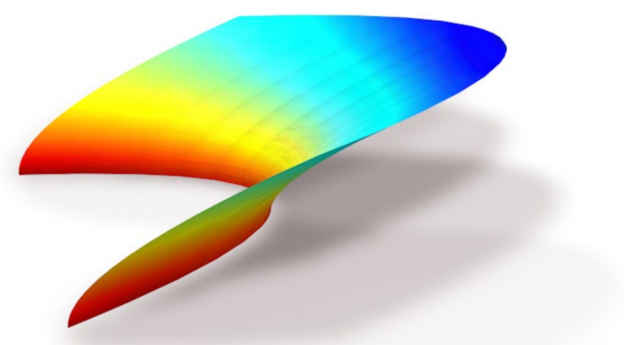
Welch Labs



# Imaginary Numbers Are Real

WORKBOOK

# Hello!

*"Imagine you had an art class in which they taught you how to paint a fence, but never showed you the great masters. Of course, you would say; 'I hate art.' You were bad at painting the fence but you wouldn't know what else there is to art. Unfortunately, that is exactly what happens with mathematics. What we study at school is a tiny little part of mathematics. I want people to discover the magic world of mathematics, almost like a parallel universe, that most of us aren't aware even exists."*

*- Edward Frenkel*

## 1. You Rock!

Thanks for checking out the Welch Labs Imaginary Numbers Are Real Workbook. Here you'll find everything you need to get the most out of my Imaginary Numbers are Real YouTube series. For each video in the series, you'll find a workbook section complete with the text and key figures from the video, more in-depth features covering interesting areas, and most importantly, exercises.

## 2. About Them Exercises.

As the quote above says much better than I can, the exercises we're given in math class are all too often terrible. At some point you've likely been assigned the kinds of fence-painting exercises where you're asked to do the same type of problem over and over and over with minor tweaks to the numbers. I hate this stuff. It's uninspiring and a bad use of your brain. That said, I promise I have done my very best to avoid this type of problem here. Each exercise is here for a reason. No fence painting.

Of course, this is only half the equation. Like anything else, to get real value out, you must put real work in. And if you do, I promise it's worth it. Imaginary numbers are rich and beautiful, and their history is fascinating. Really understanding this stuff will give you tremendous perspective on the power and beauty of modern mathematics and science. The exercises for each section are divided into 4 parts:

| Exercises | Description |
|---|---|
| Discussion | Designed to stimulate, you guessed it, discussion! No wrong or right answers. |
| Drill | Imaginary Numbers can be tough, and missing key concepts can really make this stuff way less fun - the Drills are designed to ensure your grasp of the key concepts is sound. |
| Critical Thinking | The Critical Thinking exercises are where things get good - these exercises are specifically designed to question your mathematical assumptions and grow your skills. |
| Challenge | These questions are like, hard. |

## 3. About Them Solutions.

I'm a firm believer in quick feedback when learning new concepts. This is why you'll find answers to the exercises in the back of the book. I encourage you to check your work as you go. Of course, this doesn't mean you should flip to the back of the book as soon as the going gets tough - a little suffering and uncertainty is good.

Enjoy!
@stephencwelch

# Table of Contents

*"Hey, so what's like, in this book?"*

Let's take the function

$$f(x) = x^2 + 1. \qquad (1)$$

If we graph our function, we obtain the friendly parabola of Figure 1. Now let's say we want to figure out where the equation equals zero - we want to find the roots. On our plot this should be where the function crosses the x-axis.

As we can see, our parabola actually never crosses the x-axis, so according to our plot, there are no solutions to the equation $x^2+1=0$.

But there's a small problem. A little over 200 years ago a smart guy named Gauss proved[1] that every polynomial equation of degree n has exactly n roots. Our polynomial has a highest power, or degree, of two, so we should have two roots. And Gauss' discovery is not just some random rule, today we call it the Fundamental Theorem of Algebra.

So our plot seems to disagree with something so important it's called the Fundamental Theorem of Algebra, which might be a problem. What Gauss is telling us here is that there are two perfectly good values of x that we could plug into our function, and get zero out. Where could these two missing roots be?

The short answer here is that we don't have enough numbers. We typically think of numbers existing on a one dimensional continuum - the number line. All our friends are here: zero, one, negative numbers, fractions, even irrational numbers like $\sqrt{2}$ show up.

