Hearing the Irrational

Music and the Development of the Modern Concept of Number

By Peter Pesic*

ABSTRACT

Because the modern concept of number emerged within a quadrivium that included music alongside arithmetic, geometry, and astronomy, musical considerations affected mathematical developments. Michael Stifel embedded the then-paradoxical term "irrational numbers" (*numerici irrationales*) in a musical context (1544), though his philosophical aversion to the "cloud of infinity" surrounding such numbers finally outweighed his musical arguments in their favor. Girolamo Cardano gave the same status to irrational and rational quantities in his algebra (1545), for which his contemporaneous work on music suggested parallels and empirical examples. Nicola Vicentino's attempt to revive ancient "enharmonic" music (1555) required and hence defended the use of "irrational proportions" (*proportiones inrationales*) as if they were numbers. These developments emerged in richly interactive social and cultural milieus whose participants interwove musical and mathematical interests so closely that their intense controversies about ancient Greek music had repercussions for mathematics as well. The musical interests of Stifel, Cardano, and Vicentino influenced their respective treatments of "irrational numbers." Practical as well as theoretical music both invited and opened the way for the recognition of a radically new concept of number, even in the teeth of paradox.

The arithmetician sees numbers in themselves, the musician and the algebraist indeed know numbers, but in their relation to something else.
—Guillaume Gosselin, *On the Great Art or the Hidden Part of Numbers, Commonly Called Algebra or Almucabala* (1577)

* St. John's College, 1160 Camino de la Cruz Blanca, Santa Fe, New Mexico 87505-4511; ppesic@sjcsf.edu.

I am grateful to Amir Alexander, André Barbera, H. Floris Cohen, Sven Dupré, Thomas Mathiesen, Richard Oosterhoff, Andrei Pesic, and Benjamin Wardhaugh for their helpful comments and suggestions. I thank Don Howard and the Program in History and Philosophy of Science at the University of Notre Dame for their hospitality, which enabled me to discuss these ideas in their stimulating company. Special thanks to William Donahue for his help with the Latin and to Manfred Cordes for his generous permission to use the musical score in Figure 8, for which Jörg Jacobi also gave his kind assistance. I also thank the John Simon Guggenheim Memorial Foundation for its support.

Isis, 2010, 101:501–530

0020-9903/2010/1013-0002$10.00

WE TEND TO TAKE FOR GRANTED the broad inclusiveness of our current concept of number (comprising integers along with rational and irrational quantities), as if it were inevitable and unalterable, though in fact it took its present shape only over the past few centuries.[1] Indeed, the modern concept of "real numbers" differs profoundly from the concept of number as it was commonly understood in the West from antiquity until about 1600. Given this fundamental change, what is the relation between ancient and modern concepts of number? Why and how did the concept change? And what difference do these changes make? In the history of ideas, this shift of understanding is so consequential that it deserves much further study, not least because modern mathematics depends on it, hence also much of modern science. As Plato first argued, mathematics offers a touchstone of epistemic certainty that is important for philosophy, theology, and metaphysics. To illuminate these questions, this essay will bring forward interactions between music (both in theory and in practice) and the development of the modern concept of number. These relations will help illuminate the hesitation about the nascent concept of irrational number in the work of Michael Stifel (1544), as compared to Girolamo Cardano's intermixture of irrational and rational quantities in his musical and algebraic works (1545) and Nicola Vicentino's practical use of "irrational proportions" (1555) for musical ends. The differences between them reflect their different approaches to music: Stifel's was traditional and largely theoretical, Cardano's much more practical, while Vicentino was moved primarily by a new musical ideal based as much in practice as in his theoretical advocacy of the ancient Greek enharmonic genus.

In recent years, increasing attention has been directed to the interaction between music and mathematics, especially in the work of Vincenzo Galilei (ca. 1520–1591), a lutenist and composer who took up music theory in the 1560s and wrote the *Dialogo della musica antica, et della moderna* [*Dialogue on Ancient and Modern Music*] (1581), "surely the most influential music treatise of the late sixteenth century." As the father of Galileo Galilei, Vincenzo arguably had an important influence on the nascent experimental science; Stillman Drake, for instance, suggested that Galileo may have first pondered pendulums in the context of his father's experiments, where they acted as weights swinging on the ends of monochords to test (and disconfirm) old Pythagorean legends. These experiments and discussions unfolded in a lively social context: Vincenzo was an important member of the Florentine Camerata, a group surrounding Count Giovanni de' Bardi whose members were fascinated by the accounts of the prodigies of ancient music, which led to their own further speculations and initiatives. Around 1600, the Camerata and other like-minded groups were crucial nurseries for the artistic ideas that led to the first operas, conceived as imaginative reconstructions and revivals of the legendary powers of ancient Greek tragedy, whose music had long been lost. The ensuing marriage of dramatic expressivity with musical powers consciously drawn from dissonance was of great importance to new artistic sensibilities and to the new science envisaged by Bacon, Galileo, and Descartes, as I will discuss elsewhere.[2]

[1] The epigraph is from Guillaume Gosselin, *De arte magna seu de occulta parte numerorum, quæ et Algebra et Almucabala vulgo dicitur* (Paris, 1577), p. 2r; quoted in Jacob Klein, *Greek Mathematical Thought and the Origin of Algebra*, trans. Eva Brann (New York: Dover, 1992), p. 262 n 225.

[2] The quoted assessment is the editor's remark in Vincenzo Galilei, *Dialogue on Ancient and Modern Music*, ed. and trans. Claude V. Palisca (New Haven, Conn.: Yale Univ. Press, 2003), p. xvii. See Stillman Drake, "Music and Philosophy in Early Modern Science," in *Music and Science in the Age of Galileo*, ed. Victor Coelho (Dordrecht: Kluwer, 1992), pp. 3–16; and Palisca, "Was Galileo's Father an Experimental Scientist?" *ibid.*, pp. 143–151. Cf. Nino Pirrotta, *Music and Culture in Italy from the Middle Ages to the Baroque* (Cambridge, Mass.:

In this essay, I will turn to the generations preceding Vincenzo Galilei, in which many of the ideas came to light that he and his contemporaries would ponder and set into action. Among the rich and largely unexplored avenues of that earlier period, I will focus on a seminal mathematical thread for its intrinsic importance as a missing link in the history of mathematics and the newly emergent mathematical sciences. This thread emerges in the context of discussions about the nature of ancient Greek music that grew throughout the sixteenth century in Italy, leading directly to the work of Vincenzo Galilei and the Camerata. These musical discussions were part of the revival of antiquity in many forms—artistic, literary, and philosophical—that began in the fifteenth century. But music posed an especially difficult and intriguing case: though practically all ancient Greek music was apparently lost, the philosophical corpus contained numerous references to the wonders performed by that music, which seemed capable, like Orpheus, of moving the very stones.[3]

In connecting these musical investigations with mathematics, I do not mean to be understood as making an "interdisciplinary" intervention, as the present status of mathematics as a discipline wholly distinct from music might seem to suggest. In the case at hand (and many others like it), our current disciplinary maps and boundaries are anachronistic and misleading. In the sixteenth century, music was still studied as part of the quadrivium, next to arithmetic, geometry, and astronomy, so that it would have been natural for musical and mathematical considerations to have met. The assumption that a mathematician would also be versed in music theory persisted through the work of René Descartes, Johannes Kepler, and even Isaac Newton, remaining important into the following century, as H. Floris Cohen and Benjamin Wardhaugh have discussed extensively.[4] As early modern musicians and thinkers contemplated the mysterious powers of Greek music, they were constantly engaged with the other parts of the quadrivium, for arithmetic and geometry grounded musical and astronomical theory. Yet only music connected heaven and earth, theory and experience, mathematics and feeling.

Harvard Univ. Press, 1984), pp. 219–222, which critiques Vincenzo's originality and importance in the history of music. Peter Pesic, "Music, Science, and Passion" (a book in preparation), addresses the intertwined themes of dispassionate beauty and expressive power in music and natural philosophy.

[3] See Anthony Grafton, "Renaissance Readers and Ancient Texts," in *Defenders of the Text: The Traditions of Scholarship in an Age of Science, 1450–1800* (Cambridge, Mass.: Harvard Univ. Press, 1994), pp. 23–46. The earliest printed editions of Euclid in Latin date from 1482; vernacular translations were published in the 1560s and 1570s.

[4] See Nan Cooke Carpenter, *Music in the Medieval and Renaissance Universities* (New York: Da Capo, 1972), pp. 24–27, 115–118, 313–315. For the earlier history of music in the quadrivium see Pearl Libre, "The *Quadrivium* in the Thirteenth Century Universities (with Special Reference to Paris)," in *Arts libéraux et philosophie au Moyen Âge: Actes du quatrième congrès international de philosophie médiévale* (Paris: Vrin, 1969), pp. 175–191; and Ann E. Moyer, Musica Scientia: *Musical Scholarship in the Italian Renaissance* (Ithaca, N.Y.: Cornell Univ. Press, 1992), pp. 11–35. For interactions between music and the "new philosophy" see H. F. Cohen, *Quantifying Music: The Science of Music at the First Stage of the Scientific Revolution, 1580–1650* (New York: Springer, 1984); for later developments see Benjamin Wardhaugh, *Music, Experiment, and Mathematics in England, 1653–1705* (Aldershot: Ashgate, 2008). René Descartes's *Compendium musicæ* (1618) is translated as *Compendium of Music*, trans. Walter Robert, ed. Charles Kent (Rome: American Institute of Musicology, 1961). For Kepler see Peter Pesic, "Earthly Music and Cosmic Harmony: Johannes Kepler's Interest in Practical Music, Especially Orlando di Lasso," *Journal of Seventeenth-Century Music*, 2005, *11*(1), http://www.sscm-jscm.org/jscm/v11/no1/pesic.html; for Newton see Pesic, "Isaac Newton and the Mystery of the Major Sixth: A Transcription of His Manuscript 'Of Musick' with Commentary," *Interdisciplinary Science Reviews*, 2006, *31*:291–306. See also the essays collected in Philippe Vendrix, ed., *Music and Mathematics in Late Medieval and Early Modern Europe* (Turnhout: Brepols, 2008).

ANCIENT AND MODERN CONCEPTS OF NUMBER

Greek mathematics restricted the term "number" (ἀριθμός) to integers greater than one; "ratio" (λόγος) and "rational" referred in the first instance to proportions of integers, as studied by arithmetic. In contrast, "magnitude" (μέγεθος) designated a more general quantity that was not necessarily rational, such as the length of a line in a geometrical figure.[5] Euclid considered these distinctions necessary because such a figure can have integer sides but still have diagonals that are not expressible by any ratio of integers. In the case of the diagonal of a square of unit side, there is no way to "say" its length as a ratio or number; in that sense, it is literally "unspeakable" (ἄρρητος or ἄλογος). Nor is there any indication that Euclid would have found adequate our symbol for the length of this diagonal, $\sqrt{2}$, which does not *say* what that length really is but only designates an infinite sum of fractions.[6]

By comparison, modern mathematics rests on a portmanteau concept of "real numbers" that includes "whole numbers," "rational numbers," and "irrational numbers."[7] Yet for Greek mathematics, a nonintegral number was a contradiction in terms. Euclid rigorously separated "number" from "magnitude," allowing proportions only between one integer and another or between one magnitude and another. He never alternated or "cross-multiplied" these proportions (as we allow ourselves to do), which would intermix them. By avoiding any such admixture, he sought to avoid contradictions in the foundations of mathematics as he understood them.

Sixteenth-century mathematicians struggled with these ancient distinctions. For instance, though *The Whetstone of Witte* (1557) by Robert Recorde (1510–1558) notes that "*Euclide*, *Boetius*, and other good writers" acknowledge only "*whole numbers*," Recorde also includes "nombres irrationale," approximated as closely as desired by infinite series of fractions. Thus, as Katharine Neal notes, Recorde broadened his "number concept while simultaneously using labels that signaled his awareness of the unacceptability, by traditional standards, of the new numbers." Recorde observed that his number terms draw on algebra, the "cossic art," whose solutions include both rational and irrational quantities. This art has many practical aspects, as Cardano and other Italian mathematicians had noted; Recorde dedicated his book to the "venturers" of the Muscovy Company, offering practical examples of military formations, bricklaying, and geography and promising a further book on navigation.[8]

[5] See Klein, *Greek Mathematical Thought* (cit. n. 1); Ivor Grattan-Guinness, "Numbers, Magnitudes, Ratios, and Proportions in Euclid's *Elements*: How Did He Handle Them?" *Historia Mathematica*, 1996, *23*:355–375; and Katharine Neal, *From Discrete to Continuous: The Broadening of Number Concepts in Early Modern England* (Dordrecht: Kluwer, 2002), pp. 12–27.

[6] Note that in Greek mathematics one was *not* considered a number (ἀριθμός) because it was the "monad" (μονάς) in terms of which all numbers were multiples. See Paul Tannery, "Du rôle de la musique greque dans le développement de la mathématique pure," *Mémoire Scientifique*, 1902, *3*:68–69. More recently, Árpád Szabó, *The Beginnings of Greek Mathematics* (Dordrecht: Reidel, 1978), pp. 99–184, argued that "all the important terms of the theory of proportions have their origins in the theory of music" (p. 170). See also Luigi Borzacchini, "Incommensurability, Music, and Continuum: A Cognitive Approach," *Archive for History of Exact Sciences*, 2007, *61*:273–302.

[7] Fractions were not considered equivalent to ratios in Greek mathematics, whose concept of magnitude included both what we call "algebraic" quantities, like $\sqrt{2}$, which are solutions of algebraic equations of finite degree, and "transcendental" quantities, such as π, which are not the solution of any such finite algebraic equation. See Peter Pesic, *Abel's Proof: An Essay on the Sources and Meaning of Mathematical Unsolvability* (Cambridge, Mass.: MIT Press, 2003), pp. 5–21.

[8] Robert Recorde, *The Whetstone of Witte, which is the seconde parte of Arithmetike* (London: Jhon Kyngston, 1557), sigs. Aiir, Sir: "And a third sorte there is of nombres *radicalle*, whiche commonly bee called *nombers*

Both practical and theoretical considerations moved François Viète (1541–1603) to make crucial symbolic innovations that linked these different number concepts more closely. In his *Canon mathematicus* (1579), Viète advocated the use of decimal fractions to replace the sexagesimal calculations traditionally used for astronomy; such decimals could express both rational and irrational quantities, as Simon Stevin also realized.[9] Viète's reading of Diophantus and Pappus, along with his own innovative cryptanalytic work, led him to introduce alphabetic signs for unknowns as well as for coefficients, as outlined in his *In artem analyticem Isagoge* [*Introduction to the Analytical Art*] (1591).[10] Because a symbol like x could now stand for an integer as well as for an irrational quantity, the new algebraic usage effectively unified these heretofore separate and opposed categories.

