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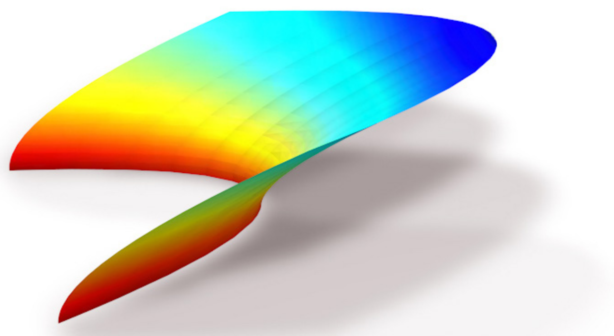


Imaginary Numbers Are Real

WORKBOOK

Hello!

"Imagine you had an art class in which they taught you how to paint a fence, but never showed you the great masters. Of course, you would say; 'I hate art.' You were bad at painting the fence but you wouldn't know what else there is to art. Unfortunately, that is exactly what happens with mathematics. What we study at school is a tiny little part of mathematics. I want people to discover the magic world of mathematics, almost like a parallel universe, that most of us aren't aware even exists."

- Edward Frenkel

1. You Rock!

Thanks for checking out the Welch Labs Imaginary Numbers Are Real Workbook. Here you'll find everything you need to get the most out of my Imaginary Numbers are Real YouTube series. For each video in the series, you'll find a workbook section complete with the text and key figures from the video, more in-depth features covering interesting areas, and most importantly, exercises.

2. About Them Exercises.

As the quote above says much better than I can, the exercises we're given in math class are all too often terrible. At some point you've likely been assigned the kinds of fence-painting exercises where you're asked to do the same type of problem over and over and over with minor tweaks to the numbers. I hate this stuff. It's uninspiring and a bad use of your brain. That said, I promise I have done my very best to avoid this type of problem here. Each exercise is here for a reason. No fence painting.

Of course, this is only half the equation. Like anything else, to get real value out, you must put real work in. And if you do, I promise it's worth it. Imaginary numbers are rich and beautiful, and their history is fascinating. Really understanding this stuff will give you tremendous perspective on the power and beauty of modern mathematics and science. The exercises for each section are divided into 4 parts:

Exercises	Description
Discussion	Designed to stimulate, you guessed it, discussion! No wrong or right answers.
Drill	Imaginary Numbers can be tough, and missing key concepts can really make this stuff way less fun - the Drills are designed to ensure your grasp of the key concepts is sound.
Critical Thinking	The Critical Thinking exercises are where things get good - these exercises are specifically designed to question your mathematical assumptions and grow your skills.
Challenge	These questions are like, hard.

3. About Them Solutions.

I'm a firm believer in quick feedback when learning new concepts. This is why you'll find answers to the exercises in the back of the book. I encourage you to check your work as you go. Of course, this doesn't mean you should flip to the back of the book as soon as the going gets tough - a little suffering and uncertainty is good.

Enjoy!
@stephencwelch

Table of Contents

"Hey, so what's like, in this book?"

Part	Title	Pages
1	Introduction	4-9
2	A Little History	10-17
3	Cardan's Problem	18-21
4	Bombelli's Solution	22-26
5	Numbers are Two Dimensional	27-31
6	The Complex Plane	32-37
7	Complex Multiplication	38-42
8	Math Wizardry	43-47
9	Closure	48-53
10	Complex Functions	54-60
11	Wandering In Four Dimensions	61-66
12	Riemann's Solution	67-73
13	Riemann Surfaces	74-80
	Solutions To Exercises	80-end

Imaginary Numbers Are Real

Part 1: Introduction

Let's take the function

$$f(x) = x^2 + 1. \tag{1}$$

If we graph our function, we obtain the friendly parabola of Figure 1. Now let's say we want to figure out where the equation equals zero - we want to find the roots. On our plot this should be where the function crosses the x-axis.

As we can see, our parabola actually never crosses the x-axis, so according to our plot, there are no solutions to the equation $x^2+1 = 0$.

But there's a small problem. A little over 200 years ago a smart guy named Gauss proved[1] that every polynomial equation of degree n has exactly n roots. Our polynomial has a highest power, or degree, of two, so we should have two roots. And Gauss' discovery is not just some random rule, today we call it the Fundamental Theorem of Algebra.

So our plot seems to disagree with something so important it's called the Fundamental Theorem of Algebra, which might be a problem. What Gauss is telling us here is that there are two perfectly good values of x that we could plug into our function, and get zero out. Where could these two missing roots be?

The short answer here is that we don't have enough numbers. We typically think of numbers existing on a one dimensional continuum - the number line. All our friends are here: zero, one, negative numbers, fractions, even irrational numbers like $\sqrt{2}$ show up.

But this system is incomplete. And our missing numbers are not just further left or right, they live in a whole new dimension. Algebraically, this new dimension has everything to do with a problem that was mathematically

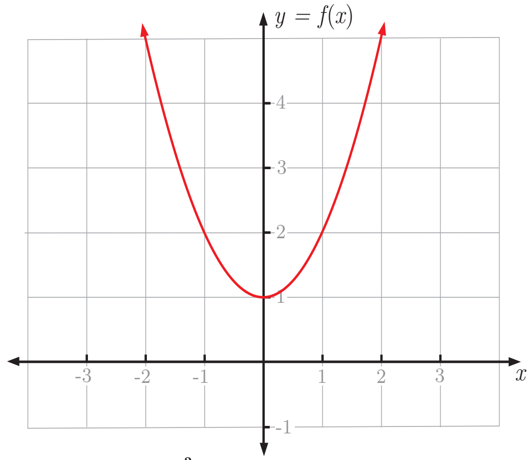


Figure 1 | Graph of $f(x) = x^2+1$.

considered impossible for over two thousand years: the square root of negative one.[2]

When we include this missing dimension in our analysis - our parabola ends up looking more like Figure 2, and needless to say, is a bit more interesting.