But this system is incomplete. And our missing numbers are not just further left or right, they live in a whole new dimension. Algebraically, this new dimension has everything to do with a problem that was mathematically considered impossible for over two thousand years: the square root of negative one.[2]

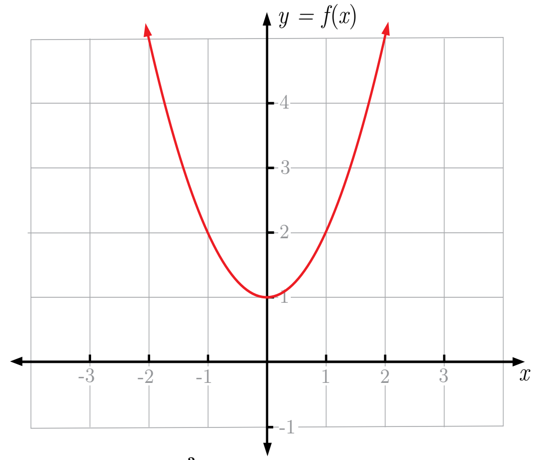



**Figure 1 | Graph of $f(x) = x^2+1$.**

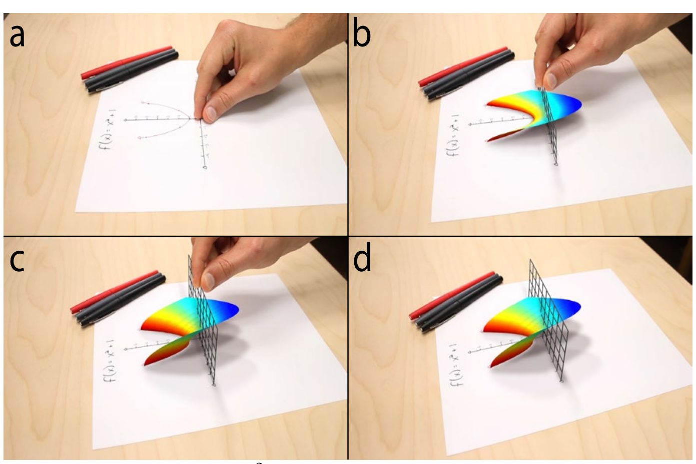



**Figure 2 | Graph of $f(x) = x^2+1$ where $x$ includes imaginary numbers.** Panels a-d show "pulling" the function out of the page.

When we include this missing dimension in our analysis - our parabola ends up looking more like Figure 2, and needless to say, is a bit more interesting.

Now that our input numbers are in their full two dimensional form, we see how our function $x^2+1$ really behaves. And we can now see that our function does cross the x-axis![3] We were just looking in the wrong dimension.

So, why is this extra dimension that numbers possess not common knowledge? Part of this reason is that it has been given a terrible, terrible name. A name that suggest that these numbers aren't ever real! [4]

In fact, Gauss himself had something to say about this naming convention:

*"That this subject [imaginary numbers] has hitherto been surrounded by mysterious obscurity, is to be attributed largely to an ill adapted notation. If, for example, +1, -1, and the square root of -1 had been called direct, inverse and lateral units, instead of positive, negative and imaginary (or even impossible), such an obscurity would have been out of the question."*

*- Carl Friedrich Gauss (1777-1855)*

So yes, this missing dimension is comprised of numbers that have been given ridiculous name imaginary. Gauss proposed these numbers should instead be given the name

1 Mostly, the full proof took a little longer

2 More on this later

3 If you're paying attention, you should be thinking "what the heck Stephen, you said the graph would cross the axis twice (there would be two solutions), and the graph in figure 2 crosses like a million times!" Great point Greg! The reason for this is we've only plotted the real part of the graph to keep things simple (ish) – we'll cover the complete solution (which does have exactly 2 answers) in Part 13. Get excited.

4 ☺

lateral[1] - so from here on, let's let lateral mean imaginary.

To get a better handle on imaginary/lateral numbers and really understand what's going on in Figure 2, let's spend a little time thinking about numbers.

Early humans really only had use for the natural numbers (1, 2, 3...). This makes sense because of how numbers were used – as a tool for counting things. So to early humans, the number line would have just been a series of dots.

