Polygon Polynomials

Topics

- Adding, subtracting, and multiplying polynomials
- Factoring common factors out from a polynomial
- Proving that polynomials are equivalent
- Patterns and formulas for polygonal numbers
- Arithmetic sequences and series

In the *Polygon Polynomials* projects, students practice operations on polynomials as they investigate patterns and relationships in *polygonal numbers*—numbers used to create diagrams of regular polygons. By exploring geometric models built from squares and dots, students create formulas to describe the patterns that they find and prove that the expressions in the formulas are equivalent. I strongly suggest allowing students to share and compare their formulas and strategies in order to help them appreciate the great variety of possibilities and to explore the relationships between them.

Sometimes, the greatest benefits of a project lie hidden beneath the surface. Apart from the obvious value in reinforcing and extending students' content knowledge, the *Polygon Polynomials* project deepens students' appreciation for the ways in which numbers, shapes, and algebra can unite to support discovery and understanding. The project also offers students many opportunities to enhance their skills in *observation* and *questioning*. There is more to each image than first meets the eye—especially as the images become progressively more complex. Allow plenty of time for open-ended conversation about each image before sharing the directions.

Note: Dr. Ron Knott's website on Polygonal numbers inspired some of the problems and images in this activity, especially in Problems #4 – 6. *After* students have completed these problems, they may enjoy exploring his site and the additional ideas and challenges that it presents.

An overview of the project

Most of the polynomials in Stages 1 and 2 are quadratic. Even though students are not working with higher-degree polynomials here, they are learning to work with 1^{st}- and 2^{nd}-degree polynomials in flexible and sophisticated ways. The theme of factoring out multiple-term factors from a polynomial appears often. In Stage 3, students will work with 3^{rd}-degree (cubic) and 4^{th}-degree (quartic) polynomials.

Stage 1

Problem #1	Triangular numbers
Problem #2	Oblong numbers (a bridge between Problems #1 and #3)
Problem #3	Square numbers
Note	The vocabulary for these numbers is not introduced until Stage 2.

Stage 2

Problem #4	A new connection between oblong and square numbers (optional)
Problems #5–7	The center of the project: A deep exploration of polygonal numbers

Stage 3

Problem #8	Application and extension: Discovering a formula for sums of squares
Problem #9	Application and extension: Counting rectangles in a grid

Stage 1

In Stage 1, students explore *triangular*, *oblong*, and *square numbers* (though you need not introduce these names until Stage 2). The square and triangular numbers are particularly important, because they are special types of *polygonal numbers*, and they form the foundation for studying general polygonal numbers in Stage 2.

Students will learn two ways to represent formulas for sums of sequences of numbers: (1) in terms of the value final addend, V, and (2) using the term number, k, which represents the position of the final value in the list. For example, in the sum

$$2 + 4 + 6 + \cdots + 20$$

$V = 20$, because 20 is the final addend, and $k = 10$, because the final addend is the 10^{th} term. You may find it helpful to share the meanings of V and k with students in advance.

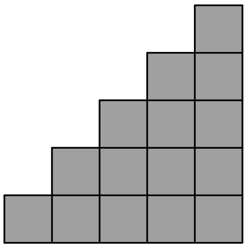
What students should know

- Create formulas to represent a set of calculations.
- Understand and apply properties of numbers and operations.
- Be familiar with the standard order of operations.
- Write simple algebraic expressions in different forms by combining like terms, manipulating simple algebraic fractions, etc.

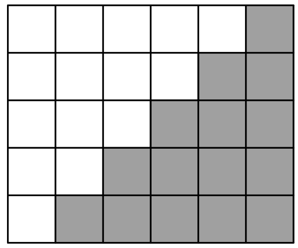
What students will learn

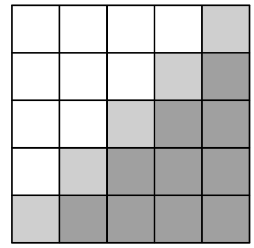
- Become more fluent with adding and subtracting simple polynomials.
- Prove the equivalence of simple polynomial expressions.
- Discover patterns for adding terms in certain well-known arithmetic sequences.
- Make connections between numeric patterns, geometric diagrams, and algebraic formulas.

Problem #1



$1 + 2 + 3 + 4 + 5$





Directions

- Add the five numbers. Then use the middle picture to do the calculation in fewer steps. Explain your thinking.
- Use the picture on the right to show another way to do the calculation.
- Draw pictures like these for $1 + 2 + 3 + 4 + 5 + 6$. Do the calculations again.
- Use the middle and right pictures to create two formulas for $1 + 2 + 3 + \cdots + k$.
- Prove that the expressions in your formulas are equivalent.

Solutions for #1

Adding the numbers

$$1 + 2 + 3 + 4 + 5 = 15$$

Using the middle picture

$$\frac{6 \cdot 5}{2} = 15$$

The two identical staircases join to form a 6 by 5 rectangle. Therefore, you multiply 6 by 5 to find the area of the rectangle and divide the result by 2.

Using the right-hand picture

Strategy 1

Find the area of the square.

$$5 \cdot 5 = 25$$

Subtract the light grey diagonal squares.

$$25 - 5 = 20$$

Divide by 2 to find the number of dark grey (or white) squares.

$$20 \div 2 = 10$$

Add the diagonal squares back in.

$$10 + 5 = 15$$

Summary:

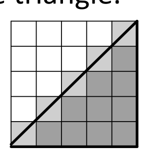
$$\frac{5 \cdot 5 - 5}{2} + 5 = 15$$

Strategy 2

Find the area of the square.

$$5 \cdot 5 = 25$$

Divide by 2 to find the area of the triangle.



$$25 \div 2 = 12.5$$

Add the 5 small half-squares back in.

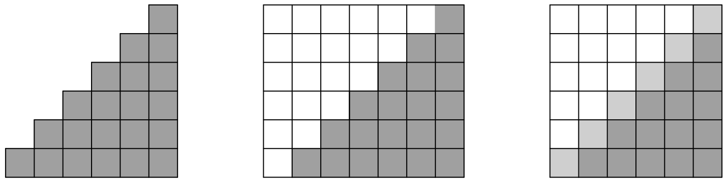
$$12.5 + 2.5 = 15$$

Summary:

$$\frac{5 \cdot 5}{2} + \frac{5}{2} = 15$$

Other strategies are possible.

Pictures for $1+2+3+4+5+6$



Sample calculations

- $1+2+3+4+5+6=21$
- $\frac{7\cdot 6}{2}=\frac{42}{2}=21$
- $\frac{6\cdot 6-6}{2}+6=\frac{30}{2}+6=15+6=21$
- $\frac{6\cdot 6}{2}+\frac{6}{2}=\frac{36}{2}+3=18+3=21$

Formulas

From the middle picture

(1) $1+2+3+\cdots+k=\frac{(k+1)k}{2}$

From the right-hand picture

(2) $1+2+3+\cdots+k=\frac{k\cdot k-k}{2}+k$

(3) $1+2+3+\cdots+k=\frac{k\cdot k}{2}+\frac{k}{2}$

Other strategies (such as using a formula to find the sum of an arithmetic series) are possible. Another likely result is

(4) $1+2+3+\cdots+k=\frac{1}{2}(k^2+k)$ or $\frac{1}{2}k^2+\frac{1}{2}k$.

Proving that the expressions are equivalent

Students will have many approaches. A few examples include:

Proving that (2) and (3) are equivalent

$\frac{k\cdot k-k}{2}+k=$	Begin with (2).
$\frac{k\cdot k}{2}-\frac{k}{2}+k=$	Decompose into two fractions.
$\frac{k\cdot k}{2}-\frac{1}{2}k+k=$	Dividing by 2 is equivalent to multiplying by $\frac{1}{2}$.
$\frac{k\cdot k}{2}+\frac{1}{2}k$	Combine like terms.
$\frac{k\cdot k}{2}+\frac{k}{2}$	Multiplying by $\frac{1}{2}$ is equivalent to dividing by 2. The result is (3).

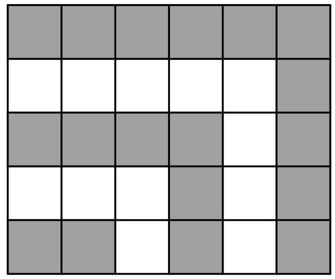
Proving that (1) and (3) are equivalent

$\frac{(k+1)k}{2} =$	Begin with (1).
$\frac{k \cdot k + k}{2} =$	Use the distributive property in the numerator.
$\frac{k \cdot k}{2} + \frac{k}{2} =$	Decompose into two fractions. The result (3).