Now that our input numbers are in their full two dimensional form, we see how our function x^2+1 really behaves. And we can now see that our function does cross the x-axis![3] We were just looking in the wrong dimension.

So, why is this extra dimension that numbers possess not common knowledge? Part of this reason is that it has been given a terrible, terrible name. A name that suggest that these numbers aren't ever real! [4]

In fact, Gauss himself had something to say about this naming convention:

"That this subject [imaginary numbers] has hitherto been surrounded by mysterious obscurity, is to be attributed largely to an ill adapted notation. If, for example, +1, -1, and the square root of -1 had been called direct, inverse and lateral units, instead of positive, negative and imaginary (or even impossible), such an obscurity would have been out of the question."

- Carl Friedrich Gauss (1777-1855)

So yes, this missing dimension is comprised of numbers that have been given ridiculous name imaginary. Gauss proposed these numbers should instead be given the name

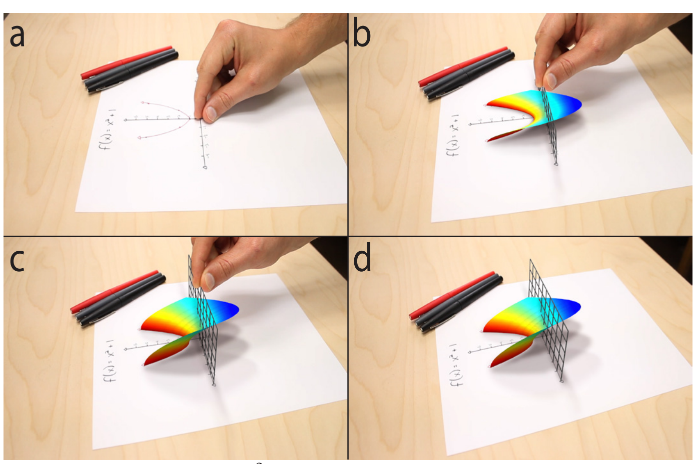


Figure 2 | Graph of $f(x) = x^2+1$ where x includes imaginary numbers. Panels a-d show "pulling" the function out of the page.

1 Mostly, the full proof took a little longer

2 More on this later

3 If you're paying attention, you should be thinking "what the heck Stephen, you said the graph would cross the axis twice (there would be two solutions), and the graph in figure 2 crosses like a million times!" Great point Greg! The reason for this is we've only plotted the real part of the graph to keep things simple (ish) – we'll cover the complete solution (which does have exactly 2 answers) in Part 13. Get excited.

4 ☺

lateral[1] - so from here on, let's let lateral mean imaginary.

To get a better handle on imaginary/lateral numbers and really understand what's going on in Figure 2, let's spend a little time thinking about numbers.

Early humans really only had use for the natural numbers (1, 2, 3...). This makes sense because of how numbers were used – as a tool for counting things. So to early humans, the number line would have just been a series of dots.

As civilizations advanced, people needed answers to more sophisticated math questions – like when to plant seeds, how to divide land, and how to keep track of financial transactions. The natural numbers just weren't cutting it anymore, so the Egyptians innovated and developed a new, high tech solution: fractions.

Fractions filled in the gaps in our number line, and were basically cutting edge technology for a couple thousand years.

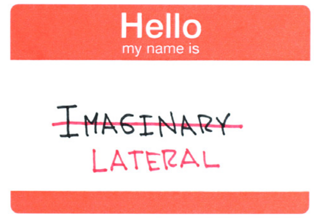


Figure 3 | From here on, lateral = imaginary. Gauss preferred the term lateral to imaginary, we'll see exactly why in Part 6.

The next big innovations to hit the number line were the number zero and negative numbers, but it took some time[2] to get everyone on board. Since it's not obvious[3] what these numbers mean or how they fit into the real world,

1 We'll explain *why* Gauss preferred the name imaginary in part six.

2 Like a lot of time. Like thousands of years time.

3 Remember 3rd grade?

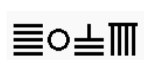
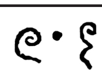
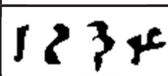
Civilization	Relevant Time	Example Numerals	Fractions	Zero As Placeholder	Zero	Negatives	Imaginary Numbers	Number Line
Prehistory	<3000 BC	| || |||	✗	✗	✗	✗	✗	1 2 3 ...
Ancient Egypt	1740BC		✓	✓	✗	✗	✗	1 2 3
Babylonia	300BC		✓	✓	✗	✗	✗	1 2 3
Olmec	700-400BC		✓	✓	✗	✗	✗	1 2 3
Greek	500BC-100AD	Σ Μ Α	✓	✗	✗	✗	✗	1 2 3
China	200BC-200AD		✓	✓	✗	✓	✗	-3 -2 -1 1 2 3
Roman	27 BC-476AD	II III VII	✗	✗	✗	✗	✗	1 2 3 ...
Cambodia	700AD		✓	✓	✓	✗	✗	1 2 3
India + Persia	600-1000AD		✓	✓	✓	✓	✗	-3 -2 -1 0 1 2 3
Medieval Europe	500-1400AD	II III VII	✗	✗	✗	✗	✗	1 2 3 ...
Renaissance Europe	1300-1700AD	1, 2, 3	✓	✓	✓	✓	✗	-3 -2 -1 0 1 2 3
Modern Era	>1700 AD	1, 2, 3	✓	✓	✓	✓	✓	i, -1, 1, $-i$

