Colorado State University
Department of Mathematics

Masters Thesis

# An Explicit Geometric Computation of the $2, 3$ Double Hurwitz Polynomial

**(in Genus 0)**

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**Abstract**

In order to better understand certain geometric spaces, one might be inclined to study maps between them. Maps from compact Riemann surfaces to the sphere are *ramified covers*. The (automorphism weighted) number of such maps from a given surface with certain fixed properties is the *Hurwitz number*. We introduce Hurwitz numbers and then a family thereof, *double Hurwitz numbers*. We then define the various moduli spaces, and objects therein, which will allow us to compute the 2,3, genus zero case of the double Hurwitz (piecewise) polynomial by using a nifty wall-crossing formula.

# 1 Introduction

Were I to speak of my new compact Riemann surface about which you knew almost nothing, you might be inclined to request that I compare it to an object you know well, say for instance, the sphere. Acquiescing to this request, I might in turn produce for you some maps from my nice new surface to your beloved sphere. A good question at this point would perhaps be,

> "What do maps from your surface to the sphere tell me about its properties?"

Maps from a compact Riemann surface onto the Riemann sphere are *ramified covers*. That is, there are a finite number of *branch points* on the sphere (the collection of which is called the *branch locus*), such that the number of points in the preimage of a branch point is less than the degree of the map. Take, for instance, figure 1. It depicts a degree 5 ramified cover of the Riemann sphere. The red "**x**" represents a point on the sphere not in the branch locus, and the five "**x**"s above it represent its preimage. The blue points on the sphere are the map's branch points, each having less than five points in its preimage. Because

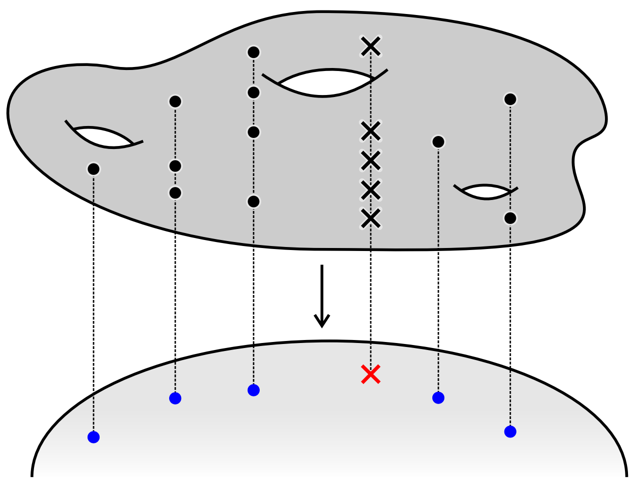


Figure 1: Ramified cover of $\mathbb{P}^1$

the number of points in the preimage of a branch point is less than the degree of our map, we count the points in the preimage with multiplicities, such that the sum of these multiplicities equals our degree. For instance, the leftmost branch point in figure 1 has only one preimage, which necessarily has a multiplicity of 5. In fact, if we gave reasonable local coordinates to neighborhoods of these two points, we would get the map $z \mapsto z^5$ as is shown in figure 2. In general, we say that the *ramification* of a point is its multiplicity minus 1. Thanks to

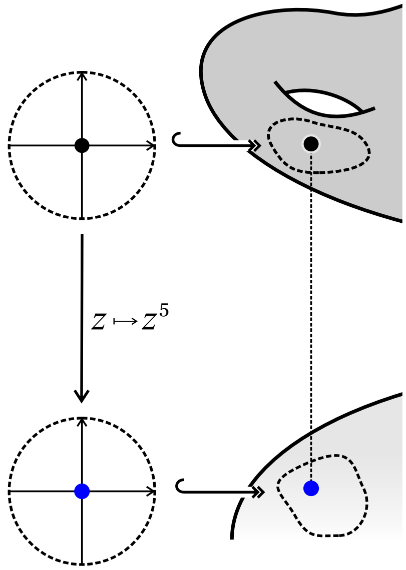



Figure 2: Ramification of a point

the Riemann-Hurwitz equation, we have a way to relate the genus of our source curve to the degree of our map and its total ramification.

Now, a reasonable question might be,

> "How many such covers are there for a given genus, degree, and fixed branch locus with prescribed ramification profiles?"

Hurwitz numbers answer just this. Geometrically, they count genus $g$, degree $d$ covers of $\mathbb{P}^1$ with prescribed ramification over a fixed branch locus.

Hurwitz numbers have played a role in a somewhat diverse collection of areas within mathematics. Stemming from the work of Hurwitz himself, they are useful for relating combinatorial objects to the geometry of elliptic curves (as in [GGN12] and [BCF+12]). They have also been important objects of study in the area of Gromov-Witten theory and in the study of the moduli space of curves. The main result connecting Hurwitz numbers to the moduli space of curves was given by Ekedahl, Lando, Shapiro, and Vainshtein (as the now dubbed "ELSV formula") in [ELSV01], which gives Hurwitz numbers as an integral over the moduli space of curves. This formula has led to many strong geometric results such as Okounkov and Pandharipande's proof of Witten's conjecture, relating tautological intersection theory on the moduli space of curves and integrable systems.

For our purposes, we concern ourselves with a specific case of Hurwitz numbers denoted *double Hurwitz numbers* (which are defined precisely in the following section, 2.1.) This family of Hurwitz numbers has generated an interest as to their geometric properties. In [GJV03], Goulden, Jackson, and Vakil prove a piecewise polynomiality result for double Hurwitz numbers. This piecewise polynomiality corresponds to a chamber structure in the domain of ramification profiles which has, in turn, been explored in genus 0 in [SSV08], and then in greater generality in [CJM10].

Cavalieri and Marcus express the double Hurwitz number as an intersection of tautological classes on $\overline{\mathcal{M}}_{g,n}$ in [CM13]. They view the double Hurwitz number as effectively being top intersections of a psi class in the moduli space of relative stable maps to an unparametrized $\mathbb{P}^1$. Psi classes on the space of relative stable maps are equal to the pullback via the stabilization morphism of psi classes in $\overline{\mathcal{M}}_{g,n}$ plus chamber dependent boundary corrections. Pushing forward via the stabilization morphism and applying projection formula, Cavalieri and Marcus express the Hurwitz number in terms of tautological intersections on $\overline{\mathcal{M}}_{g,n}$. In genus 0, however, they reach a curiosity (§5.1, [CM13]). At certain wall-crossings, divisors appear in the correction which push forward, via the stabilization morphism, to 0 in $\overline{\mathcal{M}}_{0,n}$. However, these divisors have nontrivial intersections in the space of relative stable maps. Furthermore, these intersections push forward to strata with nontrivial multiplicities in $\overline{\mathcal{M}}_{g,n}$ and, therefore, contribute to the Hurwitz polynomial. They find that ignoring such divisors produces a result that is correct only up to a sign. They take this to mean "that there are interesting and potentially useful vanishing statements hidden in the intersections that constitute the geometric wall crossing formula."

In this paper, we compute a specific case of the genus 0 Hurwitz polynomial using the methods of Cavalieri and Marcus, while also including intersections from the types of divisors which have led to their "genus 0 curiosity." In the dearth of an explanation for this curiosity, the hope is that such explicit computations might help lead to a conjecture of the cause of Cavalieri and Marcus' erroneous sign. This thesis is organized as follows:

First, in section 2.1, we introduce double Hurwitz numbers more precisely. We explain the chamber structure which corresponds to their piece-wise polynomiality. This chamber structure introduces a wall-crossing formula which will be used in computing the double Hurwitz number in each chamber. Then, we introduce the spaces and maps appearing in the central diagram, which we use to compute our piece-wise polynomial geometrically. Additionally, in section 3.6, we define the psi classes necessary for intersection theory on each of these spaces.

In section 4 we use the fact that the degree of the branch map, from the space of relative stable maps to the Losev-Manin space, is our double Hurwitz number. This allows us, in our final section (§5), to compute the double Hurwitz number by the pull back of a point (in terms of a psi class) in the Losev-Manin space. This gives us a formula in terms of $x_1\widetilde{\psi}_1$, a psi class on the space of relative stable maps. We can then compute the Hurwitz polynomial in terms of intersections of boundary divisors which appear as corrections at each wall crossing. As previously stated, we include in these calculations those divisors which caused the genus zero curiosity in [CM13] (which is simpler to do in our specific case than in the general one).

With these intersections we get an expression in each chamber which we push forward via our stabilization morphism to obtain a formula in terms of intersections on $\overline{\mathcal{M}}_{0,5}$.


### Acknowledgments

The author wishes to thank his advisor, Renzo Cavalieri, for putting up with (and advising) him through this process, as well his wife for tirelessly brewing the coffee required to see this through.