Proving that (1) and (4) are equivalent

$\frac{(k+1)k}{2} =$	Begin with (1).
$\frac{k^2+k}{2} =$	Use the distributive property (and the definition of an exponent).
$\frac{1}{2}(k^2 + k)$	Dividing by 2 is equivalent to multiplying by $\frac{1}{2}$. The result is (4).

Problem #2



$2 + 4 + 6 + 8 + 10$

Directions

- Add the five numbers. Then use the picture to calculate the sum a different way.
- Create a similar picture for $2 + 4 + 6 + 8 + 10 + 12$. Do the calculations again.
- Create a formula for $2 + 4 + 6 + \cdots + V$. Explain your thinking.
- Rewrite your formula using the term number, k, instead of V.
- Compare your results with Problem #1. Explain what you notice.

Solutions for #2

Adding the numbers

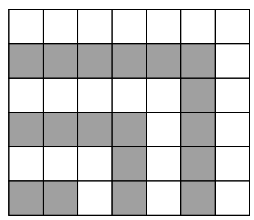
$$2 + 4 + 6 + 8 + 10 = 30$$

Using the picture to calculate the sum

$$6 \cdot 5 = 30$$

The picture forms a 6 by 5 rectangle. Multiply 6 by 5 to find the area of the rectangle.

A picture and calculations for $2 + 4 + 6 + 8 + 10 + 12$



$$2 + 4 + 6 + 8 + 10 + 12 = 42$$
$$6 \cdot 7 = 42$$

A formula for $2 + 4 + 6 + \cdots + V$

The length of the rectangle (top to bottom) is always half of V. The width of the rectangle is always 1 greater than half of V. The sum is equal to the area of the rectangle. Therefore:

$$2 + 4 + 6 + \cdots + V = \frac{V}{2}\left(\frac{V}{2} + 1\right)$$

Students may discover other equivalent ways to write to formula.

A formula using the term number, k

V is always twice the value of the term number, k. Therefore, the formula is

$$2 + 4 + 6 + \cdots + 2k = k(k + 1).$$

In other words, the k^{th} rectangular number is $k(k + 1)$. *Note*: The "k" version of the formula is helpful for purposes of this project, but each is useful in different situations.

Comparing with Problem #1

Due to the distributive property, the sum $2 + 4 + 6 + \cdots + 2k$ is always twice the sum $1 + 2 + 3 + \cdots + k$:

$$2(1 + 2 + 3 + \cdots + k) = 2 + 4 + 6 + \cdots + 2k$$

The formulas from Problem #1 and #2 reflect this fact: $2\,\frac{k(k+1)}{2} = k(k + 1)$

Problem #3

$$1 + 3 + 5 + 7 + \cdots + V$$

Directions

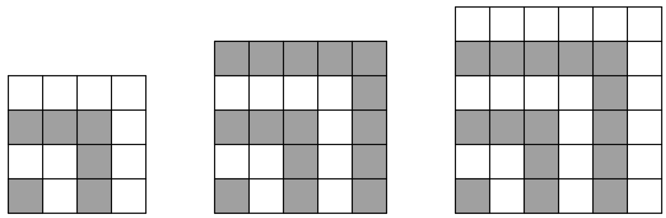
- Use Problem #2 as a guide to draw pictures for $V = 7, 9$, and 11.
- Use the pictures to show a quick way to calculate each sum. Explain your thinking.
- Apply your method to calculate the sum for $V = 99$.
- Create a formula for $1 + 3 + 5 + 7 + \cdots + V$. Explain your thinking.
- Rewrite your formula in terms of the term number, k.

Diving Deeper

- Show how to obtain the formula from Problem #1 by combining your formulas from Problems #3 and #2.

Solutions for #3

Pictures for $V = 7, 9,$ *and 11*



A quick way to calculate each sum

$1 + 3 + 5 + 7 = 4 \cdot 4 = 16$
$1 + 3 + 5 + 7 + 9 = 5 \cdot 5 = 25$
$1 + 3 + 5 + 7 + 9 + 11 = 6 \cdot 6 = 36$
Sums of consecutive odd numbers beginning at 1 appear to create square numbers!

The sum for $V = 99$

The number that you square is not the final value, V, but the final term number, k. After some experimentation and reasoning, you may discover the number of terms is half of the number that is 1 greater than the final value. For example:

In $1 + 3 + 5 + 7$, the final value, $V = 7$, is the 4^{th} term. $\left(\frac{7+1}{2}\right) = 4$
In $1 + 3 + 5 + 7 + 9$, the final value, $V = 9$, is the 5^{th} term. $\left(\frac{9+1}{2}\right) = 5$
99 appears in term number $\left(\frac{99+1}{2}\right) = 50$. Therefore, the sum is $50^2 = 2500$.

A formula for $1 + 3 + 5 + 7 + \cdots + V$

Following the pattern above:

$$1 + 3 + 5 + 7 + \cdots + V = \left(\frac{V+1}{2}\right)^2$$

A formula in terms of k.

V is always 1 less than twice the term number. Therefore:

$$1 + 3 + 5 + 7 + \cdots + (2k - 1) = k^2$$

As you would expect, the k^{th} square number is k^2.

Again, then "k" version of the formula is helpful in this project, because it emphasizes the *concept* of producing square numbers by adding consecutive odd numbers. The "V" version may be more useful for actually calculating the sums.

Stage 2

In Problem #4, students analyze connections between the *oblong numbers*

2, 6, 12, 20, 30, 42, …

from Problem #2 and the *square numbers*

1, 4, 9, 16, 25, …

from Problem #3. Introduce this terminology before students begin work. *Note:* If you prefer to launch directly into exploring polygonal numbers in Problems #5 – 7, omitting Problem #4 will not cause any difficulties.

Important notes about specific problems:

- Problem #5 contains a hint page. Let students make their best efforts without it first. However, if they continue to struggle with producing a formula for pentagonal numbers, the hints may give them some ideas for moving forward without giving too much away.
- The Problem #7 handout contains some "spoilers." Do not distribute it until students have completed Problem #6.

Problems #5, 6, and 7 lie at the heart of the *Polygon Polynomials* investigation. Students apply and extend their knowledge from Problem #1 (*triangular numbers*) and Problem #3 (*square numbers*) as they explore *pentagonal*, *hexagonal*, *heptagonal*, and general *polygonal numbers*. If they are occasionally confused between the polygonal numbers themselves and the sums that produce them, remind them of examples in Stage 1. For instance:

The sums

$1 \qquad 1+2 \qquad 1+2+3 \qquad 1+2+3+4 \qquad 1+2+3+4+5$, etc.

produce the triangular numbers

$$1, 3, 6, 10, 15, \ldots,$$

and the sums

$1 \qquad 1+3 \qquad 1+3+5 \qquad 1+3+5+7 \qquad 1+3+5+7+9$, etc.

produce the square numbers

$$1, 4, 9, 16, 25, \ldots.$$

Before students begin Stage 2, you will need to introduce them to the notation used to keep track of the various types of numbers. In this activity:

$B(k)$ stands for the k^{th} oblong number.

$P_n(k)$ stands for the k^{th} n-agonal number.

For example:

$B(4) = 20$, the 4th oblong number.
$P_3(5) = 15$, the 5th triangular number.
$P_4(3) = 9$, the 3rd square number.
$P_5(2) = 5$, the 2nd pentagonal number (which students will explore in Problem #5).

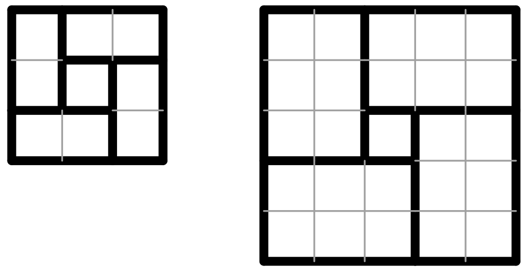
What students should know

- Understand concepts from Stage 1.

What students will learn

- Become more fluent adding, subtracting, multiplying, and simplifying polynomials.
- Recognize common factors (including multiple-term factors) in polynomial expressions and factor them out.
- Prove the equivalence of polynomial expressions.
- Discover patterns and formulas for adding terms in arithmetic sequences.
- Make connections between numeric patterns, geometric diagrams, and algebraic formulas.

