Hearing the Irrational

Music and the Development of the Modern Concept of Number

By Peter Pesic*

ABSTRACT

Because the modern concept of number emerged within a quadrivium that included music alongside arithmetic, geometry, and astronomy, musical considerations affected mathematical developments. Michael Stifel embedded the then-paradoxical term “irrational numbers” (numeri irrationales) in a musical context (1544), though his philosophical aversion to the “cloud of infinity” surrounding such numbers finally outweighed his musical arguments in their favor. Girolamo Cardano gave the same status to irrational and rational quantities in his algebra (1545), for which his contemporaneous work on music suggested parallels and empirical examples. Nicola Vicentino’s attempt to revive ancient “enharmonic” music (1555) required and hence defended the use of “irrational proportions” (proportiones irrationales) as if they were numbers. These developments emerged in richly interactive social and cultural milieus whose participants interwove musical and mathematical interests so closely that their intense controversies about ancient Greek music had repercussions for mathematics as well. The musical interests of Stifel, Cardano, and Vicentino influenced their respective treatments of “irrational numbers.” Practical as well as theoretical music both invited and opened the way for the recognition of a radically new concept of number, even in the teeth of paradox.

The arithmetician sees numbers in themselves, the musician and the algebraist indeed know numbers, but in their relation to something else.

—Guillaume Gosselin, On the Great Art or the Hidden Part of Numbers, Commonly Called Algebra or Almucabala (1577)

* St. John’s College, 1160 Camino de la Cruz Blanca, Santa Fe, New Mexico 87505-4511; ppesic@sjcsf.edu.

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We tend to take for granted the broad inclusiveness of our current concept of number (comprising integers along with rational and irrational quantities), as if it were inevitable and unalterable, though in fact it took its present shape only over the past few centuries. Indeed, the modern concept of “real numbers” differs profoundly from the concept of number as it was commonly understood in the West from antiquity until about 1600. Given this fundamental change, what is the relation between ancient and modern concepts of number? Why and how did the concept change? And what difference do these changes make? In the history of ideas, this shift of understanding is so consequential that it deserves much further study, not least because modern mathematics depends on it, hence also much of modern science. As Plato first argued, mathematics offers a touchstone of epistemic certainty that is important for philosophy, theology, and metaphysics. To illuminate these questions, this essay will bring forward interactions between music (both in theory and in practice) and the development of the modern concept of number. These relations will help illuminate the hesitation about the nascent concept of irrational number in the work of Michael Stifel (1544), as compared to Girolamo Cardano’s intermixture of irrational and rational quantities in his musical and algebraic works (1545) and Nicola Vicentino’s practical use of “irrational proportions” (1555) for musical ends. The differences between them reflect their different approaches to music: Stifel’s was traditional and largely theoretical, Cardano’s much more practical, while Vicentino was moved primarily by a new musical ideal based as much in practice as in his theoretical advocacy of the ancient Greek enharmonic genus.

In recent years, increasing attention has been directed to the interaction between music and mathematics, especially in the work of Vincenzo Galilei (ca. 1520–1591), a lutenist and composer who took up music theory in the 1560s and wrote the Dialogo della musica antica, et della moderna [Dialogue on Ancient and Modern Music] (1581), “surely the most influential music treatise of the late sixteenth century.” As the father of Galileo Galilei, Vincenzo arguably had an important influence on the nascent experimental science; Stillman Drake, for instance, suggested that Galileo may have first pondered pendulums in the context of his father’s experiments, where they acted as weights swinging on the ends of monochords to test (and disconfirm) old Pythagorean legends. These experiments and discussions unfolded in a lively social context: Vincenzo was an important member of the Florentine Camerata, a group surrounding Count Giovanni de’ Bardi whose members were fascinated by the accounts of the prodigies of ancient music, which led to their own further speculations and initiatives. Around 1600, the Camerata and other like-minded groups were crucial nurseries for the artistic ideas that led to the first operas, conceived as imaginative reconstructions and revivals of the legendary powers of ancient Greek tragedy, whose music had long been lost. The ensuing marriage of dramatic expressivity with musical powers consciously drawn from dissonance was of great importance to new artistic sensibilities and to the new science envisaged by Bacon, Galileo, and Descartes, as I will discuss elsewhere.


In this essay, I will turn to the generations preceding Vincenzo Galilei, in which many of the ideas came to light that he and his contemporaries would ponder and set into action. Among the rich and largely unexplored avenues of that earlier period, I will focus on a seminal mathematical thread for its intrinsic importance as a missing link in the history of mathematics and the newly emergent mathematical sciences. This thread emerges in the context of discussions about the nature of ancient Greek music that grew throughout the sixteenth century in Italy, leading directly to the work of Vincenzo Galilei and the Camerata. These musical discussions were part of the revival of antiquity in many forms—artistic, literary, and philosophical—that began in the fifteenth century. But music posed an especially difficult and intriguing case: though practically all ancient Greek music was apparently lost, the philosophical corpus contained numerous references to the wonders performed by that music, which seemed capable, like Orpheus, of moving the very stones.3

In connecting these musical investigations with mathematics, I do not mean to be understood as making an “interdisciplinary” intervention, as the present status of mathematics as a discipline wholly distinct from music might seem to suggest. In the case at hand (and many others like it), our current disciplinary maps and boundaries are anachronistic and misleading. In the sixteenth century, music was still studied as part of the quadrivium, next to arithmetic, geometry, and astronomy, so that it would have been natural for musical and mathematical considerations to have met. The assumption that a mathematician would also be versed in music theory persisted through the work of René Descartes, Johannes Kepler, and even Isaac Newton, remaining important into the following century, as H. Floris Cohen and Benjamin Wardhaugh have discussed extensively.4 As early modern musicians and thinkers contemplated the mysterious powers of Greek music, they were constantly engaged with the other parts of the quadrivium, for arithmetic and geometry grounded musical and astronomical theory. Yet only music connected heaven and earth, theory and experience, mathematics and feeling.


ANCIENT AND MODERN CONCEPTS OF NUMBER

Greek mathematics restricted the term “number” (ἄριθμος) to integers greater than one; “ratio” (λόγος) and “rational” referred in the first instance to proportions of integers, as studied by arithmetic. In contrast, “magnitude” (μέγεθος) designated a more general quantity that was not necessarily rational, such as the length of a line in a geometrical figure.\(^5\) Euclid considered these distinctions necessary because such a figure can have integer sides but still have diagonals that are not expressible by any ratio of integers. In the case of the diagonal of a square of unit side, there is no way to “say” its length as a ratio or number; in that sense, it is literally “unspeakable” (ἀπρήπτος or ἄλογος). Nor is there any indication that Euclid would have found adequate our symbol for the length of this diagonal, \(\sqrt{2}\), which does not say what that length really is but only designates an infinite sum of fractions.\(^6\)

By comparison, modern mathematics rests on a portmanteau concept of “real numbers” that includes “whole numbers,” “rational numbers,” and “irrational numbers.”\(^7\) Yet for Greek mathematics, a nonintegral number was a contradiction in terms. Euclid rigorously separated “number” from “magnitude,” allowing proportions only between one integer and another or between one magnitude and another. He never alternated or “cross-multiplied” these proportions (as we allow ourselves to do), which would intermix them. By avoiding any such admixture, he sought to avoid contradictions in the foundations of mathematics as he understood them.

Sixteenth-century mathematicians struggled with these ancient distinctions. For instance, though The Whetstone of Witte (1557) by Robert Recorde (1510–1558) notes that “Euclide, Boetius, and other good writers” acknowledge only “whole numbers,” Recorde also includes “nombres irrationale,” approximated as closely as desired by infinite series of fractions. Thus, as Katharine Neal notes, Recorde broadened his “number concept while simultaneously using labels that signaled his awareness of the unacceptability, by traditional standards, of the new numbers.” Recorde observed that his number terms draw on algebra, the “cossic art,” whose solutions include both rational and irrational quantities. This art has many practical aspects, as Cardano and other Italian mathematicians had noted; Recorde dedicated his book to the “venturers” of the Muscovy Company, offering practical examples of military formations, bricklaying, and geography and promising a further book on navigation.\(^8\)

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\(^6\) Note that in Greek mathematics one was not considered a number (ἀριθμός) because it was the “monad” (μονάς) in terms of which all numbers were multiples. See Paul Tannery, “Du rôle de la musique grecque dans le développement de la mathématique pure,” Mémoire Scientifique, 1902, 3:68–69. More recently, Árpád Szabó, The Beginnings of Greek Mathematics (Dordrecht: Reidel, 1978), pp. 99–184, argued that “all the important terms of the theory of proportions have their origins in the theory of music” (p. 170). See also Luigi Borzacchini, “Incommensurability, Music, and Continuum: A Cognitive Approach,” Archive for History of Exact Sciences, 2007, 61:273–302.

