Interesting Times

Topics

- Geometric sequences
- Writing and evaluating algebraic expressions
- Solving algebraic equations
- Geometric series (the sum of the terms of a geometric sequence)

The *Interesting Times* project briefly introduces students to the adult world as they investigate the ways that loan payments are structured. The main problem situation involves a home loan, something that most students will probably not deal with for some time. However, the same ideas may apply to smaller loans for cars, household items, etc. which may be coming up sooner in students' futures! A home loan was chosen as the main example for this project because certain important mathematical patterns show up more clearly when the numbers are larger.

Begin by gathering knowledge that your students already have on the topic of loans. They should become familiar with a few key words and phrases before they begin.

- *Down payment*: the amount that you pay up front
- *Principal*: the amount that you borrow/owe
- *Interest*: money that you pay the lender for the privilege of borrowing
- *Annual interest rate*: The percentage of the principal that determines the amount of interest you pay each year
- *Term* of a loan: the amount of time over which you repay the loan

Stage 1

Stage 1 contains two problems in which students begin exploring principal and interest payments for long term loans. Do not hand out Problem #2 until students have completed Problem #1. The second problem gives some things away.

To introduce the ideas, ask students how much interest they think they would pay on a loan of $100,000 at 5% annual interest over a 10-year period. They may suggest an answer of $50,000 based on the following calculations:

$\$100{,}000 \cdot 0.05$ per year = $5000 per year

$5000 per year · 10 years = $50,000

In reality, the interest would normally be much less than this, because large loans are typically paid off in monthly payments (called *installments*), and each payment consists of a combination of principal and interest. Ask students to think about why it might be useful to keep track of principal and interest separately. Reasons may include:

- Each principal payment reduces the amount you owe on the loan.
- Each principal payment increases your "ownership" (technically, your *equity*) of the thing you are purchasing.
- You may be allowed to deduct interest payments when calculating your taxes.

It is important to understand that only the *principal* portion of each payment reduces the amount that you owe. The interest portion is an extra expense that comes with the process of borrowing. Students may begin to wonder about the fairness of having to pay interest on money that they have already paid back. It may seem that the interest payments should decrease as you pay off more of the original principal. In fact, this is just what happens. Lenders use a carefully designed system that spreads the principal payments over the entire length (*term*) of the loan. Problem #1 shows an example of this. Students' task is to analyze the patterns and figure out as much as they can about how the process works.

Problem #2 may be more challenging than a typical Stage 1 problem. If it seems too difficult or distracting for the moment, you may delay it, discuss it together as a class, or even skip it. It is not needed in order to understand the later problems. Its purpose is to help students gain a deeper understanding of the connections between methods for calculating interest payments and the goal of keeping the total monthly payments constant. The notes in Stage 3 will remind you of the opportunity to have students return to the problem if they have not dealt with it yet.

What students should know:

- Be familiar with the vocabulary: *down payment, principal*, *interest*, *interest rate, down payment*, and *term* of a loan.
- Calculate with percentages.
- Recognize a geometric sequence and identify its common ratio.

What students will learn:

- Understand the relationship between monthly and annual interest rates.
- Understand why loan payments are split between principal and interest.
- Discover that principal payments form geometric sequences.
- Write algebraic expressions for the terms of a geometric sequence.
- Discover two methods for determining the interest portion of a loan payment.
- Explain why or prove that both methods give the same result (Problem #2).

Problem #1

Price of house: $220,000.00 Down payment: 20%

Monthly Payments (dollars) on a home loan

Month (k)	Principal (P_k)	Interest (I_k)
1	244.67	616.00
2	245.53	615.14
3	246.39	614.28
4	247.25	613.42
5	248.11	612.56
6	248.98	611.69
7	249.85	610.82
8	250.73	609.94
9	251.61	609.06
10	252.49	608.18
11	253.37	607.30
12	254.26	606.41
24	265.14	595.53
36	276.50	584.17
48	288.34	572.33
60	300.68	559.99

Directions

- Determine the amount that was borrowed.
- Make observations and describe as many patterns as possible in the table.
- Invent and apply a method to calculate the annual interest rate for the loan.
- If you haven't already, create a formula for the k^{th} principal payment (P_k) using the initial payment of $244.67 as a starting point. Explain your thinking.
- If you haven't already, explain how to calculate the interest payment for each month.

Conversation Starters for Problem #1

What do you notice? What do you wonder?

I notice that the principal and interest payments both change very slowly.

I notice that most of the payment each month goes toward interest!

I wonder if the principal payment will eventually become greater than the interest payment.

I notice that I can figure out the amount borrowed without using the table.

I notice that at the bottom of the table, the time increases by years instead of months.