Table 1 | A brief overview of the history of numbers. It has taken quite some time for modern numbers to come to be. Only in the last couple hundred years do we see imaginary/lateral numbers really accepted. The dates here are approximate, and keep in mind that most of these civilizations didn't actually have number lines! The point here is visualize how numbers developed over time. Finally, notice the difference between zero and zero as a placeholder. In a positional number systems like ours, the location of a digit carries meaning. The three in 23 means 3 units, while the three in 32 means 3 "tens", or thirty. We run into a problem if we need to tell the difference between 30 and 300, and we don't have zero! This is the placeholder zero, it is not a concept alone, but a notational tool.

zero and negative numbers were met with skepticism, and largely avoided or ignored. Some cultures were more suspicious that others, depending largely on how people viewed the connection between mathematics and reality. A great example is here Greek civilization – despite making huge strides in geometry, the Greeks generally didn't accept negative numbers or zeros, after all how could nothing be something?

What's even wilder is that this is not all ancient history - just a few centuries ago, mathematicians would intentionally move terms around to avoid having negatives show up in equations. Suspicion of zero and negative numbers did eventually fade - partially because negatives are useful for expressing concepts like debt, but mostly because negatives just kept sneaking into mathematics.

It turns out there's just a whole lot of math you just can't do if you don't allow negative numbers to play. Without negatives, simple algebra problems like $x + 3 = 2$ have no answer. Before negatives were accepted, this problem would have no solution, just like we thought Equation 1 had no solution.

The thing is, it's not crazy or weird to think problems like this have no solutions – to solve $x + 3 = 2$, we subtract 3 from both sides, resulting in $x = 2 - 3$. In words, this algebra problem basically says: "if I have 2 things and I take away 3, how many things do I have left?"

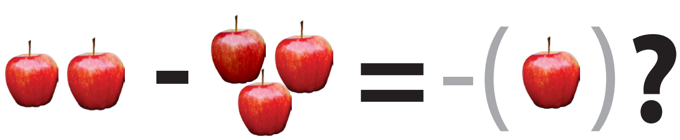


Figure 4 | Negative numbers don't always make sense. Two apples, take away three apples equals...the anti-apple?

It's not surprising that most of the people who have lived on our planet would be suspicious of questions like this. These problems don't make any sense. Even brilliant mathematicians of the 18th century, such as Leonard Euler, didn't really know what to do with negatives – he at one point wrote that negatives were greater than infinity.[1]

So it's fair to say that negative and imaginary numbers raise a lot of very good, very valid questions, such as:

- Why do we require students to understand and work with numbers that eluded the greatest mathematical minds for thousands of years?
- Why did we even come accept negative and imaginary numbers in the first place, when they don't really seem connected to anything in the real world?
- How do these extra numbers help explain the missing solutions to Equation 1?

Next time, we'll begin to address these questions by going way back to the discovery of complex numbers.

1 Sketchy.

Exercises 1

Discussion

1.1 Why do you think most people who have lived on our planet would have been suspicious of negative numbers?

1.2 Why do you think negative numbers have been so widely accepted today, despite being somewhat sketchy?

1.3 Do you think negative numbers should be taught to elementary school students? Why or why not?

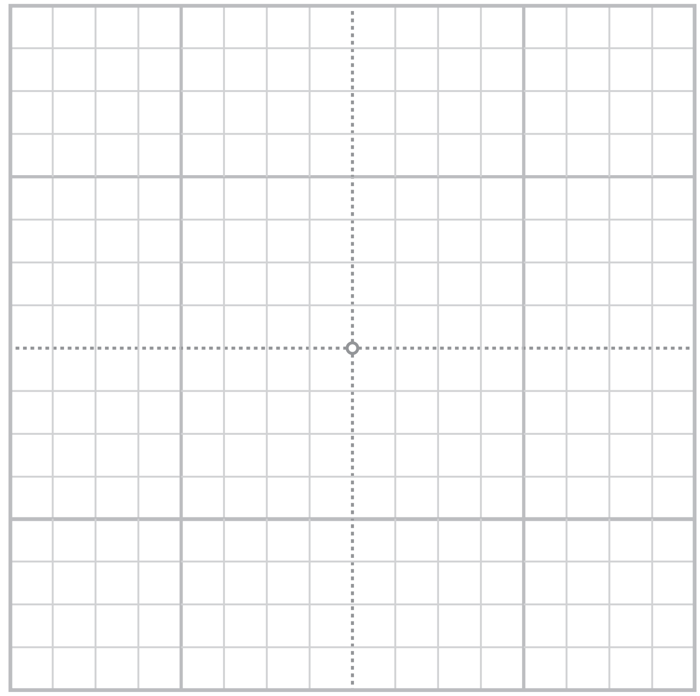
1.4 How would you explain negative numbers to a 5th grader?

