sum_i \left( \sum_j Y_{ij} \right)
\]

(6)

From (I), the last term is just \(\sum_i X_i\). Since \(Y\) and \(X_i\) are both fixed, maximising (6) is equivalent to maximising

\[
S = - \sum_{ij} Y_{ij} \log Y_{ij}
\]

(7)

which is identical to the model for probabilistic entropy given in Shannon (1929)

Jaynes (1957) notes:

"In making inferences on the basis of partial information we must use that probability distribution [in our case the set of spending flows \(Y_{ij}\)] which has maximum entropy subject to whatever is known. This is the only unbiased assumption we can make; to use any other would amount to arbitrary assumption of information which by hypothesis we do not have"

The task then is to maximise (7), subject to the constraints (I), (II) and (III).

To do so, the method of Lagrangian multipliers is used, ie solving \(\nabla \Lambda = 0\), where

\[
\Lambda(Y_{ij}, \alpha, \beta, \gamma) = - \sum_{ij} Y_{ij} \log Y_{ij} + \alpha \left( \sum_{ij} Y_{ij} \log Z_j - B \right) - \beta \left( \sum_{ij} Y_{ij} c_{ij} - C \right) \cdots
\]

\[
- \sum_i \gamma_i \left( \sum_j Y_{ij} - X_i \right).
\]

(8)

for Lagrangian multipliers \(\alpha, \beta, \gamma_i\). A full description of the method may be seen here.
With this approach \( \frac{\partial \Lambda}{\partial \alpha} = 0, \frac{\partial \Lambda}{\partial \beta} = 0 \) and \( \frac{\partial \Lambda}{\partial \gamma_i} = 0 \) fix the constraints outlined above, and:

\[
\frac{\partial \Lambda}{\partial Y_{ij}} = -\log Y_{ij} - 1 + \alpha \log Z_j - \beta c_{ij} - \gamma_i = 0 \tag{9}
\]

which leads directly to the result

\[
Y_{ij} = Z_j^\alpha \exp (-\beta c_{ij}) \exp (1 + \gamma_i) \tag{10}
\]

Further, (I) implies

\[
X_i = \sum_k Z_k^\alpha \exp (-\beta c_{ik}) \exp (1 + \gamma_i), \tag{11}
\]

so, setting a normalisation term:

\[
A_i = \frac{1}{\sum_k Z_k^\alpha \exp (-\beta c_{ik})} \tag{12}
\]

allows us to eliminate the Lagrangian multiplier \( \gamma_i \), and to rewrite (10) as in Wilson (2007):

\[
Y_{ij} = A_i X_i Z_j^\alpha \exp (-\beta c_{ij}). \tag{13}
\]

The solution to this and (12) will yield the most probable set of spending flows.

Noting that \( Z_j^\alpha \) may be written \( \exp(\alpha \log Z_j) \), another form of (13) is

\[
Y_{ij} = X_i \frac{\exp (\alpha \log Z_j - \beta c_{ij})}{\sum_k \exp (\alpha \log Z_k - \beta c_{ik})} \tag{14}
\]

by returning to the notion of attractiveness in (1), this form of \( Y_{ij} \) now makes intuitive sense. \( \exp (\alpha \log Z_j - \beta c_{ij}) \) speaks of the total advantage or pulling power of center \( j \) to area \( i \), weighted by the constants \( \alpha \) and \( \beta \). The denominator acts as a competition term, comparing the attractiveness of center \( j \) to all other centers.

So, the flow of money into a given shopping center \( j \) is its relative attractiveness to a person in residential area \( i \) times by the total spending power of \( i \), \( X_i \).

**A dynamic model for \( Z_j \)**

If \( Y_{ij} \) is known for a given time step, then \( D_j \), the money taken by shopping center \( j \), may be found from

\[
D_j = \sum_i Y_{ij} \tag{15}
\]
since the total flow of money into \( j \) must be equivalent to the sum of the flows from each residential area \( i \). One would expect that the retail floor space of a given shopping center \( Z_j \) would be a direct result of the money it takes. Based on this idea, the following assumption is made by Harris and Wilson (1978):

- If \( D_j > Z_j \), the center should grow
- If \( D_j < Z_j \), the center should decline

This forms the basis of the dynamical model, where the given value of \( Z_j \) at the current time step is changed by

\[
\Delta Z_j = \epsilon (D_j - Z_j) Z_j.
\] (16)

at the next time step, for some growth parameter \( \epsilon \). The \( Z_j \) term at the end governs the dynamics of growth near \( Z_j = 0 \); small centers grow slower than large ones. The process may then be repeated: finding \( Y_{ij} \) from (13) based on the original \( X_i \) and the latest \( Z_j \), and using (15) to find the new \( D_j \) and again update \( Z_j \).

Considering the equilibrium state \( D_j = Z_j \) for all \( j \), where no centers grow or shrink, we may use (14) in (15) to derive an explicit set of equations in \( Z_j \) and combining the Boltzmann and Lotka ,Volterra models:

\[
\sum_i \left[ \frac{X_i Z_j^\alpha \exp(-\beta c_{ij})}{\sum_k Z_k^\alpha \exp(-\beta c_{ik})} \right] = Z_j
\] (17)