I notice that when a principal payment increases by a certain amount, the interest payment decreases by the same amount.

I notice that the principal payments appear to show a linear pattern at first (an increase of 60¢ per month), but this changes over the long term.

I wonder if I could use multiplication instead of division to describe the change in principal each month.

I notice that the monthly interest rate has to be much less than the annual rate.

I wonder why the interest payment changes each month even though the overall interest rate does not change?

I notice that as the remaining principal to be paid goes down, the interest payment also goes down. *I wonder* if there is a pattern to this.

I wonder how the interest *rate* is connected to the interest *paid* each month.

I notice two ways to calculate the interest each month.

I wonder how long it will take to pay the entire loan off?

Solutions for #1

The amount borrowed

The amount borrowed was $176,000 (20% less than [or 80% of] $220,000).

Sample observations and patterns

- The interest payments start out much higher than the principal payments, but then the principal payments go up while the interest payments go down.
- When a principal payment increases by a certain amount, the interest payment for that month decreases by the same amount.
- The principal payments look linear at first, but over time they (very) gradually increase more quickly.
- The principal payments make a geometric sequence starting at $244.67 with a constant ratio of 1.0035.
- Each total monthly payment (principal plus interest) is the same: $860.67.
- Each interest payment is 0.0035 times the *remaining* principal to be paid. (This may be difficult to notice. Students may be more likely to discover it if you have discussed the idea mentioned in the introduction to Stage 1—that it does not seem fair to continue paying interest on principal that you have already paid back.)

The annual interest rate

The annual interest rate is 4.2%. (Answers may vary, but should be close to this.)

Students may have different ideas about how to calculate this percentage, because the amount on which the interest is paid changes each month! Accept all reasonable strategies and answers that students are able to justify.

Officially, the annual interest rate is determined as:

$$0.35\% \text{ per month} \times 12 \text{ months} = 4.2\% \text{ annual interest.}$$

The 0.35% value is the amount of interest paid each month on the *remaining* principal. For example:

1st month: $176{,}000 \times 0.035 = 616.00$

2nd month: $(176{,}000 - 244.67) \times 0.0035 \approx 615.14$.

3rd month: $(176{,}000 - 244.67 - 245.53) \times 0.0035 \approx 614.28$

etc.

Note: The *total* interest paid over the first year ($7334.80) is a little less than 4.2% (it's about 4.17%) of the original loan amount of $176,000, because the interest is paid on a slightly smaller principal each month.

A formula for the monthly principal payments

The principal payment, P_k, for month k is

$$P_k = 244.67(1.0035)^{k-1}.$$

In words: the first payment is $244.67, and it increases by 0.35% each month.

Some students may be able to write a general formula (assuming that payments are monthly and the interest is expressed as a decimal).

First principal payment:	P_1
Annual interest rate:	i
Formula:	$P_k = P_1(1 + \frac{i}{12})^{k-1}$

Note: It takes quite a bit of thought to figure out the amount of the *first* principal payment, P_1. (See Problem #5.)

Calculating the monthly interest payments

Method 1

First calculate the principal payment for the month using the formula above. Then subtract it from the total monthly payment ($860.67).

Method 2

First find the outstanding principal balance by subtracting all previous principal payments from the original loan amount ($176,000). Then take 0.35% of this amount. (See the examples on the previous page.)

Note: It should seem surprising that both methods always give the same answer! As students will learn in Problem #2, this happens because the principal payments form a geometric sequence.

Problem #2

In a typical long-term loan:

(1) The monthly principal payments form a geometric sequence whose common ratio is equal to monthly interest rate.
(2) Each month's interest payment is calculated from the *outstanding* principal*.

You may also have noticed that the total monthly payment never changes. This happens *automatically* because of the two conditions above!

Directions

- Prove or explain why $P_2 + I_2$ always equals $P_1 + I_1$ under these two conditions.
- Repeat the process: prove or explain why $P_3 + I_3$ always equals $P_2 + I_2$.
- Explain how you could demonstrate that *all* of the total payments must be equal.

Diving Deeper

Do some research about the *principle of mathematical induction*. Explain how it relates to this question.

*The *outstanding principal* is the principal that is still to be paid.

Conversation Starters for Problem #2

What do you notice? What do you wonder?

I notice that it makes sense for the interest payment to go down when the remaining principal to be paid goes down.

I notice that it helps to focus on the remaining principal to be paid.

You could even give it a name if you like. For example, the original principal could be B_0; the amount remaining to be paid after the first month could be B_1, meaning that $B_1 = B_0 - P_1$. Similarly, $B_2 = B_0 - P_1 - P_2 = B_1 - P_2$, etc.