1.5 Which historical civilizations do you think embraced mathematics? Which didn't?

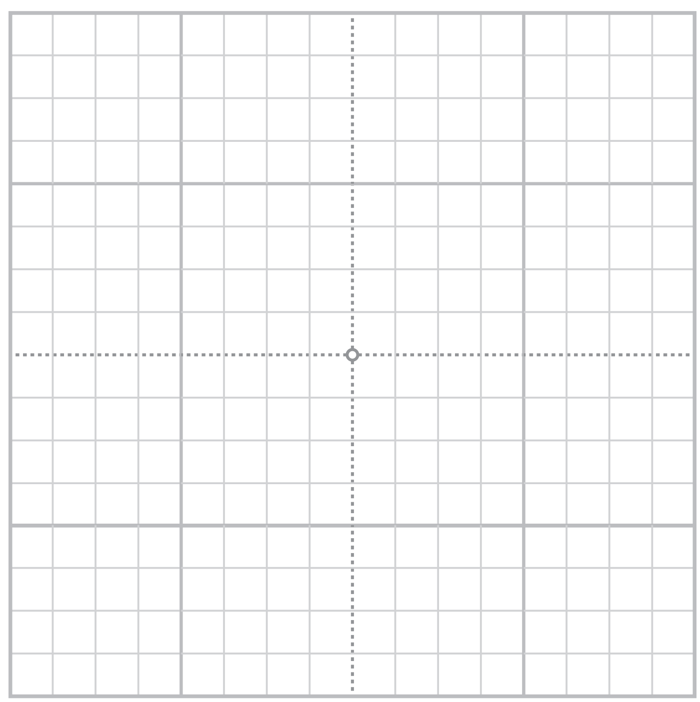
Drill

Plot each function.

1.6 $f(x) = x^2-4x+4$



1.7 $g(x) = -x^2+x+6$

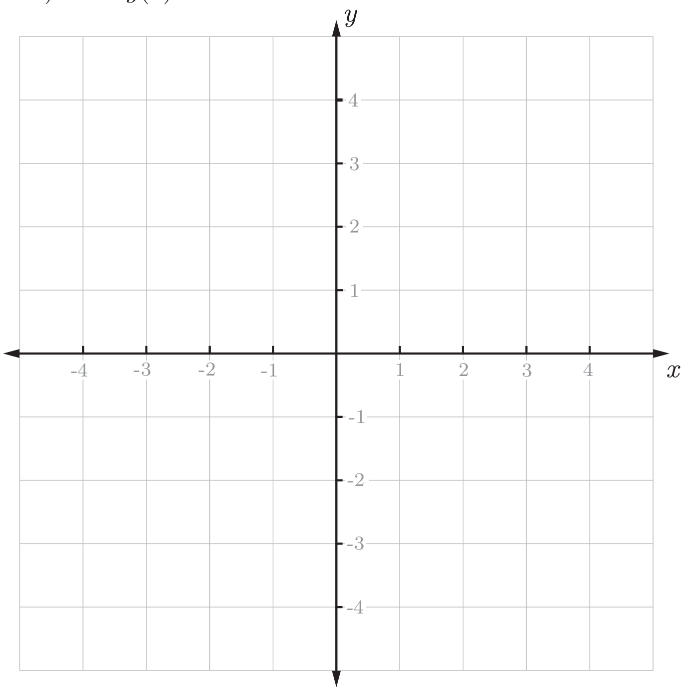


Critical Thinking

1.8 Consider the better-behaved sibling of Equation 1:

$$g(x) = x^2\text{-}1$$

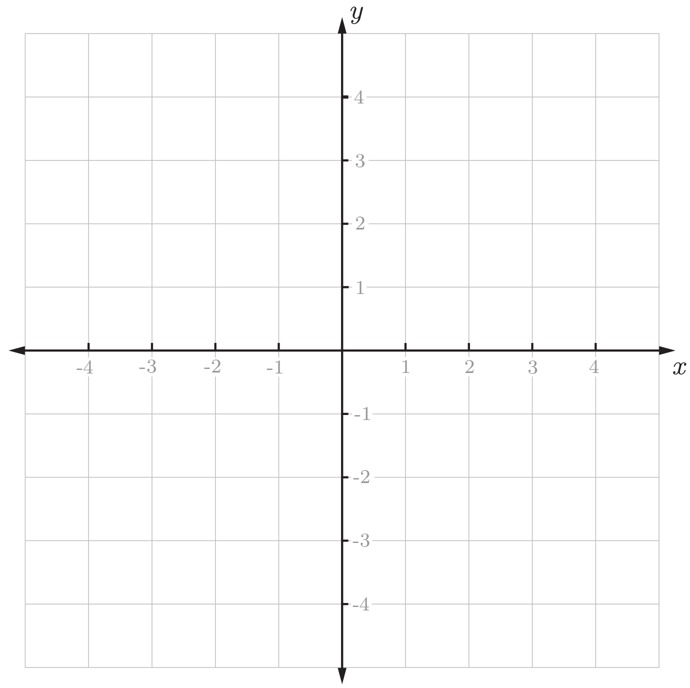
a) Plot $g(x)$.