# 2 Double Hurwitz Numbers

Fix $d, g \in \mathbb{N}$ and partitions $\mathbf{x}_0$, $\mathbf{x}_\infty$ of $d$. Double Hurwitz numbers $H_g(\mathbf{x_0}, \mathbf{x}_\infty)$ count the number of (automorphism-weighted) genus $g$, degree $d$ ramified covers of $\mathbb{P}^1$ with fixed branch points, where the ramification profile over 0 and $\infty$ are given by $\mathbf{x}_0$ and $\mathbf{x}_\infty$, respectively. We require that all other branch points have simple ramification profiles.

We consider the parts of $\mathbf{x}_\infty$ to be negative (note: this does not affect the ramification profile over $\infty$). Let $\mathbf{x}$ be the $n$-tuple of integers whose positive and negative components are given by $\mathbf{x_0}$ and $\mathbf{x}_\infty$, respectively. Then $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ and we get the relation

$$\sum_{i=1}^{n} x_i = 0 \tag{1}$$

For our purposes, we will proceed to only consider genus 0 covers and will use the notation $H(\boldsymbol{x})$ to mean $H_0(\boldsymbol{x}_0, \boldsymbol{x}_\infty)$, where $\boldsymbol{x}$ is defined as above.

If we consider all $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ in $(\mathbb{Z} \setminus \{0\})^n$ such that $\sum x_i = 0$, then double Hurwitz numbers determine a function

$$H : (\mathbb{Z} \setminus \{0\})^n \to \mathbb{Q} \tag{2}$$

which Goulden, Jackson, and Vakil showed to be a piecewise polynomial of degree $n - 3$ ($4g - 3 + n$ for any given genus) in [GJV03].

**Example 2.1.** Consider $H(\mathbf{x})$, where $\mathbf{x} \in (\mathbb{Z} \setminus \{0\})^5$. Fix $\mathbf{x}$ with the following properties:

(a) $x_1, x_2 > 0$

(b) $x_3, x_4, x_5 < 0$

(c) $x_1 > |x_i| + |x_j|$ for every $i, j \in \{2, 3, 4, 5\}$

Then, for such an $\mathbf{x}$, we have that $H(\mathbf{x}) = 6x_1^2$, which is degree 2 as expected (for the computation rendering this result, see section 5.1). Figure 3 shows the case where $\mathbf{x} = (10, 2, -3, -4, -5)$. The degree of such a map is 12, and the Riemann-Hurwitz formula gives that there must be three branch points with simple ramification profiles. There are 600 such (automorphism-weighted) ramified covers.

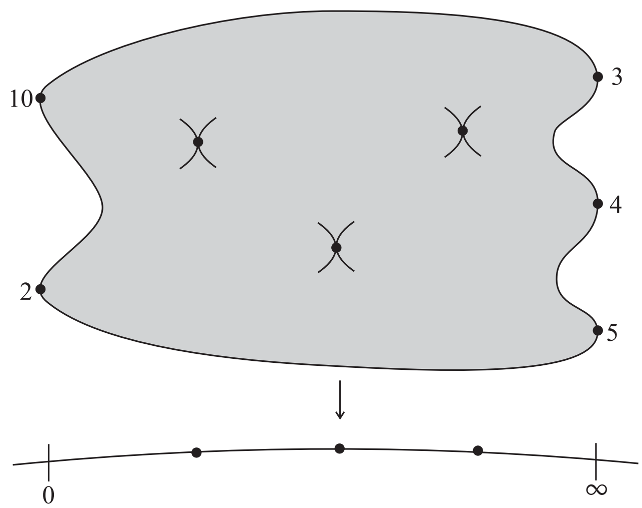



Figure 3: Ramified covers given by $\mathbf{x}_0 = (10, 1)$ and $\mathbf{x}_\infty = (-3, -4, -5)$

In our calculations, we will look only at Hurwitz numbers where the lengths of $\mathbf{x}_0$ and $\mathbf{x}_\infty$ are 2 and 3, respectively. We refer to these as *2,3 Hurwitz numbers*.

## 2.1 Chamber structure of Hurwitz polynomials

Shadrin, Shapiro, and Vainshtein, in [SSV08] showed that the piecewise nature of Hurwitz polynomials (in genus 0) corresponds to a partitioning of the domain $(\mathbb{Z} \setminus \{0\})^n$ into *chambers*. (A description of this chamber structure for arbitrary genus is given in [CJM10].) Let $I \subsetneq \{1, \ldots, n\}$. The walls which define the chambers in $(\mathbb{Z} \setminus \{0\})^n$ are defined by the hyperplanes

$$W_I = \left\{ \sum_{i \in I} x_i = 0 \right\} \tag{3}$$

in $\mathbb{R}^n$. Additionally, two adjacent chambers come with a corresponding wall-crossing polynomial. Fix $I \subset \{1, \ldots, n\}$. Let $\mathfrak{c}_1$ be a chamber along $W_I$ such that $\sum_{i \in I} x_i < 0$ and $\mathfrak{c}_2$ a chamber along $W_I$ (adjacent to $\mathfrak{c}_1$) such that $\sum_{i \in I} x_i > 0$. Let $P_1(\mathbf{x})$ be the the Hurwitz polynomial in $\mathfrak{c}_1$ (respectively $P_2(\mathbf{x})$ and $\mathfrak{c}_2$) The wall crossing is given by the polynomial

$$WC_I(\mathbf{x}) = P_2(\mathbf{x}) - P_1(\mathbf{x}) \tag{4}$$

an explicit formula for which is given in [CJM10], Theorem 1.5.

# 3 Setup

We now recall the definitions and background required to establish the central diagram, appearing in [CM13], which we use to calculate the $2, 3$ double Hurwitz polynomial (geometrically) in each chamber.

## 3.1 Moduli of $n$ marked points on $\mathbb{P}^1$

A moduli space is a space which parametrizes geometric objects up to an equivalence. In turn, the moduli space is, itself, a geometric object whose points correspond to the objects which it is parameterizing. We consider the following moduli space

$$\mathcal{M}_{0,n} = \left\{ \left(\mathbb{P}^1, (p_1, p_2, \ldots, p_n)\right) \middle| p_1, p_2, \ldots, p_n \in \mathbb{P}^1, \ p_i \neq p_j \right\} / \sim \tag{5}$$

where $\left(\mathbb{P}^1, (p_1, p_2, \ldots, p_n)\right) \sim \left(\mathbb{P}^1, (p'_1, p'_2, \ldots, p'_n)\right)$ if there exists an automorphism $\phi$ in $\mathrm{Aut}(\mathbb{P}^1)$ such that $\phi(p_i) = p'_i$ for all $i \in \{1, \ldots, n\}$. (Where there is no confusion, simply refer to the $n$ marked points as $1, 2, \ldots, n$.)

**Remark 3.1.** *Given any three points on $\mathbb{P}^1$, there exists an automorphism mapping any three points to $0$, $1$, and $\infty$.*

Consider $\left(\mathbb{P}^1 \setminus \{0, 1, \infty\}\right)^{n-3}$ and define coordinates $x_1, \ldots, x_{n-3}$ on each of the $n-3$ factors of $\mathbb{P}^1 \setminus \{0, 1, \infty\}$. Define $\Delta$ to be the set of *diagonals* where

$x_i = x_j$ for some $i, j \in \{1, \dots, n-3\}$. As a result of remark of 3.1 and the stipulation that $p_i \neq p_j$ in the definition of $\mathcal{M}_{0,n}$, it is easy to show that

$$\mathcal{M}_{0,n} \cong \left(\mathbb{P}^1 \setminus \{0, 1, \infty\}\right)^{n-3} \setminus \Delta \tag{6}$$

As a consequence, $\mathcal{M}_{0,n}$ is an affine, smooth space of dimension $n-3$.

$\mathcal{M}_{0,n}$ is not compact (consider points in $\Delta$ from equation 6). The Deligne-Mumford compactification of $\mathcal{M}_{0,n}$ is denoted

$$\overline{\mathcal{M}}_{0,n} = \left\{\left(\mathcal{C}, \mathbb{P}^1, (p_1, p_2, \dots, p_n)\right) \mid p_1, p_2, \dots, p_n \in \mathcal{C},\ p_i \neq p_j\right\} / \sim \tag{7}$$

where $\mathcal{C}$ is a nodal curve whose irreducible components are isomorphic to $\mathbb{P}^1$. The points $p_1, p_2, \dots, p_n$ lie on the smooth locus of $\mathcal{C}$. Additionally, $\mathcal{C}$ must be stable, where stability is defined by the ampleness of the log canonical divisor $K_{\mathcal{C}} + \sum_{i=1}^{n} p_i$.

Since stability is defined by the ampleness of the log canonical divisor we have that the nodal points of $\mathcal{C}$ are considered *special* points even though they do not correspond to any of the $p_i$'s; and, therefore, contribute to stability. Furthermore, $\left(\mathcal{C}, \mathbb{P}^1, (p_1, p_2, \dots, p_n)\right) \sim \left(\mathcal{C}', \mathbb{P}^1, (p_1', p_2', \dots, p_n')\right)$ if there exists an automorphism $\phi : \mathcal{C} \to \mathcal{C}'$ such that $\phi(p_i) = p_i'$. ([DM69])

We will be working with $\overline{\mathcal{M}}_{0,5}$, which we construct in 3.1.1.