I notice that the remaining principal to be paid each month (the amount on which the interest is charged) decreases by principal paid the previous month.

I notice that the principal paid the previous month is also the amount from which the increase in the current principal payment is calculated.

I wonder how to figure out the first principal payment.

Students will explore this question in Problem #5 after they have learned methods for adding the terms in a geometric sequence.

Solutions for #2

A verbal explanation

- The *principal* payment each month increases by the monthly percentage of the previous month's principal payment (because of the geometric series).
- The *interest* payment each month decreases by the monthly percentage of the previous month's principal payment (because the remaining principal to be paid decreases by this principal payment).
- The previous two changes counteract each other, leaving the total monthly payment the same.

An algebraic proof

Suppose that the original amount of the loan is B_0 (the *balance* of the principal to be paid after 0 payments have been made). B_k can then stand for the balance left to be paid after k payments have been made.

Showing why $P_2 + I_2 = P_1 + I_1$

$P_2 + I_2 =$

$P_1\left(1 + \frac{i}{12}\right) + B_1\frac{i}{12} =$ $\quad$ $P_1\left(1 + \frac{i}{12}\right)$ comes from condition (1). $B_1\frac{i}{12}$ comes from condition (2).

$P_1\left(1 + \frac{i}{12}\right) + (B_0 - P_1)\frac{i}{12} =$

$P_1 + P_1\frac{i}{12} + B_0\frac{i}{12} - P_1\frac{i}{12} =$

$P_1 + B_0\frac{i}{12} =$

$P_1 + I_1$

Showing why $P_3 + I_3 = P_2 + I_2$

$P_3 + I_3 =$

$P_2\left(1 + \frac{i}{12}\right) + B_2\frac{i}{12} =$

$P_2\left(1 + \frac{i}{12}\right) + (B_1 - P_2)\frac{i}{12} =$

$P_2 + P_2\frac{i}{12} + B_1\frac{i}{12} - P_2\frac{i}{12} =$

$P_2 + B_1\frac{i}{12} =$

$P_2 + I_2$

Explaining why all of the total monthly payments are equal

Using the same calculation pattern, you can create a list of equations:

$$P_4 + I_4 = P_3 + I_3 \qquad P_5 + I_5 = P_4 + I_4 \qquad P_6 + I_6 = P_5 + I_5 \quad \text{etc.}$$

Since each expression, $P_k + I_k$, is equal to the previous one, $P_{k-1} + I_{k-1}$, all of them are equal to each other.

Some students who are using algebraic proofs may find it helpful to do the calculation one more time in order to ensure that they understand the pattern thoroughly.

$P_4 + I_4 =$

$P_3\left(1 + \frac{i}{12}\right) + B_3\frac{i}{12} =$

$P_3\left(1 + \frac{i}{12}\right) + (B_2 - P_3)\frac{i}{12} =$

$P_3 + P_3\frac{i}{12} + B_2\frac{i}{12} - P_3\frac{i}{12} =$

$P_3 + B_2\frac{i}{12} =$

$P_3 + I_3$

Stage 2

Up to this point, students have learned how to determine the amount of a principal payment from the previous month's payment. However, they may be wondering how to establish the value of the first month's payment. If it is too low, the loan will not be paid off in time. If it is too high, the monthly payments will be higher than needed to pay the entire principal on schedule.

Students develop some of the tools grapple with this question in Stage 2. In order to answer it, they need to be able to add principal payments throughout the course of the loan. Since loans often extend over many years and are paid monthly, this can be a very tedious and time-consuming process. Fortunately, there is a fast way to add the terms of a geometric sequence*. In Problem #3, students learn this shortcut. In Problem #4, they apply it.

To help students prepare for the problems, you may want to clarify a couple of points

- Distinguish between the term of a *loan* (the amount of time scheduled to pay it off) and a term of a *sequence* (any individual number in the sequence).
- The " $\cdots$ " symbol indicates that the sequence continues according to the same pattern. (It avoids the necessity of writing out each term in a long sequence.)

What students should know

- Understand the ideas from Problem #1.

What students will learn

- Learn a method for adding the terms of a geometric sequence.
- Apply to method to calculate the length of a long-term loan.