These innovations, however ingenious and practical, skated over a foundational abyss because they subsumed irrational and rational under a single symbol. Yet these very issues about the nature of number had emerged earlier in the context of musical theory. The nature of the musical evidence, both theoretical and practical, strongly supported the necessity and legitimacy of irrational numbers. Music was ideally situated to mediate this new understanding between her sisters arithmetic and geometry.[11]

---

irrationalle: because many of them are soche, as can not bee expressed, by common nombers *Abstracte*, norther by an certain rationale nomber. Other men call them more aptly *Surde numbers*." Recorde also cites the ancient dictum that "One is no nomber" (sig. Aiir) and notes that "although there be many kindes of irrationall nombres, yet those figures that serve in *Cossike nombres*, bee the figures also of all irrationalle nombres" (sig. Siv). See Neal, *From Discrete to Continuous* (cit. n. 5), pp. 49–55; the quotation is from p. 50. Brigitte Van Wymeersch, "Qu'entend-on par 'nombre sourd'?" in *Music and Mathematics*, ed. Vendrix (cit. n. 4), pp. 97–110, discusses the use of the term "*radix surdus*" by twelfth-century writers, who did not, however, call such quantities *numeri*.

[9] For Viète's advocacy of decimals see Carl B. Boyer and Uta C. Merzbach, *A History of Mathematics*, 2nd ed. (New York: Wiley, 1991), p. 303. Simon Stevin's *De thiende* (1585), dedicated to "Astronomers, Landmeters, Measurers of Tapestry, Gaugers, Stereometers in general, Money-Masters, and to all Merchants," advocated decimal calculations so that "any man's business may be performed easily thereby." In his *L'arithmétique* (1585) Stevin drew from these decimals the implication that the numbers they represented were continuous rather than discrete, the ancient texts notwithstanding. See E. J. Dijksterhuis, *Simon Stevin: Science in the Netherlands around 1600* (The Hague: Nijhoff, 1970), pp. 16–19, 21–22, 38–39; Henk J. M. Bos, *Redefining Geometrical Exactness: Descartes' Transformation of the Early Modern Concept of Construction* (New York: Springer, 2001), pp. 119–143; and Rudolf Rasch, "Simon Stevin and the Calculation of Equal Temperament," in *Music and Mathematics*, ed. Vendrix, pp. 253–320.

[10] Klein, *Greek Mathematical Thought* (cit. n. 1), pp. 150–185, contains a seminal discussion of Viète's work *vis-à-vis* ancient mathematics. For Viète's cryptological work see Peter Pesic, "François Viète, Father of Modern Cryptanalysis—Two New Manuscripts," *Cryptologia*, 1997, *21*:1–29; Pesic, "Secrets, Symbols, and Systems: Parallels between Cryptanalysis and Algebra, 1580–1700," *Isis*, 1997, *88*:674–692; Pesic, *Labyrinth: A Search for the Hidden Meaning of Science* (Cambridge, Mass.: MIT Press, 2000), pp. 73–83; and Marco Panza, "François Viète: Between Analysis and Cryptanalysis," *Studies in History and Philosophy of Science*, 2006, *37*:269–289. In his *De recognitione æquationum* (published posthumously in 1615), Viète writes that sometimes "equations are exhibited with fractions or incommensurable numbers [*fractis numberis vel asymmetris*]. In geometrical explanations, it is true, the accident of fraction or incommensurability [*asymmetriæ*] does not usually interfere with equations . . . the more elevated the power and degree of affection, the more inexpressibility or irrationality [ἄρρησις ἢ ἀλογία] shows itself in explicating a problem": François Viète, *Opera mathematica*, ed. Joseph E. Hofmann (Hildesheim: Olms, 2001), p. 84, as translated in Viète, *The Early Theory of Equations: On Their Nature and Constitution*, trans. Robert Schmidt (Annapolis, Md.: Golden Hind, 1986), p. 3.

[11] In comparison with music, painting relies on geometry, rather than on explicit arithmetic; this perhaps explains why developments similar to those described in this essay did not happen in the visual arts. For instance, though Piero della Francesa was an important mathematician, his writings do not show any interaction between his innovations in painting and the concept of number. See J. V. Field, *The Invention of Infinity: Mathematics and Art in the Renaissance* (Oxford: Oxford Univ. Press, 1997), p. 67; and Field, *Piero della Francesca: A Mathematician's Art* (New Haven, Conn.: Yale Univ. Press, 2005), pp. 24–31, 282–284, 312–316. See also Ann E. Moyer, "Music, Mathematics, and Aesthetics: The Case of the Visual Arts in the Renaissance," in *Music and Mathematics*, ed. Vendrix (cit. n. 4), pp. 111–146. I thank Don Howard for raising this interesting question, which deserves further attention, and Mark Peterson for sharing his thoughts on Piero.

STIFEL'S TREATMENT OF IRRATIONAL NUMBERS

The earliest explicit mention of "irrational numbers" as a self-conscious term for these mathematical hybrids seems to have been in the *Arithmetica integra* (1544) of Michael Stifel (1487–1567), a former Augustinian monk who left the order and became a friend and collaborator of Martin Luther. Alongside his work as a fervent advocate of ecclesiastical reform (he anagrammatized the name of Pope Leo X to yield 666, the Number of the Beast), Stifel was arguably the most distinguished German mathematician of the sixteenth century; his methods were crucial sources for Recorde. In *Arithmetica integra* Stifel introduced the term "exponent" and used the signs +, –, and $\surd$.[12]

Stifel begins by reviewing "the nature and species of abstract numbers [*numerorum abstractorum*]." From the beginning, he embeds his novel term "irrational numbers" (*numerici irrationales*) in an extensive discussion of music.[13] In book 1 Stifel treats musical intervals in terms of the ratios of string lengths, beginning with the ancient definitions of the octave (1:2), fifth (2:3), fourth (3:4), and tone or whole step (8:9). Because the octave cannot be divided into an integral number of whole tones, the construction of scales requires dividing tones in half, as Boethius described in *De institutione musica* [*Elements of Music*], written circa 510 A.D., which transmitted ancient music theory to the West.[14] But dividing a semitone exactly in half would involve a *geometric* mean that is necessarily irrational (see Figure 1), hence impossible in the context of the pure *arithmetic* ratios of Greek musical theory.[15] Boethius avoided this problem by dividing the tone unequally into a "major semitone" and a "minor semitone," which differ by the tiny interval called the "comma."[16]

[12] Boyer and Merzbach, *History of Mathematics* (cit. n. 9), pp. 281–282; long before John Napier, Stifel seems to have invented logarithms independently. For a detailed treatment see Matthias Aubel, *Michael Stifel: Ein Mathematiker im Zeitalter des Humanismus und der Reformation* (Augsburg: Rauner, 2008). For Recorde and Stifel see Neal, *From Discrete to Continuous* (cit. n. 5), p. 49. Viète seems not to have known Stifel's work; see H. L. L. Busard, "Über einige Papiere aus Viète's Nachlaß in der Parisier Bibliothèque Nationale," *Centaurus*, 1964, *10*:65–126.

[13] Michael Stifel, *Arithmetica integra* (1544), fols. 7v (quotation), 55r–58r (here and throughout this essay, translations into English are mine unless otherwise indicated). Stifel also relies on his interpretation of book 10 of Euclid's *Elements*, which he was one of the first moderns to treat in detail. See J. E. Hofmann, *Michael Stifel (1487?–1567): Leben, Wirken, und Bedeutung für die Mathematik seiner Zeit* (Wiesbaden: Sudhoffs Archiv, 1968); W. Jentsch, "Michael Stifel—Mathematiker und Mitstreiter Martin Luthers," *Esslinger Studien*, 1989, *28*:25–50; and Aubel, *Michael Stifel*, pp. 265–325 (a particularly extensive treatment).

[14] Stifel, *Arithmetica integra*, fols. 70r–75v. Five tones undershoot the octave: $(9{:}8)^5 \approx 1.802 \ldots < 2{:}1$; six tones overshoot: $(9{:}8)^6 \approx 2.027 \ldots > 2{:}1$. The master text is Boethius, *Fundamentals of Music*, trans. Calvin M. Bower, ed. Claude V. Palisca (New Haven, Conn.: Yale Univ. Press, 1989), 3.11. For an excellent treatment of the whole problem of the transmission of ancient music theory see Thomas J. Mathiesen, *Apollo's Lyre: Greek Music and Music Theory in Antiquity and the Middle Ages* (Lincoln: Univ. Nebraska Press, 1999).

[15] Boethius, *Fundamentals of Music*, 3.11. Defining a "superparticular" interval $B{:}C$ to have the form $B{:}C :: n{+}1{:}n$, where n is an integer (and the ratio is expressed in lowest terms), then the problem of dividing such an interval into two equal subintervals is equivalent to the problem of finding a mean proportional number D such that $B{:}D :: D{:}C$. This proof also figures in the *Sectio canonis*, traditionally attributed to Euclid and translated by Giorgio Valla in 1497. See *The Euclidean Division of the Canon*, ed. and trans. André Barbera (Lincoln: Univ. Nebraska Press, 1991), props. 3, 16; it is also included in the anthology *Greek Musical Writings*, ed. Andrew Barker (Cambridge: Cambridge Univ. Press, 1989), Vol. 2, pp. 190–208. See also Barbera, "Placing *Sectio Canonis* in Historical and Philosophical Contexts," *Journal of Hellenic Studies*, 1984, *104*:157–161; and Wilbur R. Knorr, *The Evolution of the Euclidean Elements* (Dordrecht: Kluwer, 1975), Ch. 7.

[16] Modern nomenclature defines an octave as comprising 1,200 cents, of which an equal semitone would be 100 cents, a "major semitone" $(2{,}187{:}2{,}048 = 3^7{:}2^{11})$ 113.7 cents, a "minor semitone" $(256{:}243 = 2^8{:}3^5)$ 90.2 cents, a Pythagorean comma $(3^{12}/2^{19} = 531{,}441/524{,}288)$ 23.5 cents. Boethius's terminology shows that he was aware of these problems in dividing tones. See Gillian R. Evans, "Fractions and Fraction-Symbols in Boethius' *Musica*," *Centaurus*, 1982/1983, *26*:215–217; and Allison M. Peden, "*De Semitone*: Some Medieval Exercises in Arithmetic," *Studi Medievali*, 1994, *35*:368–403.

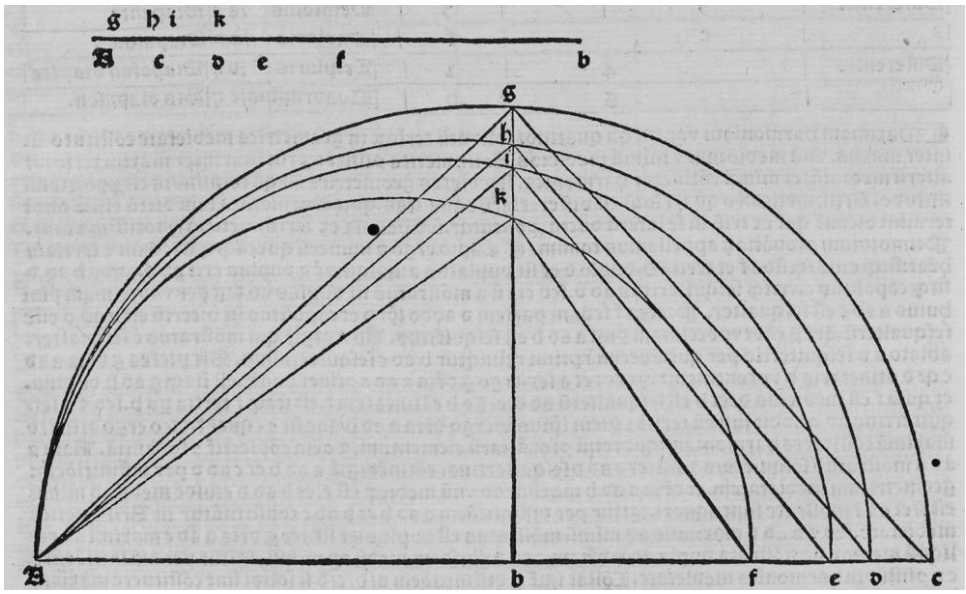


Figure 1. *Jacques Lefèvre d'Étaples's diagram from* Elementa musicalia *(1496), fol. 6gv, demonstrating that interval* Ab:bc *can be divided geometrically exactly in half (*bg*); the upper register shows these lengths deployed along a string. If the two collinear segments* Ab *and* bc *are in the ratio* Ab:bc :: 8:9*, then (Euclid,* Elements*, book 6, proposition 13) erecting the perpendicular bisector* bg *on* Ac *gives the mean proportional* Ab:bg :: bg:bc*. Thus, a string of length* bg *would sound an exact semitone higher than string* Ab*; because* 8:bg :: bg:9*, in modern notation,* bg = $\sqrt{72}$*, hence the "ratio" of a semitone is* 8:$\sqrt{72}$*. (This item is reproduced by permission of the Huntington Library, San Marino, California [Huntington, RB 67813].)*

But Stifel notes that "musicians speak of certain irrational proportions [*proportionibus quibusdam irrationalibus*]," implying that these proportions are already in current musical use (probably mainly theoretically) and hence should be mathematically acceptable. In contrast, earlier theorists had held that "music does not consider irrational proportions."[17] Stifel's statement acknowledges the new *musical* desirability of such equal division, despite its *mathematical* irrationality.

To be sure, Campanus of Novara's 1482 translation of Euclid's *Elements* used the phrase "irrational proportion" to denote "incommensurable quantities."[18] But Euclid never

[17] Stifel, *Arithmetica integra* (cit. n. 13), fol. 76r. At fol. 76r–76v, Stifel instances the schisma (half a comma) and the diaschisma (half a minor semitone) as "irrational proportions"; as with his equal division of the octave (fol. 79r), his concerns may be mainly theoretical rather than practical. The earlier theorist quoted here is the fourteenth-century Johannes de Muris; see Christoph Falkenroth, *Die Musica speculativa des Johannes de Muris* (*Archiv für Musikwissenschaft*, Vol. 34, suppl.) (Stuttgart: Steiner, 1992), pp. 292–301, on p. 294. De Muris notes that "the true half tone does not exist in nature"; see Frank Hentschel, "Die Unmöglichkeit der Teilung des Ganztones in zwei gleiche Teile und der Gegenstand der *Musica Sonora* um 1300," in *Musik—und die Geschichte der Philosophie und Naturwissenschaften in Mittelalter* (Leiden: Brill, 1998), pp. 39–60, on p. 41.