### 3.1.1 A geometric construction of $\overline{\mathcal{M}}_{0,5}$

## 3.2 Losev-Manin Spaces

We introduce a family of compactifications of $\overline{\mathcal{M}}_{0,n}$ studied by Hassett ([Has02]). Assigning weights $\{a_i\}_{i=1}^{n}$ to $p_1, \dots, p_n$, such that $\sum_{i=1}^{n} a_i > 2$, and requiring the divisor $K_{\mathcal{C}} + \sum_{i=1}^{n} a_i p_i$ to be ample, we get a new compact moduli space $\overline{\mathcal{M}}_{0,n}(a_1, a_2, \dots, a_n)$ which parametrizes weighted, nodal, stable curves of genus 0 .

Alternatively, one can give a more combinatorial description of this stability condition. Let $\mathcal{C}$ be a nodal curve with marked points $p_1, \dots, p_n$ with respective weights $a_1, \dots, a_n$. Then $\mathcal{C} = \bigcup C_j$, where the $C_j$s are the irreducible components of $\mathcal{C}$. Let $I_j \subseteq \{1, \dots, n\}$ be the set of indices of the marked points on the irreducible component $C_j$. Then we have that the curve $\mathcal{C}$ is stable if and only if for every irreducible component $C_j$ of $\mathcal{C}$,

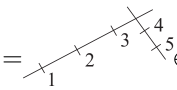
$$\sum_{i \in I_j} a_i + n_{C_j} > 2 \tag{8}$$

where $n_{C_j}$ is the number of nodes of $C_j$.

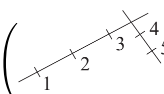
There is a natural contraction morphism

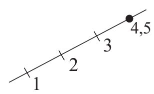
$$c : \overline{\mathcal{M}}_{0,n} \to \overline{\mathcal{M}}_{0,n}(a_1, a_2, \ldots, a_n) \tag{9}$$

which takes a point in $\overline{\mathcal{M}}_{0,n}$ to its stabilization in $\overline{\mathcal{M}}_{0,n}(a_1, a_2, \ldots, a_n)$. If $\Delta$ is a marked curve corresponding to a point in $\overline{\mathcal{M}}_{0,n}$, and $C_k$ is an irreducible component of $\Delta$; then if $C_k$ is unstable it contracts to a point in the weighted, nodal, marked curve corresponding to $c(\Delta)$.

**Example 3.1.** Consider $c : \overline{\mathcal{M}}_{0,5} \to \overline{\mathcal{M}}_{0,5}\left(1, 1, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right)$ and $\mathcal{X} :=$ 

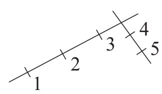
$\in \overline{\mathcal{M}}_{0,5}$. The contraction morphism contracts the irreducible component of $\mathcal{X}$, with the points marked 4 and 5, to a point, as is shown in (10).

$c\Big($ 

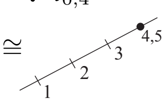
$\Big) =$ 

$\in \overline{\mathcal{M}}_{0,5}\left(1, 1, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right)$ (10)

Notice that




$$\begin{aligned} &\cong \overline{\mathcal{M}}_{0,4} \times \overline{\mathcal{M}}_{0,3} \\ &\cong \overline{\mathcal{M}}_{0,4} \times \{\text{a point}\} \\ &\cong \overline{\mathcal{M}}_{0,4} \\ &\cong \end{aligned}$$




†

For our purposes, we wish to consider the moduli space

$$\overline{\mathcal{M}}_{0,n}(1, 1, \varepsilon^{n-2}) \tag{11}$$

such that $(n-2)\varepsilon < 1$. This space parametrizes stable *chains* of projective lines, with $n$ marked, weighted points. We still consider the nodes of such a chain as *special* points with weight 1, we assign the weight 1 to one of the marked points on each of the extremal components of such a curve (which, we consider to be the points 0 and $\infty$). This moduli space is known as the *Losev-Manin space* ([LM00]).

**Remark 3.2.** *Our stability condition requires that there be at least one marked point on the smooth locus of each of the extremal rays of such a nodal curve, as the nodes only contribute a weight of* 1.

### 3.2.1 A geometric construction of $\overline{\mathcal{M}}_{0,5}(1,1,\varepsilon^3)$

For our computations we will only consider the Losev-Manin space $\overline{\mathcal{M}}_{0,5}(1,1,\varepsilon^3)/S_3$, where we work modulo the symmetric group in order to ignore the ordering of our three "light" points.

**Example 3.2.** The marked curves in figure 4 represent points in $\overline{\mathcal{M}}_{0,5}(1,1,\varepsilon^3)/S_3$, where the dots correspond to marked points of weight $\varepsilon$ (these are only two examples and by no way an exhaustive list). The dots are not labeled in order to indicate that we have chosen to ignore their orderings. Each irreducible com-

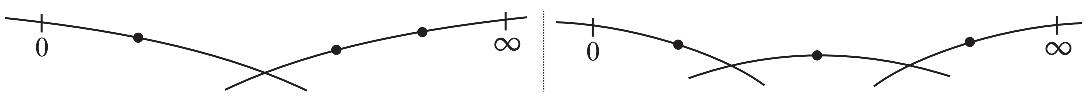



Figure 4: Two curves representing points in $\overline{\mathcal{M}}_{0,5}(1,1,\varepsilon^3)/S_3$

ponent of both of these curves has two special points of weight 1, and therefore satisfy our stability condition. †

## 3.3 The space of relative stable maps to $\mathbb{P}^1$

Given $d \in \mathbb{N}$ and partitions, $\mathbf{x}_0 := (x_1, \ldots, x_n)$ and $\mathbf{x}_\infty := (y_1, \ldots, y_m)$, of $d$ the moduli space of relative stable maps

$$\overline{\mathcal{M}}_0^{\sim}\left(\mathbb{P}^1; \mathbf{x}_0[0], \mathbf{x}_\infty[\infty]\right) \tag{12}$$

parametrizes maps $f : \mathcal{C} \to \mathcal{X}$ where

(a) $\mathcal{C}$ is a rational nodal curve.

(b) $\mathcal{X} = X_1 \cup X_2 \cup \cdots \cup X_n$ is a chain of $\mathbb{P}^1$s.

(c) $f$ is a degree $d$ cover.

(d) $\widetilde{0} \in X_1$ and $\widetilde{\infty} \in X_n$ have ramification profiles $\mathbf{x}_0$ and $\mathbf{x}_\infty$, respectively.

(e) For every node $n$ in $\mathcal{X}$, $f^{-1}(n)$ consists of nodes in $\mathcal{C}$.

(f) Consider the *normalization morphism* $\widetilde{f} : \widetilde{\mathcal{C}} \to \coprod_{i=1}^n X_i$. Fix a nodal point $n \in \mathcal{X}$. Then there exits $X_j$ and $X_{j+1}$ such that $n$ corresponds to $n_1 \in X_j$

and $n_2 \in X_{j+1}$. Fix $p \in f^{-1}(n)$. The point $p \in \mathcal{C}$ then corresponds to $p_1, p_2 \in \widetilde{\mathcal{C}}$ such that $p_1 \in \widetilde{f}^{-1}(n_1)$ and $p_2 \in \widetilde{f}^{-1}(n_2)$. Then we have that the ramification of $p_1$ over $n_1$ equals the ramification of $p_2$ over $n_2$ via the map $\widetilde{f}$. (This is called the *kissing condition*.)

(g) For every $i \in \{1, \ldots, n\}$, there is at least one branch point on the smooth locus of $X_i$. (This is our stability condition.)

(h) The two maps $(C, \mathbf{x}_0, \mathbf{x}_\infty) \to \mathcal{X}$ and $(C', \mathbf{x}'_0, \mathbf{x}'_\infty) \to \mathcal{X}$ are equivalent if there exists an isomorphism of nodal curves $\sim$ and a map $\phi \in \mathrm{Aut}(\mathcal{X})$ which preserves the marked points $\widetilde{0}$ and $\widetilde{\infty}$, such that diagram 13 commutes ([GV03]).

$$\begin{array}{ccc} (C, \mathbf{x}_0, \mathbf{x}_\infty) & \xrightarrow{\sim} & (C', \mathbf{x}'_0, \mathbf{x}'_\infty) \\ \downarrow & & \downarrow \\ \mathcal{X} & \xrightarrow{\phi} & \mathcal{X} \end{array} \tag{13}$$

**Example 3.3.** Consider $\overline{\mathcal{M}}_0^{\sim}(\mathbb{P}^1; (10, 2), (5, 4, 3))$ which parametrizes stable maps of degree 12 to $\mathbb{P}^1$. Figure 5 shows a point $[f : \mathcal{C} \to \mathcal{X}]$ in $\overline{\mathcal{M}}_0^{\sim}(\mathbb{P}^1; (10, 2), (5, 4, 3))$. The ramification profiles over $P_1$ and $P_2$ are given by $(2, 7, 3)$ and $(2, 3, 4, 3)$,