\(^7\) Fractions were not considered equivalent to ratios in Greek mathematics, whose concept of magnitude included both what we call “algebraic” quantities, like \(\sqrt{2}\), which are solutions of algebraic equations of finite degree, and “transcendental” quantities, such as \(\pi\), which are not the solution of any such finite algebraic equation. See Peter Pesic, Abel’s Proof: An Essay on the Sources and Meaning of Mathematical Unsolvability (Cambridge, Mass.: MIT Press, 2003), pp. 5–21.

\(^8\) Robert Recorde, The Whetstone of Witte, which is the seconde parte of Arithmetike (London: Jhon Kyngston, 1557), sigs. Aiir, Sir: “And a third sorte there is of nombres radicalle, whiche commonly bee called nombres
Both practical and theoretical considerations moved François Viète (1541–1603) to make crucial symbolic innovations that linked these different number concepts more closely. In his *Canon mathematicus* (1579), Viète advocated the use of decimal fractions to replace the sexagesimal calculations traditionally used for astronomy; such decimals could express both rational and irrational quantities, as Simon Stevin also realized.9 Viète’s reading of Diophantus and Pappus, along with his own innovative Stevin work, led him to introduce alphabetic signs for unknowns as well as for coefficients, as outlined in his *De recognitione æquationum* (published posthumously in 1615).10 Because a symbol like \( x \) could now stand for an integer as well as for an irrational quantity, the new algebraic usage effectively unified these heretofore separate and opposed categories.

These innovations, however ingenious and practical, skated over a foundational abyss because they subsumed irrational and rational under a single symbol. Yet these very issues about the nature of number had emerged earlier in the context of musical theory. The nature of the musical evidence, both theoretical and practical, strongly supported the necessity and legitimacy of irrational numbers. Music was ideally situated to mediate this new understanding between her sisters arithmetic and geometry.11

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**irrationalle:** because many of them are soche, as can not bee expressed, by common nombers *Abstracte*, norther by a certain rationalle number. Other men call them more aptly *Surde numbers.*” Recorde also cites the ancient dictum that “One is no nomber” (sig. Aiir) and notes that “although there be many kinds of irrationall nombres, yet those figures that serve in *Cossike nombres*, bee the figures also of all irracionalle nombres” (sig. Siv). See Neal, *From Discrete to Continuous* (cit. n. 5), pp. 49–55; the quotation is from p. 50. Brigitte Van Wy meersch, “Qu’entend-on par ‘nombre sord’?” in *Music and Mathematics*, ed. Vendrix (cit. n. 4), pp. 97–110, discusses the use of the term “radix surdus” by twelfth-century writers, who did not, however, call such quantities *numeri.*


11 In comparison with music, painting relies on geometry, rather than on explicit arithmetic; this perhaps explains why developments similar to those described in this essay did not happen in the visual arts. For instance, though Piero della Francesca was an important mathematician, his writings do not show any interaction between his innovations in painting and the concept of number. See J. V. Field, *The Invention of Infinity: Mathematics and Art in the Renaissance* (Oxford: Oxford Univ. Press, 1997), p. 67; and Field, *Piero della Francesca: A Mathematician’s Art* (New Haven, Conn.: Yale Univ. Press, 2005), pp. 24–31, 282–284, 312–316. See also Ann E. Moyer, “Music, Mathematics, and Aesthetics: The Case of the Visual Arts in the Renaissance,” in *Music and Mathematics*, ed. Vendrix (cit. n. 4), pp. 111–146. I thank Don Howard for raising this interesting question, which deserves further attention, and Mark Peterson for sharing his thoughts on Piero.
STIFEL’S TREATMENT OF IRRATIONAL NUMBERS

The earliest explicit mention of “irrational numbers” as a self-conscious term for these mathematical hybrids seems to have been in the *Arithmetica integra* (1544) of Michael Stifel (1487–1567), a former Augustinian monk who left the order and became a friend and collaborator of Martin Luther. Alongside his work as a fervent advocate of ecclesiastical reform (he anagrammatized the name of Pope Leo X to yield 666, the Number of the Beast), Stifel was arguably the most distinguished German mathematician of the sixteenth century; his methods were crucial sources for Recorde. In *Arithmetica integra* Stifel introduced the term “exponent” and used the signs +, −, and √.12

Stifel begins by reviewing “the nature and species of abstract numbers [numerorum abstractorum].” From the beginning, he embeds his novel term “irrational numbers” (numerici irrationales) in an extensive discussion of music.13 In book 1 Stifel treats musical intervals in terms of the ratios of string lengths, beginning with the ancient definitions of the octave (1:2), fifth (2:3), fourth (3:4), and tone or whole step (8:9). Because the octave cannot be divided into an integral number of whole tones, the construction of scales requires dividing tones in half, as Boethius described in *De institutione musica* [*Elements of Music*], written circa 510 A.D., which transmitted ancient music theory to the West.14 But dividing a semitone exactly in half would involve a geometric mean that is necessarily irrational (see Figure 1), hence impossible in the context of the pure arithmetic ratios of Greek musical theory.15 Boethius avoided this problem by dividing the tone unequally into a “major semitone” and a “minor semitone,” which differ by the tiny interval called the “comma.”16


15 Boethius, *Fundamentals of Music*, 3.11. Defining a “superparticular” interval B:C to have the form B:C :: n+1:n, where n is an integer (and the ratio is expressed in lowest terms), then the problem of dividing such an interval into two equal subintervals is equivalent to the problem of finding a mean proportional number D such that B:D :: D:C. This proof also figures in the *Sectio canonis*, traditionally attributed to Euclid and translated by Giorgio Valla in 1497. See *The Euclidean Division of the Canon*, ed. and trans. André Barbera (Lincoln: Univ. Nebraska Press, 1991), props. 3, 16; it is also included in the anthology *Greek Musical Writings*, ed. Andrew Barker (Cambridge: Cambridge Univ. Press, 1989), Vol. 2, pp. 190–208. See also Barbera, “Placing *Sectio Canonis* in Historical and Philosophical Contexts,” *Journal of Hellenic Studies*, 1984, 104:157–161; and Wilbur R. Knorr, *The Evolution of the Euclidean Elements* (Dordrecht: Kluwer, 1975), Ch. 7.

16 Modern nomenclature defines an octave as comprising 1,200 cents, of which an equal semitone would be 100 cents, a “major semitone” (2,048 : 2,187 ≈ 3 : 21/12) 113.7 cents, a “minor semitone” (256 : 243 ≈ 8 : 9) 90.2 cents, a Pythagorean comma (3125/219 ≈ 531,441/524,288) 23.5 cents. Boethius’s terminology shows that he was aware of these problems in dividing tones. See Gillian R. Evans, “Fractions and Fraction-Symbols in Boethius’ *Musica*,” *Centaurus*, 1982/1983, 26:215–217; and Allison M. Peden, “De Semitone: Some Medieval Exercises in Arithmetic,” *Studi Medioevali*, 1994, 35:368–403.
Stifel notes that “musicians speak of certain irrational proportions.” implying that these proportions are already in current musical use (probably mainly theoretically) and hence should be mathematically acceptable. In contrast, earlier theorists had held that “music does not consider irrational proportions.”

Stifel’s statement acknowledges the new musical desirability of such equal division, despite its mathematical irrationality.