Problem #4



Directions

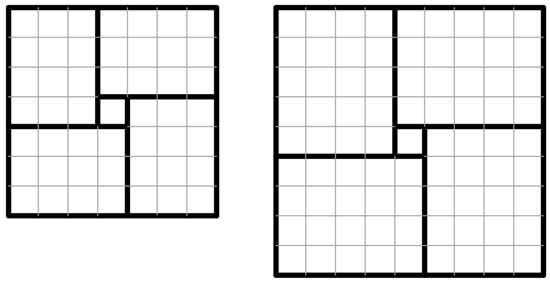
- Draw the next one or two pictures in the pattern.
- Use the pictures to create a formula relating the k^{th} oblong number, $B(k)$, to a certain square number, $P_4(x)$. (Express x in terms of k.) Explain your thinking.
- Use what you learned from Problems #2 and #3 to prove your formula.

Diving Deeper

- Explore $B_k B_{k+1}$. Describe your discovery in words and a formula. Prove your formula.

Solutions for #4

The next two pictures in the pattern



A formula relating oblong numbers to square numbers

A formula suggested by the pictures:

$$4B(k) + 1 = P_4(2k + 1)$$

Thinking Processes

- 4 copies of the rectangle representing the 1st oblong number combined with 1 unit square join to create a square representing the 3rd square number.
$$4 \cdot (1 \cdot 2) + 1 = 3^2$$
$$4 \cdot 2 + 1 = 9$$
- 4 copies of the rectangle representing the 2nd oblong number combined with 1 unit square join to create a square representing the 5th square number.
$$4 \cdot (2 \cdot 3) + 1 = 5^2$$
$$4 \cdot 6 + 1 = 25$$
- 4 copies of the rectangle representing the 3rd oblong number combined with 1 unit square join to create a square representing the 7th square number.
$$4 \cdot (3 \cdot 4) + 1 = 7^2$$
$$4 \cdot 12 + 1 = 49$$

In general, 4 times the k^{th} oblong number plus 1 equals the $(2k + 1)^{\text{th}}$ square number.

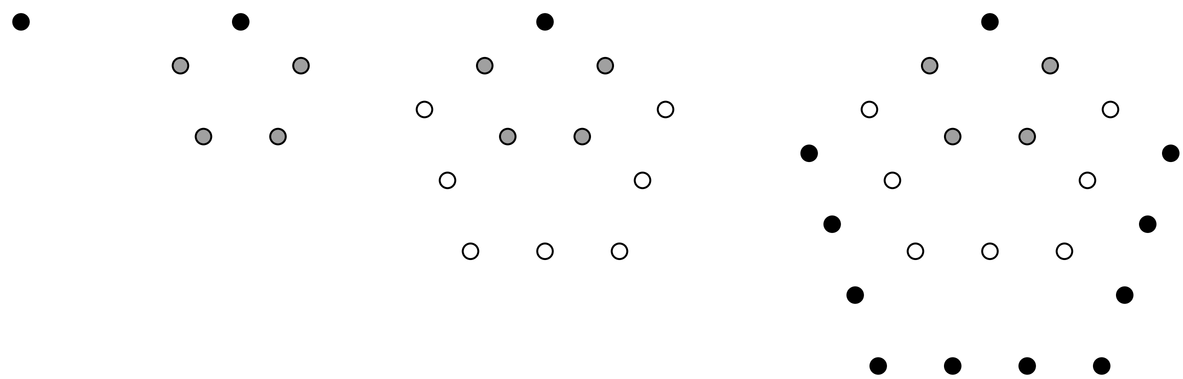
$$4B(k) + 1 = P_4(2k + 1)$$
$$4 \cdot k(k + 1) + 1 = (2k + 1)^2$$

Proof

$4 \cdot k(k + 1) + 1 =$	$(2k + 1)^2 =$
$4 \cdot (k^2 + k) + 1 =$	$(2k)^2 + 2(2k \cdot 1) + 1^2 =$
$4k^2 + 4k + 1$	$4k^2 + 4k + 1$

Since both expressions are equivalent to the same expression, they are equivalent to each other. (Alternatively, students could continue the process on the left by factoring the expression $4k^2 + 4k + 1$.)

Problem #5



Pentagonal Numbers: $1, 5, 12, 22, \cdots$

Directions

- Draw the next pentagon diagram, and find the next *pentagonal number*.
- Continue the pattern of sums. Describe the pattern, and explain what causes it.

$$1 \qquad 1+4 \qquad 1+4+7$$

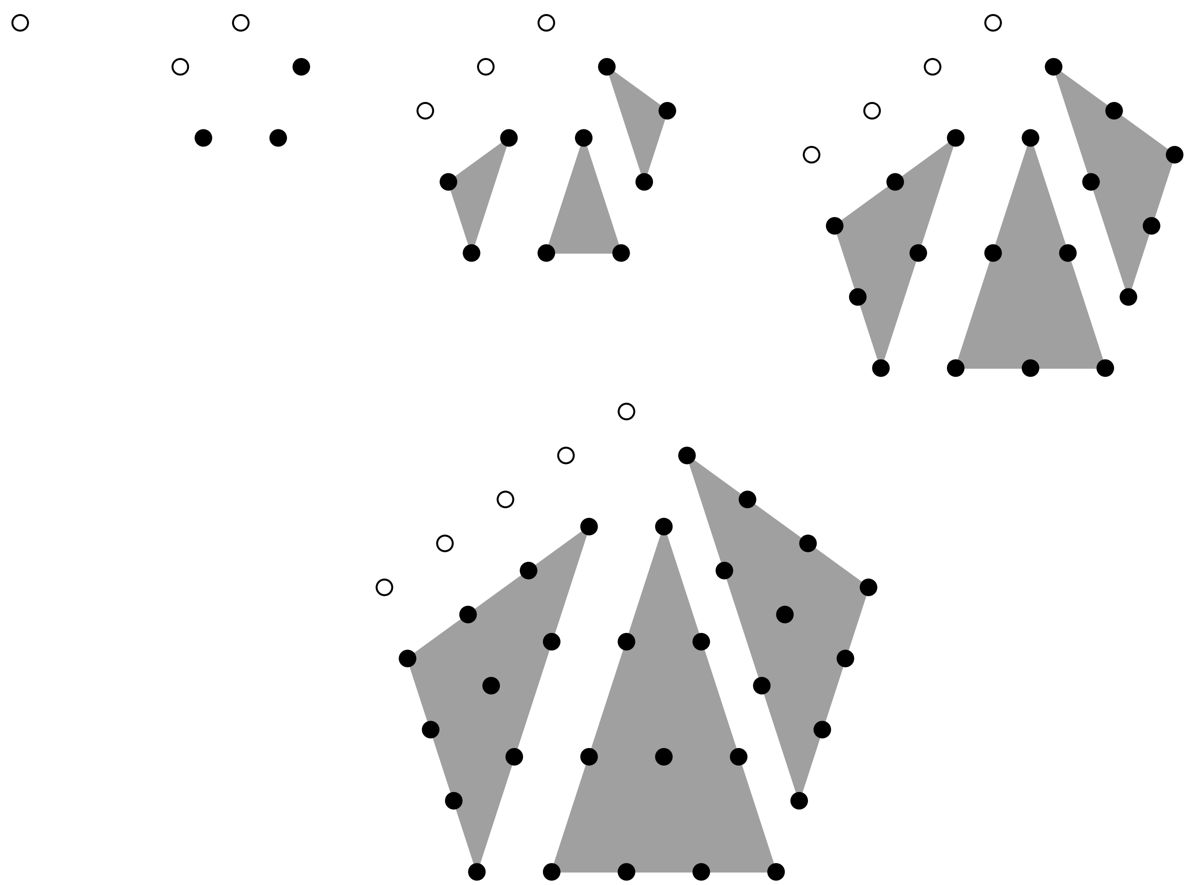
- Use the pattern to predict more numbers in the sequence.
- Find a formula for the k^{th} pentagonal number, $P_5(k)$. Explain your thinking.

Hints for Problem #5
A Formula for Pentagonal Numbers

Hint 1

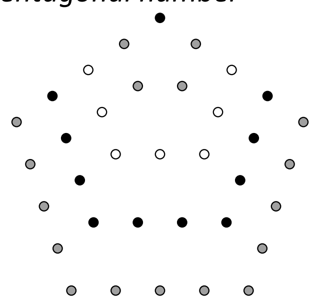
$1 + 4 + 7 + 10$ $\qquad$ $1 + 4 + 7 + 10 + 13$

Hint 2



Solutions for #5

The next pentagon diagram and pentagonal number



The next pentagonal number is 35 (the number of dots in the pentagon).