**Note*: In later math classes, students will learn that the sum of the terms in a geometric sequence is known as a *geometric series.*

Problem #3

A Mathematical Detour:
The Sum of the Numbers in a Geometric Sequence

$$3 + 12 + 48 + 192 + 768 + \cdots + 786{,}432$$

$$\begin{aligned} S\ &= 3 + 12 + 48 + 192 + 768 + \cdots + 786{,}432 \\ 4S &= \qquad 12 + 48 + 192 + 768 + \cdots + 786{,}432 + 3{,}145{,}728 \end{aligned}$$

$$\begin{aligned} 4S - S &= 3{,}145{,}728 - 3 \\ 3S &= 3{,}145{,}725 \\ S &= 1{,}048{,}575 \end{aligned}$$

Directions

- Study and explain the shortcut illustrated in the example above.
- Show how to use the shortcut to find the value of each sum.

$$1 + 3 + 9 + 27 + 81 + \cdots + 177{,}147$$
$$7 + 14 + 28 + 56 + \cdots 57{,}344$$
$$50 + 20 + 8 + 3.2 + \cdots + 0.032768$$

- Explain how this process might help you work with loans.

Conversation Starters for Problem #3

What do you notice? What do you wonder?

I notice that the second sequence is multiplied by the common ratio of the first sequence.

I notice that the terms of the two geometric series are all the same except for the first term of the S sequence and the last term of the $4S$ sequence.

I wonder how someone thought of the "trick" of multiplying the sequence by 4 (the constant ratio).

I notice that it is not necessary to write down every part of the process in the example at the top of the page.

I notice that the amount you divide by at the end is always 1 less than the common ratio.

This comes from subtracting the sum of the original sequence, which you may think of as $1S$.

I notice that the sum depends on just three features of the sequence: the term after the last one, the first term, and the common ratio.

I wonder if this process still works when the geometric sequence contains negative terms.

Solutions for #3

An explanation of the shortcut

The example shows a shortcut for adding the numbers in the geometric sequence,

$$3 + 12 + 48 + 192 + 768 + \cdots + 786{,}432.$$

The first term (a_1) in the sequence is 3, and the common ratio (r) is 4.

The details of the process:

- Multiply each term in the sum by r (4, in this case):
  $$12 + 48 + 192 + 768 + \cdots + 786{,}432 + 3{,}145{,}728$$
- All terms but the first term of the original sum and the last term of the second sum are the same. Therefore, when you subtract the sums, only these two terms remain:
  $$3{,}145{,}728 - 3$$
- Since you have subtracted the original sum (S) from 4 times the sum ($4S$), the result (3,145,725) is 3 times the sum ($3S$). Therefore, when you divide this number by 3, you obtain the sum, S.
  $$3{,}145{,}725 \div 3 = 1{,}048{,}575$$

You may check that this result is correct by adding all ten numbers in the sum.

The work above shows both the process *and why it works*. When you are using the shortcut, it is not necessary to show all of these details. Since you know that all of the terms in the middle will subtract away, you may simplify the process as follows.

(1) Find the next term in the sum. (Multiply the last term by r.)
(2) Subtract the first term.
(3) Divide the result by 1 less than the common ratio ($r - 1$).

Students may need to create a few more of their own examples to understand where each step of this process comes from. Some may be able to write a formula for the sum:

$$S = \frac{a_1 r^n - a_1}{r-1} \text{ or } S = \frac{a_1(r^n - 1)}{r-1},$$

where the original sequence has n terms.

Finding the sum of the terms of the first sequence

$1 + 3 + 9 + 27 + 81 + \cdots + 177{,}147$ ($a_1 = 1$ and $r = 3$.)

(1) Find the next term. $177{,}147 \cdot 3 = 531{,}441.$

(2) Subtract the first term. $531{,}441 - 1 = 531{,}440$

(3) Divide by $r - 1$. $531{,}440 \div 2 = 265{,}720$

The sum is 265,720.

Finding the sum of the terms of the second sequence

$7 + 14 + 28 + 56 + \cdots 57{,}344$ ($a_1 = 7$ and $r = 2$.)

(1) Find the next term. $57{,}344 \cdot 2 = 114{,}688$

(2) Subtract the first term. $114{,}688 - 7 = 114{,}681$

(3) Divide by $r - 1$. No change.

The sum is 114,681.

Finding the sum of the terms of the third sequence

$50 + 20 + 8 + 3.2 + \cdots + 0.032768$ ($a_1 = 50$ and $r = 0.4$.)

(1) Find the next term. $0.032768 \cdot 0.4 = 0.0131072$

(2) Subtract the first term. $0.0131072 - 50 = -49.9868928$

(3) Divide by $r - 1$. $-49.9868928 \div -0.6 = 83.311488$

The sum is 83.311488.

Using the process to work with loans

Since the monthly principal payments form a geometric sequence, you could use this process to quickly find the total amount of principle paid after k months. There is no need to add them directly!

Problem #4

When you know an efficient method for adding the terms of a geometric sequence, you may quickly find the total amount of principal paid so far on a loan.

Directions

- For the loan in Problem #1, determine the total amount of principal paid after 20 years (T_{20}). Explain your thinking.
- Write a formula for the total amount of principal paid after k months.
- Determine the *term* of the loan in Problem #1 (the number of years needed to pay it off).