To be sure, Campanus of Novara’s 1482 translation of Euclid’s Elements used the phrase “irrational proportion” to denote “incommensurable quantities.” But Euclid never

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**Figure 1.** Jacques Lefèvre d’Étапles’s diagram from Elementa musicalia (1496), fol. 6gv, demonstrating that interval $Ab:bc$ can be divided geometrically exactly in half ($bg$); the upper register shows these lengths deployed along a string. If the two collinear segments $Ab$ and $bc$ are in the ratio $Ab:bc :: 8:9$, then (Euclid, Elements, book 6, proposition 13) erecting the perpendicular bisector $bg$ on $Ac$ gives the mean proportional $Ab:bg :: bg:bc$. Thus, a string of length $bg$ would sound an exact semitone higher than string $Ab$; because $8:bg :: bg:9$, in modern notation, $bg = \sqrt{72}$, hence the “ratio” of a semitone is $8:\sqrt{72}$. (This item is reproduced by permission of the Huntington Library, San Marino, California [Huntington, RB 67813].)
overstepped the boundary between rational and irrational, as Stifel does. For example, Stifel divides the tone into equal semitones following the construction given by Jacques Lefèvre d’Étuples (Jacobus Faber Stapulensis), which is based on the Euclidean mean proportional (see Figure 1). Likewise, Stifel applies various arithmetic operations to musical proportions, noting that “in these ways irrational proportions of irrational terms can be computed by rational numbers through this beautiful reckoning [pulchra ratione],” including his explicit halving of the tone (see Figure 2). To my knowledge, this is the earliest printed statement that combines a rational (arithmetic) proportion (8:9) with its irrational (geometric) mean (8:√72). Acknowledging the controversy about them, Stifel still asserts that

these halvings are so certain that no one can deny them. Neither Jordanus [Nemorarius] nor Stapulensis nor any other among the learned has denied anything else concerning this question, except that a tone can be divided in two equal parts by means of a certain and constituted number (as they themselves put it), that is, a rational number. Moreover, they do not deny that a tone can be divided by an uncertain number and that is constituted by no assembly of units, that is, by an irrational number. And because any part you please of the aforementioned halvings consists of a certain term or rational [number], and from a term that is uncertain and
unknown, or irrational, therefore also the parts themselves individually are uncertain and unknown, or irrational, proportions. 21

Thus, in this musical context, Stifel treats these irrational “halved ratios” as though they are as valid as rational proportions.

Yet when Stifel returns to the larger question of “the essence of irrational numbers,” his attitude shifts:

It is properly debated whether irrational numbers are true numbers or fictions. For if we lack rational numbers in geometrical figures, their place is taken by irrationals, which prove precisely those things that rational numbers could not; certainly from the demonstrations they show us we are moved and compelled to admit that they [irrational numbers] really exist from their effects, which we perceive to be real, sure, and constant.

On the other hand, other things move us to a different assertion, namely that we are forced to deny that irrational numbers are numbers. Namely, where we might try to subject them to numeration and to make them proportional to rational numbers, we find that they flee perpetually, so that none of them in itself can be precisely grasped: a fact that we perceive in the resolving of them, as I will show below in its place. Moreover, it is not possible to call that a true number which is such as to lack precision and which has no known proportion to true numbers. Just as an infinite number is not a number, so an irrational number is not a true number and is hidden under a sort of

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21 Ibid., fol. 79v.
Here, the “cloud of infinity” is the infinite sum of fractions needed to represent an irrational quantity; such an “actual infinite” was rejected by Aristotle. Stifel’s distaste for this infinitude finally outweighs his geometrical and musical arguments that irrational quantities can take the place of rational numbers in every effective respect. Thus, even though his musical arguments had led him to affirm irrational numbers, his arithmetic concern to avoid the infinite ultimately moved him to demote them from the class of “true numbers.”

**RATIONALIZING THE IRRATIONAL**

Stifel’s arguments show the effect of musical considerations on mathematical concerns, indicating the possibility of shifting and surprising alliances between the various parts of the quadrivium: geometric irrationalities, formerly excluded from arithmetic, could find a place in music. Though Stifel himself finally gave precedence to an Aristotelian rejection of the actual infinite, others would take these arguments in a different direction precisely by placing a new and different emphasis on the musical side.

Among these, the famous mathematician, physician, and polymath Girolamo Cardano (1501–1576) has special importance, even though his writings on music are less well known than the rest of his vast output. Though Cardano’s *De musica* was published only in his *Opera omnia* (1663), among his works on arithmetic and geometry, he wrote this manuscript during the period (ca. 1546) surrounding the appearance of his most famous mathematics book, *De arte magna* (1545), which announced the general solutions of the cubic and quartic equations (in the midst of notorious controversies about priority and disclosure), a landmark in the development of modern algebra. Thus Cardano’s presentation of what he modestly called “this most abstruse and clearly unsurpassed treasury of the entire arithmetic” should be read next to his contemporaneous musical work, which is notable for its emphasis on practical techniques related to musical instruments as well as its theoretical considerations.

Cardano sang and played several instruments, including the recorder and the lyra, and was a skilled composer, as is shown by several compositions he includes in *De musica* and his careful accounts of instrumental techniques. Cardano’s awareness of such modes of ornamentation as trills and vibrato draws attention to microtonal shifts that singers and instrumentalists used to decorate their melodies. He particularly emphasizes the unusual interval of a “diesis,” a quarter tone (half a semitone) that produces “such a movement [that] titillates the ear and increases its pleasure.” As Clement Miller notes, “his affection

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22 Ibid., fol. 103r.
for this tonal embellishment was very great and his description of the beauty and pleasantness of the effect sometimes borders on the ecstatic.\textsuperscript{24}

Cardano’s predilection for the diesis led him to put forward new opinions about its definition and that of the semitone, though he cites the mathematical problem of the exact divisibility of ratios. To divide a tone into two equal semitones (or a semitone into two equal quarter tones) “correctly and arithmetically [verè & arithmetice],” he acknowledges that a “true calculation” involves an irrational root, for which he accepts a rational approximation that is “closer in perception.” For these “true” irrational intervals (whether of semitone or diesis), he empirically substitutes a simple rational approximation, thus conflating the geometrically irrational with the arithmetically rational. He treats the result as a “correct” system of tuning, not merely a stopgap or approximation; in fact, the application of his calculation of his approximate semitone (18/17) to fretting a lute was “the first really practical approximation of equal temperament,” later (incorrectly) attributed to Vincenzo Galilei.\textsuperscript{25} Cardano treats rational and irrational intervals on the same footing primarily because of musical considerations: he chooses between rational approximations for the diesis not on the basis of closeness of numerical value (which would lead to 35/34) but on perception as judged musically (leading to 36/35). He calls “true” (verù) both the irrational “true diesis” and its rational, musical equivalent, which is true “arithmetically.”

Though in De arte magna Cardano often refers to “numbers” with the sense of “integers” and never uses the term “irrational numbers,” he uses the phrase “the numbers [numerici] that were to be found” to refer to specifically irrational expressions. Cardano’s “golden rule” shows how to find what he calls the “closest approximation” through finding the integers, “greater and less, which most nearly satisfy the equation,” then generating a series of “differences” between the values those integers generate when substituted in the equation, from which a further refined estimate can be made, leading to what he takes to be a converging series of approximations. Through this procedure, “you will undoubtedly arrive at an insensible [insensibilium] difference” compared with the true value: “This is universal reasoning and needs no other rule.”\textsuperscript{26} His procedures here seem to reflect the

\textsuperscript{24} Cardano might well have been in contact with his father’s colleague Gafori, the celebrated music theorist mentioned above, who commissioned translations of Aristides Quintilianus and Ptolemy; under the influence of Ficino and Neo-Platonism, Gafori engaged in extensive speculation on musical cosmology but was also, like Cardano, interested in practical music. See Cardan, \textit{Writings on Music}, p. 15 n 1; and Moyer, \textit{Musica Scientia}, pp. 67–92. For Cardano on the diesis see \textit{De tranquitillitate}, in Cardano, \textit{Opera omnia}, Vol. 2, p. 337, cited in Cardan, \textit{Writings on Music}, p. 22 n 36, which also includes the quotation from Miller.