The pattern of sums

$1 \qquad 1+4 \qquad 1+4+7 \qquad 1+4+7+10 \qquad 1+4+7+10+13$ etc.

Each addend is three greater than the preceding one. The pattern creates the sequence of pentagonal numbers

$$1, 5, 12, 22, 35, \ldots$$

Because of the pattern in the sums, the differences between neighboring pairs of terms increase by 3 as you move to the right: 4, 7, 10, 13, ….

Why the differences always increase by 3

The numbers (1, 4, 7, 10, 13, …) in the sums describe the number of dots you join in order to create the next pentagon in the sequence. Students may think of many ways to explain why this number always increases by 3. For example:

- You create each new diagram by joining three new sides toward the bottom of the previous diagram. Each side has 1 more dot than the corresponding side in the previous picture, resulting in three additional dots. (It does not matter that the process counts the corner dots twice, because it does so for both diagrams.)

- The k^{th} pentagon has $3k-2$ dots on the bottom three sides, because each of the three sides contains k dots, but the two dots on the vertices are counted twice. For the same reason, the bottom three sides of the $(k+1)^{\text{th}}$ pentagon contain $3(k+1)-2$ dots. The difference between these expressions represents the amount by which the number of dots added increases each time.

$$(3(k+1)-2)-(3k-2)=$$
$$3k+3-2-3k+2=$$
$$3$$

Predicting more numbers in the sequence

$$1, 5, 12, 22, 35, 51, 70, 92, 117, \ldots$$

$1 + 4 = 5$	$5 + 7 = 12$	$12 + 10 = 22$	$22 + 13 = 35$
$35 + 16 = 51$	$51 + 19 = 70$	$70 + 22 = 92$	$92 + 25 = 117$

A formula for the k^{th} pentagonal number, $P_5(k)$

Students may write the formula in many forms. Possibilities include:

$$P_5(k) = \frac{k}{2}(3k - 1) \qquad P_5(k) = \frac{3}{2}k^2 - \frac{1}{2}k \qquad P_5(k) = \frac{1}{2}(3k^2 - k)$$

Sample strategy 1 (from Hint 1 on the *Hints for Problem #5* handout)

Use the sum $1 + 4 + 7 + \cdots + (3k - 2)$

You may determine the expression, $3k - 2$, for the final term using the geometric reasoning on the previous page or by recognizing that the term comes from starting at 1 and adding $k - 1$ groups of 3:

$$1 + 3(k - 1) = 1 + 3k - 3 = 3k - 2$$

Each pair of numbers in the Hint 1 has the same sum, because as you move toward the center, you subtract 3 from the term on the right and add 3 to the term on the left. The sum of each pair is equal to the sum of the first and last term:

$$1 + (3k - 2) =$$
$$3k - 1$$

When k is even, there are $\frac{k}{2}$ groups of $3k - 1$, resulting in a combined sum of

$$\frac{k}{2}(3k - 1).$$

When k is odd, the middle term is always half of the sum of each pair. Thus, there remain $\frac{k-1}{2}$ groups of $3k - 1$ plus the middle term, $\frac{3k-1}{2}$. The entire sum is

$$\frac{k - 1}{2}(3k - 1) + \frac{3k - 1}{2} =$$
$$\frac{3k - 1}{2}((k - 1) + 1) =$$
$$\frac{3k - 1}{2}(k)$$

which is equivalent to the expression for an even number of terms.

Sample strategy 2 (from Hint 2 on the *Hints for Problem #5* handout)

Hint 2 suggests that the dots in the k^{th} pentagon decompose into one side of the pentagon (containing k dots) plus three triangles (each containing the $(k-1)^{\text{th}}$ triangular number of dots). Thus, the total number of dots in the k^{th} pentagon is

$$P_5(k) = k + 3\frac{(k-1)k}{2} =$$

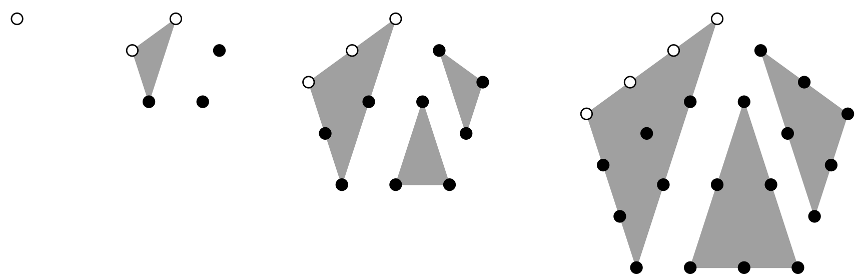
$$\frac{2k}{2} + \frac{3(k-1)k}{2} =$$

$$\frac{k}{2}(2 + 3(k-1)) =$$

$$\frac{k}{2}(2 + 3k - 3) =$$

$$\frac{k}{2}(3k - 1)$$

Note: If you join the white dots with the triangle on the left, you have the k^{th} triangular number plus *two* copies of the $(k-1)^{\text{th}}$ triangular number—yet another method for developing the formula.



Sample strategy 3 (from the standard formula for summing an arithmetic sequence)

Students who know and remember a formula for the sum, S_k, of the arithmetic sequence $a_1, a_2, a_3, \cdots, a_k$ may apply it directly.

$$S_k = \frac{k(a_1 + a_k)}{2} =$$

$$S_k = \frac{k(1 + (3k-2))}{2} =$$

$$\frac{k}{2}(3k - 1)$$

The idea behind this formula mirrors the approach in Strategy 1.

Students who have learned a summation formula but do not notice the connection to this problem may benefit from applying their own strategies first and later reviewing the summation formula in order to connect their strategies to it.

Problem #6

Hexagonal numbers	$P_6(k)$	$1, 6, 15, 28, \cdots$
Heptagonal numbers	$P_7(k)$	$1, 7, 18, 34, \cdots$

Directions

- Carry out a full investigation for each sequence as in problem #5.
- Find patterns *between* your formulas. Extend the patterns backwards to triangular and square numbers. Compare your results to Problems #1 and #3.
- Use the patterns to write a general formula for the k^{th} n-agonal number, $P_n(k)$.

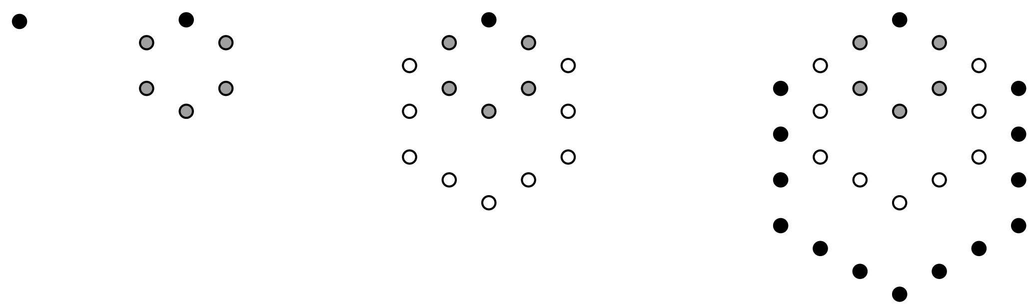
Diving Deeper

- Compare the hexagonal numbers to the triangular numbers. What do you notice? Why does this happen?

Solutions for #6

In the Solutions for Problem #6, I provide the almost the same level of detail for hexagonal numbers as I did for pentagonal numbers in order to highlight patterns in the processes. For heptagons, I give just a summary of the results, because the patterns for pentagons and hexagons make it clear to how to proceed.

Hexagonal number pictures



The pattern of sums

$1 \qquad 1+5 \qquad 1+5+9 \qquad 1+5+9+13 \qquad 1+5+9+13+17$ etc.

Each addend is four greater than the preceding one. The pattern creates the sequence of hexagonal numbers

$$1, 6, 15, 28, 45, \ldots$$

The first difference is 5, and each difference increases by 4.