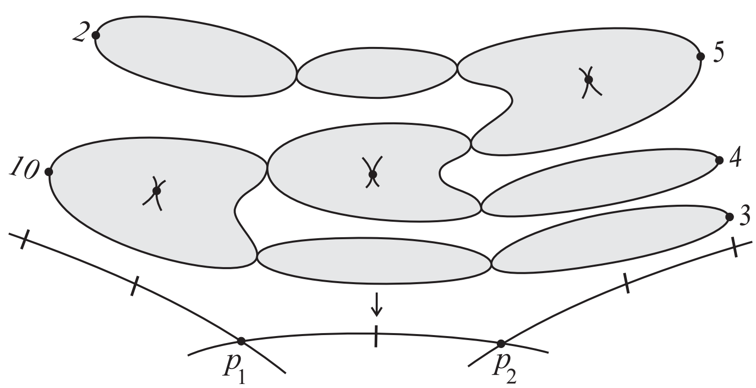



Figure 5: $[f : \mathcal{C} \to \mathcal{X}]$

respectively. The ramification locus of these two points correspond to nodes in $\mathcal{C}$. †

For our computations we will be using the space

$$\overline{\mathcal{M}}_0^{\sim}(\mathbb{P}^1;\mathbf{x}) := \overline{\mathcal{M}}_0^{\sim}(\mathbb{P}^1;(x_1,x_2),(x_3,x_4,x_5)) \tag{14}$$

where we treat the ramification indices as variables.

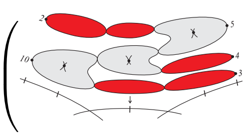
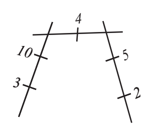
### 3.4 Stabilization morphism

The stabilization morphism

$$\text{stab} : \overline{\mathcal{M}}_0^{\sim}(\mathbb{P}^1;(x_1,x_2),(x_3,x_4,x_5)) \to \overline{\mathcal{M}}_{0,5} \tag{15}$$

takes the map $[f : \mathcal{C} \to \mathcal{X}]$ to the stabilization of the source curve $\mathcal{C}$ in $\overline{\mathcal{M}}_{0,5}$. The marked points on $\text{stab}([f : \mathcal{C} \to \mathcal{X}])$ correspond to the points with special ramification (over $\widetilde{0}$ and $\widetilde{\infty}$) in $\mathcal{C}$, and are thus labeled $x_1, x_2, x_3, x_4$, and $x_5$. (a more general description of this morphism can be found in [GV03]):

**Example 3.4.** Figure 6 shows again the same point in $\overline{\mathcal{M}}_0^{\sim}(\mathbb{P}^1;(10,2),(5,4,3))$ from example 12, but with the components of $\mathcal{C}$ which are unstable in $\overline{\mathcal{M}}_{0,5}$ colored red. The stabilization morphism contracts these components, and we get that

stab (  ) =  $\in \overline{\mathcal{M}}_{0,5}$ (16)



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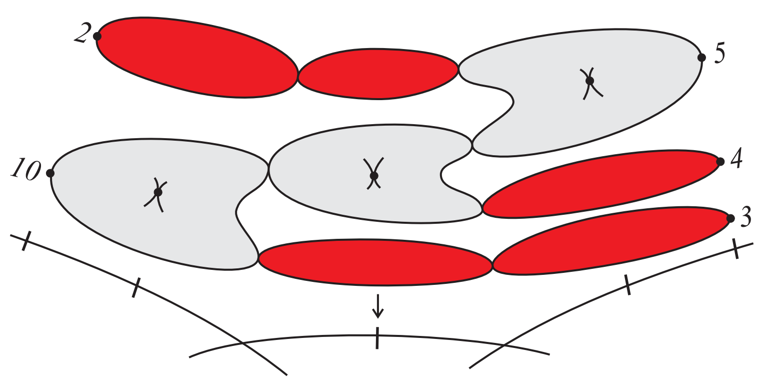




Figure 6: $[f : \mathcal{C} \to \mathcal{X}]$

### 3.5 Branch Morphism

The branch morphism

$$\mathrm{br} : \overline{\mathcal{M}}_0^{\sim}(\mathbb{P}^1; (x_1, x_2), (x_3, x_4, x_5)) \to \overline{\mathcal{M}}_{0,5}(1, 1, \varepsilon^3)/S_3 \tag{17}$$

takes the the target curve $\mathcal{X}$ of the map $[f : \mathcal{C} \to \mathcal{X}]$ to $\bar{\mathcal{X}} \in \overline{\mathcal{M}}_{0,5}(1, 1, \varepsilon^3)/S_3$, such that

(a) $\bar{\mathcal{X}}$ is a chain of projective lines isomorphic to $\mathcal{X}$.

(b) The points $\widetilde{0}$ and $\widetilde{\infty}$ in $\mathcal{X}$, which have special ramification, are sent to 0 and $\infty$, respectively, on the extremal rays of $\bar{\mathcal{X}}$.

(c) The branch points in $\mathcal{X}$ with simple ramification profiles are sent to the "light" points in $\overline{\mathcal{M}}_{0,5}(1, 1, \varepsilon^3)/S_3$.

where the points $\widetilde{0}$ and $\widetilde{\infty}$ in $\mathcal{X}$ correspond to the marked points 0 and $\infty$ (respectively) in $\overline{\mathcal{M}}_{0,5}(1, 1, \varepsilon^3)/S_3$ which are given weight 1, and all other marked points in $\mathcal{X}$ correspond to the 3 other "light" points.

In order to compute double Hurwitz numbers geometrically we will use the fact that the double Hurwitz number $H(\mathbf{x})$ is the degree of the branch morphism (in the same manner as Cavalieri and Marcus in [CM13]).

### 3.6 Psi classes

In order to compute non-transverse intersections on our moduli spaces, we introduce psi classes. We can define psi classes using euler classes in the following way (a more comprehensive introduction to psi classes can be found in [Koc01]):

Consider the *forgetful morphism*

$$\pi_{n+1} : \overline{\mathcal{M}}_{0,n+1} \to \overline{\mathcal{M}}_{0,n} \tag{18}$$

which forgets the marked point $p_{n+1}$ and stabilizes. This gives us a universal family over $\overline{\mathcal{M}}_{0,n}$ and the relative dualizing sheaf $\omega_\pi$. Additionally, we have the $n$ canonical sections

$$\sigma_i : \overline{\mathcal{M}}_{0,n} \to \overline{\mathcal{M}}_{0,n+1} \tag{19}$$

which are represented in red in figure 7 for the case where $n = 5$.

Each of these canonical sections comes with a corresponding cotangent line bundle $\mathbb{L}_i := \sigma_i^* \omega_\pi$. $\mathbb{L}_i$ is dual to the tangent line bundle $\mathbb{T}_i$ which is isomorphic to the normal bundle supported on each section. This is because the tangent space at each point in a section is determined by a normal vector (represented

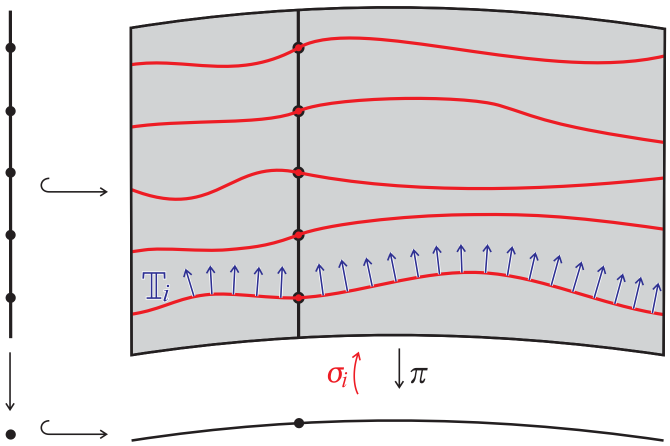



Figure 7: A family of 5-pointed curves over a 1-dimensional base

by the blue vectors on the "bottom" section in figure 7). We define the $i^{\text{th}}$ psi class to be the Euler class of $\mathbb{L}_i$. That is,

$$\psi_i := e\left(\mathbb{L}_i\right) \tag{20}$$

**Remark 3.3.** *Because $\mathbb{L}_i$ is dual to the tangent line bundle $\mathbb{T}_i$,*

$$e\left(\mathbb{T}_i\right) = -e\left(\mathbb{L}_i\right) = -\psi_i$$

#### 3.6.1 Computing Psi Classes and Intersections on $\overline{\mathcal{M}}_{0,5}$

Let $D_{i,n+1}$ be the divisor in $\overline{\mathcal{M}}_{0,n+1}$ which represents a curve where the marked points $p_i$ and $p_{n+1}$ lie together on an irreducible component, and all other marked points lie on the other irreducible component, as is shown in figure 8.