\textsuperscript{25} Cardan, \textit{Writings on Music}, p. 45, cites Lefèvre. Cardano’s argument proceeds: “But if I wish to divide a proportion by itself, as the whole tone 9/8, I can do this correctly and arithmetically [verè & arithmetice] only if both numerator and denominator are doubled, making 18/16. Then you take a number equidistant from each, 17 in this instance, and the proportion 18/16 [sic; this should read 17/16 or 18/17] is a semitone or one-half of 9/8. According to this a diesis will be 36/35 or 35/34, as you wish, but it is closer in perception [sensibilium] to 36/35, although the true [vera] diesis is closer to 35/34. I said closer because a true calculation of one-half of 9/8 is made by multiplying 9 times 8 to make 72, and by taking its square root or $\sqrt{72}$, and the latter’s proportion to 8 or $\sqrt{72}/8$ is the true semitone. In the same way a diesis will be RR4608/8, which is very close to 239/232 or to 35/34, and closer to 35/34 than to 36/35.” \textit{Ibid.}, pp. 47–48 (Cardano, \textit{Opera omnia}, Vol. 10, p. 108). The textual error noted in this quotation is verified by Cardano’s statement on p. 45 that the “small semitone” is 18/17. He writes ratios as fractions, though Greek mathematics emphatically did not treat them as equivalent, and notates $\sqrt{R}$ for square roots, RR for fourth roots. The quotation about the dating of the resultant temperament comes from J. Murray Barbour, \textit{Tuning and Temperament} (East Lansing: Univ. Michigan Press, 1951), p. 7.

\textsuperscript{26} For the phrase “the numbers [numerici] that were to be found” see Cardano, \textit{Opera omnia}, Vol. 4, p. 281, translated in Cardano, \textit{Great Art of the Rules of Algebra} (cit. n. 23), p. 203. Cardano calls irrational quantities “irrationales” in the 1545 edition but “alogi” in the 1663 edition (pp. 35 n 10, 41 n 24, 47); he does not use these terms as adjectives for “numerici.” For his comments about the “insensible difference” and “universal reasoning” see pp. 182, 185.
practical sense that comes to the fore in his musical writings: “true” geometric (irrational) and “true” arithmetic (rational) quantities sound the same; their differences are “insensible.” The sole example he has of this kind of mediation between “perceptual” and “true” quantities is music; he lacks any other kind of mathematical physics (as it would later come to be called) that could have confronted mathematical idealizations with physical reality. But music was sufficient. Musical judgments intermixed rational and irrational quantities, supporting and paralleling Cardano’s working equivalence of the two in his algebraic art and providing it with crucial examples.

THE TRIAL OF VICENTINO

The grounds on which Nicola Vicentino (1511–ca. 1576) treated irrational quantities as numbers had more to do with issues of melodic style and musical practice, as for Cardano, than with the more purely theoretical questions about the elementary intervals that concerned Stifel. Boethius had enumerated three ancient “genera” of melody, each genus designating a separate set of basic intervals on which music could be constructed. The most familiar is the diatonic genus, based on the pattern of a semitone followed by a tone and another tone: S T T. (Modern major and minor scales are diatonic, though neither was among the ancient modes.) The other two genera are more unfamiliar. The chromatic genus has the pattern S S S3, where S3 stands for a “trihemitone” (an interval composed of three semitones). According to Boethius, this genus is “called ‘colored’ since it is the first alteration from” the diatonic genus, which “is somewhat more severe and natural [duirus et naturalius], whereas the chromatic departs from natural inflection and becomes more sensual [mollius].” The name “chromatic” persists even today to describe music that makes extensive use of consecutive semitones, sometimes (as Boethius suggested) to evoke greater sensuality or expressivity.

No such parallel remains in our music corresponding to the enharmonic genus, which Boethius considers “even more closely joined” than the chromatic, in the sense that the enharmonic genus uses the quarter tone (the diesis, abbreviated D), according to the pattern D D T2, where T2 denotes a “ditone,” an interval composed of two whole steps. Apart from some self-conscious attempts to recreate such music that we will come to and some experimental music of the twentieth century, the diesis fell out of use in Western music. Yet Boethius does not treat it as exotic but only remarks that it “is beautifully and fittingly yoked together”; indeed, its Greek name (ἀρμονία) is the general word that has come down to us as “harmony,” suggesting that the enharmonic genus was considered harmonious par excellence.

27 Boethius, Fundamentals of Music (cit. n. 14), 1.21. The ancient Dorian species of the octave used this pattern in the form E F-G-A-B/C-D-e, in which ‘ marks a semitone, - a tone (hence, in this case, S T T T S T T); on the modern keyboard it would be sounded by the sequential white keys starting on E. Note that the diatonic pattern S T T is not rigidly repeated but sets a general design for the octave species: in the diatonic genus, only tones and semitones are used and every semitone is surrounded by two sequential tones on either side. The other ancient octave species follow this same diatonic pattern of tones and semitones, though beginning at a different point along the pattern; thus, the ancient Phrygian octave species (now usually stated beginning on the note D) runs T S T T S T (D-E-F-G-A-B/C-d).

This modern representation is anachronistic not only in its use of present note names (hence also raising questions of the underlying temperament) but also in stating this mode as a “scale” ranging over an octave; ancient and medieval music theory characteristically stated modes in terms of tetrachords (four sequential notes) or hexachords (six-note scalewise groups), not octaves, as we tend to do. Note also that Boethius uses for his semitone 243:256, the interval between a ditone and a perfect fourth, which is not an exact equal division of the tone. The trihemitone is 294.1 cents, slightly smaller than the modern minor third, 300 cents.

28 Ibid. The ditone is 407.8 cents, slightly larger than the modern major third, 400 cents. The Czech composer
Before the sixteenth century, only the diatonic genus seems to have evoked commentary, perhaps because it was used in musical practice, such as chant; in general, it may be that Boethius’s passing mention of the genera was taken to indicate that this was an obscure or recondite matter, hence usually neglected. New translations brought these genera to prominence—for example, Carlo Valgulio’s Latin translation (1507) of the *De musica* commonly (but incorrectly) attributed to Plutarch, from which Vincenzo Galilei later drew much of his information on Greek music.29 Pseudo-Plutarch emphasized the superiority of the enharmonic genus and complained that “the musicians of our times, though, disdained the most beautiful genus of all and the most fitting, which the ancients cherished for its majesty and severity.”30 In his book *L’antica musica ridotta alla moderna prattica* [Ancient Music Adapted to Modern Practice] (Rome, 1555), Vicentino identified the enharmonic as the secret behind those extraordinary, lost powers of ancient music, which he decided to revive.

Vicentino was a practicing musician and a composer with deep interests in the theory of music (see Figure 3). Born in Vicenza, he came under the influence of the humanist Giovanni Giorgio Trissino, who in 1524 had already described the enharmonic and the chromatic as “two genera that our age does not know.” After studies with Adrian Willaert, the great Venetian composer, Vicentino came to Ferrara at the behest of Cardinal Ippolito II d’Este, whom he then accompanied to Rome. In 1546 Vicentino published his first book of madrigals, but around 1534 he had already begun thinking about the ancient genera.31 In 1551 he became embroiled in a public controversy that shows the extent to which these matters provoked hot contention among the educated elite throughout Europe.

The argument started in June 1551 after a performance of a motet at the home of Bernardo Acciaioli in Rome, when those present began discussing what genera of melody were used in the composition. The Portuguese composer and theorist Vincente Lusitano maintained that the motet used only the diatonic genus, whereas Vicentino argued that it used elements of all three genera. What was at stake went beyond this single work to all of contemporary practice: What was the true status of those ancient genera in contemporary music? The broader implications of this question concerned not only whether modern music had kept or broken faith with its ancient heritage but also the character and integrity of the cosmos, which was widely assumed to be regulated by musical intervals.

The debate began with a wager of two gold scudi and quickly became formal and public. Over a period of five days (2–7 June), Vicentino and Lusitano presented their arguments at the Vatican to “an audience of many learned men,” in the presence of Cardinal Ippolito and

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“judges” who were singers in the chapel of Pope Julius III. This tribunal found against Vicentino in a statement that reads (given the ecclesiastical authority of the presiding cardinal

Figure 3. Portrait of Nicola Vicentino, aged forty-four, as the frontispiece of his book L’antica musica ridotta alla moderna prattica [Ancient Music Adapted to Modern Practice] (Rome, 1555). In the outer border, his motto reads: “You have revealed to me the uncertain and hidden things of Thy science.” In the inner border, he is identified as “inventor of the archicembalo and also of the practical division of the chromatic and enharmonic genera.”