Conversation Starters for Problem #4

What do you notice? What do you wonder?

I notice that I have to be careful to express the years as months.

I notice that the exponent on 1.0035 lags behind the number of months by 1.

This makes sense, because the original principal payment (in month 1) is not multiplied by 1.0035.

I wonder what has to happen in order for the loan to be paid off.

The total of the *principal* payments has to equal the original loan amount ($176,000).

I notice that it is useful to have a formula in order to find the term of the loan.

I wonder if I could use my strategy for this problem to calculate the total *interest* paid over a certain number of months.

No, because the interest payments do not form a geometric sequence.

Solutions for #4

The amount of principal paid after 20 years

The total amount of principal paid after 20 years is $91,874.27

20 years = 240 months. The principal payment for this month is

$$P_{20} = P_1 \cdot \left(1 + \frac{i}{12}\right)^{240-1} = 244.67(1.0035^{239}) \approx 563.94$$

To find the total principal paid up to this time:

(1) Find the next term. $563.94 \cdot 1.0035 = 565.914940356$

(2) Subtract the first term. $565.914940356 - 244.67 = 321.244940356$

(3) Divide by $r - 1$. $321.244940356 \div 0.0035 \approx 91{,}874.27$

Note: Students could shorten this process by calculating the number in Step (1) directly: $244.67(1.0035^{240}) \approx 565.91$.

Sample formulas for the total amount of principal paid after k months (T_k)

$$T_k = \frac{244.67(1.0035^k) - 244.67}{0.0035}$$

or

$$T_k = \frac{244.67(1.0035^k - 1)}{0.0035}$$

Some students may be able to find a more general formula such as

$$T_k = \frac{P_1\left(\left(1 + \frac{i}{12}\right)^k - 1\right)}{\frac{i}{12}}$$

The term of the loan

The term of the loan is 30 years.

The loan is paid off once the total *principal* of $176,000 is paid. Some students may guess, test, and revise based on the three-step process or one of the formulas, trying to make $T_k = 176{,}000$. Others may set one of the expressions for T_k equal to 176,000 and try to solve it for k. For example:

$$\frac{244.67(1.0035^k - 1)}{0.0035} = 176{,}000$$

$$244.67(1.0035^k - 1) = 616$$

$$1.0035^k - 1 = \frac{616}{244.67}$$

$$1.0035^k = \frac{616}{244.67} + 1 \approx 3.5176769$$

At this point (until they learn about logarithms), students do not have the tools to isolate k. They may now guess, test, and revise until they discover that $k \approx$ 360 months, or 30 years.

Note: If you substitute 360 for k in the expression above, you obtain

$$\frac{244.67(1.0035^{360} - 1)}{0.0035} = 175{,}999.84$$

meaning that, according to this calculation, you are 16¢ short of repaying the principal on the loan! To fix any discrepancy, the lender could make a small adjustment to final principal payment.

Stage 3

Because Problem #5 is not quite as challenging as a typical Stage 3 problem, and because it will help students consolidate their learning from the first two stages, it may be a good idea to encourage most of your students attempt it. The goal is to calculate principal and interest payments for a car loan from the "ground up." Students are given the basic facts about a loan (price of a car, down payment percentage, annual interest rate, and loan term). From this information, they calculate the details necessary to set up the loan payments*. They also compare the effects of choosing different down payments.

Problem #6 is very open-ended. Using information about a potential borrower's financial situation, students decide whether it is a good time for her to buy a home, and if so, how expensive a house she can afford, how much of a down payment she should make, etc. The problem is more realistic than earlier ones, because it includes property taxes and insurance, which are usually included in monthly house payments. Feel free to share with students the standard rule of thumb that house payments should be no more than about 30% of a person's monthly income.

By the way, if students did not complete Problem #2, this might be a good time to have them go back and try it! If they answered the question verbally, ask them to take their understanding to a new level by proving it algebraically. This is a great opportunity to

(1) Learn how to use a clearly defined set of assumptions to reach a conclusion.
(2) Transform one algebraic expression into an equivalent one.
(3) Gain informal experience with the concept of *proof by mathematical induction* (an idea usually addressed in advanced algebra or pre-calculus courses).

What students should know

- The concepts and procedures from Stage 1 and Stage 2.

What students will learn

- Apply knowledge from the first two stages to analyze new loan situations.
- Draw conclusions about the effects of different down payments.
- Make decisions about a home purchase based on a person's financial situation.