Why the differences always increase by 4

- You create each new diagram by joining four new sides toward the bottom of the previous diagram. Each side has 1 more dot than the corresponding side in the previous picture, resulting in four additional dots.
- The k^{th} hexagon has $4k-3$ dots on the bottom four sides. The bottom four sides of the $(k+1)^{\text{th}}$ hexagon contain $4(k+1)-3$ dots. The difference represents the amount by which the number of dots added increases each time.

$$(4(k+1)-3)-(4k-3) =$$
$$4k+4-3-4k+3 =$$
$$4$$

Predicting more numbers in the sequence

$$1, 6, 15, 28, 45, 66, 91, 120, 153, \ldots$$

$1+5=6$	$6+9=15$	$15+13=28$	$28+17=45$
$45+21=66$	$66+25=91$	$91+29=120$	$120+33=153$

A formula for the k^{th} hexagonal number, $P_6(k)$

Students may write the formula in many forms. Possibilities include:

$$P_6(k) = \frac{k}{2}(4k-2) \qquad P_6(k) = k(2k-1) \qquad P_6(k) = 2k^2 - k$$

Sample strategy 1

Use the sum $1 + 5 + 9 + \cdots + (4k-3)$

Determine the expression, $4k-3$, for the final term by using the geometric reasoning on the previous page or by recognizing that the term comes from starting at 1 and adding $k-1$ groups of 4:

$$1 + 4(k-1) = 1 + 4k - 4 = 4k - 3$$

Again, the numbers form pairs having the same sum. The sum of each pair is equal to the sum of the first and last term:

$$1 + (4k-3) = 4k - 2$$

When k is even, there are $\frac{k}{2}$ groups of $4k-2$, resulting in a combined sum of

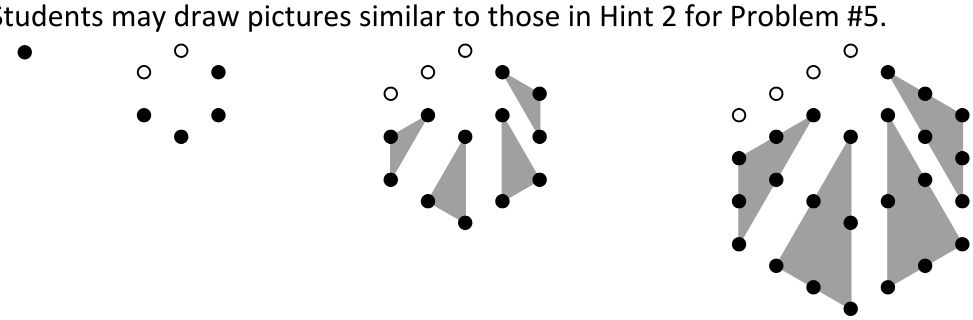
$$\frac{k}{2}(4k-2) = k(2k-1).$$

When k is odd, the middle term is again half of the sum of each pair. There remain $\frac{k-1}{2}$ groups of $4k-2$ plus the middle term, $\frac{4k-2}{2} = 2k-1$. The total is

$$\frac{k-1}{2}(4k-2) + (2k-1) =$$
$$(k-1)\frac{1}{2}(4k-2) + (2k-1) =$$
$$(k-1)(2k-1) + (2k-1) =$$
$$(2k-1)((k-1)+1) =$$
$$(2k-1)k.$$

Sample strategy 2

Students may draw pictures similar to those in Hint 2 for Problem #5.



The dots in the k^{th} hexagon decompose into one side of the hexagon (containing k dots) plus four triangles (each containing the $(k-1)^{\text{th}}$ triangular number of dots). Thus, the total number of dots in the k^{th} pentagon is

$$P_6(k) = k + 4\frac{(k-1)k}{2} =$$

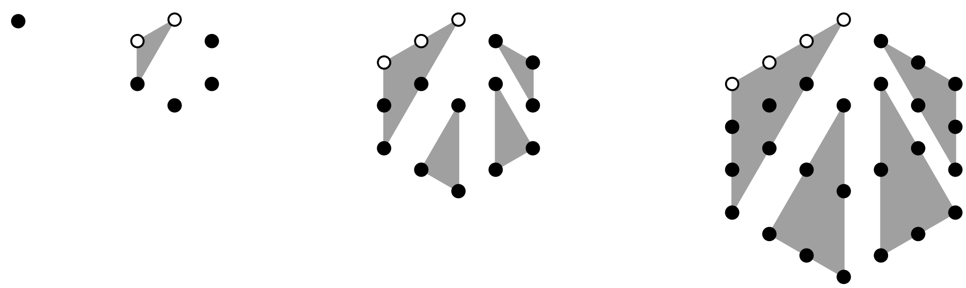
$$\frac{2k}{2} + \frac{4(k-1)k}{2} =$$

$$\frac{k}{2}(2 + 4(k-1)) =$$

$$\frac{k}{2}(2 + 4k - 4) =$$

$$\frac{k}{2}(4k-2) = k(2k-1)$$

Note: Again, if you join the white dots with the triangle on the left, you have the k^{th} triangular number plus *three* copies of the $(k-1)^{\text{th}}$ triangular number—suggesting a slightly different approach to finding the formula.



Sample strategy 3 (from the standard formula for summing an arithmetic sequence)

For the sum, S_k, of the arithmetic sequence $a_1, a_2, a_3, \cdots, a_k$:

$$S_k = \frac{k(a_1 + a_k)}{2} =$$

$$S_k = \frac{k(1 + (4k-3))}{2} =$$

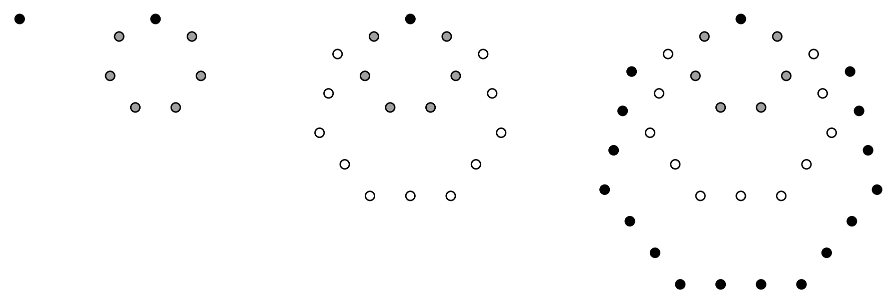
$$\frac{k}{2}(4k-2) = k(2k-1)$$

As before, the summation formula mirrors the approach in Strategy 1.

A summary of the results for heptagonal numbers

Pictures

I am including pictures for heptagons in case they are helpful, but drawing them accurately is challenging. Students may choose not to create them unless they need them in order to make further progress.



Sums and a list of heptagonal numbers

$1 \qquad 1+6 \qquad 1+6+11 \qquad 1+6+11+16 \qquad 1+6+11+16+21$ etc.

$$1, 7, 18, 34, 55, \ldots$$

The first difference is 6, and each successive difference increases by 5.

Sample formulas for the k^{th} heptagonal number, $P_7(k)$

$$P_7(k) = \frac{k}{2}(5k-3) \qquad P_7(k) = \frac{1}{2}(5k^2-3k) \qquad P_7(k) = \frac{5}{2}k^2 - \frac{3}{2}k$$

Examples of patterns between the formulas

Pentagonal numbers	$P_5(k) = \frac{k}{2}(3k-1)$	$P_5(k) = \frac{3}{2}k^2 - \frac{1}{2}k$
Hexagonal numbers	$P_6(k) = \frac{k}{2}(4k-2)$	$P_6(k) = \frac{4}{2}k^2 - \frac{2}{2}k$
Heptagonal numbers	$P_7(k) = \frac{k}{2}(5k-3)$	$P_7(k) = \frac{5}{2}k^2 - \frac{3}{2}k$

The expressions for hexagonal numbers are left unsimplified in order to make it easier to see the patterns.

Left column: $\frac{k}{2}$ is factored out in every case. The expression in parentheses is of the form $ak - b$, where a is 2 less than n, and b is 4 less than n.

Right column: Every expression appears in standard quadratic form. The leading coefficient is half of the number that is 2 less than n. The linear coefficient is 1 less than the leading coefficient, and the linear term is subtracted from the quadratic term.

Extending the pattern backwards (left-hand column):

Square numbers $P_4(k) = \frac{k}{2}(2k - 0) = \frac{k}{2}(2k) = k^2$

Triangular numbers $P_3(k) = \frac{k}{2}(1k - (-1)) = \frac{k}{2}(k + 1) = \frac{k(k+1)}{2}$

Problem #1 was about triangular numbers (even though we had not named them yet), and Problem #3 was about square numbers. The formulas above are the same as those discovered in Problems #1 and #3!

Note: Extending the patterns in the right-hand column leads to equivalent results.