[18] Campanus glosses the definitions to book 5 by noting that *proportio* could apply to sounds (*sonis*) through music; see H. L. L. Busard, *Campanus of Novara and Euclid's* Elements (Wiesbaden: Steiner, 2005), Vol. 1. pp. 160–161. Though the fourteenth-century mathematician Nicole Oresme refers to *proportiones rationales* and *irrationales* and also includes a significant treatment of music, he never connects number with irrationality because "no irrational ratio [*proportio irrationalis*] is found in numbers." See Nicole Oresme, *De proportionibus proportionum and Ad pauca respicientes*, ed. Edward Grant (Madison: Univ. Wisconsin Press, 1966), esp. pp. 60–65, 304–309; and Grant, ed., *Nicole Oresme and the Kinematics of Circular Motion* (Madison: Univ.

overstepped the boundary between rational and irrational, as Stifel does. For example, Stifel divides the tone into equal semitones following the construction given by Jacques Lefèvre d'Étaples (Jacobus Faber Stapulensis), which is based on the Euclidean mean proportional (see Figure 1).[19] Likewise, Stifel applies various arithmetic operations to musical proportions, noting that "in these ways irrational proportions of irrational terms can be computed by rational numbers through this beautiful reckoning [*pulchra ratione*]," including his explicit halving of the tone (see Figure 2).[20] To my knowledge, this is the earliest printed statement that combines a rational (arithmetic) proportion (8:9) with its irrational (geometric) mean (8:$\sqrt{72}$). Acknowledging the controversy about them, Stifel still asserts that

> these halvings are so certain that no one can deny them. Neither Jordanus [Nemorarius] nor Stapulensis nor any other among the learned has denied anything else concerning this question, except that a tone can be divided in two equal parts by means of a certain and constituted number (as they themselves put it), that is, a rational number. Moreover, they do not deny that a tone can be divided by an uncertain number and that is constituted by no assembly of units, that is, by an irrational number. And because any part you please of the aforementioned halvings consists of a certain term or rational [number], and from a term that is uncertain and

Wisconsin Press, 1971), pp. 78–161, 296–305, on p. 297. See also the valuable discussion in Oscar João Abdounar, "Ratios and Music in the Late Middle Ages: A Preliminary Survey," in *Music and Mathematics*, ed. Vendrix (cit. n. 4), pp. 23–69.

[19] Jacobus Faber Stapulensis, *Elementa musicalia* (Paris, 1496), fol. g6v, cited in Stifel, *Arithmetica integra* (cit. n. 13), fol. 76v. Erasmus of Höritz (1506), Heinrich Schreiber (also known as Grammateus) (1518), and Franchino Gafori (Gaffurio) (1518) also referred to this result. See Claude V. Palisca, "Music and Scientific Discovery," in *Music and Ideas in the Sixteenth and Seventeenth Centuries* (Urbana: Univ. Illinois Press, 2006), pp. 131–160, esp. p. 145; and Franchino Gaffurio, *De harmonia instrumentorum opus, liber secundus* (Milan, 1518), fol. XXVv. I thank Thomas Mathiesen for pointing out that, long before Euclid's geometry had been translated, this argument about the divisibility of the tone was well known to students of Boethius. For instance, Jacques of Liège (Jacobus Leodiensis) (ca. 1260–ca. 1330) noted in his *Speculum musicæ* (ca. 1330), the most extended surviving medieval treatise on music, that "musical consonant proportions depend on geometry, not arithmetic; whence the divisibility of a consonance into two equal parts is founded on the divisibility of a proportion into two equal geometrical parts": *Jacobi Leodiensis Speculum musicæ*, ed. Roger Bragard (Corpus Scriptorum de Musica, 3[2]) ([Rome]: American Institute of Musicology, 1961), p. 54. See also F. Joseph Smith, "A Medieval Philosophy of Number: Jacques de Liège and the *Speculum musicæ*," in *Arts libéraux et philosophie au Moyen Âge* (cit. n. 4), pp. 1023–1039. Cf. Abdounar, "Ratios and Music in the Late Middle Ages," p. 50, which holds that "irrational numbers and/or incommensurable magnitudes were arising in musical contexts . . . for the first time with Nicolas of Cusa, who asserts in his *Idiota de Mente* of 1450 that the musical half-tone is derived by geometric division of the whole-tone, and hence is defined as an irrational number." Likewise, David Paul Goldman, "Nicholas Cusanus' Contribution to Music Theory," *Rivista Internazionale di Musica Sacra*, 1989, *10*:308–338, considered Cusanus to have been Lefèvre's source. Yet Cusanus's text notes only that "from the relation of a half-tone [to a full tone]—and from the relation of half a double [proportion], this relation being that of the side of a square to its diagonal—I behold a number that is simpler than our mind's reason can grasp [*numerum simpliciorem intueor quam nostræ mentis ratio attingere queat*]. For [this] relation is not understood without number; yet, that number would have to be both even and odd. A lengthy and very delightful discourse could be held on this topic if we were not hastening onwards toward other points": Jasper Hopkins, *Nicholas of Cusa on Wisdom and Knowledge* (Minneapolis: Banning, 1996), p. 552. Though intrigued, Nicholas did not go further than this general remark.

[20] Stifel computes the result of *divisio per minutias* (division into smaller parts), as when "8 tones and 3 minor semitones are to be divided by 8/5 or (what is the same thing) 8 tones and 3 minor semitones are multiplied by 5/8: this operation makes 5 tones and 15/8 minor semitones. You see here sufficiently that 1/8 of a minor semitone is wanting of the perfection of one diapason [i.e., 16/8]. . . . He who, on the other hand, accepts a schisma, that is, half a comma, can then certainly by that same token accept a half tone, [for] it is clear that a schisma taken with a minor semitone is really half a tone. For two schismata make one comma, but a comma taken with two minor semitones makes up a whole tone": Stifel, *Arithmetica integra* (cit. n. 13), fol. 78v. In his list of symbols for musical intervals (such as – for a semitone, | for a tone), the sign for the halved interval stands on the same footing as the sign for the whole interval, uniting rational and irrational quantities in a single symbology, which is an interesting step toward a kind of algebraic symbolization, though here not alphabetic.

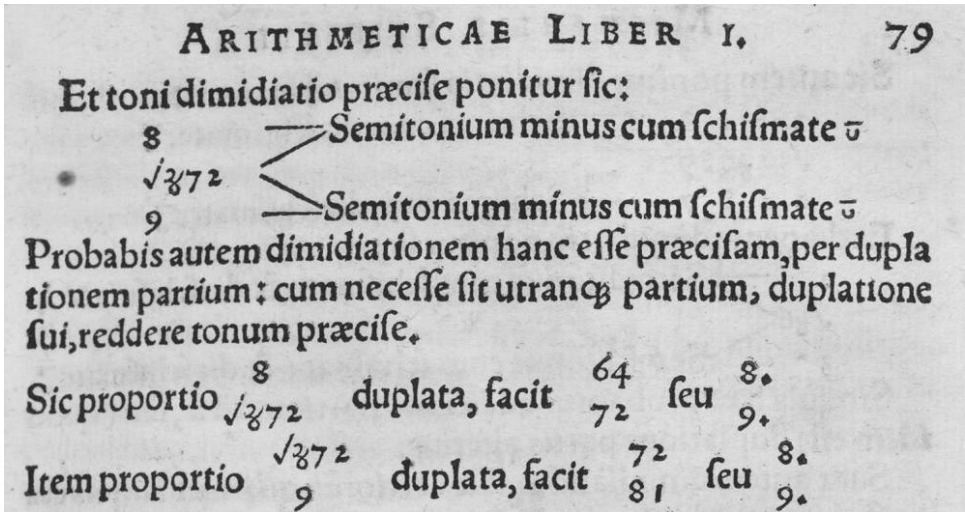


Figure 2. *Michael Stifel's diagram showing the equal division of a whole tone, from* Arithmetica integra *(1544), fol. 79r. Stifel uses the sign* $\sqrt{\ }$ *to denote a square root and describes a schisma as an "irrational proportion," namely "a comma [531,441:524,288] divided in half." Text: "And the tone can precisely taken as halved thus: 8:*$\sqrt{72}$*, minor semitone with a schisma* $\overline{\upsilon}$ *;* $\sqrt{72}$*:9, minor semitone with a schisma* $\overline{\upsilon}$ *. Moreover, you will prove this halving is precise by duplicating the parts, because it is necessary that either part when duplicated gives precisely a tone. Thus, the proportion 8:*$\sqrt{72}$ *duplicated gives 64:72 or 8:9. Also the proportion* $\sqrt{72}$*:9 duplicated gives 72:81 or 8:9." (This item is reproduced by permission of the Huntington Library, San Marino, California [Huntington, RB 707254].)*

> unknown, or irrational, therefore also the parts themselves individually are uncertain and unknown, or irrational, proportions.[21]

Thus, in this musical context, Stifel treats these irrational "halved ratios" as though they are as valid as rational proportions.

Yet when Stifel returns to the larger question of "the essence of irrational numbers," his attitude shifts:

> It is properly debated whether irrational numbers are true numbers or fictions. For if we lack rational numbers in geometrical figures, their place is taken by irrationals, which prove precisely those things that rational numbers could not; certainly from the demonstrations they show us we are moved and compelled to admit that they [irrational numbers] really exist from their effects, which we perceive to be real, sure, and constant.
>
> On the other hand, other things move us to a different assertion, namely that we are forced to deny that irrational numbers are numbers. Namely, where we might try to subject them to numeration and to make them proportional to rational numbers, we find that they flee perpetually, so that none of them in itself can be precisely grasped: a fact that we perceive in the resolving of them, as I will show below in its place. Moreover, it is not possible to call that a true number which is such as to lack precision and which has no known proportion to true numbers. Just as an infinite number is not a number, so an irrational number is not a true number and is hidden under a sort of

[21] *Ibid.*, fol. 79v.

> cloud of infinity. And thus the ratio of an irrational number to a rational number is no less uncertain than that of an infinite to a finite.[22]

Here, the "cloud of infinity" is the infinite sum of fractions needed to represent an irrational quantity; such an "actual infinite" was rejected by Aristotle. Stifel's distaste for this infinitude finally outweighs his geometrical and musical arguments that irrational quantities can take the place of rational numbers in every effective respect. Thus, even though his musical arguments had led him to affirm irrational numbers, his arithmetic concern to avoid the infinite ultimately moved him to demote them from the class of "true numbers."

RATIONALIZING THE IRRATIONAL

Stifel's arguments show the effect of musical considerations on mathematical concerns, indicating the possibility of shifting and surprising alliances between the various parts of the quadrivium: geometric irrationalities, formerly excluded from arithmetic, could find a place in music. Though Stifel himself finally gave precedence to an Aristotelian rejection of the actual infinite, others would take these arguments in a different direction precisely by placing a new and different emphasis on the musical side.

Among these, the famous mathematician, physician, and polymath Girolamo Cardano (1501–1576) has special importance, even though his writings on music are less well known than the rest of his vast output. Though Cardano's *De musica* was published only in his *Opera omnia* (1663), among his works on arithmetic and geometry, he wrote this manuscript during the period (ca. 1546) surrounding the appearance of his most famous mathematics book, *De arte magna* (1545), which announced the general solutions of the cubic and quartic equations (in the midst of notorious controversies about priority and disclosure), a landmark in the development of modern algebra. Thus Cardano's presentation of what he modestly called "this most abstruse and clearly unsurpassed treasury of the entire arithmetic" should be read next to his contemporaneous musical work, which is notable for its emphasis on practical techniques related to musical instruments as well as its theoretical considerations.[23]

Cardano sang and played several instruments, including the recorder and the lyra, and was a skilled composer, as is shown by several compositions he includes in *De musica* and his careful accounts of instrumental techniques. Cardano's awareness of such modes of ornamentation as trills and vibrato draws attention to microtonal shifts that singers and instrumentalists used to decorate their melodies. He particularly emphasizes the unusual interval of a "diesis," a quarter tone (half a semitone) that produces "such a movement [that] titillates the ear and increases its pleasure." As Clement Miller notes, "his affection

[22] *Ibid.*, fol. 103r.

[23] *Hieronymi Cardani Mediolensis opera omnia* (Lyon, 1663; rpt., New York: Johnson, 1967), Vol. 10, p. 222, as translated in Girolamo Cardano, *The Great Art of the Rules of Algebra*, ed. and trans. T. Richard Witmer (Mineola, N.Y.: Dover, 2007), p. 1. For the background to these problems see Pesic, *Abel's Proof* (cit. n. 7), pp. 23–40; for the larger context see Anthony Grafton, *Cardano's Cosmos: The Worlds and Works of a Renaissance Astrologer* (Cambridge, Mass.: Harvard Univ. Press, 2000). For the dating of Cardano's musical writings (which include a later manuscript also entitled *De musica*) see Jerome Cardan, *Writings on Music*, ed. and trans. Clement A. Miller ([Rome]: American Institute of Musicology, 1973), p. 19; and Moyer, Musica Scientia (cit. n. 4), pp. 158–168. Oystein Ore, *Cardano: The Gambling Scholar* (New York: Dover, 1965), p. 49, notes that in the description of his works Cardano included his book on music in the category of mathematics.

for this tonal embellishment was very great and his description of the beauty and pleasantness of the effect sometimes borders on the ecstatic."[24]

Cardano's predilection for the diesis led him to put forward new opinions about its definition and that of the semitone, though he cites the mathematical problem of the exact divisibility of ratios. To divide a tone into two equal semitones (or a semitone into two equal quarter tones) "correctly and arithmetically [*verè & arithmeticè*]," he acknowledges that a "true calculation" involves an irrational root, for which he accepts a rational approximation that is "closer in perception." For these "true" irrational intervals (whether of semitone or diesis), he empirically substitutes a simple rational approximation, thus conflating the geometrically irrational with the arithmetically rational. He treats the result as a "correct" system of tuning, not merely a stopgap or approximation; in fact, the application of his calculation of his approximate semitone (18/17) to fretting a lute was "the first really practical approximation of equal temperament," later (incorrectly) attributed to Vincenzo Galilei.[25] Cardano treats rational and irrational intervals on the same footing primarily because of musical considerations: he chooses between rational approximations for the diesis *not* on the basis of closeness of numerical value (which would lead to 35/34) but on perception as judged musically (leading to 36/35). He calls "true" (*verè*) *both* the irrational "true diesis" and its rational, musical equivalent, which is true "arithmetically."

Though in *De arte magna* Cardano often refers to "numbers" with the sense of "integers" and never uses the term "irrational numbers," he uses the phrase "the numbers [*numerici*] that were to be found" to refer to specifically irrational expressions. Cardano's "golden rule" shows how to find what he calls the "closest approximation" through finding the integers, "greater and less, which most nearly satisfy the equation," then generating a series of "differences" between the values those integers generate when substituted in the equation, from which a further refined estimate can be made, leading to what he takes to be a converging series of approximations. Through this procedure, "you will undoubtedly arrive at an insensible [*insensibiliem*] difference" compared with the true value: "This is universal reasoning and needs no other rule."[26] His procedures here seem to reflect the

[24] Cardano might well have been in contact with his father's colleague Gafori, the celebrated music theorist mentioned above, who commissioned translations of Aristides Quintilianus and Ptolemy; under the influence of Ficino and Neo-Platonism, Gafori engaged in extensive speculation on musical cosmology but was also, like Cardano, interested in practical music. See Cardan, *Writings on Music*, p. 15 n 1; and Moyer, Musica Scientia, pp. 67–92. For Cardano on the diesis see *De tranquillitate*, in Cardano, *Opera omnia*, Vol. 2, p. 337, cited in Cardan, *Writings on Music*, p. 22 n 36, which also includes the quotation from Miller.