Problem #5

Price of car: $23,000
Down payment: 20%
Annual Interest rate: 2.4%
Loan term: 5 years

Directions

- Determine the total monthly payment for this car loan.
- Determine the principal and interest payments each month for the first half-year.
- Figure out how it would affect your monthly payment and the total interest paid if you put only 10% down.

Conversation Starters for Problem #5

What do you notice? What do you wonder?

I notice that I need to calculate the original interest and principal payments separately in order to figure out the total monthly payment.

I notice that once I know the first principal and interest payments, I can just use the patterns I discovered in Problem #1 to find the remaining payments.

I notice that, unlike the case of the home loan, the first principal payment is much greater than the first interest payment.

I wonder why this happens.

Solutions for #5

The total monthly payment

The total monthly principal/interest payment will be \$325.74.

The amount of the loan:

20% of 23,000 = 0.2(23,000) = 4,600

23,000 – 4,600 = 18,400

The amount to be borrowed will be \$18,400.

The first interest payment (I_1):

The monthly interest rate is $\frac{i}{12} = \frac{2.4\%}{12} = \frac{0.024}{12} = 0.002$ (or 0.2%).

The first interest payment is 0.2% of full loan amount of \$18,400.

0.002(\$18,400) = \$36.80.

The first principal payment (P_1)—two strategies:

Find the value of P_1 so that the full principal of \$18,400 is paid in 5 years (60 months).

Strategy 1 (Guess, test, and revise.)

Suppose that the first principal payment is \$300.

(1) Find the "61st payment": $300(1.002^{60}) \approx 338.208522$

(2) Subtract the first payment: $388.208522 - 300 \approx 38.208522$

(3) Divide by $r - 1$: $38.208522 \div 0.002 \approx 19{,}104.26$

This is greater than 18,400, so the \$300 guess is too high.

Suppose that the first principal payment is \$250.

(1) Find the "61st payment": $250(1.002^{60}) \approx 281.840435$

(2) Subtract the first payment: $281.840435 - 250 \approx 31.840435$

(3) Divide by $r - 1$: $31.840435 \div 0.002 \approx 15{,}290.22$

This is less than 18,400, so the \$250 guess is too low.

Continue revising your guesses until you find the correct value: $P_1 = 288.94$.

Strategy 2 (Solve a T_{60} formula for P_1.)

The necessary relationships are in a formula from Problem #4,

$$T_k = \frac{P_1\left(\left(1+\frac{i}{12}\right)^k - 1\right)}{\frac{i}{12}}$$

which adds the principal payments in the geometric sequence.

In this case, $\frac{i}{12} = 0.002$, $k = 60$, and $T_{60} = 18{,}400$. P_1 is the only unknown!

$$18{,}400 = \frac{P_1(1.002^{60} - 1)}{0.002}$$

$$P_1 = \frac{18{,}400(0.002)}{1.002^{60} - 1} \approx 288.94$$

The total monthly principal/interest payment ($P_1 + I_1$):

$$P_1 + I_1 = 288.94 + 36.80 = 325.74$$

Principal and interest payments for the first half-year

Month (k)	Principal (P_k)	Interest (I_k)
1	288.94	36.80
2	289.52	36.22
3	290.10	35.64
4	290.68	35.06
5	291.26	34.48
6	291.84	33.90

You can find each principal payment by multiplying the previous payment by 1.002. Afterwards, you can find the interest payment by subtracting this principal payment from the total monthly payment of 325.74. You may check this number by taking 0.2% of the *remaining* principal balance.

Example (Month 2)

$$288.94(1.002) = 289.52$$
$$325.74 - 289.52 = 36.22$$
Check: $0.002(18{,}400 - 288.94) \approx 36.22$

The effect on the monthly payment when you put only 10% down

If you put only 10% down, the loan amount will be

$$23{,}000 - 0.1(23{,}000) = 0.9(23{,}000) = 20{,}700.$$

The initial interest payment is

$$I_1 = 0.002(20{,}700) = 41.40.$$

The initial principal payment is

$$P_1 = \frac{20{,}700(0.002)}{1.002^{60} - 1} \approx 325.06$$

The total monthly payment is $P_1 + I_1 = 325.06 + 41.40 = 366.46.$

This is an increase of 366.46 – 325.74 = $40.72 in each monthly payment compared to the 20% down payment case (or a percentage increase of $\frac{40.72}{325.74} \approx 12.5\%$).

The effect on the total interest paid when you put only 10% down

Total interest paid in 20% down payment case:

Total interest paid =
Total payments – amount borrowed
= 325.74 x 60 – 18,400 = 19,544.40 – 18,400 = 1144.40

Total interest paid in 10% down payment case:

Total interest paid =
Total payments – amount borrowed
= 366.46 x 60 – 20,700 = 21,987.60 – 20,700 = 1287.60

In the end, you pay an additional $1287.60 – $1144.40 = $143.20 in interest if you make the smaller down payment.