G*eneral formulas for the* k^{th} n*-agonal number,* $P_n(k)$

Recall the pattern for the left column:

$\frac{k}{2}$ is factored out in every case. The expression in parentheses is of the form $ak - b$, where a is 2 less than n, and b is 4 less than n.

Writing these observations in the language of algebra leads directly to a formula.

$$P_n(k) = \frac{k}{2}((n-2)k - (n-4))$$

Writing out the patterns in the right-hand column leads to the equivalent formula

$$P_n(k) = \frac{n-2}{2}k^2 - \frac{n-4}{2}k.$$

Problem #7

Polygonal Numbers $P_n(k)$

		k						
		1	**2**	**3**	**4**	**5**	**6**	**7**
	3	1	3	6	10	15	21	28
	4	1	4	9	16	25	36	49
n	**5**	1	5	12	22	35	51	70
	6	1	6	15	28	45	66	91
	7	1	7	18	34	55	81	112
	8	1	8	21	40	65	96	133

Directions

- Find and describe patterns in the table.
- Create and prove a formula that calculates any square number, $P_4(k)$, from two consecutive triangle numbers.
- Create a formula that calculates *any* polygonal number, $P_n(k)$, from two consecutive triangle numbers. Explain your thinking.
- Prove that this formula is equivalent to your general formula from Problem #6.

Solutions for #7

Patterns in the table

Students may recognize the patterns in the rows from their earlier work in the project. Each row is a list of n-agonal numbers. The first difference is always one less than n, and each subsequent difference increases by two less than n.

The columns have patterns that students are unlikely to have noticed as yet. Each column forms an arithmetic sequence whose first number is the k^{th} triangular number and whose common difference is the $(k-1)^{\text{th}}$ triangular number. *Note:* This means that you can calculate *any* polygonal number from triangular numbers alone!

A formula relating square numbers and triangular numbers

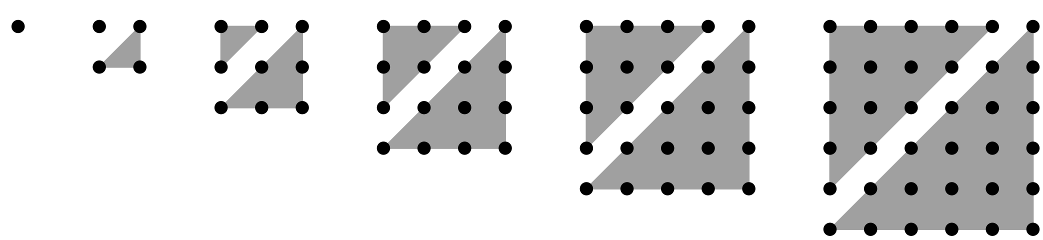
$$P_4(k) = P_3(k) + P_3(k-1)$$

In words: The k^{th} square number is the sum of the k^{th} triangular number and the $(k-1)^{\text{th}}$ triangular number. You can easily see this pattern in the table! Whenever you add two consecutive numbers in the top row, you get a number in the second row (directly below the larger addend).

Algebraic proof

$$P_3(k) + P_3(k-1) =$$
$$\frac{k(k+1)}{2} + \frac{(k-1)k}{2} =$$
$$\frac{k^2 + k + k^2 - k}{2} = \frac{2k^2}{2} = k^2 =$$
$$P_4(k)$$

Note: You can also *see* the formula with pictures!



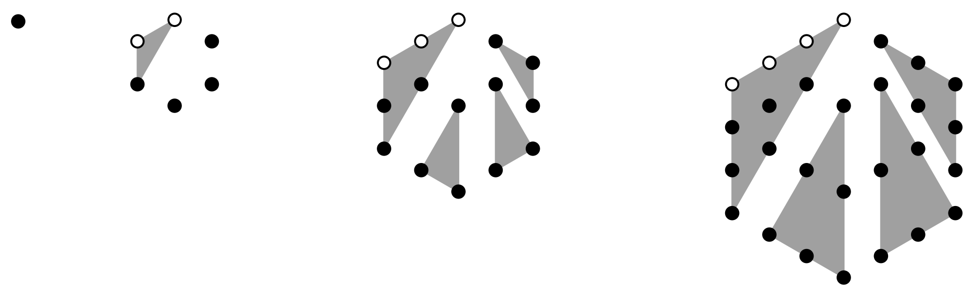
A formula that relates any *polygonal number to a pair of consecutive triangular numbers*

$$P_n(k) = P_3(k) + (n-3)P_3(k-1)$$

In words: The k^{th} n-agonal number is the sum of the k^{th} triangular number and $n-3$ groups of the $(k-1)^{\text{th}}$ triangular number. You can see this pattern in the table as well. Any number in the table is equal to the sum of the triangular number in the column above it and $n-3$ times the triangular number in the column to its left.

The formula is a direct consequence of the fact that the columns are arithmetic sequences. It also describes pictures from Problems # 5 and #6!

For example, when $n = 6$:



Algebraic proof

$$P_3(k) + (n-3)P_3(k-1) =$$

$$\frac{k(k+1)}{2} + (n-3)\frac{(k-1)k}{2} =$$

$$\frac{k}{2}\big((k+1) + (n-3)(k-1)\big) =$$

$$\frac{k}{2}\big((k+1) + (nk - n - 3k + 3)\big) =$$

$$\frac{k}{2}\big((1+n-3)k + (1-n+3)\big) =$$

$$\frac{k}{2}\big((n-2)k + (4-n)\big) =$$

$$\frac{k}{2}\big((n-2)k - (n-4)\big) =$$

$$P_n(k)$$

Stage 3

In Stage 3, students apply their knowledge of triangular and square numbers to solve challenging problems that lead to third-and fourth-degree polynomials. In Problem #8, they develop a formula for sums of squares,

$$1^2 + 2^2 + \cdots + k^2,$$

by analyzing a complex diagram. In problem #9, build on what they learned in Problem #8 in order to count rectangles on a grid, find patterns, and create formulas for grids of varying sizes.

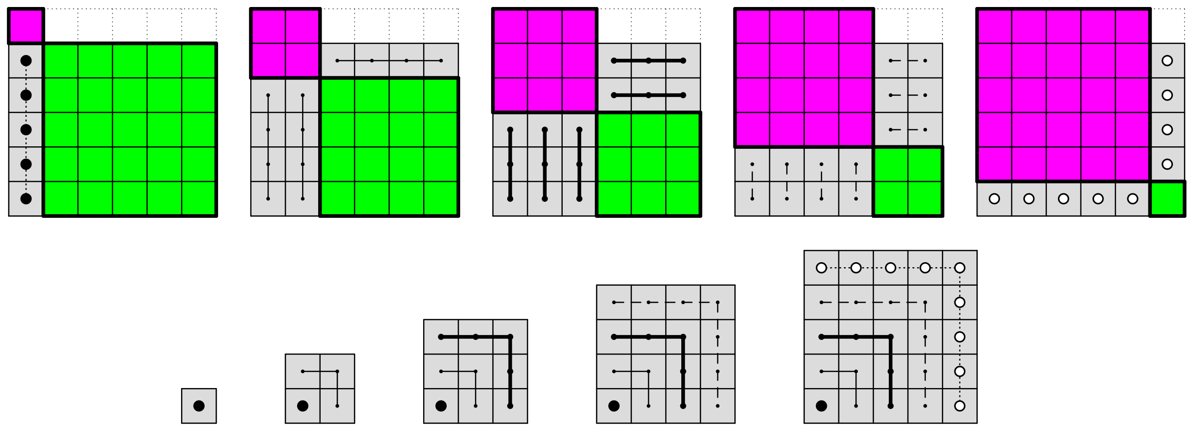
What students should know

- Understand concepts from Stages 1 and 2.

What students will learn

- Become more fluent adding, subtracting, multiplying, and simplifying higher-degree polynomials.
- Recognize common factors (including multiple-term factors) in polynomial expressions, and factor them out.
- Prove the equivalence of higher-degree polynomial expressions.
- Develop and use a formula for sums of squares.
- Discover a formula for sums of cubes.
- Make connections between numeric patterns, geometric diagrams, and algebraic formulas.

Problem #8



Exploring $1^2 + 2^2 + \cdots + k^2$ when $k = 5$

Directions

- Observe and describe patterns in the diagram.
- Draw a similar diagram for $k = 4$ and/or $k = 6$.
- Use the diagrams to find a formula for $1^2 + 2^2 + \cdots + k^2$. Explain your thinking.