As civilizations advanced, people needed answers to more sophisticated math questions – like when to plant seeds, how to divide land, and how to keep track of financial transactions. The natural numbers just weren't cutting it anymore, so the Egyptians innovated and developed a new, high tech solution: fractions.

Fractions filled in the gaps in our number line, and were

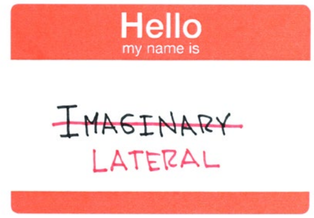



**Figure 3 | From here on, lateral = imaginary.** Gauss preferred the term lateral to imaginary, we'll see exactly why in Part 6.

basically cutting edge technology for a couple thousand years.

The next big innovations to hit the number line were the number zero and negative numbers, but it took some time[2] to get everyone on board. Since it's not obvious[3] what these numbers mean or how they fit into the real world,

1 We'll explain *why* Gauss preferred the name imaginary in part six.

2 Like a lot of time. Like thousands of years time.

3 Remember 3rd grade?

| Civilization | Relevant Time | Example Numerals | Fractions | Zero As Placeholder | Zero | Negatives | Imaginary Numbers | Number Line |
|---|---|---|---|---|---|---|---|---|
| Prehistory | <3000 BC | I II III | ✗ | ✗ | ✗ | ✗ | ✗ | 1 2 3 ... |
| Ancient Egypt | 1740BC | | ✓ | ✓ | ✗ | ✗ | ✗ | 1 2 3 |
| Babylonia | 300BC | | ✓ | ✓ | ✗ | ✗ | ✗ | 1 2 3 |
| Olmec | 700-400BC | | ✓ | ✓ | ✗ | ✗ | ✗ | 1 2 3 |
| Greek | 500BC-100AD | ΣΜΑ | ✓ | ✗ | ✗ | ✗ | ✗ | 1 2 3 |
| China | 200BC-200AD | | ✓ | ✓ | ✗ | ✓ | ✗ | -3 -2 -1 1 2 3 |
| Roman | 27 BC-476AD | II III VII | ✗ | ✗ | ✗ | ✗ | ✗ | 1 2 3 ... |
| Cambodia | 700AD | | ✓ | ✓ | ✓ | ✗ | ✗ | 1 2 3 |
| India + Persia | 600-1000AD | | ✓ | ✓ | ✓ | ✓ | ✗ | -3 -2 -1 0 1 2 3 |
| Medieval Europe | 500-1400AD | II III VII | ✗ | ✗ | ✗ | ✗ | ✗ | 1 2 3 ... |
| Renaissance Europe | 1300-1700AD | 1, 2, 3 | ✓ | ✓ | ✓ | ✓ | ✗ | -3 -2 -1 0 1 2 3 |
| Modern Era | >1700 AD | 1, 2, 3 | ✓ | ✓ | ✓ | ✓ | ✓ | $i$, -1, 1, $-i$ |

**Table 1 | A brief overview of the history of numbers.** It has taken quite some time for modern numbers to come to be. Only in the last couple hundred years do we see imaginary/lateral numbers really accepted. The dates here are approximate, and keep in mind that most of these civilizations didn't actually have number lines! The point here is visualize how numbers developed over time. Finally, notice the difference between zero and zero as a placeholder. In a positional number systems like ours, the location of a digit carries meaning. The three in 23 means 3 units, while the three in 32 means 3 "tens", or thirty. We run into a problem if we need to tell the difference between 30 and 300, and we don't have zero! This is the placeholder zero, it is not a concept alone, but a notational tool.

zero and negative numbers were met with skepticism, and largely avoided or ignored. Some cultures were more suspicious that others, depending largely on how people viewed the connection between mathematics and reality. A great example is here Greek civilization – despite making huge strides in geometry, the Greeks generally didn't accept negative numbers or zeros, after all how could nothing be something?

What's even wilder is that this is not all ancient history - just a few centuries ago, mathematicians would intentionally move terms around to avoid having negatives show up in equations. Suspicion of zero and negative numbers did eventually fade - partially because negatives are useful for expressing concepts like debt, but mostly because negatives just kept sneaking into mathematics.

It turns out there's just a whole lot of math you just can't do if you don't allow negative numbers to play. Without negatives, simple algebra problems like $x + 3 = 2$ have no answer. Before negatives were accepted, this problem would have no solution, just like we thought Equation 1 had no solution.