Problem #6

Madison wants to buy a house. She has been saving for a down payment.

Madison's desired price range	\$150,000 to \$200,000
Options for down payments	20% or 30%
Options for loan terms	15-year loan: 3.75% annual interest rate
	30-year loan: 4.00% annual interest rate
Annual salary	\$46,000.
Savings	\$50,000
Estimated homeowner's insurance	\$1200 annually
Annual property taxes	1.5% of the value of the house

Directions

- Design some loan scenarios for Madison.
- Help Madison decide if this is a good time to buy. If so, explain why and choose a good loan scenario for her. If not, explain why you think she should wait.

Diving Deeper

You may have noticed that the first principal payment is sometimes greater than the first interest payment. Observe the conditions under which this happens. What are the two key factors? Which formula—and which parts of it—control this phenomenon? How could you calculate a "cutoff" point at which the principal and interest are approximately equal in the first payment?

Conversation Starters for Problem #6

What do you notice? What do you wonder?

I wonder what are the most important things to focus on in making decisions.

Possibilities include: having enough money to make a down payment, keeping some money in savings, keeping the monthly payments low enough (30% or less on monthly income), keeping the total cost of interest as low as possible, spending enough to get the type of house you need, etc.

I wonder how to choose scenarios that will help me make decisions.

It may help to choose extreme examples—for instance, the lowest and highest prices. It may also help to change just one feature in an example at a time (for example, the term of the loan) so that you can investigate how the change affects down payments, monthly payments, total loan cost, etc.

I wonder how realistic these data are.

Most of them are probably realistic, though home prices, interest rates, annual salaries, options for loan terms, etc. all vary quite a bit over time and place. Additional options may be available. For instance, you may not be restricted to 15- and 30- year loans or to 20% and 30% down payments.

I wonder if there is a clear “best answer” to this problem.

Problems as open-ended as this one usually have many reasonable answers. The important things are to place yourself in Madison’s position, use mathematics to select relevant data and make accurate calculations that inform your decision, have good reasons for your conclusions, and communicate your thinking clearly.

Solutions for #6

Some important things to consider

Madison probably won't want to use all of her savings. She should keep her monthly payments manageable (no more than 30% of her monthly income), and she might want to consider how much interest she will pay over the entire life of the loan.

Of course, there are also elements in her decision that cannot be addressed with the information from this problem. For example: Will a $150,000 house have the features that she needs? How many other monthly expenses does she have? Does she have other financial resources? What are the housing market conditions?

A few sample loan scenarios

1. 30% down payment on a $150,000 house

   Down payment: $45,000 Loan amount: $105,000.

   15-year (180 month) loan at 3.75% annual interest

   First interest payment

   $$105{,}000\left(\frac{0.0375}{12}\right) = 105{,}000(0.003125) = 328.125$$

   First principal payment

   $$\frac{105{,}000(0.003125)}{1.003125^{180} - 1} \approx 435.46$$

   Total principal and interest: $328.125 + 435.458 \approx 763.58$

   Insurance and taxes per month:

   $$\frac{1200 + 0.015(150{,}000)}{12} = 287.50$$

   Total monthly payment: $763.58 + 287.50 = \mathbf{1051.08}$

   30-year (360 month) loan at 4.0% annual interest

   First interest payment

   $$105{,}000\left(\frac{0.04}{12}\right) = 105{,}000(0.00\overline{3}) = 350.00$$

   First principal payment

   $$\frac{105{,}000(0.00\overline{3})}{1.00\overline{3}^{360} - 1} \approx 151.29$$

   Total principal and interest: $350.00 + 151.29 = 501.29$

   Insurance and taxes per month:

   $$\frac{1200 + 0.015(150{,}000)}{12} = 287.50$$

   Total monthly payment: $501.29 + 287.50 = \mathbf{788.79}$

2. 20% down payment on a $150,000 house
   Down payment: $30,000 Loan amount is $120,000

15-year (180 month) loan at 3.75% annual interest

First interest payment

$$120{,}000\left(\frac{0.0375}{12}\right) = 120{,}000(0.003125) = 375.00$$

First principal payment

$$\frac{120{,}000(0.003125)}{1.003125^{180} - 1} \approx 497.67$$

Total principal and interest: $375.00 + 497.67 \approx 872.67$

Insurance and taxes per month:

$$\frac{1200 + 0.015(150{,}000)}{12} = 287.50$$

Total monthly payment: $872.67 + 287.50 = \mathbf{1160.17}$

30-year (360 month) loan at 4.0% annual interest

First interest payment

$$120{,}000\left(\frac{0.04}{12}\right) = 120{,}000(0.00\overline{3}) = 400.00$$

First principal payment

$$\frac{120{,}000(0.00\overline{3})}{1.00\overline{3}^{360} - 1} \approx 172.90$$

Total principal and interest: $350.00 + 151.29 = 572.90$

Insurance and taxes per month:

$$\frac{1200 + 0.015(150{,}000)}{12} = 287.50$$

Total monthly payment: $572.90 + 287.50 = \mathbf{860.40}$

3. 20% down payment on a \$200,000 house
   Down payment: \$40,000 Loan amount is \$160,000.