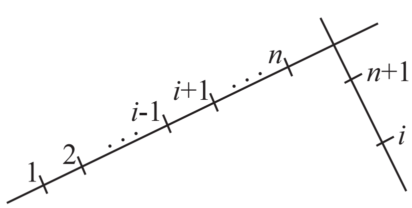



Figure 8: $D_{i,n+1}$

**Lemma 3.4.** *Consider the forgetful morphism $\pi_{n+1}$. We have the following* *identity*

$$\psi_i = \pi_{n+1}^* \psi_i + D_{i,n+1} \tag{21}$$

*relating psi classes in* $\overline{\mathcal{M}}_{0,n}$ *(on the right side of the equation) with psi classes in* $\overline{\mathcal{M}}_{0,n+1}$ *(on the left side of the equation).*

*Proof.* Found as proof of lemma 1.3.1 in [Koc01].

■

Using Lemma 3.4 we can now compute psi classes in terms of boundary divisors in $\overline{\mathcal{M}}_{0,n}$. Let $\Delta_{(n,\{1\},\{2,3\})}$ denote the sum of all divisors in $\overline{\mathcal{M}}_{0,n}$ representing curves where $p_1$ lies on one irreducible component of the curve, and $p_2, p_3$ lie on the other irreducible component. Our claim is that

$$\psi_1 = \Delta_{(n,\{1\},\{2,3\})} \tag{22}$$

*Proof of equation 22.* First, we note that since $\overline{\mathcal{M}}_{0,3}$ is just a point

$$\Delta_{(3,\{1\},\{2,3\})} = 0$$

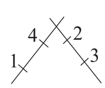
and our claim holds trivially. Now, assume that $\psi_1 = \Delta_{(n-1,\{1\},\{2,3\})}$ in $\overline{\mathcal{M}}_{0,n-1}$. Then, by induction and lemma 3.4, we have that

$$\begin{aligned} \psi_1 &= \pi_n^* \psi_1 + D_{1,n} \\ &= \pi_n^* \Delta_{(n-1,\{1\},\{2,3\})} + D_{1,n} \\ &= \Delta_{(n,\{1\},\{2,3\})} \end{aligned}$$

as required.

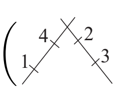
■

**Example 3.5.** Consider the case where $n = 5$. Then we have that

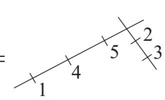
$\Delta_{(4,\{1\},\{2,3\})} =$  (23)



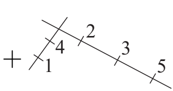
and that

$\pi_5^*\Big($ 



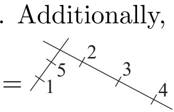
$\Big) =$ 



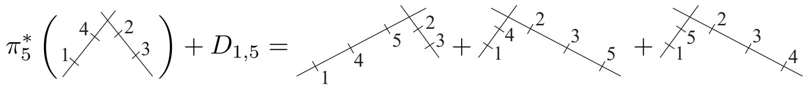
$+$ 



where there are two irreducible components on which the $5^{\text{th}}$ marked point could have lied before being "forgotten". Additionally, we note that

$D_{1,5} =$ 

Summing, as in the proof of equation 22, gives

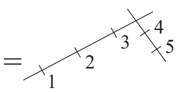


$$= \Delta_{(5,\{1\},\{2,3\})}$$

as expected.

†

Using psi classes, we can now compute non-transverse intersections on $\overline{\mathcal{M}}_{05,}$. We begin by an example of a self-intersection.

**Example 3.6.** Consider the divisor $D =$  $\in \overline{\mathcal{M}}_{0,5}$. We wish to compute the intersection $D \cap D$. A set-theoretic self-intersection would give $D$, which is of the wrong co-dimension (that is, of co-dimension 1). In order to compute this non-transverse intersection, we must use psi classes.

First, we note that

$$D \cong \overline{\mathcal{M}}_{0,4} \times \overline{\mathcal{M}}_{0,3}$$

where we consider the marked points on $\overline{\mathcal{M}}_{0,4}$ to be $\{1, 2, 3, \bullet\}$; and the marked points on $\overline{\mathcal{M}}_{0,3}$ to be $\{4, 5, \star\}$. There are two natural projections $\pi_\bullet$ and $\pi_\star$ of $D$ onto its factors

$$\begin{array}{ccc} & D & \\ & \wr\Vert & \\ & \overline{\mathcal{M}}_{0,4} \times \overline{\mathcal{M}}_{0,3} & \\ \swarrow^{\pi_\bullet} & & \searrow^{\pi_\star} \\ \overline{\mathcal{M}}_{0,4} & & \overline{\mathcal{M}}_{0,3} \end{array}$$

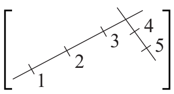
Let $N$ be the normal bundle of $D$ in $\overline{\mathcal{M}}_{0,5}$. That is

$$N := N_{D/\overline{\mathcal{M}}_{0,5}}\big|_D$$

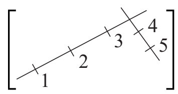
By proposition 3.31 in [HM98]

$$N = \mathbb{T}_\bullet \otimes \mathbb{T}_\star$$

Then we have that




$\cap$



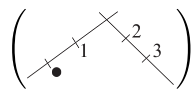

$$= e\left(N\right)$$

$$= +e\left(\pi_{\bullet}^{*}\left(\mathbb{T}_{\bullet}\right)\right) + e\left(\pi_{\star}^{*}\left(\mathbb{T}_{\star}\right)\right)$$

$$= -e\left(\pi_{\bullet}^{*}\left(\mathbb{L}_{\bullet}\right)\right) - e\left(\pi_{\star}^{*}\left(\mathbb{L}_{\star}\right)\right)$$

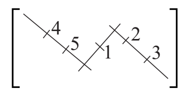
$$= -\pi_{\bullet}^{*}\left(\psi_{\bullet}\right) - \pi_{\star}^{*}\left(\psi_{\star}\right)$$

$= -\pi_{\bullet}^{*}$




$- 0$

$= -$




$$= -\left\{pt.\right\} \in \overline{\mathcal{M}}_{0,5}$$

which is co-dimension 2 as required.

†

In general, if $D$ is an irreducible divisor in $\overline{\mathcal{M}}_{0,5}$, and $\delta$ is a boundary stratum such that $\delta \subseteq D$ then we get the following intersection formula

$$\delta \cap D := e\left(N_{D/\overline{\mathcal{M}}_{0,5}}\big|_{\delta}\right) \tag{24}$$

**Remark 3.5.** $\left(\psi_i\right)^2 = \{pt.\}$ *in* $\overline{\mathcal{M}}_{0,5}$.

### 3.6.2 Psi classes on our other spaces

We also wish to compute non-transverse intersections on the other moduli spaces which appear in the central diagram. To this end we introduce the psi classes $\widehat{\psi}$ and $\widetilde{\psi}$.

There are cotangent line bundle classes at the fully weighted points 0 and $\infty$ on the Losev-Manin space $\overline{\mathcal{M}}_{0,5}(1,1,\varepsilon^3)$, which we denote $\widehat{\psi}_0$ and $\widehat{\psi}_\infty$, respectively. Additionally we get the following relations between psi classes:

$$c^*\left(\widehat{\psi}_0\right) = \psi_0 \tag{25}$$

and

$$c^*\left(\widehat{\psi}_\infty\right) = \psi_\infty$$

where $c^*$ is the pullback via the contraction morphism from equation 9. From this and remark 3.5 we get that

$$\widehat{\psi}_0^2 = \{pt.\} \in \overline{\mathcal{M}}_{0,5}(1,1,\varepsilon^3) \tag{26}$$

The pre-images of $\widetilde{0}$ and $\widetilde{\infty}$ on the target curve in $\overline{\mathcal{M}}_0^{\sim}\left(\mathbb{P}^1;(x_1,x_2),(x_3,x_4,x_5)\right)$, which correspond to each of the $x_i$'s, also have cotangent line bundle classes which we denote $\widetilde{\psi}_i$. Fix a point corresponding to some $x_i$ in $f^{-1}\left(\widetilde{0}\right)$. We get the following correspondence via the branch morphism

$$\mathrm{br}^*\left(\widehat{\psi}_0\right) = x_i\widetilde{\psi}_i \tag{27}$$

(We get a similar correspondence between $\widehat{\psi}_\infty$ and the psi-classes of points over $\widetilde{\infty}$.) Furthermore, via the stabilization morphism, we get that

$$\mathrm{stab}_*\left(\widetilde{\psi}_i\right) = \psi_i \tag{28}$$

# 4 The Central Diagram

Diagram 29 is a special case of the central diagram appearing as equation 12 in [CM13]. We use it to give $H(\mathbf{x})$ as an intersection number on $\overline{\mathcal{M}}_{0,5}$ as equation 30.

$$\begin{array}{ccc} \overline{\mathcal{M}}_0^{\sim}\left(\mathbb{P}^1;\mathbf{x}\right) & \xrightarrow{\text{stab}} & \overline{\mathcal{M}}_{0,5} \\ \Big\downarrow{\scriptstyle \mathrm{br}} & & \\ \overline{\mathcal{M}}_{0,5}(1,1,\varepsilon^3)/S_3 & & \end{array} \tag{29}$$

From the degree of the branch morphism and equation 26 we get that

$$\frac{1}{3!}H(\mathbf{x}) = \mathrm{br}^*\left(\widehat{\psi}_0^2\right)$$

where the factor of $\frac{1}{3!}$ comes from working with the Losev-Manin space modulo the symmetric group. Then, using equations 27 and 25, we get the following:

$$\begin{aligned} H(\mathbf{x}) &= 6\left(\mathrm{br}^*\left(\widehat{\psi}_0^2\right)\right) \\ &= 6\left(\widetilde{\psi}_1 x_1\right)^2 \\ &= 6\ \mathrm{stab}_*\left(\psi_1 x_1 + \sum D_i\right)^2 \end{aligned}$$

where $\sum D_i$ is the sum of divisors of $\overline{\mathcal{M}}_0^{\sim}\left(\mathbb{P}^1;\mathbf{x}\right)$ which parametrize maps where the point with ramification $x_i$ is supported on a trivial component of the source curve. Such divisors appear as corrections at walls from the pull-back of $\psi_1$ via the branch morphism.