Vicentino, Ancient Music Adapted to Modern Practice, pp. 302–314, gives Vicentino’s account; the passage cited comes from p. 303. The “judges” were Bartolomeo Escobedo and Ghiselin Danckerts, who, though he was not present, later studied statements prepared by both sides; see also Moyer, Musica Scientia (cit. n. 4), pp. 168–184. The young Orlando di Lasso may have been in attendance, according to Kaufmann, Life and Works of Nicola Vicentino, p. 24 n 5; but he may have arrived later in 1551, as claimed by Horst Leuchtmann, Orlando di Lasso: Music der Renaissance am Münchner Fürstenhof (Wiesbaden: Reichert, 1982), p. 112. In either case, Lasso’s extraordinary chromatic Prophetiae Sibyllarum dates from 1550–1552 and may well show the influence of Vicentino and the controversy surrounding him.
and the papal offices of the judges) like a legal anathematization, concluding that “the said Don Nicola must be condemned, as we sentence him in the wager made between them” (see Figure 4). Vicentino’s own account treats quite seriously what he considered a grave injustice, whether or not we take the scene as an auto-da-fé for “heretical pravity” that anticipates the trials of Bruno or Galileo. Perhaps Vicentino’s wounded pride kept him from taking the less serious tone others may have adopted. But even a high-spirited imitation of inquisitorial proceedings presided over by an eminent cardinal seems ominous. The Church, especially the Jesuits, condemned any alterations to the foundations of mathematics that would undermine its epistemic certainty and hence the unchanging rational foundations of Christian doctrine. We will shortly consider Vicentino’s argument that experience was the “mistress” of musical and mathematical theory—rather than pure reason, as Boethius taught and the Church insisted.34

The confrontation at the Vatican led Vicentino to publish his defense for a larger public interested in the case and willing to pay to read about it. By way of amplifying and illustrating his assertions, Vicentino described his newly invented archicembalo, an “arch-harpsichord” whose specially designed keyboard could play the complex variety of semitones and dieses necessary to execute chromatic and enharmonic compositions (see Figure 5). Vicentino’s keyboard mechanized the playing of quarter tones, heretofore

33 Vicentino, Ancient Music Adapted to Modern Practice, pp. 313–314. Danckerts later wrote a treatise explaining his reasoning; see Paul Anthony Luke Boncella, “Denying Ancient Music’s Power: Ghiselin Danckerts’ Essays in the ‘Generi Insusitati,’” Tijdschrift van de Vereniging voor Nederlandse Muziekgeschiedenis, 1988, 38:59–80. Danckerts’ reasoning was that Vicentino’s definition of the genera was overly broad, encompassing more examples than really followed the genera strictly, and he insisted that a proper example of the enharmonic genus must use the diesis. See also Timothy R. McKinney, “Point/Counterpoint: Vicentino’s Musical Rebuttal to Lasiano,” Early Music, 2005, 33:393–411.


35 See Marco Tiella, “The Archicembalo of Nicola Vicentino,” English Harpsichord Magazine, 1975, 1:134–144; Rudolf Rasch, “Why Were Enharmonic Keyboards Built? From Nicola Vicentino (1555) to Michael Bulyowski (1699),” Schweizer Jahrbuch für Musikwissenschaft, 2002, 22:35–93; Denzil Wright, “The Cimbal Cromatico and Other Italian String Keyboard Instruments with Divided Accidents,” ibid., pp. 105–136; and Patrizio Barbieri, “The Evolution of Open-Chain Enharmonic Keyboards, c 1480–1650,” ibid., pp. 145–184. The split black keys of Vicentino’s instrument were not unique; other keyboards had earlier used this device, for nonequal temperaments distinguish notes like G♭ and F♯ that equal temperament conflates into what is now called a single “enharmonically equivalent” pitch G♭/F♯, though this modern term here indicates an effective equivalence between these pitches that reverses the ancient meaning of this word (and its denotation of quarter-step distinctions). Vicentino’s intervals are played on a modern archiorgano in the CD attached to Cordes, Nicola Vicentinos Enharmonik (cit. n. 29).
The sentence passed against Vicentino, as recorded in L’antica musica ridotta alla moderna prattica, fol. 98v.
laboriously measured and sounded one by one. His new instrument enabled accurate renditions of enharmonic music, but tuning it required deciding the exact interval of a diesis. To do so, Vicentino needed to unearth the work of ancient theorists who addressed these musical and mathematical questions.

36 In *De subtilitate* (1550), Cardano described with interest Vicentino’s new instruments, along with the innovations of Lucretia Todescha, a young girl from Bologna, who added six strings to the lute, allowing new possibilities of intonation; see Cardan, *Writings on Music* (cit. n. 23), pp. 194–195.
THE CONFLICTING CLAIMS OF HEARING AND REASON

In the mid-sixteenth century, the problem of dividing the tone was not yet solved uniquely; several conflicting definitions of the semitone remained in use. This aggravated the problem of defining the diesis: how could one define a quarter tone if the half tone remained so contentious? The obvious approach was to define unequal major and minor dieses by dividing up major and minor semitones, but this would lead to an endless recurrence of the problem of dividing intervals, mise en abîme. New clarity was sought in the ancient sources. What follows, then, is not a digression into antiquity but an account of how ancient problems returned to life.

Boethius had not given the enharmonic diesis a precise ratio, perhaps because of its very smallness. Plato and Aristotle considered the diesis a kind of element, analogous to a vowel or consonant; Aristoxenus of Tarentum (fourth century b.c.), a pupil of Aristotle, judged that “the voice cannot distinctly produce an interval even smaller than the smallest diesis, nor can the hearing detect one, in such a way as to grasp what part it is either of a diesis or of any of the other intervals which are known.” The diesis, it was concluded, may be so small an interval that strict, secure definition is elusive. This judgment may also reflect the material circumstances and difficulties surrounding the production of this interval. Aristides Quintilianus (first century A.D.) noted that the enharmonic “has gained approval by those most distinguished in music; but for the multitude, it is impossible. On this account, some gave up melody by diesis because they assumed through their own weakness that the interval was wholly unsingable.” Thus, even in ancient times the diesis involved discrimination and virtuosity, as in the quarter-tonal “bending” of pitch possible on the aulos (a pipe with finger holes and a reed mouthpiece, often played in pairs). Aristotle described the aulos as “orgiastic,” its shrill wails often associated with Bacchic and Corybantic rites; Longinus wrote that the aulos could send its listeners out of their...
minds and set their feet tapping to its rhythms. Such associations would not militate toward fussiness in intonation, if indeed the diesis was an ecstatic “bending” of a pitch not really to be measured by any ratio but only by the inspired frenzy of the Dionysian virtuoso. Though this aural interpretation goes against Pythagorean tradition and its ratios, its ancient champion was Aristoxenus, the great contrarian voice in Greek music theory. Where the Pythagoreans exalted reason over sensual judgment, Aristoxenus emphasized the fundamental role of the senses (here showing Aristotle’s influence): “Through hearing we assess the magnitude of intervals, and through reason we apprehend their functions.”

Most of the information we have on the enharmonic diesis comes from him, suggesting that this interval may have fit particularly well into his thesis that discriminative hearing, rather than predetermined, fixed ratios, really determines musical intervals.

Aristoxenus stood at a critical point in the problem of subdividing intervals, which (as we have seen) involves irrational magnitudes if the divisions are to be strictly equal. In the face of this paradox, Aristoxenus opened the liberating possibility that we might weaken or abandon the demand that every interval be strictly rational through his reliance on the sense of hearing, rather than on reason. He himself never seems to remark that his line of argument would imply the possibility of quantities that might bridge the rational and irrational; on the contrary, he explicitly maintains a sharp “division...in respect of the differences between the rational and the irrational.”