   15-year (180 month) loan at 3.75% annual interest
   First interest payment
$$160{,}000\left(\frac{0.0375}{12}\right) = 160{,}000(0.003125) = 500.00$$
   First principal payment
$$\frac{160{,}000(0.003125)}{1.003125^{180} - 1} \approx 663.56$$
   Total principal and interest: $500.00 + 663.56 \approx 1163.56$

   Insurance and taxes per month:
$$\frac{1200 + 0.015(200{,}000)}{12} = 350.00$$

   Total monthly payment: $1163.56 + 350.00 = \mathbf{1513.56}$

   30-year (360 month) loan at 4.0% annual interest
   First interest payment
$$160{,}000\left(\frac{0.04}{12}\right) = 160{,}000(0.00\overline{3}) \approx 533.33$$
   First principal payment
$$\frac{160{,}000(0.00\overline{3})}{1.00\overline{3}^{360} - 1} \approx 230.53$$
   Total principal and interest: $533.33 + 230.53 = 763.86$

   Insurance and taxes per month:
$$\frac{1200 + 0.015(200{,}000)}{12} = 350.00$$

   Total monthly payment: $763.86 + 350.00 = \mathbf{1113.86}$

4. 30% down on a \$200,000 house
   Down payment: \$60,000 dollars.

   Madison does not have enough money in savings to make the down payment.

Making a decision

Option 4 (30% down payment on a $200,000 house) is eliminated immediately, because Madison does not have enough in savings to make the down payment.

If she sticks with the usual recommendation to keep her monthly payments to no more than 30% of her monthly income, her payments should not exceed $1150.

$$\frac{46{,}000}{12}(0.3) = 1150$$

This eliminates the 15-year loan for both option 2 (20% down on a $150,000 house) and option 3 (the 20% down payment on a $200,000 house).

The 30-year loan for option 3 is possible, but Madison would be stretching the limits of her resources in three ways:

1. She is leaving only $10,000 in her savings account
2. Her monthly payment ($1113.86) is close to the suggested limit of $1150.
3. A 30-year loan is much more expensive over the long run than a 15-year loan.

If Madison considers the previous option to have too many points against it, the remaining possibilities are the 30-years loans for options 1 and 2, and the 15-year loan for option 1. (The $200,000 houses may be a bit out of her price range, at least for now.) Advantages and disadvantages for these possibilities include:

15-year loan for option 1 (30% down on a $150,000 house):

- Leaves her only $5000 in savings
- Has a fairly high but probably manageable monthly payment of $1051.08
- Has the lowest total interest cost of any option. Total interest charges are:

$$\$763.58 \text{ per month} \cdot 180 \text{ months} - \$105{,}000 = \$32{,}444.40$$

30-year loan for option 1 (30% down on a $150,000 house):

- Leaves her only $5000 in savings
- Has the lowest monthly payment of any option at $788.79
- *Much* higher total interest cost than the 15-year option. Total interest charges:

$$\$501.29 \text{ per month} \cdot 360 \text{ months} - \$105{,}000 = \$75{,}464.40$$

30-year loan for option 2 (20% down on a $150,000 house):

- Leaves her much more in savings ($20,000) than the other options
- Has a low monthly payment of $860.40
- Has the highest total interest cost of these options. Total interest charges:

$$\$572.90 \text{ per month} \cdot 360 \text{ months} - \$120{,}000 = \$86{,}244.00$$

Balancing all of these considerations may be challenging for Madison. If she does not have a lot of other monthly expenses, a 15-year loan may be the way to go, because it costs so much less in the long run. She may even go with the 15-year loan for option 2 (which we did not consider above). The monthly payment is only about $10 over the recommended limit, and it has big advantages in terms of leaving her a lot of savings and having a much lower overall interest cost.

If the $150,000 houses are not ideal for her needs, she may be better off waiting until she has saved more for a down payment and/or is making a higher salary so that the monthly payments would be more comfortable. Of course, there may be houses *between* $150,000 and $200,000 that would not stretch her resources quite so much and would be closer to what she is looking for.