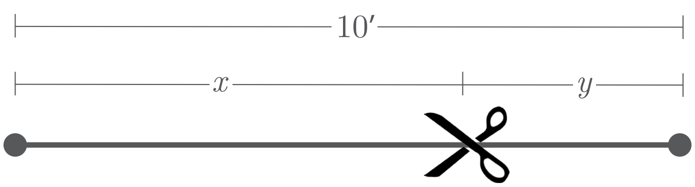
b) For what x-values does $g(x)$=0? $g(x)$=0 when x =________ or x =__________. These are called the **roots, zeros, or x-intercepts of *g*.**

c) The roots of g are much easier to find than the roots of f (Equation 1), why is this the case?

1.9 So far we've seen a function with 2 roots, g, and a function with no obvious roots, f (Equation 1). Find and plot a highest power of two (quadratic) function with one obvious root. Call it $h(x)$. Or else.



1.10 You cut x feet from a 10 foot rope, leaving y feet of remaining rope.



a) Write an equation relating x and y.

b) Solve for y when x=7', x=10', and x=13'.

c) One of your answers from part b should be negative. Is this result meaninfgul? What does a negative answer tell you about your remaining rope? Does the specific value of your answer matter, or is knowing your answer is negative enough to reach a conclusion about the remaining rope?

1.11 You have exactly 10 minutes to get to class for your big test on lateral numbers! Your friend Gus (who's weirdly in to this sort of thing) says that class is exactly 10,000 feet away. Let's say you travel to class at a rate of r feet/minute. You would like to get there a few minutes early. More specifically, let's say you will arrive t minutes early.

a) Write an equation that relates r and t.

b) Compute t for r=1250 feet/minute, r=1000 feet/minute, and r=625 feet/minute.

c) One of your answers from part b should be negative. Is this result meaningful? What does a negative answer tell you about when you will arrive to class?

Challenge

1.12 Consider the function:

$$p(x) = x^4\text{-}10x^3+35x^2\text{-}50x+24$$

a) According to the fundamental theorem of algebra, how many roots should $p(x)$ have?

b) Find all the roots of $p(x)$. You may use technology if you would like, but it's not necessary to complete the problem.

Imaginary Numbers Are Real

Part 2: A Little History

Last time, we left off wondering how imaginary/lateral numbers could help us find the roots of our equation x^2+1, and further, how imaginary and negative numbers even became a part of modern mathematics after being avoided and ignored for a couple of thousand years, because, let's be honest here – they don't really make that much sense.[1]

But something happened in Europe around five centuries ago that would no longer allow mathematicians to ignore these numbers. An Italian mathematician, Scipione del Ferro, was trying to solve a problem not that different than ours.

At some point you've likely seen the quadratic formula. This formula is super useful because gives us the roots[2] of any equation with a highest power of two[3]- all you have to do is plug in a, b, and c,and out pops the answer:

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}, \tag{2}$$

where:

$$0 = ax^2 + bx + c.$$

Del Ferro was trying to find a formula like this for equations with a highest power of three - cubics. The general case is pretty tough:

$$0 = ax^3 + bx^2 + cx + d,$$

so Del Ferro first considered the case where the x^2 term is missing[4], and the last term is negative:

$$0 = ax^3 + cx - d.$$

In the 16th century, negative terms were way to sketchy to write, so del Ferro wrote his cubic as

$$x^3 + cx = d, \tag{3}$$

and required c and d to be positive.[5]

Now that we have our equation set up, the game here is to get x by itself on one side, and all the constants[6] on the

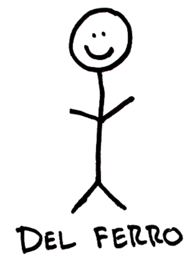


Figure 5 | Highly-paid professional artist rendering of Scipione del Ferro. 1465-1562.

other side. This is pretty easy in linear[7] equations, we can just add, multiply, divide, or subtract until we get x alone.

Quadratics are a bit harder – you may have learned to do this in school – it requires some cleverness, and factoring by completing the square.

Del Ferro was trying to do the same thing for his cubic equation, and through some very clever substitution, he eventually found a solution: [8]

$$x = \sqrt[3]{\frac{d}{2} + \sqrt{\frac{d^2}{4} + \frac{c^3}{27}}} + \sqrt[3]{\frac{d}{2} - \sqrt{\frac{d^2}{4} + \frac{c^3}{27}}}. \tag{4}$$

Just like the quadratic formula, del Ferro's new formula allowed him to find the solution to cubic equations by simply plugging in values. Table 2 shows a summary of these types of polynomial equations and solutions.

Now, for some reason, the way mathematicians earned money in the 16th century was through challenging other mathematicians to what were basically "math duels" – so del Ferro kept his new formula a secret to use in his next duel.

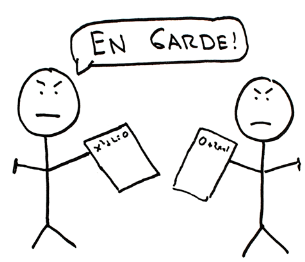


Figure 6 | High-stakes math duel. Because that's obviously what math is for. Dueling.

What happens next is a bit of a long story – here's the quick version. Del Ferro kept the formula secret until he was on his death bed, when he *finally* told his student

1 Imaginary numbers: "Hey, we're having a little get together next week, we're hoping you can make it!" Mathematicians: "ehhh, we're a little busy doing *real* math."
2 Aka solutions, aka zeros – let's just say it lets' you find x!
3 Quadratic, hence the name…
4 Making the equation easier to solve, this is called a "depressed cubic"
5 Notice we lost the *"a"* here as well. We're allowed to do this by dividing through by a, and letting the "new" c be c/a and the "new" d be d/a. After all, they're just constants!
6 constants = a, b, c

7 highest power 1, this shown in more detail in the first row of Table 2.
8 If you think del Ferro wasn't that clever, try solving equation for x yourself. A full derivation is available at welchlabs.com/blog.