This “Aristoxenian turn” from regarding intervals as ratios to regarding them as “pitch distances” can be more clearly discerned in Claudius Ptolemy’s Harmonics (second century A.D.), in which the great astronomer connects stars, planets, and musical genera. In Ptolemy’s view, Aristoxenus constructed the enharmonic genus by essentially assigning the unit of 6 to each diesis, in units where the tone is 24 units. If so, each interval is no


43 Barker, ed., Greek Musical Writings (cit. n. 15), Vol. 2, pp. 150. Though ancient Roman scholars referred to Aristoxenus simply as “the musician,” most of his numerous works were lost and his Elements of Harmonics was not available until Antonio Gogava’s Latin translation of 1562—hence after Vicentino’s book appeared—though Valgulio had quoted a few fragments from him (1507) and spoken in his praise. Boethius generally followed mainstream Pythagorean tradition; he discusses Aristoxenus briefly and dismissively, noting that “since he attached little value to reason but yielded to aural judgment, he does not indicate numbers for pitches as a means of obtaining their ratios. Instead, he estimates the differences between them. Very incautiously he considered that he knew the differences between pitches for which he had established no magnitude or measure”:

44 For Aristoxenus’s references to the diatonic see Barker, ed., Greek Musical Writings, Vol. 2, pp. 135, 140, 143, 145, 154, 165–166, 182, 184. Barker argues that “though Aristoxenus insists that the smallest usable interval is the enharmonic diesis, a quarter-tone, he is perfectly prepared to employ smaller intervals, down to one twelfth of a tone, in his theoretical calculations” (p. 135 n 50). Aristoxenus writes that “the student should try to accept each of these [definitions of note, intervals, and their combination in a systēma or concatenation of intervals] in the right spirit, without quibbling over whether the account offered of each is exact or only rather approximate....For it is difficult, perhaps, in all cases where we are dealing with things that stand at the beginning to articulate an account that contains an exhaustive and accurate interpretation” (p. 135).

45 Ibid., p. 137. Note also the discussion of rational and irrational rhythms in Aristoxenus’s fragmentary Elementa rhytmica: ibid., p. 188.

46 Ptolemy, in Harmonics, expresses intervals as ratios, contra Aristoxenus, as noted in Barker’s commentary:
longer a ratio but some multiple of a unit fixed essentially arbitrarily (here, by assigning 24 of them to a tone). By breaking away from Pythagorean ratios, Aristoxenus took a crucial step toward treating an arbitrary musical quantity as a unit unto itself, apart from whether it is or is not rational with respect to the initial integral units of string length. His demonstrations recall Euclid, who had shown in the *Elements* that magnitudes are rational or irrational only relative to other magnitudes, not in any absolute sense. The diagonal of a square is incommensurable with its side but may be perfectly commensurable with other lines (for instance, with the sides of another square built on that diagonal). If Ptolemy is correct that Aristoxenus treated the diesis as a unit on which tone and semitone are built, that would ignore the inherent incommensurability of tone and semitone or quarter tone, as discussed above. At the very least, by abandoning the idealized fiction of a pure ratio underlying every note Aristoxenus was able to bring forward the practical “commensurability” of every pitch sung or played on real strings: because we can hear those intervals, he implies, they must de facto be commensurable.

At the beginning of his book, Vicentino acknowledges both these ancient authorities even as he proposes to go beyond them:

> Aristoxenus, who depended solely on sense, denied reason, whereas the Pythagoreans, in contrast, governed themselves solely by reason, not sense. Ptolemy more sanely embraced both sense and reason, and his opinion has satisfied many people up to now. In this work, however, you will recognize many cases in which reason is not a friend to sense, and sense is not receptive to reason. And I shall give you detailed information as to how compatible sense and reason can be, thus enabling you to assess the dearth of sweet musical concords in the past. Therefore, with experience [experientia] as the mistress of things, it will be easy to judge the difference between ancient and modern music by considering examples of both.

Paradoxically, Vicentino’s project seeks to outstrip the “modern” practice of music by reviving the “ancient,” specifically through the retrieval of the lost enharmonic genus he considers “more sweet and smooth than the other two genera.” The emphasis on experientia means that musical practice is the new touchstone that can outweigh older arguments about rationality.

Vicentino’s “Aristoxenian turn” also drew on an older contemporary who had enjoyed the patronage of the d’Este family and who seems to have been his immediate source on ancient tunings and modern temperament. In his *Musica theorica* (1529), Lodovico Fogliano (before 1500–after 1538) had taken an Aristotelian, Aristoxenian position that

Barker, ed., *Greek Musical Writings*, Vol. 2, pp. 384–391 (for Ptolemy), 270 (commentary regarding Aristoxenus). I thank Thomas Mathiesen for pointing out to me that Ptolemy’s expansion of Aristoxenian parts of the fourth from 30 to 60 may well indicate that Ptolemy had not actually read Aristoxenus but was relying on an epitome, whether by Cleonides or someone else.


48 Vicentino, *Ancient Music Adapted to Modern Practice* (cit. n. 31), pp. 6, 12 [fols. 3r, 4v]. Even though Vicentino did not have access to Aristoxenus’s full treatise in 1555, he had gathered enough of the essential content of Aristoxenus’s arguments that he referred to them often. Valgulio’s translation of pseudo-Plutarch included many citations of Aristoxenus, especially concerning the enharmonic genus; see Barker, ed., *Greek Musical Writings*, Vol. 1, pp. 205–257, which discusses the enharmonic genus at pp. 215–218. Also, in 1497 Giorgio Valla had brought out a Latin translation of Cleonides’ compendium of Aristoxenus; see Claude V. Palisca, “Musical Change and Intellectual History,” in *Music and Ideas in the Sixteenth and Seventeenth Centuries* (cit. n. 19), pp. 1–12, esp. p. 6. Though Ptolemy’s *Harmonics* did not appear until Gogava’s translation of 1562, the fifth book of Boethius’s *Fundamentals of Music* would have given Vicentino a close account of Ptolemy’s book 1.
declined to subordinate music to mathematics. In the course of setting out a new system of temperament, Fogliano found himself, like so many other theorists before him, confronting the problem of dividing a musical interval into two equal pieces—this time the syntonic comma (81:80), which he found necessary in order to define the major third consistently within his system. To do so, he used the same construction used by Lefèvre (see Figure 1), here applied to divide this tiny interval (see Figure 6). Fogliano addresses himself both to “musici practici” and to “Theoretici” who, for their several needs, have recourse to instruments and sounding strings and hence find themselves confronting proportions that are “irrational and surd.” His geometrically exact result departs from a rational approximation by a practically inaudible difference. In the process of evenly

Figure 6. Diagram from Lodovico Fogliano, Musica theorica (1529), fol. 36r, showing how to obtain a mean proportional BC between AB and BD, hence dividing exactly in half the comma 81:80; this differs from a rational approximation 80½ (the arithmetic mean of 80 and 81) by only 1/30 of a cent, a practically inaudible difference.

Maria Rika Maniates notes that, in Vicentino’s book, “all 14 allusions to Lodovico Fogliano are tacit, even though Fogliano was probably Vicentino’s source on matters of ancient tuning and modern temperament”; see Vicentino, Ancient Music Adapted to Modern Practice, pp. xxvi. Francesco Patrizi mentioned Fogliano and Vicentino together in his list of those the house of d’Este had sponsored in its capacity as “regeneratrici della Musica”; see Kaufmann, Life and Works of Nicola Vicentino (cit. n. 31), pp. 19 n 26, 48. See also Moyer, Musica Scientia (cit. n. 4), pp. 141–147.

Lodovico Fogliano, Musica theoretica (Venice, 1529), fol. 36r. For full details see Claude V. Palisca, “Music and Scientific Discovery,” in Music and Ideas in the Sixteenth and Seventeenth Centuries (cit. n. 19), pp. 131–160, esp. pp. 143–144. Essentially, Fogliano’s problem was that following the Pythagorean option the major third would turn out too large (81:64), but following the Ptolemaic option it would be too small (80:64). His compromise involved splitting the difference, the interval 81:80 known as the syntonic comma.
dividing ever smaller musical intervals, such results further geometrized what had here-tofore been considered the purely numerical domain of musical intervals, tacitly over-stepping the ancient separation between geometry and arithmetic through the commonsensical identification of a physical string length with the corresponding line in a geometric diagram (such as Figure 6), rather than with a purely numerical ratio, considered apart from any sounding body.