Since we wish to compute $H(\mathbf{x})$ as an intersection on $\overline{\mathcal{M}}_{0,5}$, we push forward via the stabilization morphism to get

$$H(\mathbf{x}) = 6 \text{ stab}_* \left(x_1^2 \text{stab}^* \psi_1^2 + 2x_1 \text{stab}^* \psi_1 \left(\sum D_i\right) + \sum \left(D_i{}^2\right)\right) \tag{30}$$

# 5 Computations

In this section we compute the $2,3$ double Hurwitz number geometrically, using equation 30, in each chamber. We cross the following walls in their given order:

1. $W_1 : x_2 + x_3 = 0$
2. $W_2 : x_2 + x_4 = 0$
3. $W_3 : x_2 + x_5 = 0$
4. $W_4 : x_2 + x_3 + x_4 = 0$
5. $W_5 : x_2 + x_3 + x_5 = 0$
6. $W_6 : x_2 + x_4 + x_5 = 0$

We refer to the totally negative chamber where $x_1 > |x_i| + |x_j|$, for every $i, j \in \{2, 3, 4, 5\}$, as $\mathfrak{C}_{x_1}$; and where $x_2 > |x_i| + |x_j|$ for every $i, j \in \{1, 3, 4, 5\}$ as $\mathfrak{C}_{x_2}$. Additionally, we refer to to the chamber reached after crossing the $i^{\text{th}}$ wall as $\mathfrak{c}_{i+1}$. We continue with the notation introduced in 2.1, referring to the Hurwitz polynomial in the $i^{\text{th}}$ chamber as $P_i(\mathbf{x})$.

## 5.1 $\mathfrak{C}_{x_1}$

In this totally negative chamber, the size of $x_2$ and our stability condition prevent there from being any boundary strata. Therefore,

$$\begin{aligned} P_{x_1}(\mathbf{x}) &= 6 \text{ stab}_* \left(x_1^2 \text{stab}^* \psi_1^2 + 2x_1 \text{stab}^* \psi_1 \left(\sum D_i\right) + \sum \left(D_i{}^2\right)\right) \\ &= 6 \text{ stab}_* \left(x_1^2 \text{stab}^* \psi_1^2\right) \\ &= 6x_1^2 \end{aligned}$$

## 5.2 Crossing $W_1$

In this chamber, there is one new boundary divisor $D_1$ which is shown in figure 9. This gives us the following formula for the Hurwitz polynomial in $\mathfrak{c}_1$

$$P_2(\mathbf{x}) = 6\ \mathrm{stab}_* \left(x_1^2 \mathrm{stab}^* \psi_1^2 + 2x_1 \mathrm{stab}^* \psi_1 \left(D_1\right) + \left({D_1}^2\right)\right) \tag{31}$$

In order to compute the second term of $P_2(\mathbf{x})$, we must first consider $\mathrm{stab}_*(D_1)$

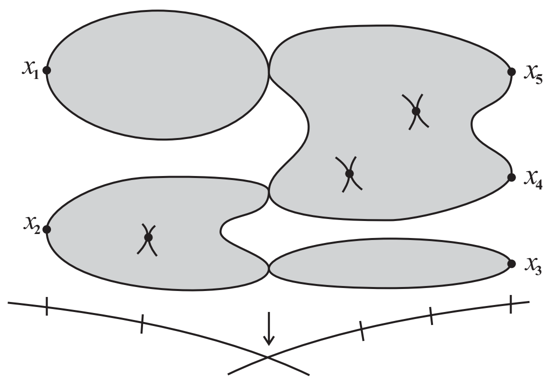


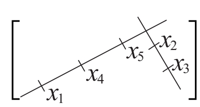

Figure 9: $D_1$

which will be a multiple of a divisor in $\overline{\mathcal{M}}_{0,5}$. This factor is given by automorphism-weighted “gluing factors” on the irreducible components of $D_1$. Consider the following as a statements about coarse moduli spaces.

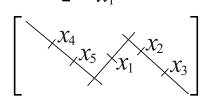
$$\begin{aligned} D_1 &\cong \mathcal{M}_0(-x_3; x_3) \times \mathcal{M}_0(x_2; -(x_2+x_3), x_3) \\ &\quad \times \mathcal{M}_0(x_1, (x_2+x_3); x_4, x_5) \times \mathcal{M}_0(x_1; -x_1) \\ &\cong \mathcal{M}_0(x_2; -(x_2+x_3), x_3) \times \mathcal{M}_0(x_1, (x_2+x_3); x_4, x_5) \end{aligned}$$

However, since we are interested in pushing forward this stratum as a fine moduli space, we must take into consideration gluing factors and automorphisms of $D_1$. In this case, the only gluing factor which is not canceled by automorphisms is $x_2 + x_3$. Using this and the intersection formula from section 3.6.1, we can compute the second term of $P_2(\mathbf{x})$ as

$$\begin{aligned} \mathrm{stab}_* \left(2x_1 \left(\mathrm{stab}^* \widetilde{\psi}_1\right) D_1\right) &= 2x_1 \psi_1 \cdot (\mathrm{stab}_* D_1) \\ &= 2x_1 \left(x_2 + x_3\right) \psi_1 \cdot \left[\ \cdots\ \right] \end{aligned}$$




$$= 2x_1 \left(x_2 + x_3\right) \left[\ \cdots\ \right]$$




$$\begin{aligned} &= 2x_1 \left(x_2 + x_3\right) \{\mathrm{pt.}\} \\ &= 2x_1 x_2 + 2x_1 x_3 \end{aligned}$$

For the third term of $P_2(\mathbf{x})$, we must compute the non-transverse intersection $D_1^2$. To this end, consider the map

$$\begin{array}{c}\overline{\mathcal{M}}_0(x_2; -(x_2+x_3), x_3) \times \overline{\mathcal{M}}_0(x_1, (x_2+x_3); x_4, x_5) \\ \Big\downarrow \text{br}\times\text{br} \\ \overline{\mathcal{M}}_0(1, 1_\bullet, \varepsilon) \times \left(\overline{\mathcal{M}}_0(1, 1_\star, \varepsilon^2)/S_2\right)\end{array} \tag{32}$$

which allows us to compute $D_1^2$ in terms of the psi classes over $1_\bullet$ and $1_\star$. Using formula 24, we get

$$\begin{aligned}
D_1^2 &= D_1 \cdot e\left(N_{D/\mathcal{M}}\right) \\
&= D_1 \cdot \left(e\left(\mathbb{T}_\bullet\right) + e\left(\mathbb{T}_\star\right)\right) \\
&= D_1 \cdot \left(-\widehat{\psi}_\bullet - \widehat{\psi}_\star\right) \\
&= -D_1 \cdot \widehat{\psi}_\bullet - D_1 \cdot \widehat{\psi}_\star \\
&= 0 - D_1 \cdot \widehat{\psi}_\star \\
&= -\frac{1}{2!} 2x_1 \left(\overline{\mathcal{M}}_0(x_1, (x_2+x_3); x_4, x_5) \times \{\text{pt.}\}\right)
\end{aligned}$$

which gives

$$\text{stab}_* \left(x_1 \left(\overline{\mathcal{M}}_0(x_1, (x_2+x_3); x_4, x_5) \times \{\text{pt.}\}\right)\right) = -x_1x_2 - x_1x_3 \tag{33}$$

We can now sum terms to compute the Hurwitz polynomial in the second chamber:

$$\begin{aligned}
P_2(\mathbf{x}) &= 6 \ \text{stab}_* \left(x_1^2 \text{stab}^* \psi_1^2 + 2x_1 \text{stab}^* \psi_1 \left(D_1\right) + \left({D_1}^2\right)\right) \\
&= 6\left(x_1^2 + (2x_1x_2 + 2x_1x_3) - (x_1x_2 + x_1x_3)\right) \\
&= 6\left(x_1^2 + x_1x_2 + x_1x_3\right) \\
&= 6x_1\left(x_1 + x_2 + x_3\right)
\end{aligned}$$