[25] Cardan, *Writings on Music*, p. 45, cites Lefèvre. Cardano's argument proceeds: "But if I wish to divide a proportion by itself, as the whole tone 9/8, I can do this correctly and arithmetically [*verè & arithmeticè*] only if both numerator and denominator are doubled, making 18/16. Then you take a number equidistant from each, 17 in this instance, and the proportion 18/16 [*sic*; this should read 17/16 or 18/17] is a semitone or one-half of 9/8. According to this a diesis will be 36/35 or 35/34, as you wish, but it is closer in perception [*sensibilis*] to 36/35, although the true [*vera*] diesis is closer to 35/34. I said closer because a true calculation of one-half of 9/8 is made by multiplying 9 times 8 to make 72, and by taking its square root or R72, and the latter's proportion to 8 or R72/8 is the true semitone. In the same way a diesis will be RR4608/8, which is very close to 239/232 or to 35/34, and closer to 35/34 than to 36/35." *Ibid.*, pp. 47–48 (Cardano, *Opera omnia*, Vol. 10, p. 108). The textual error noted in this quotation is verified by Cardano's statement on p. 45 that the "small semitone" is 18/17. He writes ratios as fractions, though Greek mathematics emphatically did not treat them as equivalent, and notates R for square roots, RR for fourth roots. The quotation about the dating of the resultant temperament comes from J. Murray Barbour, *Tuning and Temperament* (East Lansing: Univ. Michigan Press, 1951), p. 7.

[26] For the phrase "the numbers [*numerici*] that were to be found" see Cardano, *Opera omnia*, Vol. 4, p. 281, translated in Cardano, *Great Art of the Rules of Algebra* (cit. n. 23), p. 203. Cardano calls irrational quantities "*irrationales*" in the 1545 edition but "*alogi*" in the 1663 edition (pp. 35 n 10, 41 n 24, 47); he does not use these terms as adjectives for "*numerici*." For his comments about the "insensible difference" and "universal reasoning" see pp. 182, 185.

practical sense that comes to the fore in his musical writings: "true" geometric (irrational) and "true" arithmetic (rational) quantities *sound* the same; their differences are "insensible." The sole example he has of this kind of mediation between "perceptual" and "true" quantities is music; he lacks any other kind of mathematical physics (as it would later come to be called) that could have confronted mathematical idealizations with physical reality. But music was sufficient. Musical judgments intermixed rational and irrational quantities, supporting and paralleling Cardano's working equivalence of the two in his algebraic art and providing it with crucial examples.

THE TRIAL OF VICENTINO

The grounds on which Nicola Vicentino (1511–ca. 1576) treated irrational quantities as numbers had more to do with issues of melodic style and musical practice, as for Cardano, than with the more purely theoretical questions about the elementary intervals that concerned Stifel. Boethius had enumerated three ancient "genera" of melody, each genus designating a separate set of basic intervals on which music could be constructed. The most familiar is the diatonic genus, based on the pattern of a semitone followed by a tone and another tone: S T T. (Modern major and minor scales are diatonic, though neither was among the ancient modes.) The other two genera are more unfamiliar. The chromatic genus has the pattern S S S^3, where S^3 stands for a "trihemitone" (an interval composed of three semitones). According to Boethius, this genus is "called 'colored' since it is the first alteration from" the diatonic genus, which "is somewhat more severe and natural [*durius et naturalius*], whereas the chromatic departs from natural inflection and becomes more sensual [*mollius*]."[27] The name "chromatic" persists even today to describe music that makes extensive use of consecutive semitones, sometimes (as Boethius suggested) to evoke greater sensuality or expressivity.

No such parallel remains in our music corresponding to the enharmonic genus, which Boethius considers "even more closely joined" than the chromatic, in the sense that the enharmonic genus uses the quarter tone (the diesis, abbreviated D), according to the pattern D D T^2, where T^2 denotes a "ditone," an interval composed of two whole steps. Apart from some self-conscious attempts to recreate such music that we will come to and some experimental music of the twentieth century, the diesis fell out of use in Western music. Yet Boethius does not treat it as exotic but only remarks that it "is beautifully and fittingly yoked together"; indeed, its Greek name (ἁρμονία) is the general word that has come down to us as "harmony," suggesting that the enharmonic genus was considered harmonious *par excellence*.[28]

[27] Boethius, *Fundamentals of Music* (cit. n. 14), 1.21. The ancient Dorian species of the octave used this pattern in the form E^F-G-A-B^C-D-e, in which ^ marks a semitone, - a tone (hence, in this case, S T T T S T T); on the modern keyboard it would be sounded by the sequential white keys starting on E. Note that the diatonic pattern S T T is not rigidly repeated but sets a general design for the octave species: in the diatonic genus, only tones and semitones are used and every semitone is surrounded by two sequential tones on either side. The other ancient octave species follow this same diatonic pattern of tones and semitones, though beginning at a different point along the pattern; thus, the ancient Phrygian octave species (now usually stated beginning on the note D) runs T S T T T S T (D-E^F-G-A-B^C-d). This modern representation is anachronistic not only in its use of present note names (hence also raising questions of the underlying temperament) but also in stating this mode as a "scale" ranging over an octave; ancient and medieval music theory characteristically stated modes in terms of tetrachords (four sequential notes) or hexachords (six-note scalewise groups), not octaves, as we tend to do. Note also that Boethius uses for his semitone 243:256, the interval between a ditone and a perfect fourth, which is not an exact equal division of the tone. The trihemitone is 294.1 cents, slightly smaller than the modern minor third, 300 cents.

[28] *Ibid*. The ditone is 407.8 cents, slightly larger than the modern major third, 400 cents. The Czech composer

Before the sixteenth century, only the diatonic genus seems to have evoked commentary, perhaps because it was used in musical practice, such as chant; in general, it may be that Boethius's passing mention of the genera was taken to indicate that this was an obscure or recondite matter, hence usually neglected. New translations brought these genera to prominence—for example, Carlo Valgulio's Latin translation (1507) of the *De musica* commonly (but incorrectly) attributed to Plutarch, from which Vincenzo Galilei later drew much of his information on Greek music.[29] Pseudo-Plutarch emphasized the superiority of the enharmonic genus and complained that "the musicians of our times, though, disdained the most beautiful genus of all and the most fitting, which the ancients cherished for its majesty and severity."[30] In his book *L'antica musica ridotta alla moderna prattica* [*Ancient Music Adapted to Modern Practice*] (Rome, 1555), Vicentino identified the enharmonic as the secret behind those extraordinary, lost powers of ancient music, which he decided to revive.

Vicentino was a practicing musician and a composer with deep interests in the theory of music (see Figure 3). Born in Vicenza, he came under the influence of the humanist Giovanni Giorgio Trissino, who in 1524 had already described the enharmonic and the chromatic as "two genera that our age does not know." After studies with Adrian Willaert, the great Venetian composer, Vicentino came to Ferrara at the behest of Cardinal Ippolito II d'Este, whom he then accompanied to Rome. In 1546 Vicentino published his first book of madrigals, but around 1534 he had already begun thinking about the ancient genera.[31] In 1551 he became embroiled in a public controversy that shows the extent to which these matters provoked hot contention among the educated elite throughout Europe.

The argument started in June 1551 after a performance of a motet at the home of Bernardo Acciaioli in Rome, when those present began discussing what genera of melody were used in the composition. The Portuguese composer and theorist Vincente Lusitano maintained that the motet used only the diatonic genus, whereas Vicentino argued that it used elements of all three genera. What was at stake went beyond this single work to all of contemporary practice: What was the true status of those ancient genera in contemporary music? The broader implications of this question concerned not only whether modern music had kept or broken faith with its ancient heritage but also the character and integrity of the cosmos, which was widely assumed to be regulated by musical intervals.

The debate began with a wager of two gold scudi and quickly became formal and public. Over a period of five days (2–7 June), Vicentino and Lusitano presented their arguments at the Vatican to "an audience of many learned men," in the presence of Cardinal Ippolito and

---

Alois Hába (1893–1973) wrote a number of works using quarter tones (and other microtonal possibilities), as did Charles Ives (1874–1954); Béla Bartók occasionally indicated quarter tones in his works and, as a pioneering ethnomusicologist, endeavored to indicate microtonal nuances in transcriptions of his fieldwork. Note that the Greek word ἁρμονία did not originally designate simultaneously sounding pitches, as does the modern word "harmony," but, rather, describes the quality of a single melodic line.

[29] See the detailed discussion in Claude V. Palisca, *Humanism in Italian Renaissance Musical Thought* (New Haven, Conn.: Yale Univ. Press, 1985), pp. 88–110; he comments on Vincenzo on p. 10 n 35. For an outstanding account of the theoretical issues see Karol Berger, *Theories of Chromatic and Enharmonic Music in Late Sixteenth Century Italy* (Ann Arbor, Mich.: UMI Research Press, 1980). For a detailed discussion of Vicentino's music see Manfred Cordes, *Nicola Vicentinos Enharmonik: Musik mit 31 Tönen* (Graz: Akademische Druck- und Verlangsanstalt, 2007).

[30] Cited from Palisca, *Humanism in Italian Renaissance Musical Thought*, p. 109.

[31] Note the variant spelling "Vincentino" found in some sources. The quotation from Trissino comes from Palisca, *Humanism in Italian Renaissance Musical Thought*, p. 119. See esp. the detailed account in Henry William Kaufmann, *The Life and Works of Nicola Vicentino (1511–c. 1576)* (Washington, D.C.: American Institute of Musicology, 1966). For the date 1534 see Nicola Vicentino, *Ancient Music Adapted to Modern Practice*, trans. Maria Rika Maniates, ed. Claude V. Palisca (New Haven, Conn.: Yale Univ. Press, 1996), p. 33 n 25.

Figure 3. *Portrait of Nicola Vicentino, aged forty-four, as the frontispiece of his book* L'antica musica ridotta alla moderna prattica *[*Ancient Music Adapted to Modern Practice*] (Rome, 1555). In the outer border, his motto reads: "You have revealed to me the uncertain and hidden things of Thy science." In the inner border, he is identified as "inventor of the archicembalo and also of the practical division of the chromatic and enharmonic genera."*

"judges" who were singers in the chapel of Pope Julius III.[32] This tribunal found against Vicentino in a statement that reads (given the ecclesiastical authority of the presiding cardinal

[32] Vicentino, *Ancient Music Adapted to Modern Practice*, pp. 302–314, gives Vicentino's account; the passage cited comes from p. 303. The "judges" were Bartolomeo Escobedo and Ghiselin Danckerts, who, though he was not present, later studied statements prepared by both sides; see also Moyer, Musica Scientia (cit. n. 4), pp. 168–184. The young Orlando di Lasso may have been in attendance, according to Kaufmann, *Life and Works of Nicola Vicentino*, p. 24 n 5; but he may have arrived later in 1551, as claimed by Horst Leuchtmann, *Orlando di Lasso: Music der Renaissance am Münchner Fürstenhof* (Wiesbaden: Reichert, 1982), p. 112. In either case, Lasso's extraordinary chromatic *Prophetiæ Sibyllarum* dates from 1550–1552 and may well show the influence of Vicentino and the controversy surrounding him.

and the papal offices of the judges) like a legal anathematization, concluding that "the said Don Nicola must be condemned, as we sentence him in the wager made between them" (see Figure 4).[33] Vicentino's own account treats quite seriously what he considered a grave injustice, whether or not we take the scene as an *auto-da-fé* for "heretical pravity" that anticipates the trials of Bruno or Galileo. Perhaps Vicentino's wounded pride kept him from taking the less serious tone others may have adopted. But even a high-spirited imitation of inquisitorial proceedings presided over by an eminent cardinal seems ominous. The Church, especially the Jesuits, condemned any alterations to the foundations of mathematics that would undermine its epistemic certainty and hence the unchanging rational foundations of Christian doctrine. We will shortly consider Vicentino's argument that experience was the "mistress" of musical and mathematical theory—rather than pure reason, as Boethius taught and the Church insisted.[34]

The confrontation at the Vatican led Vicentino to publish his defense for a larger public interested in the case and willing to pay to read about it. By way of amplifying and illustrating his assertions, Vicentino described his newly invented archicembalo, an "arch-harpsichord" whose specially designed keyboard could play the complex variety of semitones and dieses necessary to execute chromatic and enharmonic compositions (see Figure 5).[35] Vicentino's keyboard mechanized the playing of quarter tones, heretofore

[33] Vicentino, *Ancient Music Adapted to Modern Practice*, pp. 313–314. Danckerts later wrote a treatise explaining his reasoning; see Paul Anthony Luke Boncella, "Denying Ancient Music's Power: Ghiselin Danckerts' Essays in the 'Generi Inusitati,'" *Tijdschrift van de Vereniging voor Nederlandse Muziekgeschiedenis*, 1988, *38*:59–80. Danckerts's reasoning was that Vicentino's definition of the genera was overly broad, encompassing more examples than really followed the genera strictly, and he insisted that a proper example of the enharmonic genus must use the diesis. See also Timothy R. McKinney, "Point/Counterpoint: Vicentino's Musical Rebuttal to Lusitano," *Early Music*, 2005, *33*:393–411.

[34] I thank Amir Alexander for pointing out these issues. For the Jesuits' reliance on the certainty of mathematics see Romano Gatto, *Tra scienza e immaginazione: Le mathematiche presso il collegio gesuitico napoletano (1552–1670 ca.)* (Florence: Olschki, 1994), pp. 17–64. Their mathematical curriculum included the study of music following Lefèvre, as noted in Ladislaus Lukacs and Giuseppe Cosentino, *Church, Culture, and Curriculum: Theology and Mathematics in the Jesuit* Ratio Studiorum (Philadelphia: St. Joseph's Univ. Press, 1999), pp. 50–51; see also Peter Dear, *Discipline and Experience: The Mathematical Way in the Scientific Revolution* (Chicago: Univ. Chicago Press, 1995), p. 39. In 1641 Paolo Guldin, S.J., attacked Bonaventura Cavalieri's theory of indivisibles as a geometry of the eye, rather than of the intellect. See Carl B. Boyer, *The History of the Calculus and Its Conceptual Development* (New York: Dover, 1959), pp. 121–124; and Enrico Giusti, *Bonaventura Cavalieri and the Theory of Indivisibles* (Bologna: Cremonese, 1980), pp. 45, 55–65, esp. p. 63. Indivisibles verged on a mathematical atomism considered heretical and inconsistent with Christian doctrines, especially with the Eucharist. See Egidio Festa, "La querelle de l'atomisme: Galilee, Cavalieri, et les jesuites," *Recherche*, 1990, *21*:1038–1047; and Festa, "Quelques aspects de la controverse sur les indivisibles," in *Geometria et atomismo nella scuola Galileana*, ed. M. Bucciantini and M. Torrini (Florence: Olschki, 1992), pp. 193–207. These matters were vehemently suspect even outside Catholic lands: in 1591, Robert Parsons denounced the English algebraist Thomas Harriott as an Epicurean atheist and conjurer because of his mathematical atomism. See Neal, *From Discrete to Continuous* (cit. n. 5), pp. 25–27; and Robert Kargon, *Atomism in England from Harriott to Newton* (Oxford: Clarendon, 1966), p. 27.