Diving Deeper

- Explore the pattern of equations below. How does it relate to this problem? What causes the pattern? Will it continue forever? How can you be sure?

$$1 + 2 = 3$$
$$4 + 5 + 6 = 7 + 8$$
$$9 + 10 + 11 + 12 = 13 + 14 + 15$$
$$16 + 17 + 18 + 19 + 20 = 21 + 22 + 23 + 24$$

Solutions for #8

Patterns in the diagram

General observations

- The diagram shows magenta and green squares of different sizes arranged corner-to-corner in the opposite order within a set of five 6-by-6 squares.
- The magenta squares show one copy of the sum $1^2 + 2^2 + 3^2 + 4^2 + 5^2$.
- The green squares show a second copy of the sum $1^2 + 2^2 + 3^2 + 4^2 + 5^2$.
- The grey squares show a third copy of $1^2 + 2^2 + 3^2 + 4^2 + 5^2$ decomposed and rearranged to fill "gaps" in the 6-by-6 squares. (Some parts are left unfilled.)
- The five grey squares on the bottom show how to decompose the squares.
- After gaps are filled in by all of the grey unit squares, the 6-by-6 squares are still missing 5, 4, 3, 2, and 1 unit squares respectively as you look from left to right.
- The diagram shows that $3(1^2 + 2^2 + 3^2 + 4^2 + 5^2) = 31 + 32 + 33 + 34 + 35$.
- The first number, 31, is 5 less than $6 \cdot 6$. The last number, 35, is 1 less than $6 \cdot 6$.

The details of decomposing and rearranging

- Each grey square in the bottom row decomposes into odd numbers according to the pattern in Problem #3:

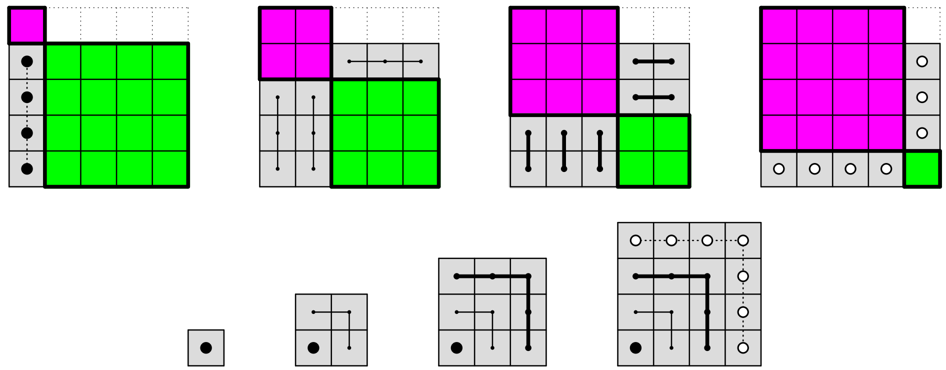
$$1 \qquad 1+3 \qquad 1+3+5 \qquad 1+3+5+7 \qquad 1+3+5+7+9$$

- The grey unit squares fill gaps in the 6-by-6 squares according to the patterns:

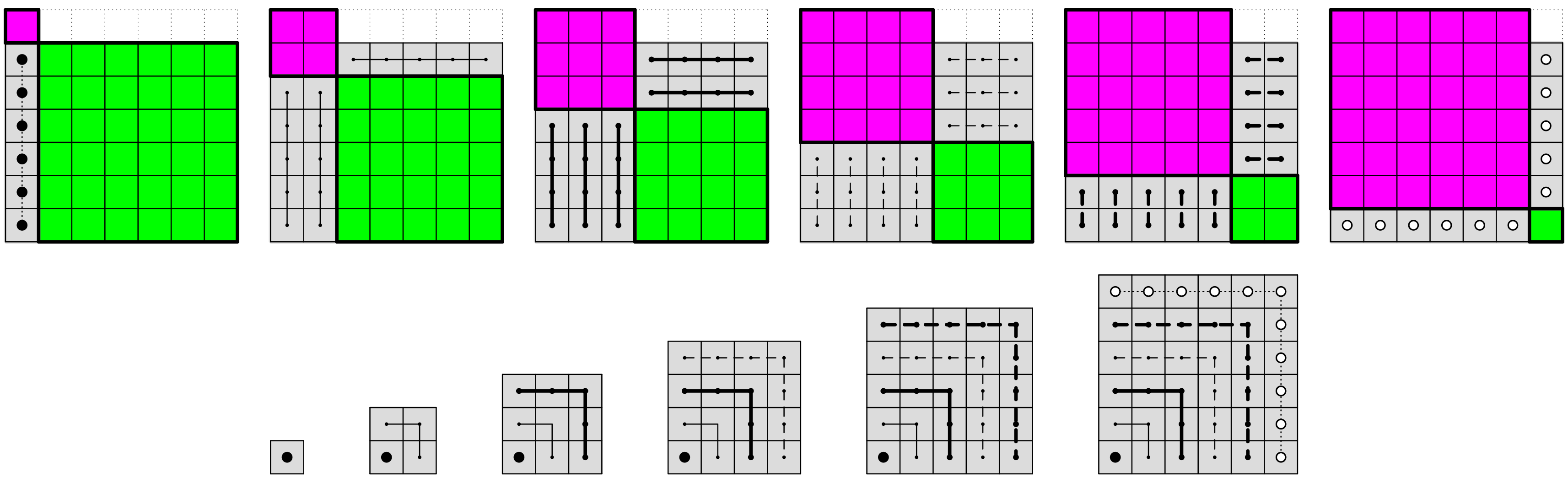
Lower set of grey squares	**Upper set of grey squares**	**Original unfilled area**
5 groups of 1	1 group of 5	2 groups of 5 (square 1)
4 groups of 3	3 groups of 4	4 groups of 4 (square 2)
3 groups of 5	5 groups of 3	6 groups of 3 (square 3)
2 groups of 7	7 groups of 2	8 groups of 2 (square 4)
1 group of 9	9 groups of 1	10 groups of 1 (square 5)

- The number of groups in the *Upper set of grey squares* is always one less than the number of groups in the *Original unfilled area*. The remaining unfilled area accounts for the missing 5, 4, 3, 2, and 1 unit squares described above.
- These observations taken together show a predictable way to extend the pattern in order to investigate other sums of the form $1^2 + 2^2 + \cdots + k^2$.

A diagram for $k = 4$



A diagram for $k = 6$



Sample formulas for $1^2+2^2+\cdots+k^2$

$$1^2+2^2+\cdots+k^2=\frac{k(k+1)(2k+1)}{6}$$

or

$$1^2+2^2+\cdots+k^2=\frac{1}{3}k^3+\frac{1}{2}k^2+\frac{1}{6}k$$

Thinking processes

Joining three copies of $1^2+2^2+\cdots+k^2$ creates k squares having side lengths of $(k+1)$ units^2 with $1+2+\cdots+k$ missing unit squares. Therefore,

$$3(1^2+2^2+\cdots+k^2)=k(k+1)^2-(1+2+\cdots+k).$$

Since $1+2+\cdots+k$ is the k^{th} triangular number, you may rewrite and simplify the formula.

$$3(1^2+2^2+\cdots+k^2)=k(k+1)^2-\frac{k(k+1)}{2}=$$

$$3(1^2+2^2+\cdots+k^2)=k(k+1)\left((k+1)-\frac{1}{2}\right)=$$

$$3(1^2+2^2+\cdots+k^2)=k(k+1)\left(k+\frac{1}{2}\right)=$$

$$1^2+2^2+\cdots+k^2=\frac{k(k+1)\left(k+\frac{1}{2}\right)}{3}=$$

$$1^2+2^2+\cdots+k^2=\frac{k(k+1)(2k+1)}{6}=$$

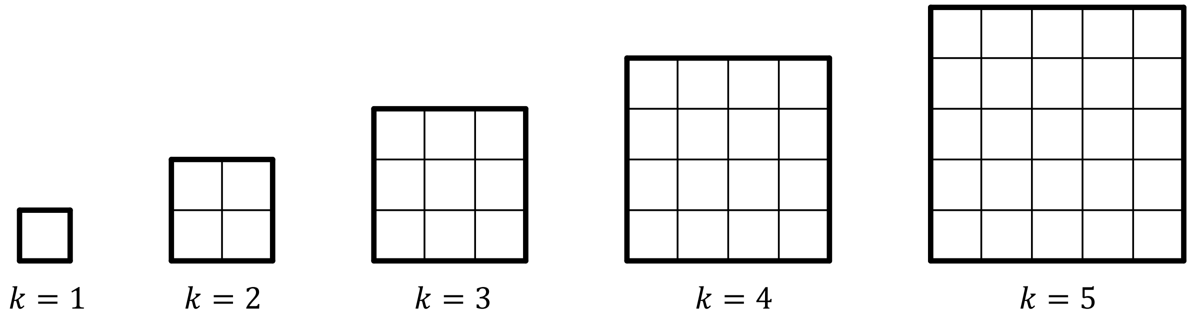
Students who multiply the factors and combine like terms may get the equation