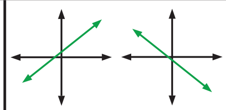
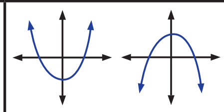
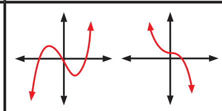
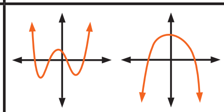
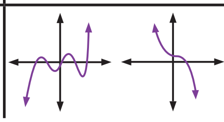
Type of Equation	Highest Power	General Form	Graphs Look Like	General Solution	Solution Discovered
Linear	1	$ax + b = 0$		$x = \dfrac{-b}{a}$	A long time ago
Quadratic	2	$ax^2 + bx + c = 0$		$x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$	~2000 BC
Cubic	3	$ax^3 + bx^2 + cx + d = 0$ $x^3 = cx + d$ *		$x = \sqrt[3]{\frac{d}{2} + \sqrt{\frac{d^2}{4} - \frac{c^3}{27}}} + \sqrt[3]{\frac{d}{2} - \sqrt{\frac{d^2}{4} - \frac{c^3}{27}}}$	Early 1500s
Quartic	4	$ax^4 + bx^3 + cx^2 + dx + e = 0$		[illegible]	1540 AD
Quintic	5	$ax^5 + bx^4 + cx^3 + dx^2 + ex + f = 0$		Proven not to exist in 1824!	

Table 2 | Polynomials and solutions. *For cubic functions, the general solution shown is to this simplified case. Note that as we increase our highest power, polynomials become significantly harder to solve! The quartic case gets a little ridiculous - just one of the 4 solutions to the quartic equation is shown here - as you can see - it doesn't quite fit.

Figure 7 | Niccolò Fontana Tartaglia. 1500-1557. If you think your nickname sucks, it doesn't – this guy got his jaw sliced by a soldier as a kid, leading to a stammer for the rest of his life, and being called "stammerer" (Tartaglia) – even in the equations he helped develop.

Antionio Fior. Foir immediately thought he was invincible, or at least invincible in a math duel,[1] and challenged a way more skilled mathematician, Fontana Tartaglia, to a duel. Tartaglia had successfully solved similar cubics, but had thus far been unable to solve cubics of del Ferro's form. Suspecting Foir would be able to solve these tougher problems, Tartaglia freaked out before the math-off and figured out how to solve the equation at the last minute, and proceeded to completely dominate Fior.[2]

Tartaglia then went on to share the formula with the world! Not really, he kept it super secret so he could keep kicking butt in math duels.[3] That is, until a very talented mathematician named Girolamo Cardano heard about the formula, and pressured Tartaglia to share – he eventually went along, but only after Cardan swore to an oath of secrecy. Fortunately for us, after Cardan came across the surviving work of the original discoverer, del Ferro,[4] he figured that it wasn't such a big secret, and published the formula in his book Ars Magna. [5]

Cardan went on to improve on his borrowed formula, even making it work for cubics that included an x^2 term.[6] However, along the way Cardan came across a problem. In a slightly different version of the equation written as $x^3 = cx + d$, under certain values of c and d[7], the formula would break.

Let's take the innocent looking[8]

$$x^3 = 15x + 4, \tag{5}$$

when we plug into Cardan's formula we get a result that involves the square root of negative numbers.[9]

$$x = \sqrt[3]{\frac{d}{2} + \sqrt{\frac{d^2}{4} - \frac{c^3}{27}}} + \sqrt[3]{\frac{d}{2} - \sqrt{\frac{d^2}{4} - \frac{c^3}{27}}}$$
$$x = \sqrt[3]{\frac{4}{2} + \sqrt{\frac{4^2}{4} - \frac{15^3}{27}}} + \sqrt[3]{\frac{4}{2} - \sqrt{\frac{4^2}{4} - \frac{15^3}{27}}}$$
$$x = \sqrt[3]{2 + \sqrt{4 - 125}} + \sqrt[3]{2 - \sqrt{4 - 125}}$$
$$x = \sqrt[3]{2 + \sqrt{-121}} + \sqrt[3]{2 - \sqrt{-121}}$$

Figure 8 | Plugging into Cardan's formula. When we try to use Cardan's formula to evaluate the simple cubic of Equation 5, we run into a small problem.