**VICENTINO’S ACCOUNT OF “IRRATIONAL RATIOS”**

Vicentino expresses this new situation as it emerges in specifying the diesis:

The enharmonic genus contains a semitonal division [i.e., the diesis] that is disproportioned and irrational [sproportionata et inrationale]. And the other parts accompanying this division cannot contain proportioned and accurate leaps because they must correspond to this irrational ratio [proportione inrationale]. Let no one be astonished [si marauiglia] that the nature and division of the genus allows this to happen. The nature of the diatonic genus goes along with its own steps and leaps, which are true in their ratios [proportione]. But the nature of the division of the chromatic genus permits the disruption [si rompi] of the diatonic order and the creation of two semitones from the whole tone as well as the step of the incomplete trihemitone [three semitones]. Likewise, the nature of the enharmonic genus disrupts the order of both the diatonic and chromatic and permits the creation of steps and leaps beyond the rational [gradi & i salti fuore di ogni ragione]. For this reason [cagione] such a division is called an irrational ratio [si domanda proportione inrationale].

In using the phrase “irrational ratio,” Vicentino is the first (as far as I can determine) to try to state in some positive (if paradoxical) way the status and character of musical intervals that are formed through irrational, geometric means (as in the Fogliano example) but at the same time are incorporated into the rational arithmetic of music theory. As so often in the history of science, Vicentino’s contribution may have been to state as clearly as possible, in common words, not only the inherent paradox of a new concept but its necessity and the functionality that justifies our embracing it despite and even through its paradoxicity.

Vicentino’s unfolding argument presents us with both sides of the paradox: the diesis, as constructed geometrically, is irrational but, functioning as the smallest “unit” within a framework of numerical ratios, is in that sense also effectively a “proportion.” Such a hybrid concept, like a centaur, needs to be grasped in its inherent duality, considered as its essence rather than as grounds to refute its existence. Vicentino’s premise—that the enharmonic genus exists and is superlatively important—was attested by many ancient sources. Therefore its basis, the diesis, must also exist, and so we should take in stride whatever paradoxical qualities it may have. Vicentino does anticipate that we might rightly be “astonished” or wonder at what at first seems a prodigy or monster, this “rational irrational”—rather in the way that we might consider a centaur monstrous if we were not familiar with examples of wise centaurs such as Chiron, “who included music.

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51 Vicentino, *Ancient Music Adapted to Modern Practice* (cit. n. 31), p. 207 [fol. 66v]. Though rare, the spelling “inrazionale” is found as a Latinized variant in the *Vocabolario della Crusca* (1612); John Florio’s *Queen Mary’s World of Words* (London, 1610) includes both spellings with the same definition (pp. 258, 268). I thank Alexander Bevilaqua, Franco Ligabue, Thomas Mathiesen, and Marco Potenza for helpful discussions of the orthography of this word.
among the first arts he taught Achilles at a tender age, and who wanted him to play the harp before he dirtied his hands with Trojan blood.”

Vicentino’s argument also builds cunningly on the successive examples of the three genera, as if in a kind of rhetorical crescendo. The diatonic sets the point of departure, the purely rational (proportione). From our earlier discussion, we know that the semitone of the diatonic genus already contains the latent problem of its relation to the tone: namely, that no rational semitone can be exactly half of a tone. The stratagem of devising major and minor semitones merely conceals this problem without solving it; it is a stopgap solution that serves to make the diatonic appear wholly rational. The two successive semitones in the chromatic genus reiterate the problem: which semitones are they to be, major or minor? In the chromatic genus, the latent problems hidden in the diatonic come forward sufficiently to cause “disruption”: incipient instability and growing theoretical uncertainty. Only with the enharmonic genus does this simmering instability disrupt both the diatonic and chromatic genera and “permit the creation of steps and leaps beyond all reason.” This, Vicentino tells us, is the cause that should move us not only to call them “irrational ratios” but also, by so naming them, to install them in mathematics and music jointly as having equal existential force with the “rational ratios” we learned from arithmetic and the “irrational magnitudes” from geometry. For Vicentino, music is the intermediate ground on which arithmetic and geometry meet in such hybrid concepts as “irrational ratios,” shared between mathematics and music.

THE MATHEMATICIAN AS COMPOSER

The attitudes of Stifel, Cardano, and Vicentino about these mathematical issues reflect their respective musical projects. As we saw, Stifel’s closest approach to affirming that “irrational numbers” were “real” or “true” came in the context of his musical theorizing. Yet this did not prove sufficient for him to maintain this position in the face of the actual infinitude of fractional sums, the “cloud of infinity,” perhaps because his involvement with music remained largely theoretical and restricted. Stifel’s main foray into practical music was his thirty-two-strophe song to propagate Luther’s teachings, “Johannes thu¨t uns schreiben” (1522), based on the popular tune “Bruder Veyt.” Stifel’s composition led him into a polemical war of song and countersong with the theologian Thomas Murner, both always keeping this same melody for their new lyrics (see Figure 7). Though Stifel’s song was very popular and went through many printings, even serving as an important early example of the power of music that may have inspired Luther himself, its purely melodic component was very simple and completely derivative. Stifel merely provided new words to an old tune; he had no vision of reforming the elements of music that would


53 Karol Berger stresses “the transformational nature of the relationship between the three genera” as “one of the most fascinating aspects of the whole Vicentinian system and one which is constantly emphasized by the theorist,” by which he means a “three-level hierarchy in which: a) the enharmonic level necessarily presupposes the more basic chromatic level which in turn presupposes the fundamental diatonic; b) the diatonic contains the possibility of the chromatic subdivision and the chromatic may in turn be subdivided enharmonically”: Berger, *Theories of Chromatic and Enharmonic Music* (cit. n. 29), pp. 15–16.


compare with Vicentino’s ambitious project to (re)create a whole new genus of music. By comparison with Stifel’s song, Cardano’s extant compositions are very ambitious, including a five-voice perpetual canon and a tour de force of four simultaneous three-voice canons (twelve voices in all). Among Vicentino’s much larger output, his motet “Musica prisca caput” (see Figure 8) dramatizes the emergence of the enharmonic genus to glorify his patron: its first verse is in the diatonic genus (measures 1–16) and the second in the chromatic (mm. 16–30), while the final verse (mm. 29–48) is in the enharmonic, dramatically reserving the introduction of the diesis to produce a special aura around the name of Cardinal Ippolito (m. 31). This motet’s pointed delineation of all three genera provides yet another demonstration and justification of Vicentino’s views to refute his critics and contest his condemnation. He tells us, as well, that the whole d’Este family, including the Cardinal and the Prince of Ferrara, sang this daring new music, quarter tones and all, “with the most exceptional diligence.” Vicentino had evidently persuaded them that, in contrast to the public uses of diatonic music “in communal places for the benefit of coarse ears,” such enharmonic music was “reserved . . . to praise great personages and heroes for the benefit of refined ears amid the private diversions of lords and princes.” Thus Vicentino brought his polemic on behalf of enharmonic music not only to experts but also to the powerful amateurs whose princely involvement he considered capital for his cause. By so doing, he carried his case to an alternative and (in his view) superior social milieu, whose approval and validation he took as definitive. In addition, he positioned himself so that his theories would be highly visible and readily available to another aristocratic set that would take up his ideas in the next generation—the Camerata and Vincenzo Galilei, whose admiring advocacy indeed vindicated Vicentino posthumously.