[35] See Marco Tiella, "The Archicembalo of Nicola Vicentino," *English Harpsichord Magazine*, 1975, *1*:134–144; Rudolf Rasch, "Why Were Enharmonic Keyboards Built? From Nicola Vicentino (1555) to Michael Bulyowsky (1699)," *Schweizer Jahrbuch für Musikwissenschaft*, 2002, *22*:35–93; Denzil Wraight, "The *Cimbalo Cromatico* and Other Italian String Keyboard Instruments with Divided Accidentals," *ibid.*, pp. 105–136; and Patrizio Barbieri, "The Evolution of Open-Chain Enharmonic Keyboards, *c* 1480–1650," *ibid.*, pp. 145–184. The split black keys of Vicentino's instrument were not unique; other keyboards had earlier used this device, for nonequal temperaments distinguish notes like G♭ and F# that equal temperament conflates into what is now called a single "enharmonically equivalent" pitch G♭/F#, though this modern term here indicates an effective equivalence between these pitches that reverses the ancient meaning of this word (and its denotation of quarter-step distinctions). Vicentino's intervals are played on a modern archiorgano in the CD attached to Cordes, *Nicola Vicentinos Enharmonik* (cit. n. 29).

LIBRO QVARTO

SENTENTIA.

In Christi nomine inuocato &c. Noi sopradetti Bartholomeo Esgobedo, & Ghisilino Dancharts, per questa nostra diffinitiua sententia & laude in presentia della detta congregatione, & delli sopra detti Don Nicola, & Don Vincentio, presenti intelligenti, audienti, & per la detta sententia instanti. Pronontiamo, sententiamo il predetto Don Nicola non hauer in uoce, ne in scritto prouato sopra che sia fondata la sua intentione della sua proposta. Immo per quanto par in uoce & in scriptis il detto Don Vincentio hà prouato, che lui per uno competentemente cognosce & intende di qual Genere sia la compositione che hoggi communamente i Compositori compongono, & si canta ogni di, come ogniuno chiaramente disopra nelle loro informationi potrà uedere. Et per questo il detto Don Nicola douer essere condennato, come lo condenniamo nella scommessa fatta fra loro, come disopra. Et cosi noi Bartholomeo & Ghisilino soprascritti ci sotto scriuiamo di nostra mano propria. Datum Romæ in Palatio Apostolico, et Capella prædetta, Die VII. Iunij. Anno suprascripto Pontificatus S. D. N. D. Iulij. PP. III. Anno secundo: & laudamo.

Pronuntiaui ut supra. Ego Bartholomeus Esgobedo, et de manu propria me subscripsi.

Pronuntiaui ut supra. Ego Ghisilinus Dancherts, & manu propria me subscripsi.

Io Don Iacobo Martelli faccio fede, come la sententia et le due polize sopra notate sono fidelmente impresse & copiate dalla Copia della medesima sententia de i sopra detti Giudici.

Io Vincenzo Ferro confirmo quanto di sopra.

Io Stefano Bettini detto il Fornarino, confirmo quanto di sopra.

Io Antonio Barrè confirmo quanto di sopra.

Io non uoglio dire cosa alcuna circa la sopra stampata sententia, perche questa cura lasciero giudicare al mondo, & al gran Iddio, ilquale è somma giustitia, che per suo mezzo farà cognoscere à ogniuno la ragione & il torto, come ha inspirato il sopradetto Don Vincentio, che publichi al mondo le mie raggioni con la sua opera stampata, contra de lui & delli Giudici.

Fine del quarto Libro.

Figure 4. *The sentence passed against Vicentino, as recorded in* L'antica musica ridotta alla moderna prattica, *fol. 98v.*

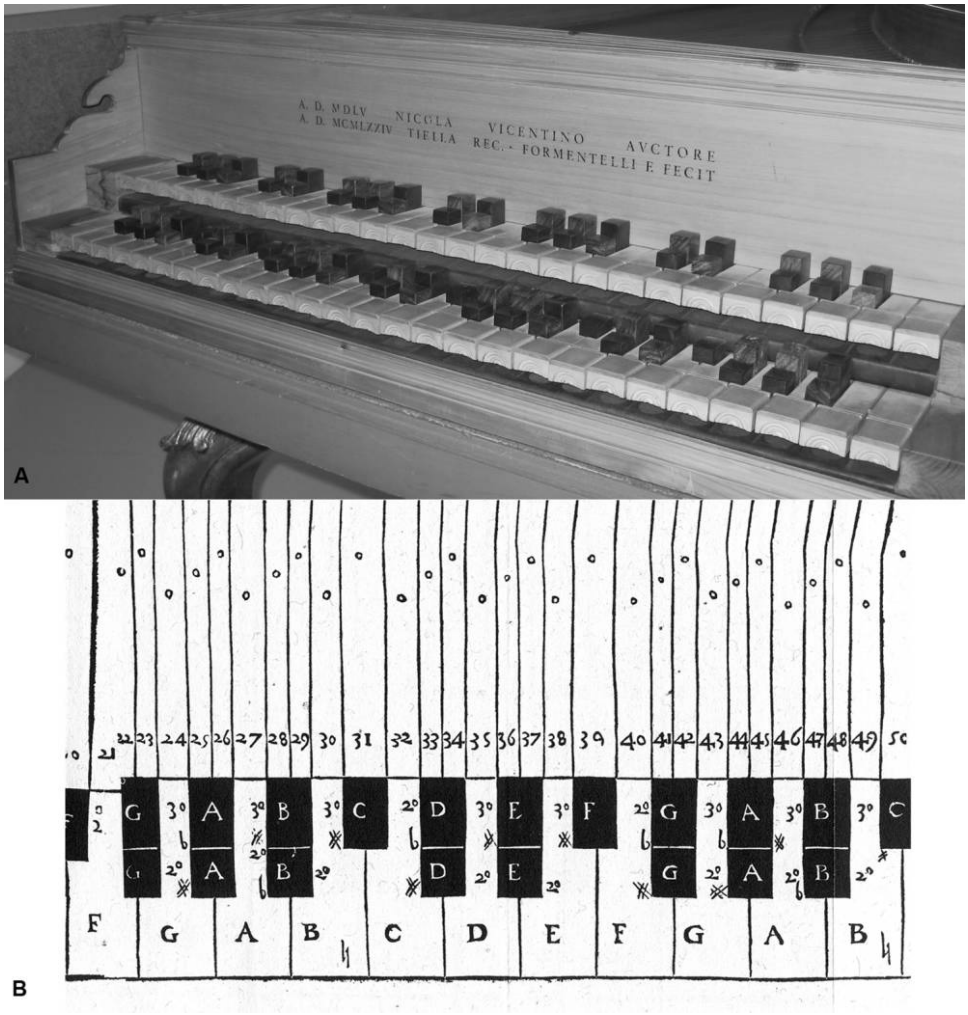


Figure 5. *Vicentino's archicembalo. (A) Detailed view of the split keys of an archicembalo reconstructed in 1974, following Vicentino's design, by Marco Tiella and made in the workshop of the organ builder Barthélemy Formentelli (Pedemonte, Verona). (Courtesy of Marco Tiella.) (B) Schematic layout of a section of a keyboard shown in a fold-out page from Vicentino,* L'antica musica ridotta alla moderna prattica. *The split black keys allow the playing of intervals of a diesis.*

laboriously measured and sounded one by one.[36] His new instrument enabled accurate renditions of enharmonic music, but tuning it required deciding the exact interval of a diesis. To do so, Vicentino needed to unearth the work of ancient theorists who addressed these musical and mathematical questions.

[36] In *De subtilitate* (1550), Cardano described with interest Vicentino's new instruments, along with the innovations of Lucretia Todescha, a young girl from Bologna, who added six strings to the lute, allowing new possibilities of intonation; see Cardan, *Writings on Music* (cit. n. 23), pp. 194–195.

THE CONFLICTING CLAIMS OF HEARING AND REASON

In the mid-sixteenth century, the problem of dividing the tone was not yet solved uniquely; several conflicting definitions of the semitone remained in use.[37] This aggravated the problem of defining the diesis: how could one define a quarter tone if the half tone remained so contentious? The obvious approach was to define unequal major and minor dieses by dividing up major and minor semitones, but this would lead to an endless recurrence of the problem of dividing intervals, *mise en abîme*.[38] New clarity was sought in the ancient sources. What follows, then, is not a digression into antiquity but an account of how ancient problems returned to life.

Boethius had not given the enharmonic diesis a precise ratio, perhaps because of its very smallness.[39] Plato and Aristotle considered the diesis a kind of element, analogous to a vowel or consonant; Aristoxenus of Tarentum (fourth century B.C.), a pupil of Aristotle, judged that "the voice cannot distinctly produce an interval even smaller than the smallest diesis, nor can the hearing detect one, in such a way as to grasp what part it is either of a diesis or of any of the other intervals which are known."[40] The diesis, it was concluded, may be so small an interval that strict, secure definition is elusive. This judgment may also reflect the material circumstances and difficulties surrounding the production of this interval. Aristides Quintilianus (first century A.D.) noted that the enharmonic "has gained approval by those most distinguished in music; but for the multitude, it is impossible. On this account, some gave up melody by diesis because they assumed through their own weakness that the interval was wholly unsingable."[41] Thus, even in ancient times the diesis involved discrimination and virtuosity, as in the quarter-tonal "bending" of pitch possible on the aulos (a pipe with finger holes and a reed mouthpiece, often played in pairs). Aristotle described the aulos as "orgiastic," its shrill wails often associated with Bacchic and Corybantic rites; Longinus wrote that the aulos could send its listeners out of their

[37] Vicentino's contemporaries were most familiar with just intonation, in which the "just diatonic semitone" (16:15 ≈ 1.067, 111.7 cents) was a common solution, but there were other competing possibilities, such as the "just chromatic semitone," sometimes called the "minor semitone of the minor tone" (25:24 ≈ 1.042, 70.7 cents).

[38] Vicentino defined a minor diesis as "one-half of the minor semitone" and the major diesis as "identical" to a minor semitone, but then we still have to divide that minor semitone exactly in half; and, as Karol Berger points out, this leaves much still unspecified about how these two dieses should then be used to make up the enharmonic genus. See Vicentino, *Ancient Music Adapted to Modern Practice* (cit. n. 31), pp. 59–62 [fols. 17r–18v]; and Berger, *Theories of Chromatic and Enharmonic Music* (cit. n. 29), pp. 7–18. For a helpful treatment of another *seicento* author concerned with the division of a tone see Tito M. Tonietti, "The Mathematical Contributions of Francesco Maurolico to the Theory of Music of the Sixteenth Century (The Problems of a Manuscript)," *Centaurus*, 2006, *48*:149–200.

[39] Boethius cites "Philolaus, a Pythagorean," who divided the tone unequally; confusingly, Boethius here uses the term "diesis" as a synonym for "minor semitone," *not* "quarter tone," as he does elsewhere. See Boethius, *Fundamentals of Music* (cit. n. 14), pp. 96–97; and the discussion of Philolaus's arguments in M. L. West, *Ancient Greek Music* (Oxford: Clarendon, 1992), pp. 235–236. Boethius argues that the Pythagorean comma is the "ultimate interval heard which can really be perceived": Boethius, *Fundamentals of Music*, pp. 96–97.

[40] See Plato, *Republic* 531a; Aristotle, *Posterior Analytics* 84b37–39; and Aristotle, *Metaphysics* 1016b18–24: in Barker, ed., *Greek Musical Writings* (cit. n. 15), Vol. 2, pp. 55 (Plato), 70, 72 (Aristotle), 135 (Aristoxenus). See also Aristides Quintilianus, *On Music, in Three Books*, trans. Thomas J. Mathiesen (New Haven, Conn.: Yale Univ. Press, 1983), pp. 81, 95. Plato, *Timaeus* 34b–36d, describes the diatonic order imposed by the demiurge on the cosmos as emerging naturally from Archytas's proportion theory (note Barker's comment in Barker, ed., *Greek Musical Writings*, Vol. 2, p. 217 n 31).

[41] Aristides Quintilianus, *On Music*, p. 84; this passage is also included in Oliver Strunk, ed., *Source Readings in Music History*, rev. Leo Treitler (New York: Norton, 1978), p. 57. In 1560, Pierre de Ronsard wrote King François II that the enharmonic "for its difficulty was never in use," whereas the chromatic "for its lasciviousness was by the ancients banished from the republic," while the diatonic was "by all approved, as approaching nearest to the melody of the macrocosm": *ibid.*, p. 301.

minds and set their feet tapping to its rhythms.[42] Such associations would not militate toward fussiness in intonation, if indeed the diesis was an ecstatic "bending" of a pitch not really to be measured by any ratio but only by the inspired frenzy of the Dionysian virtuoso.

Though this aural interpretation goes against Pythagorean tradition and its ratios, its ancient champion was Aristoxenus, the great contrarian voice in Greek music theory. Where the Pythagoreans exalted reason over sensual judgment, Aristoxenus emphasized the fundamental role of the senses (here showing Aristotle's influence): "Through hearing we assess the magnitude of intervals, and through reason we apprehend their functions."[43] Most of the information we have on the enharmonic diesis comes from him, suggesting that this interval may have fit particularly well into his thesis that discriminative hearing, rather than predetermined, fixed ratios, really determines musical intervals.[44]

Aristoxenus stood at a critical point in the problem of subdividing intervals, which (as we have seen) involves irrational magnitudes if the divisions are to be strictly equal. In the face of this paradox, Aristoxenus opened the liberating possibility that we might weaken or abandon the demand that every interval be strictly rational through his reliance on the sense of hearing, rather than on reason. He himself never seems to remark that his line of argument would imply the possibility of quantities that might bridge the rational and irrational; on the contrary, he explicitly maintains a sharp "division . . . in respect of the differences between the rational and the irrational."[45]

This "Aristoxenian turn" from regarding intervals as ratios to regarding them as "pitch distances" can be more clearly discerned in Claudius Ptolemy's *Harmonics* (second century A.D.), in which the great astronomer connects stars, planets, and musical genera. In Ptolemy's view, Aristoxenus constructed the enharmonic genus by essentially assigning the unit of 6 to each diesis, in units where the tone is 24 units.[46] If so, each interval is no

[42] The word δίεσις originally meant a "letting through," suggesting the performance practices of wind instruments. See Aristotle, *Politics* 134a21; and Longinus, *On the Sublime* 39.2: in West, *Ancient Greek Music* (cit. n. 39), pp. 81–107, esp. pp. 105–106.