The thing is, it's not crazy or weird to think problems like this have no solutions – to solve $x + 3 = 2$, we subtract 3 from both sides, resulting in $x = 2 - 3$. In words, this algebra problem basically says: "if I have 2 things and I take away 3, how many things do I have left?"

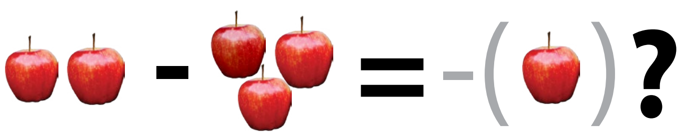


**Figure 4 | Negative numbers don't always make sense.** Two apples, take away three apples equals...the anti-apple?

It's not surprising that most of the people who have lived on our planet would be suspicious of questions like this. These problems don't make any sense. Even brilliant mathematicians of the 18th century, such as Leonard Euler, didn't fully know what to do with negatives – he at one point wrote that negatives were greater than infinity.[1]

So it's fair to say that negative and imaginary numbers raise a lot of very good, very valid questions, such as:

- Why do we require students to understand and work with numbers that eluded the greatest mathematical minds for thousands of years?
- Why did we even come accept negative and imaginary numbers in the first place, when they don't really seem connected to anything in the real world?
- How do these extra numbers help explain the missing solutions to Equation 1?

Next time, we'll begin to address these questions by going way back to the discovery of complex numbers.

1 Sketchy.

# Exercises 1

*Discussion*

1.1 Why do you think most people who have lived on our planet would have been suspicious of negative numbers?

1.2 Why do you think negative numbers have been so widely accepted today, despite being somewhat sketchy?

1.3 Do you think negative numbers should be taught to elementary school students? Why or why not?

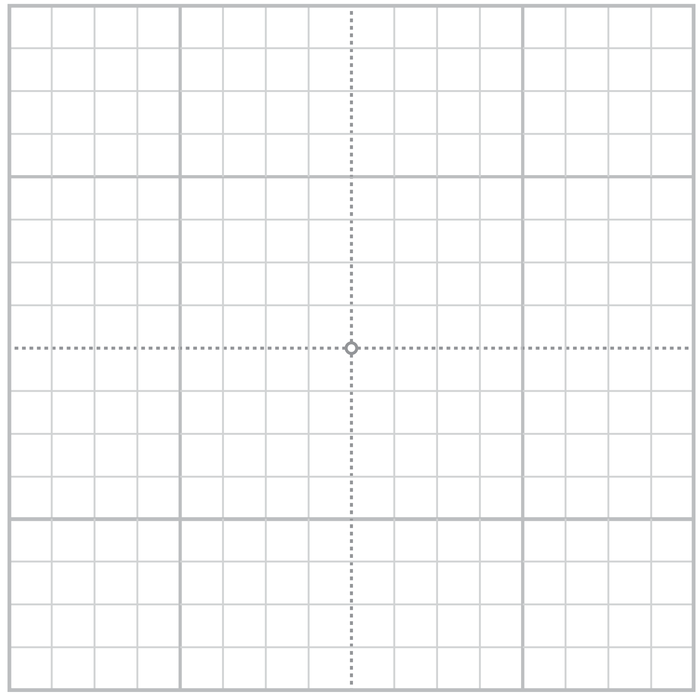
1.4 How would you explain negative numbers to a $5^{th}$ grader?