1 So not really invincible
2 This paragraph is a bit different in the accompanying video, where I got this detail wrong. I said that Tartaglia had falsely claimed to be able to solve these problems, the corrected story above is what actually happened.
3 Mwhahahaha
4 Remember him? ...from like...the last page...
5 Ars Magna = The Great Art (referring to algebra, instead of the lesser arithmetic). Fontana was not so happy about Cardan sharing his formula and accused him of plagiarism and such. Drama ensued.
6 Cardan did this through clever substitution. Given $f(x) = x^3 + bx^2 + cx + d$, substitute $x = x - b/3$.
7 $d^2/4 - c^3/27 < 0$
8 Cardan used this example in Ars Magna
9 Note that we're plugging into Cardan's modified version (shown in Table 2)

The square roots of a negative number created enough of problem to stop Cardan in his tracks. Square roots ask us to find a number, that when multiplied by itself, yield the number inside the root sign. The square root of nine is three because three times three is nine. Importantly, the square root of nine is also negative three, because negative three times negative three is also positive nine.

$$\sqrt{9} = 3, -3$$

But what about roots of negative numbers? What is the square root of negative nine? Positive three won't work, and neither will negative three, so we're stuck.

$$\sqrt{-9} = ?$$

Cardan was stuck too - he didn't know of any numbers, that when multiplied by themselves resulted in a negative. [1]

Now this certainly wasn't the first time the square root of a negative had shown up – usually mathematicians would interpret this as the problem's way of saying there are no solutions, and in many cases this is true.[2] However, in this case we *know* there is at least one solution, because of the way cubics are shaped.[3]

Regardless of their coefficients, cubic functions will always cross the x-axis at least once, meaning that our equation $x^3 = 15x+4$ will have at least one real solution.

So what we have here is a problem that must have an answer, a formula that has been proven to work. But when we put these together – and try to solve the problem with our formula – we quickly arrive at what appears to be impossible - the square roots of negative numbers.

Sometimes, when things break in math and science it means just that - they're broken – but there are other, more interesting situations in which broken mathematics give us the keys to unlock new insights. The way in which Cardan's' formula was broken turned out to be incredibly important to mathematics and science, and that's what we'll begin to discuss next time.

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of del Ferro's original equation. Cardan's modification allows us to solve cases involving negative values of del Ferro's constant, c.

1 He actually did kinda know about these, but wasn't sure how to apply them here. See exercise 3.17.

2 See Exercises 2.11 and 2.14

3 in fact, one solution to Equation 5 is just 4. Check out Figure 13 for an example of how cubics are shaped.

Exercises 2

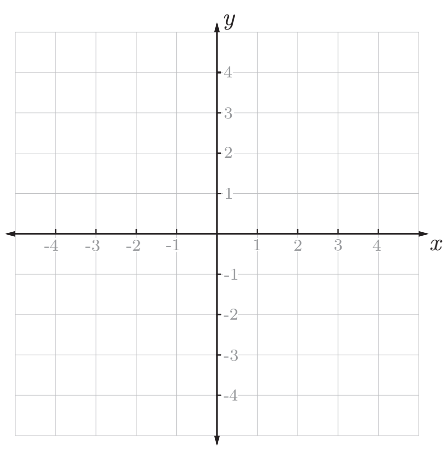
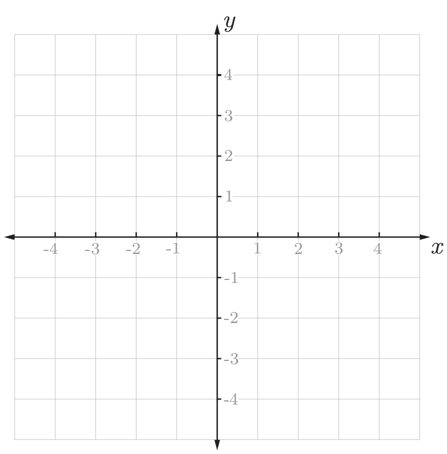
Discussion

2.1 Why did del Ferro keep his discovery secret?

2.2 Why are problems like $\sqrt{-16}$ strange?

2.3 Why do you think math duels are way less popular today than in the 16th century?

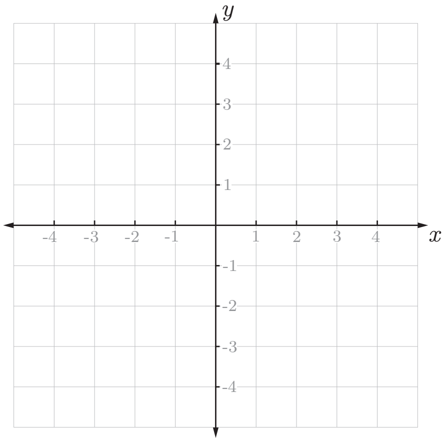
Drill

	Plot	**Solve by Factoring (If Possible)**	**Solve by Quadratic Formula (If Possible)**
2.4	$f(x) = x^2 - x - 2$ x | y -2 -1 0 1 2 3 	$0 = x^2 - x - 2$	$0 = x^2 - x - 2$
2.5	$g(x) = x^2 + 3x + 2$ x | y -3 -2 -1 0 1 	$0 = x^2 + 3x + 2$	$0 = x^2 + 3x + 2$

	Plot	Solve by Factoring (If Possible)	Solve by Quadratic Formula (If Possible)
2.6	$f(x) = x^2 - 2x + 1$	$0 = x^2 - 2x + 1$	$0 = x^2 - 2x + 1$
2.7	$g(x) = x^2 + x - 4$	$0 = x^2 + x - 4$	$0 = x^2 + x - 4$
2.8	$h(x) = x^2 + 1$	$0 = x^2 + 1$	$0 = x^2 + 1$