[43] Barker, ed., *Greek Musical Writings* (cit. n. 15), Vol. 2, pp. 150. Though ancient Roman scholars referred to Aristoxenus simply as "the musician," most of his numerous works were lost and his *Elements of Harmonics* was not available until Antonio Gogava's Latin translation of 1562—hence after Vicentino's book appeared—though Valgulio had quoted a few fragments from him (1507) and spoken in his praise. Boethius generally followed mainstream Pythagorean tradition; he discusses Aristoxenus briefly and dismissively, noting that "since he attached little value to reason but yielded to aural judgment, he does not indicate numbers for pitches as a means of obtaining their ratios. Instead, he estimates the differences between them. Very incautiously he considered that he knew the differences between pitches for which he had established no magnitude or measure": Boethius, *Fundamentals of Music* (cit. n. 14), p. 173. For the later reception of Arixtoxenus see Claude V. Palisca, "Sense over Reason: The Anti-Theoretical Reaction," in *Music and Ideas in the Sixteenth and Seventeenth Centuries* (cit. n. 19), pp. 29–48; and Palisca, "Aristoxenus Redeemed in the Renaissance," in *Studies in the History of Italian Music Theory* (Oxford: Clarendon, 1994), pp. 189–199, which discusses Valgulio's defense of Aristoxenus (pp. 190–191), though noting that Aristoxenus "received little attention from music theorists until the 1580s and 1590s" (p. 189).

[44] For Aristoxenus's references to the diatonic see Barker, ed., *Greek Musical Writings*, Vol. 2, pp. 135, 140, 143, 145, 154, 165–166, 182, 184. Barker argues that "though Aristoxenus insists that the smallest usable interval is the enharmonic diesis, a quarter-tone, he is perfectly prepared to employ smaller intervals, down to one twelfth of a tone, in his theoretical calculations" (p. 135 n 50). Aristoxenus writes that "the student should try to accept each of these [definitions of note, intervals, and their combination in a *systēma* or concatenation of intervals] in the right spirit, without quibbling over whether the account offered of each is exact or only rather approximate. . . . For it is difficult, perhaps, in all cases where we are dealing with things that stand at the beginning to articulate an account that contains an exhaustive and accurate interpretation" (p. 135).

[45] *Ibid.*, p. 137. Note also the discussion of rational and irrational rhythms in Aristoxenus's fragmentary *Elementa rhythmica*: *ibid.*, p. 188.

[46] Ptolemy, in *Harmonics*, expresses intervals as ratios, *contra* Aristoxenus, as noted in Barker's commentary:

longer a ratio but some multiple of a unit fixed essentially arbitrarily (here, by assigning 24 of them to a tone). By breaking away from Pythagorean ratios, Aristoxenus took a crucial step toward treating an arbitrary musical quantity *as a unit unto itself, apart from whether it is or is not rational* with respect to the initial integral units of string length. His demonstrations recall Euclid, who had shown in the *Elements* that magnitudes are rational or irrational only relative to other magnitudes, not in any absolute sense. The diagonal of a square is incommensurable with its side but may be perfectly commensurable with other lines (for instance, with the sides of another square built on that diagonal).[47] If Ptolemy is correct that Aristoxenus treated the diesis as a unit on which tone and semitone are built, that would ignore the inherent incommensurability of tone and semitone or quarter tone, as discussed above. At the very least, by abandoning the idealized fiction of a pure ratio underlying every note Aristoxenus was able to bring forward the practical "commensurability" of every pitch sung or played on real strings: because we can *hear* those intervals, he implies, they must *de facto* be commensurable.

At the beginning of his book, Vicentino acknowledges both these ancient authorities even as he proposes to go beyond them:

> Aristoxenus, who depended solely on sense, denied reason, whereas the Pythagoreans, in contrast, governed themselves solely by reason, not sense. Ptolemy more sanely embraced both sense and reason, and his opinion has satisfied many people up to now. In this work, however, you will recognize many cases in which reason is not a friend to sense, and sense is not receptive to reason. And I shall give you detailed information as to how compatible sense and reason can be, thus enabling you to assess the dearth of sweet musical concords in the past. Therefore, with experience [*experientia*] as the mistress of things, it will be easy to judge the difference between ancient and modern music by considering examples of both.

Paradoxically, Vicentino's project seeks to outstrip the "modern" practice of music by reviving the "ancient," specifically through the retrieval of the lost enharmonic genus he considers "more sweet and smooth than the other two genera."[48] The emphasis on *experientia* means that musical practice is the new touchstone that can outweigh older arguments about rationality.

Vicentino's "Aristoxenian turn" also drew on an older contemporary who had enjoyed the patronage of the d'Este family and who seems to have been his immediate source on ancient tunings and modern temperament. In his *Musica theorica* (1529), Lodovico Fogliano (before 1500–after 1538) had taken an Aristotelian, Aristoxenian position that

---

Barker, ed., *Greek Musical Writings*, Vol. 2, pp. 384–391 (for Ptolemy), 270 (commentary regarding Aristoxenus). I thank Thomas Mathiesen for pointing out to me that Ptolemy's expansion of Aristoxenian parts of the fourth from 30 to 60 may well indicate that Ptolemy had not actually read Aristoxenus but was relying on an epitome, whether by Cleonides or someone else.

[47] Barker notes the parallel between these demonstrations in Aristoxenus's book 3 and Euclid's *Elements*: Barker, ed., *Greek Musical Writings*, Vol. 2, p. 170 n 1. Socrates draws a square on the diagonal of a unit square in his conversation with the slave boy in Plato's *Meno* 84d–85c.

[48] Vicentino, *Ancient Music Adapted to Modern Practice* (cit. n. 31), pp. 6, 12 [fols. 3r, 4v]. Even though Vicentino did not have access to Aristoxenus's full treatise in 1555, he had gathered enough of the essential content of Aristoxenus's arguments that he referred to them often. Valgulio's translation of pseudo-Plutarch included many citations of Aristoxenus, especially concerning the enharmonic genus; see Barker, ed., *Greek Musical Writings*, Vol. 1, pp. 205–257, which discusses the enharmonic genus at pp. 215–218. Also, in 1497 Giorgio Valla had brought out a Latin translation of Cleonides' compendium of Aristoxenus; see Claude V. Palisca, "Musical Change and Intellectual History," in *Music and Ideas in the Sixteenth and Seventeenth Centuries* (cit. n. 19), pp. 1–12, esp. p. 6. Though Ptolemy's *Harmonics* did not appear until Gogava's translation of 1562, the fifth book of Boethius's *Fundamentals of Music* would have given Vicentino a close account of Ptolemy's book 1.

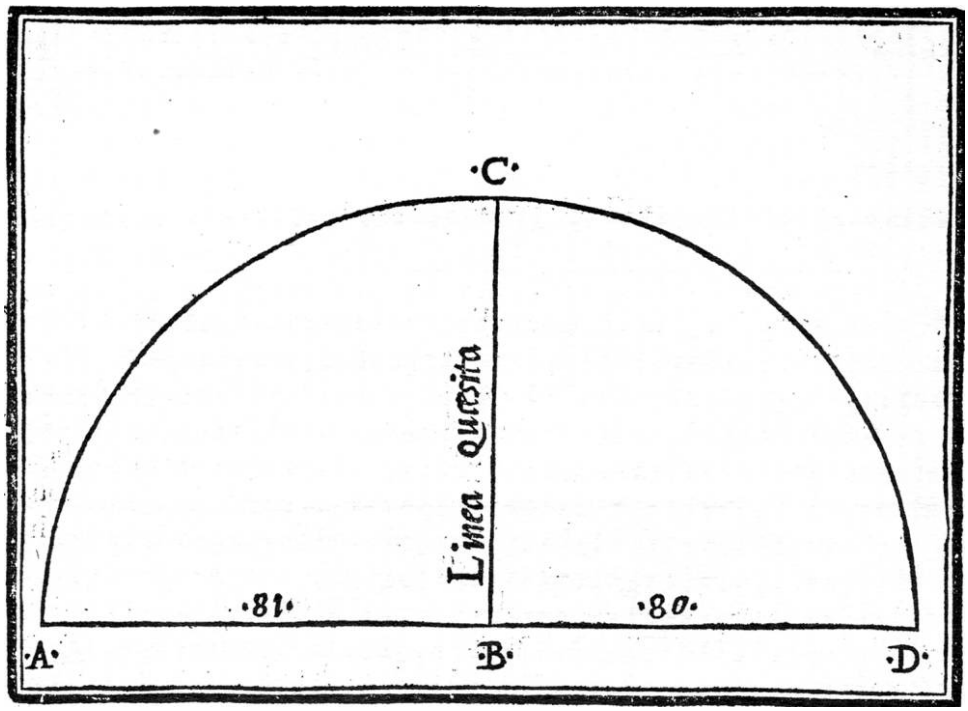


Figure 6. *Diagram from Lodovico Fogliano,* Musica theorica *(1529), fol. 36r, showing how to obtain a mean proportional* BC *between* AB *and* BD*, hence dividing exactly in half the comma 81:80; this differs from a rational approximation 80½ (the arithmetic mean of 80 and 81) by only 1/30 of a cent, a practically inaudible difference.*

declined to subordinate music to mathematics.[49] In the course of setting out a new system of temperament, Fogliano found himself, like so many other theorists before him, confronting the problem of dividing a musical interval into two equal pieces—this time the syntonic comma (81:80), which he found necessary in order to define the major third consistently within his system. To do so, he used the same construction used by Lefèvre (see Figure 1), here applied to divide this tiny interval (see Figure 6). Fogliano addresses himself both to "musici practici" and to "Theoretici" who, for their several needs, have recourse to instruments and sounding strings and hence find themselves confronting proportions that are "irrational and surd."[50] His geometrically exact result departs from a rational approximation by a practically inaudible difference. In the process of evenly

[49] Maria Rika Maniates notes that, in Vicentino's book, "all 14 allusions to Lodovico Fogliano are tacit, even though Fogliano was probably Vicentino's source on matters of ancient tuning and modern temperament"; see Vicentino, *Ancient Music Adapted to Modern Practice*, pp. xxvi. Francesco Patrizi mentioned Fogliano and Vicentino together in his list of those the house of d'Este had sponsored in its capacity as "regenerator of music [*rigeneratrice della Musica*]"; see Kaufmann, *Life and Works of Nicola Vicentino* (cit. n. 31), pp. 19 n 26, 48. See also Moyer, Musica Scientia (cit. n. 4), pp. 141–147.

[50] Lodovico Fogliano, *Musica theoretica* (Venice, 1529), fol. 36r. For full details see Claude V. Palisca, "Music and Scientific Discovery," in *Music and Ideas in the Sixteenth and Seventeenth Centuries* (cit. n. 19), pp. 131–160, esp. pp. 143–144. Essentially, Fogliano's problem was that following the Pythagorean option the major third would turn out too large (81:64), but following the Ptolemaic option it would be too small (80:64). His compromise involved splitting the difference, the interval 81:80 known as the syntonic comma.

dividing ever smaller musical intervals, such results further geometrized what had heretofore been considered the purely numerical domain of musical intervals, tacitly overstepping the ancient separation between geometry and arithmetic through the commonsensical identification of a physical string length with the corresponding line in a geometric diagram (such as Figure 6), rather than with a purely numerical ratio, considered apart from any sounding body.

VICENTINO'S ACCOUNT OF "IRRATIONAL RATIOS"

Vicentino expresses this new situation as it emerges in specifying the diesis:

> The enharmonic genus contains a semitonal division [i.e., the diesis] that is disproportioned and irrational [*sproportionata et inrationale*]. And the other parts accompanying this division cannot contain proportioned and accurate leaps because they must correspond to this irrational ratio [*proportione inrationale*]. Let no one be astonished [*si marauiglia*] that the nature and division of the genus allows this to happen. The nature of the diatonic genus goes along with its own steps and leaps, which are true in their ratios [*proportione*]. But the nature of the division of the chromatic genus permits the disruption [*si rompi*] of the diatonic order and the creation of two semitones from the whole tone as well as the step of the incomplete trihemitone [three semitones]. Likewise, the nature of the enharmonic genus disrupts the order of both the diatonic and chromatic and permits the creation of steps and leaps beyond the rational [*gradi & i salti fuore di ogni ragione*]. For this reason [*cagione*] such a division is called an irrational ratio [*si domanda proportione inrationale*].

In using the phrase "irrational ratio," Vicentino is the first (as far as I can determine) to try to state in some positive (if paradoxical) way the status and character of musical intervals that are formed through *irrational*, geometric means (as in the Fogliano example) but at the same time are incorporated into the *rational* arithmetic of music theory.[51] As so often in the history of science, Vicentino's contribution may have been to state as clearly as possible, in common words, not only the inherent paradox of a new concept but its necessity and the functionality that justifies our embracing it despite and even through its paradoxicality.

Vicentino's unfolding argument presents us with both sides of the paradox: the diesis, as constructed geometrically, is irrational but, functioning as the smallest "unit" within a framework of numerical ratios, is in that sense also effectively a "proportion." Such a hybrid concept, like a centaur, needs to be grasped in its inherent duality, considered as its essence rather than as grounds to refute its existence. Vicentino's premise—that the enharmonic genus exists and is superlatively important—was attested by many ancient sources. Therefore its basis, the diesis, must also exist, and so we should take in stride whatever paradoxical qualities it may have. Vicentino does anticipate that we might rightly be "astonished" or wonder at what at first seems a prodigy or monster, this "rational irrational"—rather in the way that we might consider a centaur monstrous if we were not familiar with examples of wise centaurs such as Chiron, "who included music

[51] Vicentino, *Ancient Music Adapted to Modern Practice* (cit. n. 31), p. 207 [fol. 66v]. Though rare, the spelling "*inrazionale*" is found as a Latinized variant in the *Vocabolario della Crusca* (1612); John Florio's *Queen Mary's World of Words* (London, 1610) includes both spellings with the same definition (pp. 258, 268). I thank Alexander Bevilaqua, Franco Ligabue, Thomas Mathiesen, and Marco Potenza for helpful discussions of the orthography of this word.

among the first arts he taught Achilles at a tender age, and who wanted him to play the harp before he dirtied his hands with Trojan blood."[52]

Vicentino's argument also builds cunningly on the successive examples of the three genera, as if in a kind of rhetorical crescendo.[53] The diatonic sets the point of departure, the purely rational (*proportione*). From our earlier discussion, we know that the semitone of the diatonic genus already contains the latent problem of its relation to the tone: namely, that no rational semitone can be exactly half of a tone. The stratagem of devising major and minor semitones merely conceals this problem without solving it; it is a stopgap solution that serves to make the diatonic appear wholly rational. The two successive semitones in the chromatic genus reiterate the problem: *which* semitones are they to be, major or minor? In the chromatic genus, the latent problems hidden in the diatonic come forward sufficiently to cause "disruption": incipient instability and growing theoretical uncertainty. Only with the enharmonic genus does this simmering instability disrupt *both* the diatonic and chromatic genera and "permit the creation of steps and leaps beyond all reason."[54] This, Vicentino tells us, is the cause that should move us not only to call them "irrational ratios" but also, by so naming them, to install them in mathematics and music jointly as having equal existential force with the "rational ratios" we learned from arithmetic and the "irrational magnitudes" from geometry. For Vicentino, music is the intermediate ground on which arithmetic and geometry meet in such hybrid concepts as "irrational ratios," shared between mathematics and music.