For instance, Gioseffo Zarlino (ca. 1517–1590), the preeminent theorist of the sixteenth

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56 These compositions can be found in Cardan, Writings on Music (cit. n. 23), pp. 139, 154–171.
57 Vicentino, Ancient Music Adapted to Modern Practice (cit. n. 31), p. 33 [fol. 10v]. For further commentary on the musical details of his motets vis-à-vis his critics see Maniates’s commentary, ibid., pp. li–liii, which treats “Musica prisca caput” on pp. lvii–lvii; recordings of this and his other motets from this book are available on the CD accompanying Cordes, Nicola Vicentinos Enharmonik (cit. n. 29).
century and Vincenzo’s teacher, incorporated these irrational ratios in his representation of the tuning of a lute (see Figure 9), which showed how a geometric construction can dictate the placement of the frets that thereby divide intervals equally and set up an equal temperament. In this way, geometry set a template that could be mechanically reproduced without having to duplicate its geometric construction.58

Even so, the use of these irrational ratios remained controversial. Though Zarlino’s student Giovanni Maria Artusi (1546–1613) accepted his teacher’s geometric construction for tuning instruments, he balked at applying such irrational ratios to vocal music. Writing in 1603, Artusi objected that Claudio Monteverdi, the great exponent of the new operatic art sponsored by the Camerata, was applying irrational ratios, “according to the doctrine of Lodovico Fogliano,” in order to generate for expressive purposes what Artusi considered intervals “false for singing,” particularly the diminished seventh and diminished fourth (see Figure 10). Artusi complained that the use of such “irrational” intervals showed that Monteverdi had no “rational” understanding of music, as he put it. Though it was possible to play the intervals on the fretted lute, “the natural voice is not suited to negotiate such unnatural intervals by means of natural ones, not having a preset stopping place like an artificial instrument. . . . It cannot justly divide the tone into two equal parts.”59 Artusi’s objections blend mathematical uneasiness about “unnatural” irrational ratios with his concomitant aversion to Monteverdi’s expressive use of those same intervals.

Such objections show the deep and long-lasting anxieties provoked by irrational ratios, anxieties that reflect both musical and mathematical considerations. Stifel was content to remain in the realm of conventional (and popular) music, and so his “irrational numbers” drew on no particular musical justification that might help defend or support them against traditional philosophical objections. In contrast, Cardano’s strong interest in composition and performance and Vicentino’s reliance on “irrational ratios” in his music seem to have helped them sustain their effective use as numbers, to the extent that they advanced them as musical and hence mathematical necessities.

The comparison of these three figures as theorists, composers, and mathematicians illuminates ways in which musical concerns, both practical and theoretical, influenced the acceptance of novel mathematical concepts, which in turn bore on musical matters. Long before, Plato had pointed out that astronomy connected the ideal and the visible phenomena shown through the motion of the heavenly bodies, noting also that “as the eyes are fixed on astronomy, so the ears are fixed on harmonic movement, and these two kinds of knowledge are in a way akin.” The pioneering studies of the development of mathematics tended to recognize only the visible side of this insight, as when Jacob Klein emphasized the intimate connection between Viète’s mathematical and astronomical works.60 Klein’s seminal treatment did not recognize the significance of music as a consequential meeting ground between mathematics and perception. In light of the innovations of Stifel, Car-

58 Interestingly, Cardano was critical of Vicentino’s scheme for tuning, which he found “not unserviceable, but it is not entirely accurate”; see the passage cited in note 36, above. Regarding Zarlino see also Moyer, Musica Scientia (cit. n. 4), pp. 202–225.

59 See Berger, Theories of Chromatic and Enharmonic Music (cit. n. 29), pp. 44–56 (Zarlino), 88–95 (Artusi); he notes that Artusi, like Vincenzo Galilei, found the use of equal temperament for the harpsichord “strange” (p. 92). The quotations from Giovanni Maria Artusi, Discorso secondo musicale di Antonio Bracchino da Todi (Venice, 1608), are taken from Mark Lindley, “Chromatic Systems (or Non-Systems) from Vicentino to Monteverdi,” Early Music History, 1982, 2:377–404, on pp. 400–404.

60 Plato, Republic 530d; and Klein, Greek Mathematical Thought (cit. n. 1), p. 151.
Dimostrazione d'una composizione fatta tutti tre i generi [...]
Figure 8. Vicentino’s Latin motet “Musica prisca caput,” which he included in L’antica musica ridotta alla moderna prattica, fols. 69v–70v, as a “Demonstration of a composition made from all three genera” of music; this modern score follows Vicentino’s convention that notes with a dot above them are raised in pitch by a diesis. Text: “Ancient music of late has raised her head out of the darkness, / So that, with antique and sweet numbers, to compete with ancient deeds, / Your great deeds, Hyppolitus, she might send high above the heavens.” (Courtesy Manfred Cordes.)
dano, and Vicentino discussed here, we need to reconsider his judgment that their mathematical work, however technically ingenious, finally lacked understanding of the underlying symbolic project.\textsuperscript{61} To Klein’s account of the emergence of algebraic symbol-

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure9.png}
\caption{Gioseffo Zarlino, Sopplimenti musicali (Venice, 1588), p. 211, showing the geometric construction to fret a lute in equal temperament using the construction shown by Lefèvre and Fogliano. The caption reads: “The equal division of the consonance of a diapason into twelve semitones.”}
\end{figure}

\textsuperscript{61} Klein, Greek Mathematical Thought, pp. 147–148: “the ‘algebra’ which has Arabic sources is continually elaborated in respect to techniques of calculation, for instance by the introduction of ‘negative,’ ‘irrational,’ and even so-called ‘imaginary’ magnitudes (numeri ‘absurdi’ or ‘ficti,’ ‘irrationales’ or ‘surdi,’ ‘impossibiles’ or ‘sophistici’), by the solution of cubic equations, and in its whole mode of operating with numbers and number signs, its self-understanding fails to keep pace with these technical advances.” See Aubel, Michael Stifel (cit. n. 12), pp. 321–325, for a reconsideration of Stifel in light of Klein’s work.
ism we now should add the striking developments that prepared the way for hybrids of the rational and the irrational through the musical meeting of arithmetic and geometry. In so doing, we may draw more closely together the study of “audible culture” and symbolic structures hitherto considered apart and unto themselves. This rapprochement may call for and guide an enriched new phenomenology that would go beyond the cool bracketing of “objectivity” by synthesizing mathematical and musical perceptions, constructs, and symbolic forms, amalgamating the theoretic, the epistemic, and the practical in a new history and philosophy of science embedded more vividly in perception, feeling, and thought.62

As Guillaume Gosselin noted in 1577, “the musician and the algebraist indeed know numbers” precisely through “their relation to something else,” in contrast with the arithmetician, who only “sees numbers in themselves.”63 His implication is that musicians

![Figure 10. Claudio Monteverdi, La favola d’Orfeo (first performed 1607), Act II, measures 274–279 (from the edition of Venice, 1615). The messenger is recounting Euridice’s dying words: “and calling on you, Orfeo, Orfeo, after a deep sigh expired in these arms.” Artusi objected to the “irrational” diminished seventh between voice (B♭) and continuo (C#) at the word “grave” (indicated by arrows), expressing the depth of her sigh.](image-url)

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63 Klein, Greek Mathematical Thought (cit. n. 1), p. 262 n 225, notes that “Gosselin in his algebraic work
were essential companions on the road that led from arithmetic to algebra in that they reached beyond numbers to the “something else” manifest through music. If so, the struggle to “hear” the mathematical irrational was indeed consequential on many levels. Mediating between the realms of mathematics and felt experience, music evoked and justified new concepts of number.

already adds ‘algebra’ to the ancient ‘Pythagorean’ division of mathematics into geometry and astronomy, arithmetic and music, assigning it, together with ‘music,’ to the realm of relational quantity. . . . This is, so to speak, the first ‘official’ introduction of algebra into the system of sciences recognized by the schools. Up to that point the ‘ars rei et census’ (art of the thing and its power, i.e., the unknown and its square) was a more or less obscure curiosity. It was even considered suitable for public exhibitions in the form of contests and aroused the wonder of the crowd much as did acrobatic or magic tricks”; he instances the turbulent contest between Niccolò Tartaglia and Luigi Ferrari, Cardano’s pupil. Klein comments on Gosselin’s classification of number on pp. 291–292 n 305; see also H. Bosmans, “Le ‘De arte magna’ de Guillaume Gosselin,” Bibliotheca Mathematica, 3rd Ser., 1906, 7:44–66 and Giovanna Cleanice Cifoletti, “Mathematics and Rhetoric: Peletier and Gosselin and the Making of the French Algebraic Tradition” (PhD diss., Princeton University, 1992). Gosselin translated into French L’arithmetique (Paris, 1578), by Tartaglia, who also experimented with the division of a tone into equal semitones; see Moyer, Musica Scientia (cit. n. 4), pp. 126–134, which also connects Tartaglia with Bartolomeo Zamberti’s comments on music in his edition of Euclid (1505).