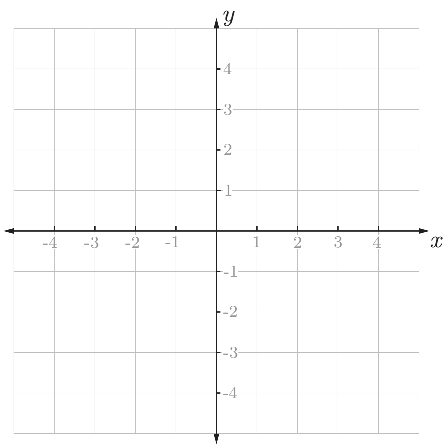
2.6 $f(x) = x^2 - 2x + 1$

x	y
-1	
0	
1	
2	
3	



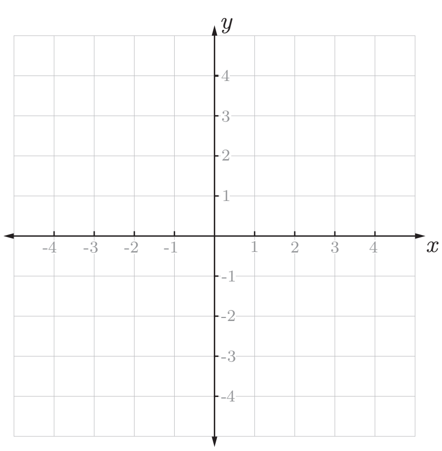
2.7 $g(x) = x^2 + x - 4$

x	y
-3	
-2	
-1	
0	
1	
2	



2.8 $h(x) = x^2 + 1$

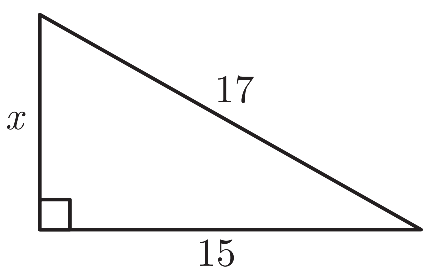
x	y
-2	
-1	
0	
1	
2	



Critical Thinking

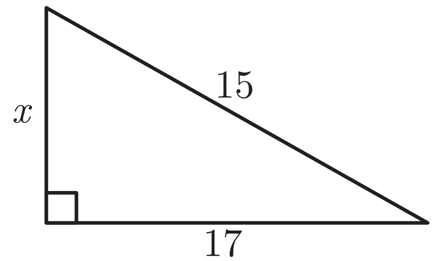
2.9 Why is the Quadratic Formula useful? Does it allow you to solve any types of problems that you couldn't otherwise? (Hint: compare exercise 2.7 to 2.4-2.6)

2.10 Solve for x:



2.11 Sometimes problems like 2.10 are less straightforward.

a) Solve for x:



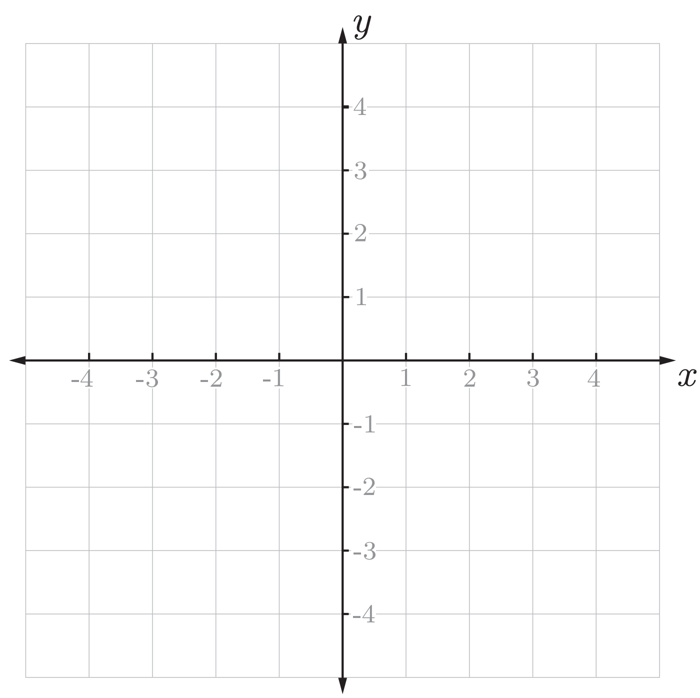
b) How did part a go? :) If you used the Pythagorean Theorem to solve for x, the result should have been the square root of a negative number. Sketchy. What's going on here?

2.12 Where do the parabola $y = x^2$-4 and the line y = $2x$-1 intersect?

2.13 Where do the parabola $y = x^2$-4 and the line y = $2x$-3 intersect?

2.14 a) Where do the parabola $y = x^2$-1 and the line y = $2x$-3 intersect?

b) Using the Quadratic Formula to solve part a should have resulted in the square root of negative number. What does this mean about the problem? Plot the parabola and line from part a below.



2.15 Solve the equation $x^3 = 8$ using del Ferro's formula (Equation 4).

2.16 Solve the equation $x^3 + 6x = 20$ using del Ferro's formula (Equation 4).

2.17 Solve the equation $x^3 = x + 2$ using Cardan's modified version of Del Ferro's formula (shown in the 3rd row of Table 2).

Challenge

2.18 Derive the Quadratic Formula from $\mathrm{a}x^2+\mathrm{b}x+\mathrm{c}=0$ by completing the square.

2.19 Derive del Ferro's formula from $x^3+\mathrm{c}x=\mathrm{d}$ by witchcraft.

2.20 What exact values of c and d result in the square roots of negative numbers in Cardan's formula (row 3 of, column 5 of Table 2)? Exactly how many real solutions must this type of cubic have?