This gives us a wall crossing formula of

$$\begin{aligned}
WC_1(\mathbf{x}) &= 6x_1\left(x_1 + x_2 + x_3\right) - 6x_1^2 \\
&= 6x_1\left(x_2 + x_3\right)
\end{aligned}$$

## 5.3 Crossing $W_2$

In order to compute $P_3(\mathbf{x})$ we introduce the following notation:

$D_{\text{new}}$ := the formal sum of all boundary divisors which appear in the new chamber

$D_{\text{old}}$ := the formal sum of all boundary divisors which have already appeared in former chambers

This notation and our wall-crossing formula allow us to rewrite equation 30 in this chamber as

$$\begin{aligned} P_3(\mathbf{x}) &= 6\ \mathrm{stab}_*\left(\psi_1 x_1 + D_{\mathrm{old}} + D_{\mathrm{new}})^2\right) \\ &= 6\ \mathrm{stab}_*\left((x_1 \mathrm{stab}^*\psi_1 + D_{\mathrm{old}})^2 + D_{\mathrm{new}}^2 + 2\left(D_{\mathrm{new}}\left(x_1 \mathrm{stab}^*\psi_1 + D_{\mathrm{old}}\right)\right)\right) \\ &= P_2(\mathbf{x}) + WC_2(\mathbf{x}) \\ &= P_2(\mathbf{x}) + 6\ \mathrm{stab}_*\left(D_{\mathrm{new}}^2 + 2\left(D_{\mathrm{new}}\left(x_1 \mathrm{stab}^*\psi_1 + D_{\mathrm{old}}\right)\right)\right) \end{aligned}$$

In $\mathfrak{c}_3$ there is only one new divisor, which we denote $D_2$. As a combinatorial object it is the same as $D_1$ (as shown in figure 9), with the points labeled $x_4$ and $x_3$ swapped. Thus, we can compute $D_2^2$ by an action of $(3\ 4) \in S_5$ on $D_1^2$ and get that

$$\mathrm{stab}_* D_2^2 = -x_1x_2 - x_1x_4 \tag{34}$$

Similarly, by the same action on $\mathrm{stab}_*\left(2x_1\left(\mathrm{stab}^*\psi_1\right)D_1\right)$ we get that

$$\mathrm{stab}_*\left(2x_1\left(\mathrm{stab}^*\psi_1\right)D_2\right) = 2x_1x_2 + 2x_1x_4 \tag{35}$$

Finally, using that fact that $D_1 \cdot D_2 = \varnothing$, we sum our terms to get

$$\begin{aligned} P_3(\mathbf{x}) &= P_2(\mathbf{x}) + WC_2(\mathbf{x}) \\ &= P_2(\mathbf{x}) + 6\ \mathrm{stab}_*\left(D_{\mathrm{new}}^2 + 2\left(D_{\mathrm{new}}\left(x_1 \mathrm{stab}^*\psi_1 + D_{\mathrm{old}}\right)\right)\right) \\ &= P_2(\mathbf{x}) + 6\left(-x_1x_2 - x_1x_4 + 2x_1x_2 + 2x_1x_4\right) \\ &= P_2(\mathbf{x}) + 6\left(x_1x_2 + x_1x_4\right) \\ &= 6\left(x_1^2 + x_1x_2 + x_1x_3 + x_1x_2 + x_1x_4\right) \\ &= 6x_1\left(x_1 + 2x_2 + x_3 + x_4\right) \end{aligned}$$

## 5.4 Crossing $W_3$

Again, we only have one new boundary divisor $D_3$. $D_3$ can be defined by the action of $(3\ 5) \in S_5$ on $D_1$. Additionally, $D_1 \cdot D_3 = \varnothing$ and $D_2 \cdot D_3 = \varnothing$, allowing us to compute the terms of $P_4(\mathbf{x})$ as we did the terms of $P_3(\mathbf{x})$, giving

$$\begin{aligned} P_4(\mathbf{x}) &= P_3(\mathbf{x}) + WC_3(\mathbf{x}) \\ &= P_3(\mathbf{x}) + 6\ \mathrm{stab}_*\left(D_{\mathrm{new}}^2 + 2\left(D_{\mathrm{new}}\left(x_1 \mathrm{stab}^*\psi_1 + D_{\mathrm{old}}\right)\right)\right) \\ &= P_3(\mathbf{x}) + 6(-x_1x_2 - x_1x_5 + 2x_1x_2 + 2x_1x_5) \\ &= P_3(\mathbf{x}) + 6(x_1x_2 + x_1x_5) \\ &= 6\left(x_1^2 + 3x_1x_2 + x_1x_3 + x_1x_4 + x_1x_5\right) \\ &= 6x_1(2x_2) \end{aligned}$$

### 5.5 Crossing $W_4$

In $\mathfrak{c}_5$ there are two new boundary divisors $D_4$ (shown as figure 10) and $D_5$ (shown as figure 11), both of new combinatorial types.

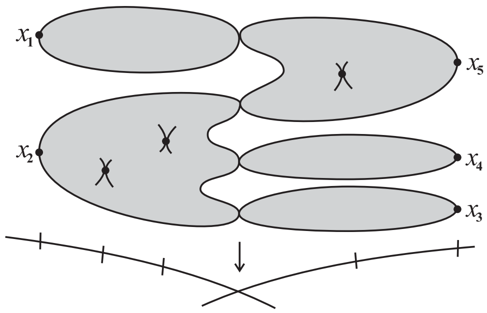


Figure 10: $D_4$

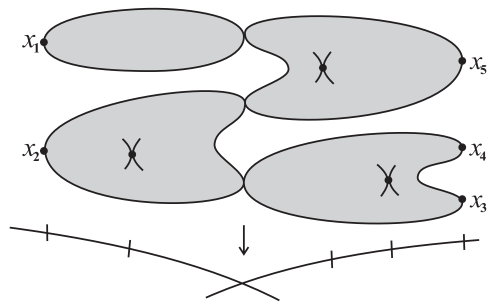


Figure 11: $D_5$

Again, we use our former notation to get that

$$\begin{aligned} P_4(\mathbf{x}) &= P_3(\mathbf{x}) + WC_4(\mathbf{x}) \\ &= P_3(\mathbf{x}) + 6 \text{ stab}_* \left(D_{\text{new}}^2 + 2\left(D_{\text{new}}\left(x_1 \text{stab}^* \psi_1 + D_{\text{old}}\right)\right)\right) \end{aligned}$$

To compute $D_{\text{new}}^2$, we first consider $D_4 \cdot D_5$, which is given as figure 12. This has only two non-trivial gluing factors, namely, $(x_1+x_2+x_5)$ and $(x_2+x_3+x_4)$. This gives

$$\begin{aligned} \text{stab}^*(D_4 \cdot D_5) &= (x_1 + x_2 + x_5)\,(x_2 + x_3 + x_4)\,\{pt.\} \\ &= x_2^2 + x_1x_2 + x_2x_3 + x_2x_4 + x_2x_5 + x_1x_3 + x_3x_5 + x_1x_4 + x_4x_5 \end{aligned}$$

Next, for $D_{\text{new}}^2$ we get

$$\begin{aligned} \text{stab}_* D_4^2 &= \text{stab}^* \left(-x_2 \left(\overline{\mathcal{M}}_0\left(x_2; (x_2+x_3+x_4), x_3, x_4\right)\right)\right) \times \{pt.\} \\ &= -x_2\,(x_2+x_3+x_4) \times \{pt.\} \\ &= -x_2^2 - x_2x_3 - x_2x_4 \end{aligned}$$

and

$$\begin{aligned} \text{stab}_* D_5^2 =& \text{stab}^*\big( - \big(\overline{\mathcal{M}}_0(x_2; (x_2+x_3+x_4), (-x3-x4)) \\ & \qquad \times \overline{\mathcal{M}}_0(x_1, (x_2+x_3+x_4); x_5) \\ & \qquad \times \overline{\mathcal{M}}_0((-x_3-x_4); x_3, x_4)\big)\big) \\ =& -(x_2+x_3+x_4)\,(-x_3-x_4)\,\{pt.\} \\ =& x_3^2 + x_4^2 + x_2x_3 + x_2x_4 + 2x_3x_4 \end{aligned}$$

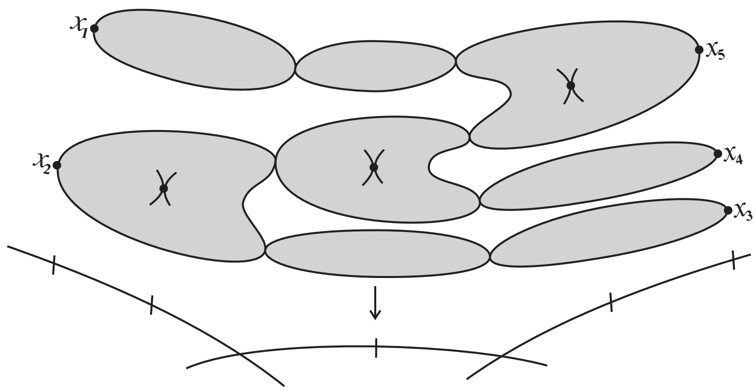



Figure 12: $D_4 \cdot D_5$

Next, because $\mathrm{stab}_*(D_4) \cdot \psi_1 = \varnothing$ and $\mathrm{codim}(\mathrm{stab}_*(d_5)) = 2$, we have that $\mathrm{stab}_*(2(D_{\mathrm{new}} \cdot x_1 \mathrm{stab}^* \psi_1)) = 0$.