$$1^2+2^2+\cdots+k^2=\frac{1}{3}k^3+\frac{1}{2}k^2+\frac{1}{6}k$$

or

$$1^2+2^2+\cdots+k^2=\frac{1}{6}(2k^3+3k^2+k)$$

Problem #9



Directions

- Count the number of squares* within each figure. Explain your thinking.
- Count the number of rectangles (including squares) within each figure. Explain.
- Predict the numbers of squares and rectangles for $k = 6$ and 7. Explain.
- Create one formula in terms of k for the number of squares and another for the number of rectangles. Explain your thinking.
- Create a formula for the number of *non*-square rectangles. Explain your thinking.

*The sides of the squares and other rectangles must lie along the grid lines.

Solutions for #9

Counting the squares

$k = 1$	1 square	1 (1-by-1)
$k = 2$	5 squares	1 (2-by-2) + 4 (1-by-1)
$k = 3$	14 squares	1 (3-by-3) + 4 (2-by-2) + 9 (1-by-1)
$k = 4$	30 squares	1 (4-by-4) + 4 (3-by-3) + 9 (2-by-2) + 16 (1-by-1)
$k = 5$	55 squares	1 (5-by-5) + 4 (4-by-4) + 9 (3-by-3) + 16 (2-by-2) + 25 (1-by-1)

Counting the rectangles

The square rectangles are already counted. It remains to find the non-square ones.

Rectangles whose horizontal side is longer:

$k = 1$	0				
$k = 2$	2	2 (1-by-2)			
$k = 3$	11	2 (2-by-3)	+ 3 (1-by-3)		
			+ 6 (1-by-2)		
$k = 4$	35	2 (3-by-4)	+ 3 (2-by-4)	+ 4 (1-by-4)	
			+ 6 (2-by-3)	+ 8 (1-by-3)	
				+ 12 (1-by-2)	
$k = 5$	85	2 (4-by-5)	+ 3 (3-by-5)	+ 4 (2-by-5)	+ 5 (1-by-5)
			+ 6 (3-by-4)	+ 8 (2-by-4)	+ 10 (1-by-4)
				+ 12 (2-by-3)	+ 15 (1-by-3)
					+ 20 (1-by-2)

For each n-by-n square, every "horizontal" rectangle has a corresponding vertical rectangle. Therefore, the total number of non-square rectangles within each square is:

$k = 1$	$0 \cdot 2 = 0$
$k = 2$	$2 \cdot 2 = 4$
$k = 3$	$11 \cdot 2 = 22$
$k = 4$	$35 \cdot 2 = 70$
$k = 5$	$85 \cdot 2 = 170$

The total number of rectangles

To find the total number of rectangles, add the number of squares to the number of non-square rectangles.

	Squares	Non-square rectangles	Total
$k = 1$	1	0	1
$k = 2$	5	4	9
$k = 3$	14	22	36
$k = 4$	30	70	100
$k = 5$	55	170	225

Predicting the counts of squares and rectangles for $n = 6$ and $n = 7$

Predicting the number of squares

The number of squares follows the "sum-of-squares" pattern from Problem #8.

$1^2 = 1$
$1^2 + 2^2 = 5$
$1^2 + 2^2 + 3^2 = 14$
$1^2 + 2^2 + 3^2 + 4^2 = 30$
$1^2 + 2^2 + 3^2 + 4^2 + 5^2 = 55$

Extending the pattern gives:

$1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2 = 91$
$1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2 + 7^2 = 140$

Predicting the number of rectangles

The total number of rectangles always appears to be a square number. In fact, the results so far are equal to 1^2, 3^2, 6^2, 10^2, and 15^2: the squares of the triangular numbers! Based on this observation, the results for $n = 6$ *and* $n = 7$ should be:

$k = 6 \qquad (1 + 2 + 3 + 4 + 5 + 6)^2 = 21^2 = 441$
$k = 7 \qquad (1 + 2 + 3 + 4 + 5 + 6 + 7)^2 = 28^2 = 784$

Some students may also recognize that the numbers are sums of cubes!

$1^3 = 1 \qquad 1^3 + 2^3 = 9 \qquad 1^3 + 2^3 + 3^3 = 36 \qquad 1^3 + 2^3 + 3^3 + 4^3 = 100$

A formula for the number of squares

The formula for the number of squares is simply the "sum-of-squares" formula from Problem #8!

$$1^2 + 2^2 + \cdots + k^2 = \frac{k(k+1)(2k+1)}{6}$$

or

$$1^2 + 2^2 + \cdots + k^2 = \frac{1}{3}k^3 + \frac{1}{2}k^2 + \frac{1}{6}k.$$

A formula for the number of rectangles

The pattern appears to be that the number of rectangles (let's call it $R(k)$) is equal to the square of the k^{th} triangular number. Therefore, the formula should be:

$$R(k) =$$
$$\left(P_3(k)\right)^2 =$$
$$\left(\frac{k(k+1)}{2}\right)^2$$

Students may want to verify that this formula works with each of the values discovered so far. By multiplying everything out, some students may obtain the equivalent formula

$$R(k) = \frac{1}{4}k^4 + \frac{1}{2}k^3 + \frac{1}{4}k^2$$

A formula for the number of non-square rectangles

Some students may try to discover the formula by analyzing patterns in the sequence

0, 4, 22, 70, 170, …

for the number of non-square rectangles. However, once you know formulas the total numbers of rectangles and number of squares, you may simply subtract them!

Using the factored forms:

Number of Squares – Number of Rectangles =

$$\left(\frac{k(k+1)}{2}\right)^2 - \frac{k(k+1)(2k+1)}{6}$$

The expression is correct as it is, but you may write it in a simpler form.

$$\left(\frac{k(k+1)}{2}\right)^2 - \frac{k(k+1)(2k+1)}{6} =$$
$$\frac{k(k+1)}{2}\left(\frac{k(k+1)}{2} - \frac{2k+1}{3}\right) =$$
$$\frac{k(k+1)}{2}\left(\frac{3k(k+1)}{6} - \frac{2(2k+1)}{6}\right) =$$
$$\frac{k(k+1)}{2}\left(\frac{3k^2+3k-4k-2}{6}\right) =$$
$$\frac{k(k+1)}{2}\left(\frac{3k^2-k-2}{6}\right) =$$
$$\frac{k(k+1)}{2}\left(\frac{(3k+2)(k-1)}{6}\right) =$$
$$\frac{(k-1)k(k+1)(3k+2)}{12}$$

This process illustrates the usefulness of simplification. While it takes quite a bit of effort to rewrite the original expression, the resulting expression is easier to use! In fact, students may easily verify that it correctly produces all of the values in the sequence

0, 4, 22, 70, 170,

and beyond.

Note: If students have not yet learned how to factor the expression $3k^2 - k - 2$, they may leave it as it is, or you may share the factorization with them and let them check that it is valid.

Using the standard forms:

The subtraction process is probably easier in this case, though the resulting expression is not quite as easy to use as the one above.

Number of Squares – Number of Rectangles =

$$\left(\frac{1}{4}k^4 + \frac{1}{2}k^3 + \frac{1}{4}k^2\right) - \left(\frac{1}{3}k^3 + \frac{1}{2}k^2 + \frac{1}{6}k\right) =$$

$$\frac{1}{4}k^4 + \left(\frac{1}{2}k^3 - \frac{1}{3}k^3\right) + \left(\frac{1}{4}k^2 - \frac{1}{2}k^2\right) - \frac{1}{6}k =$$

$$\frac{1}{4}k^4 + \frac{1}{6}k^3 - \frac{1}{4}k^2 - \frac{1}{6}k$$

or, if you prefer

$$\frac{1}{12}(3k^4 + 2k^3 - 3k^2 - 2k)$$

Some students may challenge themselves to prove algebraically that the factored expression and the standard-form expression are equivalent.