THE MATHEMATICIAN AS COMPOSER

The attitudes of Stifel, Cardano, and Vicentino about these mathematical issues reflect their respective musical projects. As we saw, Stifel's closest approach to affirming that "irrational numbers" were "real" or "true" came in the context of his musical theorizing. Yet this did not prove sufficient for him to maintain this position in the face of the actual infinitude of fractional sums, the "cloud of infinity," perhaps because his involvement with music remained largely theoretical and restricted. Stifel's main foray into practical music was his thirty-two-strophe song to propagate Luther's teachings, "Johannes thüt uns schreiben" (1522), based on the popular tune "Bruder Veyt." Stifel's composition led him into a polemical war of song and countersong with the theologian Thomas Murner, both always keeping this same melody for their new lyrics (see Figure 7).[55] Though Stifel's song was very popular and went through many printings, even serving as an important early example of the power of music that may have inspired Luther himself, its purely melodic component was very simple and completely derivative. Stifel merely provided new words to an old tune; he had no vision of reforming the elements of music that would

[52] Gioseffo Zarlino, *Istituioni harmoniche* (1558), cited in Strunk, ed., *Source Readings in Music History* (cit. n. 41), p. 299. See also Guido Mambella, "Corpo sonoro, geometria e temperamenti: Zarlino e la crisi del fondamento numerico della musica," in *Music and Mathematics*, ed. Vendrix (cit. n. 4), pp. 185–233.

[53] Karol Berger stresses "the transformational nature of the relationship between the three genera" as "one of the most fascinating aspects of the whole Vicentinian system and one which is constantly emphasized by the theorist," by which he means a "three-level hierarchy in which: a) the enharmonic level *necessarily* presupposes the more basic chromatic level which in turn presupposes the fundamental diatonic; b) the diatonic contains the *possibility* of the chromatic subdivision and the chromatic may in turn be subdivided enharmonically": Berger, *Theories of Chromatic and Enharmonic Music* (cit. n. 29), pp. 15–16.

[54] Vicentino, *Ancient Music Adapted to Modern Practice* (cit. n. 31), p. 207 [fol. 66v].

[55] See Rebecca Wagner Oettinger, "Thomas Murner, Michael Stifel, and Songs as Polemic in the Early Reformation," *Journal of Musicological Research*, 2003, *22*:45–100.

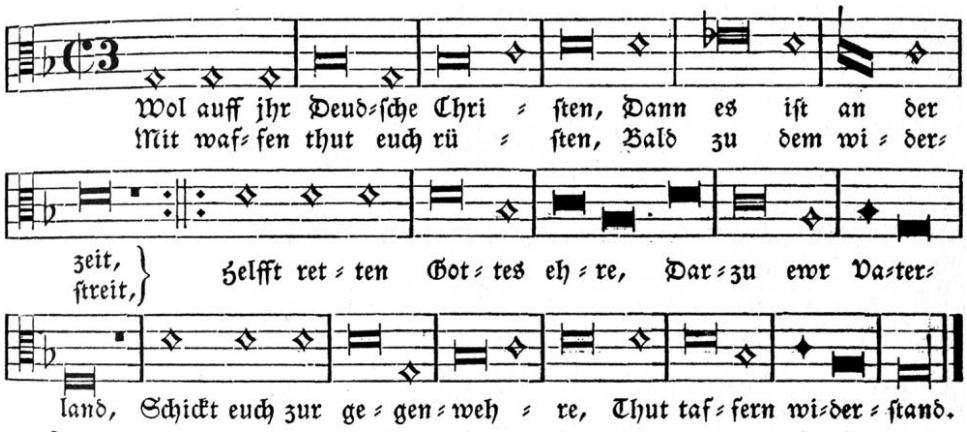


Figure 7. *The popular tune "Bruder Veyt," on which Stifel based his song "Johannes thüt uns schreiben" (1522), from Franz M. Böhme,* Altdeutsches Liederbuch *(Leipzig, 1877), p. 499 (the clef indicates middle C).*

compare with Vicentino's ambitious project to (re)create a whole new genus of music. By comparison with Stifel's song, Cardano's extant compositions are very ambitious, including a five-voice perpetual canon and a tour de force of four simultaneous three-voice canons (twelve voices in all).[56] Among Vicentino's much larger output, his motet "Musica prisca caput" (see Figure 8) dramatizes the emergence of the enharmonic genus to glorify his patron: its first verse is in the diatonic genus (measures 1–16) and the second in the chromatic (mm. 16–30), while the final verse (mm. 29–48) is in the enharmonic, dramatically reserving the introduction of the diesis to produce a special aura around the name of Cardinal Ippolito (m. 31). This motet's pointed delineation of all three genera provides yet another demonstration and justification of Vicentino's views to refute his critics and contest his condemnation. He tells us, as well, that the whole d'Este family, including the Cardinal and the Prince of Ferrara, sang this daring new music, quarter tones and all, "with the most exceptional diligence." Vicentino had evidently persuaded them that, in contrast to the public uses of diatonic music "in communal places for the benefit of coarse ears," such enharmonic music was "reserved . . . to praise great personages and heroes for the benefit of refined ears amid the private diversions of lords and princes."[57] Thus Vicentino brought his polemic on behalf of enharmonic music not only to experts but also to the powerful amateurs whose princely involvement he considered capital for his cause. By so doing, he carried his case to an alternative and (in his view) superior social milieu, whose approval and validation he took as definitive. In addition, he positioned himself so that his theories would be highly visible and readily available to another aristocratic set that would take up his ideas in the next generation—the Camerata and Vincenzo Galilei, whose admiring advocacy indeed vindicated Vicentino posthumously.

For instance, Gioseffo Zarlino (ca. 1517–1590), the preeminent theorist of the sixteenth

[56] These compositions can be found in Cardan, *Writings on Music* (cit. n. 23), pp. 139, 154–171.

[57] Vicentino, *Ancient Music Adapted to Modern Practice* (cit. n. 31), p. 33 [fol. 10v]. For further commentary on the musical details of his motets *vis-à-vis* his critics see Maniates's commentary, *ibid.*, pp. li–lviii, which treats "Musica prisca caput" on pp. lvii–lviii; recordings of this and his other motets from this book are available on the CD accompanying Cordes, *Nicola Vicentinos Enharmonik* (cit. n. 29).

century and Vincenzo's teacher, incorporated these irrational ratios in his representation of the tuning of a lute (see Figure 9), which showed how a geometric construction can dictate the placement of the frets that thereby divide intervals equally and set up an equal temperament. In this way, geometry set a template that could be mechanically reproduced without having to duplicate its geometric construction.[58]

Even so, the use of these irrational ratios remained controversial. Though Zarlino's student Giovanni Maria Artusi (1546–1613) accepted his teacher's geometric construction for tuning instruments, he balked at applying such irrational ratios to vocal music. Writing in 1603, Artusi objected that Claudio Monteverdi, the great exponent of the new operatic art sponsored by the Camerata, was applying irrational ratios, "according to the doctrine of Lodovico Fogliano," in order to generate for expressive purposes what Artusi considered intervals "false for singing," particularly the diminished seventh and diminished fourth (see Figure 10). Artusi complained that the use of such "irrational" intervals showed that Monteverdi had no "rational" understanding of music, as he put it. Though it was possible to play the intervals on the fretted lute, "the natural voice is not suited to negotiate such unnatural intervals by means of natural ones, not having a preset stopping place like an artificial instrument. . . . It cannot justly divide the tone into two equal parts."[59] Artusi's objections blend mathematical uneasiness about "unnatural" irrational ratios with his concomitant aversion to Monteverdi's expressive use of those same intervals.

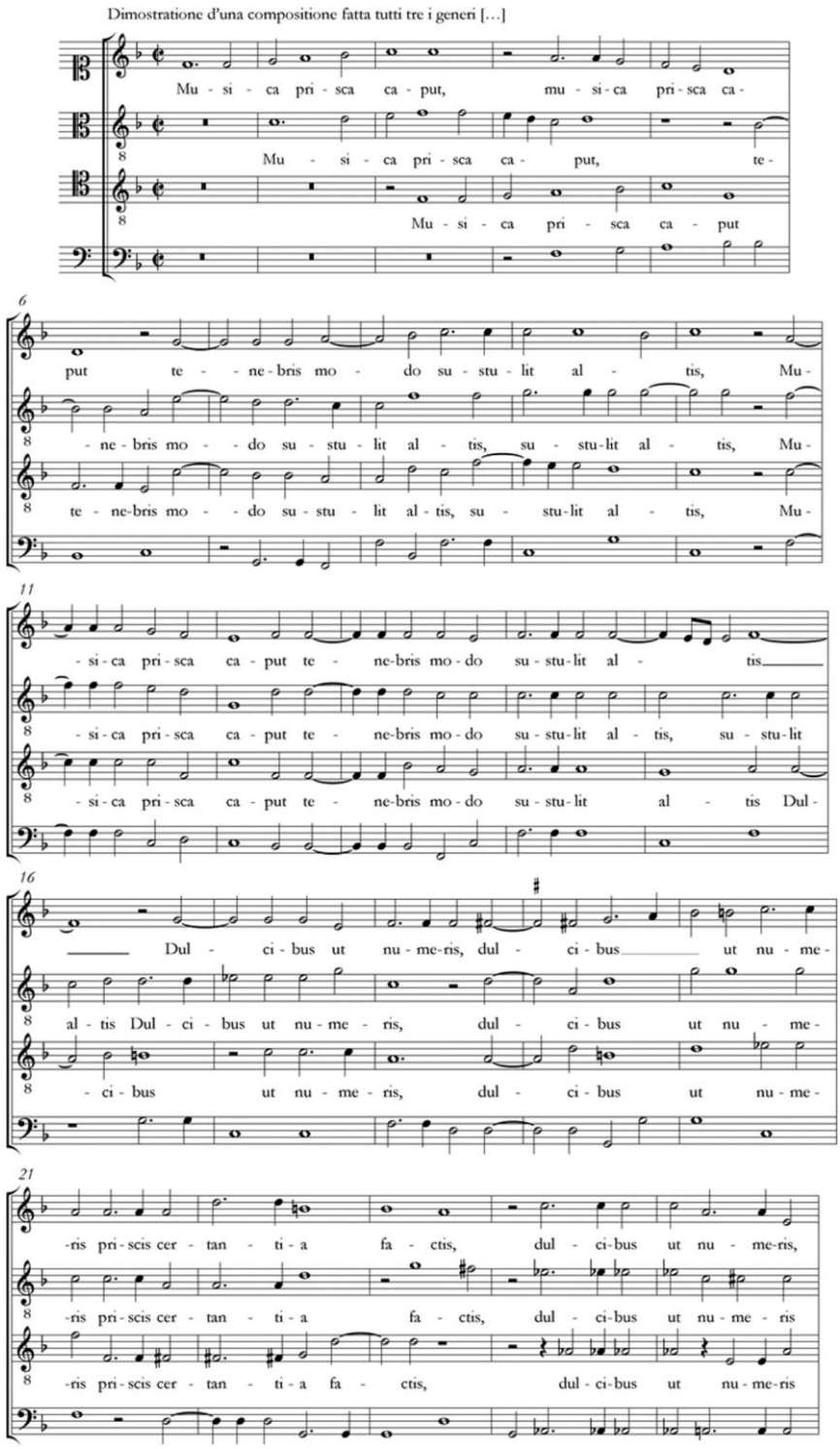
Such objections show the deep and long-lasting anxieties provoked by irrational ratios, anxieties that reflect both musical and mathematical considerations. Stifel was content to remain in the realm of conventional (and popular) music, and so his "irrational numbers" drew on no particular musical justification that might help defend or support them against traditional philosophical objections. In contrast, Cardano's strong interest in composition and performance and Vicentino's reliance on "irrational ratios" in his music seem to have helped them sustain their effective use *as numbers*, to the extent that they advanced them as musical and hence mathematical necessities.

The comparison of these three figures as theorists, composers, and mathematicians illuminates ways in which musical concerns, both practical and theoretical, influenced the acceptance of novel mathematical concepts, which in turn bore on musical matters. Long before, Plato had pointed out that astronomy connected the ideal and the visible phenomena shown through the motion of the heavenly bodies, noting also that "as the eyes are fixed on astronomy, so the ears are fixed on harmonic movement, and these two kinds of knowledge are in a way akin." The pioneering studies of the development of mathematics tended to recognize only the visible side of this insight, as when Jacob Klein emphasized the intimate connection between Viète's mathematical and astronomical works.[60] Klein's seminal treatment did not recognize the significance of music as a consequential meeting ground between mathematics and perception. In light of the innovations of Stifel, Car-

[58] Interestingly, Cardano was critical of Vicentino's scheme for tuning, which he found "not unserviceable, but it is not entirely accurate"; see the passage cited in note 36, above. Regarding Zarlino see also Moyer, Musica Scientia (cit. n. 4), pp. 202–225.

[59] See Berger, *Theories of Chromatic and Enharmonic Music* (cit. n. 29), pp. 44–56 (Zarlino), 88–95 (Artusi); he notes that Artusi, like Vincenzo Galilei, found the use of equal temperament for the harpsichord "strange" (p. 92). The quotations from Giovanni Maria Artusi, *Discorso secundo mvsicale di Antonio Bracchino da Todi* (Venice, 1608), are taken from Mark Lindley, "Chromatic Systems (or Non-Systems) from Vicentino to Monteverdi," *Early Music History*, 1982, *2*:377–404, on pp. 400–404.

[60] Plato, *Republic* 530d; and Klein, *Greek Mathematical Thought* (cit. n. 1), p. 151.