Finally, we look at $\mathrm{stab}_*(2(D_{\mathrm{new}} \cdot D_{\mathrm{old}}))$. While $D_5$ does not intersect any old divisors, $D_4$ has two such non-trivial intersections, namely $D_1 \cdot D_4$ (figure 13) and $D_2 \cdot D_4$. First, we compute $\mathrm{stab}_*(D_1 \cdot D_4)$ by considering the only two non-

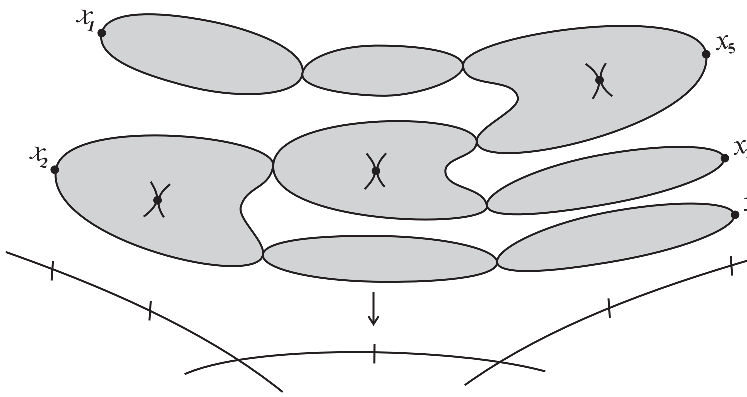



Figure 13: $D_1 \cdot D_4$

trivial gluing factors on $D_1 \cdot D_4$, giving

$$\begin{aligned}\mathrm{stab}_*(D_1 \cdot D_4) &= (x_2 + x_3)(x_2 + x_3 + x_4)\{pt.\} \\ &= x_2^2 + 2x_2x_3 + x_2x_4 + x_3^2 + x_3x_4\end{aligned}$$

And, since $D_2 \cdot D_4$ is combinatorially the same as $D_1 \cdot D_4$ in figure 13 up to an action of $(3\ 4) \in S_5$, we also get that

$$\mathrm{stab}_*(D_2 \cdot D_4) = x_2^2 + 2x_2x_4 + x_2x_3 + x_4^2 + x_3x_4$$

Therefore,

$$\mathrm{stab}_*(2(D_{\mathrm{new}} \cdot D_{\mathrm{old}})) = 4x_2^2 + 6x_2x_3 + 6x_2x_4 + 2x_3^2 + 4x_3x_4 + 2x_4^2$$

We now sum all of our terms to get

$$
\begin{aligned}
P_5(\mathbf{x}) =& P_4(\mathbf{x}) + WC_4(\mathbf{x}) \\
=& 6x_1(2x_2) \\
& \quad + 6(x_2^2 + x_1x_2 + x_2x_3 + x_2x_4 + x_2x_5 + x_1x_3 + x_3x_5 + x_1x_4 + x_4x_5 \\
& \qquad - x_2^2 - x_2x_3 - x_2x_4 \\
& \qquad + x_3^2 + x_4^2 + x_2x_3 + x_2x_4 + 2x_3x_4 \\
& \qquad + 4x_2^2 + 6x_2x_3 + 6x_2x_4 + 2x_3^2 + 4x_3x_4 + 2x_4^2) \\
=& 6x_1(2x_2) + 6x_2(x_2 + x_3 + x_4) \\
=& 6x_2(2x_1 + x_2 + x_3 + x_4)
\end{aligned}
$$

## 5.6 Crossing $W_5$

In the $6^{\text{th}}$ chamber there are two new boundary divisors, $D_6$ and $D_7$. $D_6$ is the same as $D_4$ (figure 10) and $D_7$ is of the same type as $D_5$ (figure 11), both up to an action of $(4\ 5) \in S_5$. Then, by our intersections on $D_4$ and $D_5$ as well as this symmetry, we get the following terms for $D_{\text{new}}^2$:

- $\text{stab}_* (D_6 \cdot D_7) = x_1x_3 + x_3x_4 + x_1x_5 + x_4x_5$
- $\text{stab}_* \left(D_6^2\right) = -x_2^2 - x_2x_3 - x_2x_5$
- $\text{stab}_* \left(D_7^2\right) = x_3^2 + x_5^2 + x_2x_3 + x_2x_5 + 2x_3x_5$

To compute $D_{\text{new}} \cdot D_{\text{old}}$, we again use the symmetry from the last chamber. As with $D_5$, $D_7$ does not intersect any old boundary divisors, while $D_6$ intersects both $D_1$ and $D_3$, which we can compute by an action of $(4\ 5) \in S_5$ on $\text{stab}_* (D_1 \cdot D_4)$ and $\text{stab}_* (D_2 \cdot D_4)$, respectively, giving the following:

- $\text{stab}_* (D_1 \cdot D_6) = x_2^2 + 2x_2x_3 + x_2x_5 + x_3^2 + x_3x_5$
- $\text{stab}_* (D_3 \cdot D_6) = x_2^2 + 2x_2x_5 + x_2x_3 + x_5^2 + x_3x_5$

So we get the following in $\mathfrak{c}_6$

$$\begin{aligned}
P_6(\mathbf{x}) =& P_5(\mathbf{x}) + WC_5(\mathbf{x}) \\
=& 6x_2(2x_1 + x_2 + x_3 + x_4) \\
&+ 6\ (-x_2^2 - x_2x_3 - x_2x_5 \\
&\quad + x_3^2 + x_5^2 + x_2x_3 + x_2x_5 + 2x_3x_5 \\
&\quad + x_1x_3 + x_3x_4 + x_1x_5 + x_4x_5 \\
&\quad + 4x_2^2 + 6x_2x_3 + 6x_2x_5 + 2x_3^2 + 4x_3x_5 + 2x_5^2) \\
=& 6x_2(2x_1 + x_2 + x_3 + x_4) + 6x_2(x_2 + x_3 + x_5) \\
=& 6x_2(x_1 + x_2 + x_3)
\end{aligned}$$

## 5.7 Crossing $W_6$

Finally, in the $\mathfrak{C}_{x_2}$ there are again two new boundary divisors, $D_8$ and $D_9$. As before, $D_8$ is the same as $D_4$ (figure 10) and $D_9$ is of the same type as $D_5$ (figure 11), both up to an action of $(3\ 5) \in S_5$. Again, by our intersections on $D_4$ and $D_5$ as well as this symmetry, we get the following terms for $D_{\text{new}}^2$:

- $\text{stab}_* (D_8 \cdot D_9) = x_1x_4 + x_3x_4 + x_1x_5 + x_3x_5$
- $\text{stab}_* \left(D_8^2\right) = -x_2^2 - x_2x_4 - x_2x_5$
- $\text{stab}_* \left(D_9^2\right) = x_4^2 + x_5^2 + x_2x_4 + x_2x_5 + 2x_4x_5$

We compute $D_{\text{new}} \cdot D_{\text{old}}$ as we did in the last chamber to get

- $\text{stab}_* (D_2 \cdot D_8) = x_2^2 + 2x_2x_5 + x_2x_4 + x_5^2 + x_4x_5$
- $\text{stab}_* (D_3 \cdot D_8) = x_2^2 + 2x_2x_4 + x_2x_5 + x_4^2 + x_4x_5$

...and we finally reach $\mathfrak{C}_{x_2}$ with

$$\begin{aligned}
P_{x_2}(\mathbf{x}) =& P_6(\mathbf{x}) + WC_6(\mathbf{x}) \\
=& 6x_2(x_1 + x_2 + x_3) \\
&+ 6\ (-x_2^2 - x_2x_4 - x_2x_5 \\
&\quad + x_4^2 + x_5^2 + x_2x_4 + x_2x_5 + 2x_4x_5 \\
&\quad + x_1x_4 + x_3x_4 + x_1x_5 + x_3x_5 \\
&\quad + 4x_2^2 + 6x_2x_4 + 6x_2x_5 + 2x_4^2 + 4x_4x_5 + 2x_5^2) \\
=& 6x_2(x_1 + x_2 + x_3) + 6x_2(x_2 + x_4 + x_5) \\
=& 6x_2(x_1 + 2x_2 + x_3 + x_4 + x_5) \\
=& 6x_2^2
\end{aligned}$$

as we expect from the symmetry with $\mathfrak{C}_{x_1}$.

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