Dimostratione d'una compositione fatta tutti tre i generi […]
Mu - si - ca pri - sca ca - put, mu - si - ca pri - sca ca -
Mu - si - ca pri - sca ca - put, te -
Mu - si - ca pri - sca ca - put
6
put te - - ne - bris mo - do su - stu - lit al - tis, Mu -
- ne - bris mo - - do su - stu - lit al - tis, su - stu - lit al - tis, Mu -
te - ne - bris mo - - do su - stu - lit al - tis, su - stu - lit al - tis, Mu -
11
- si - ca pri - sca ca - put te - ne - bris mo - do su - stu - lit al - tis
- si - ca pri - sca ca - put te - ne - bris mo - do su - stu - lit al - tis, su - stu - lit
- si - ca pri - sca ca - put te - ne - bris mo - do su - stu - lit al - tis Dul -
16
Dul - ci - bus ut nu - me - ris, dul - ci - bus ut nu - me -
al - tis Dul - ci - bus ut nu - me - ris, dul - ci - bus ut nu - me -
- ci - bus ut nu - me - ris, dul - ci - bus ut nu - me -
21
- ris pri - scis cer - tan - ti - a fa - ctis, dul - ci - bus ut nu - me - ris,
- ris pri - scis cer - tan - ti - a fa - ctis, dul - ci - bus ut nu - me - ris
- ris pri - scis cer - tan - ti - a fa - ctis, dul - ci - bus ut nu - me - ris

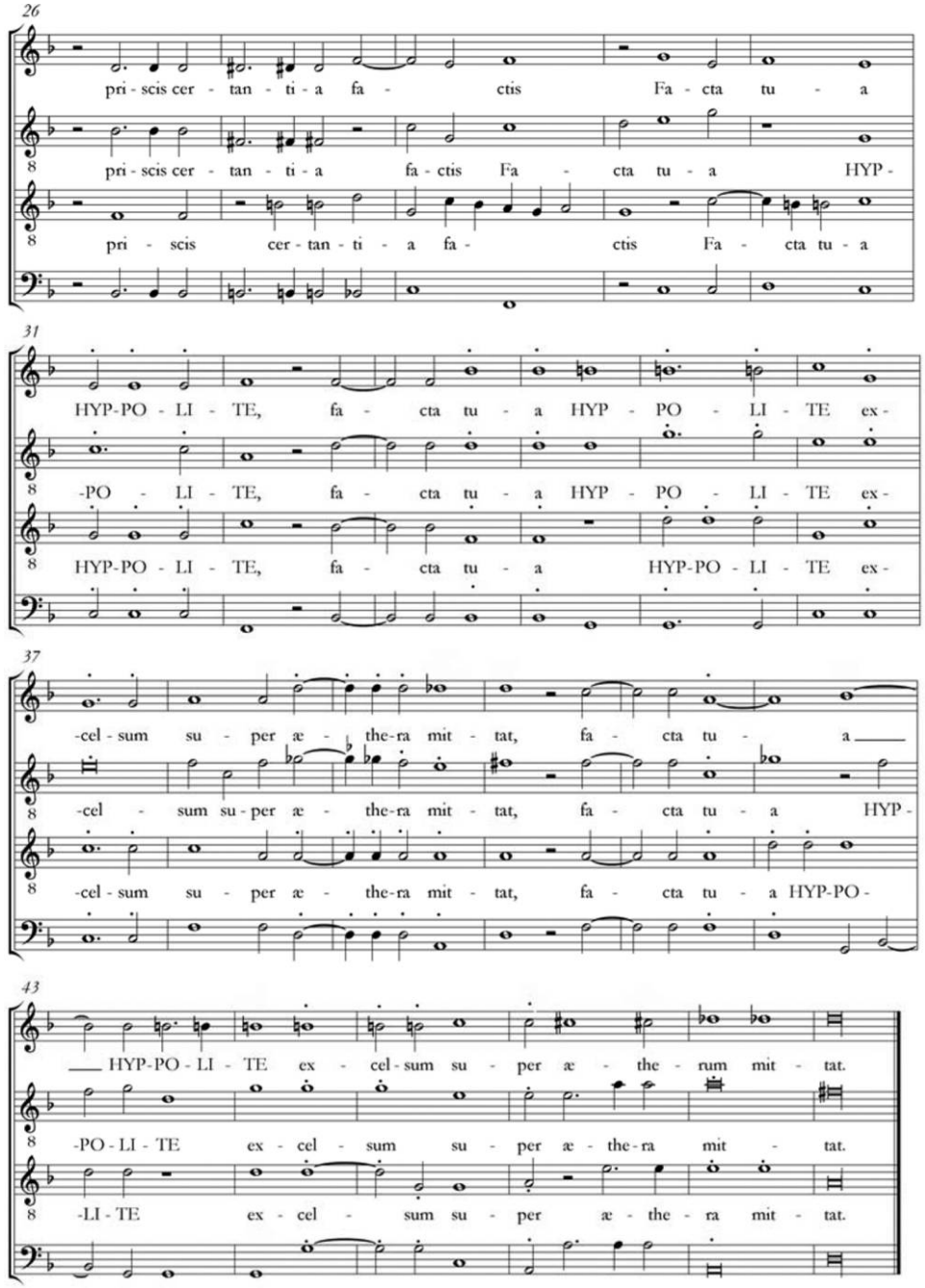


Figure 8. *Vicentino's Latin motet "Musica prisca caput," which he included in* L'antica musica ridotta alla moderna prattica*, fols. 69v–70v, as a "Demonstration of a composition made from all three genera" of music; this modern score follows Vicentino's convention that notes with a dot above them are raised in pitch by a diesis. Text: "Ancient music of late has raised her head out of the darkness, / So that, with antique and sweet numbers, to compete with ancient deeds, / Your great deeds, Hyppolitus, she might send high above the heavens." (Courtesy Manfred Cordes.)*

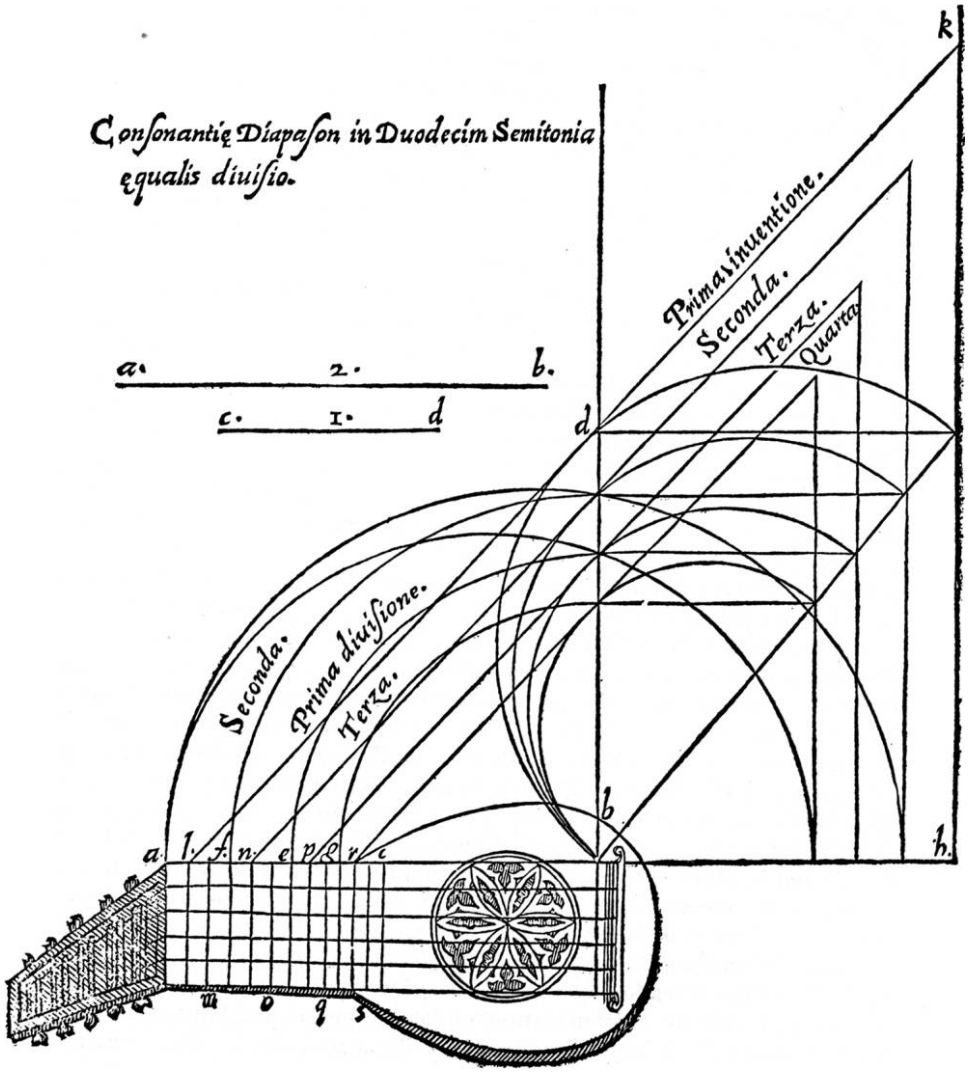


Figure 9. *Gioseffo Zarlino,* Sopplimenti musicali *(Venice, 1588), p. 211, showing the geometric construction to fret a lute in equal temperament using the construction shown by Lefèvre and Fogliano. The caption reads: "The equal division of the consonance of a diapason into twelve semitones."*

dano, and Vicentino discussed here, we need to reconsider his judgment that their mathematical work, however technically ingenious, finally lacked understanding of the underlying symbolic project.[61] To Klein's account of the emergence of algebraic symbol-

[61] Klein, *Greek Mathematical Thought*, pp. 147–148: "the 'algebra' which has Arabic sources is continually elaborated in respect to techniques of calculation, for instance by the introduction of 'negative,' 'irrational,' and even so-called 'imaginary' magnitudes (numeri 'absurdi' or 'ficti,' 'irrationales' or 'surdi,' 'impossibiles' or 'sophistici'), by the solution of cubic equations, and in its whole mode of operating with numbers and number signs, its self-understanding fails to keep pace with these technical advances." See Aubel, *Michael Stifel* (cit. n. 12), pp. 321–325, for a reconsideration of Stifel in light of Klein's work.

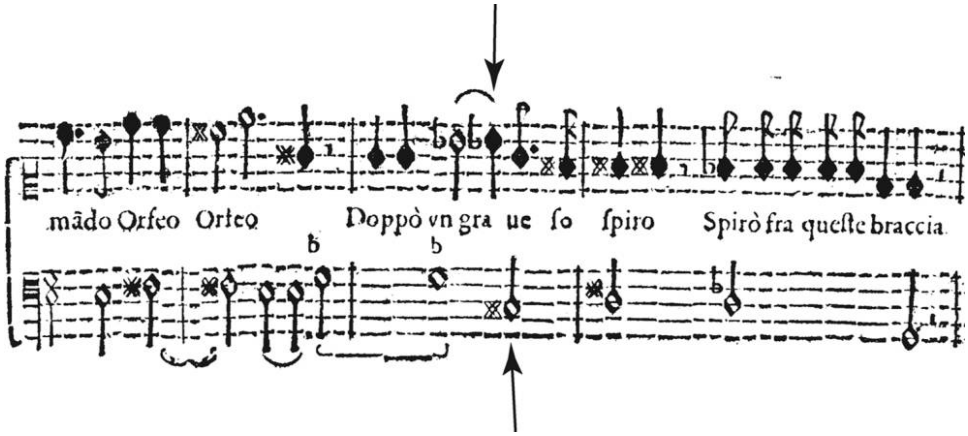


Figure 10. *Claudio Monteverdi,* La favola d'Orfeo *(first performed 1607), Act II, measures 274–279 (from the edition of Venice, 1615). The messenger is recounting Euridice's dying words: "and calling on you, Orfeo, Orfeo, after a deep sigh expired in these arms." Artusi objected to the "irrational" diminished seventh between voice (B♭) and continuo (C#) at the word "*grave*" (indicated by arrows), expressing the depth of her sigh.*

ism we now should add the striking developments that prepared the way for hybrids of the rational and the irrational through the musical meeting of arithmetic and geometry. In so doing, we may draw more closely together the study of "audible culture" and symbolic structures hitherto considered apart and unto themselves. This rapprochement may call for and guide an enriched new phenomenology that would go beyond the cool bracketing of "objectivity" by synthesizing mathematical and musical perceptions, constructs, and symbolic forms, amalgamating the theoretic, the epistemic, and the practical in a new history and philosophy of science embedded more vividly in perception, feeling, and thought.[62]

As Guillaume Gosselin noted in 1577, "the musician and the algebraist indeed know numbers" precisely through "their relation to something else," in contrast with the arithmetician, who only "sees numbers in themselves."[63] His implication is that musicians

[62] Such an endeavor would consult, revise, and transcend Edmund Husserl's initiatives, applied in Jacob Klein, "Phenomenology and the History of Science," in Klein, *Lectures and Essays*, ed. Robert B. Williamson and Elliott Zuckerman (Annapolis, Md.: St. John's College Press, 1985), pp. 65–84. Among recent efforts to reengage these larger connections see Lorraine Daston and Peter Galison, *Objectivity* (New York: Zone, 2007). Concerning the synthetic vocation of the history of science see Gerald Holton, "George Sarton, His *Isis*, and the Aftermath," *Isis*, 2009, *100*:79–88. Concerning the concept of *experientia* see Dear, *Discipline and Experience* (cit. n. 34); and Gianna Pomata and Nancy G. Siraisi, eds., *Historia: Empiricism and Erudition in Early Modern Europe* (Cambridge, Mass.: MIT Press, 2005). For explorations of "audible culture" see Friedrich Kittler, *Gramophone, Film, Typewriter*, trans. Geoffrey Winthrop-Young and Michael Wautz (Stanford, Calif.: Stanford Univ. Press, 1999); Kittler, *Musik und Mathmatik: Hellas*, Vol. 1: *Aphrodite* (Paderborn: Fink, 2006); Emily Thompson, *The Soundscape of Modernity: Architectural Acoustics and the Culture of Listening in America, 1900–1933* (Cambridge, Mass.: MIT Press, 2002); Jonathan Sterne, *The Audible Past: Cultural Origins of Sound Production* (Durham, N.C.: Duke Univ. Press, 2003); Myles W. Jackson, *Harmonious Triads: Physicists, Musicians, and Instrument Makers in Nineteenth-Century Germany* (Cambridge, Mass.: MIT Press, 2006); Alexandra Evonne Hui, "Hearing Sound as Music: Psychophysical Studies of Sound Sensation and the Music Culture of Germany, 1860–1910" (PhD diss., Univ. of California, Los Angeles, 2008); and David Pantalony, *Altered Sensations: Rudolph Koenig's Acoustical Workshop in Nineteenth-Century Paris* (New York: Springer, 2009).

[63] Klein, *Greek Mathematical Thought* (cit. n. 1), p. 262 n 225, notes that "Gosselin in his algebraic work

were essential companions on the road that led from arithmetic to algebra in that they reached beyond numbers to the "something else" manifest through music. If so, the struggle to "hear" the mathematical irrational was indeed consequential on many levels. Mediating between the realms of mathematics and felt experience, music evoked and justified new concepts of number.

---

already adds 'algebra' to the ancient 'Pythagorean' division of mathematics into geometry and astronomy, arithmetic and music, assigning it, together with 'music,' to the realm of relational quantity. . . . This is, so to speak, the first 'official' introduction of algebra into the system of sciences recognized by the schools. Up to that point the 'ars rei et census' (art of the thing and its power, i.e., the unknown and its square) was a more or less obscure curiosity. It was even considered suitable for public exhibitions in the form of contests and aroused the wonder of the crowd much as did acrobatic or magic tricks"; he instances the turbulent contest between Niccolò Tartaglia and Luigi Ferrari, Cardano's pupil. Klein comments on Gosselin's classification of number on pp. 291–292 n 305; see also H. Bosmans, "Le 'De arte magna' de Guillaume Gosselin," *Bibliotheca Mathematica*, 3rd Ser., 1906, *7*:44–66 and Giovanna Cleanice Cifoletti, "Mathematics and Rhetoric: Peletier and Gosselin and the Making of the French Algebraic Tradition" (PhD diss., Princeton University, 1992). Gosselin translated into French *L'arithmetique* (Paris, 1578), by Tartaglia, who also experimented with the division of a tone into equal semitones; see Moyer, Musica Scientia (cit. n. 4), pp. 126–134, which also connects Tartaglia with Bartolomeo Zamberti's comments on music in his edition of Euclid (1505).