1.5 Which historical civilizations do you think embraced mathematics? Which didn't?

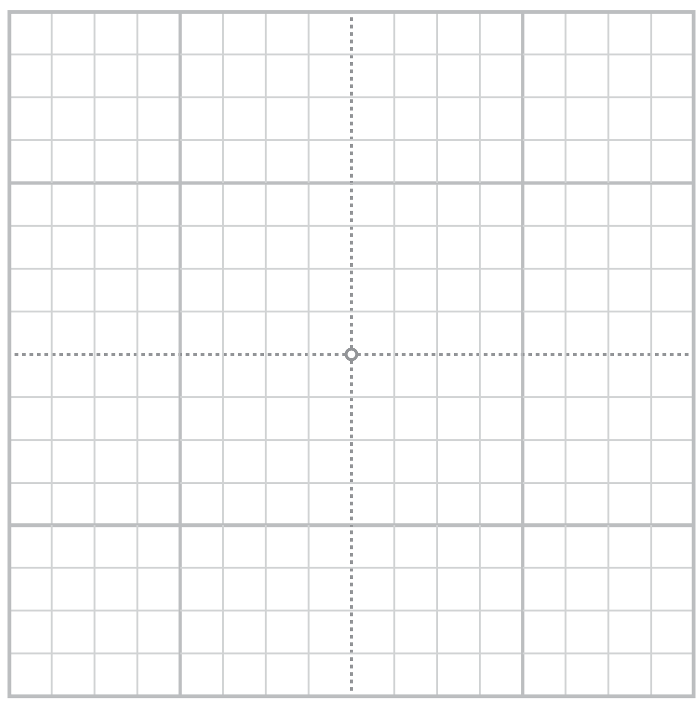
*Drill*

Plot each function.

1.6 $f(x) = x^2$-4x+4



1.7 $g(x) = -x^2+x+6$

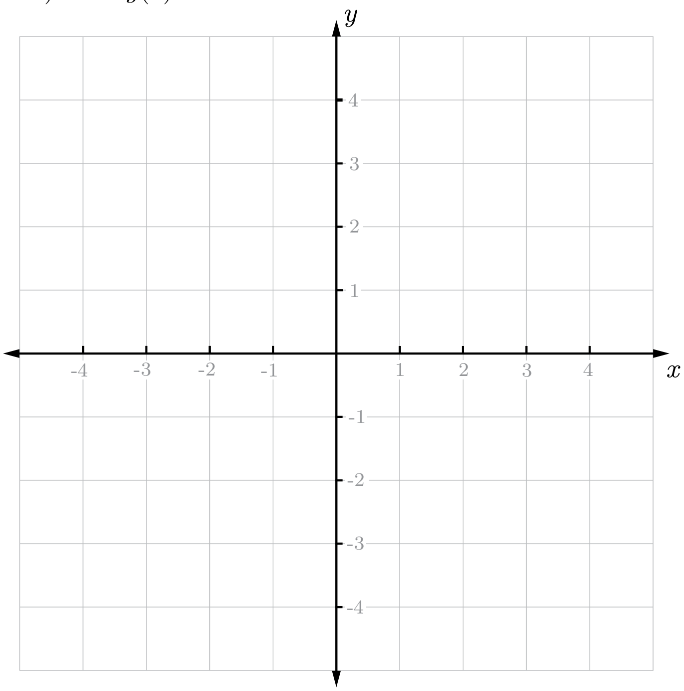


*Critical Thinking*

1.8 Consider the better-behaved sibling of Equation 1:

$$g(x) = x^2\text{-}1$$

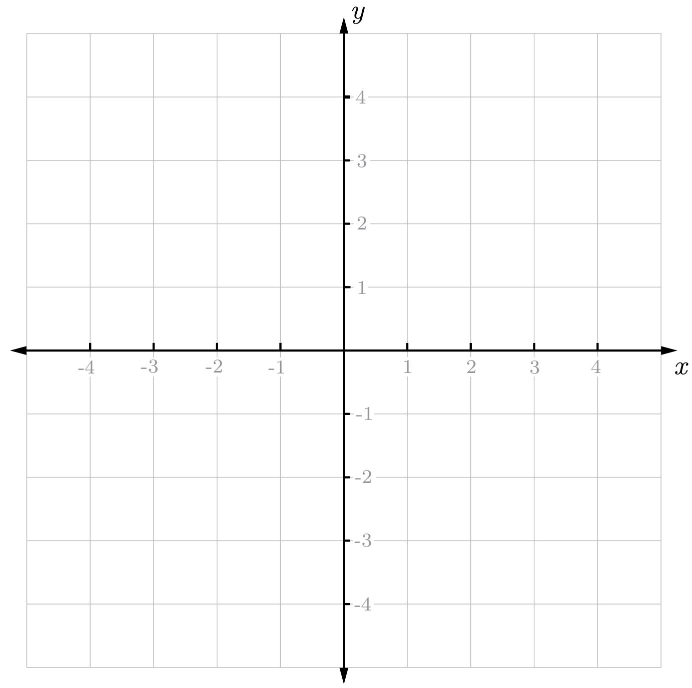
a) Plot $g(x)$.



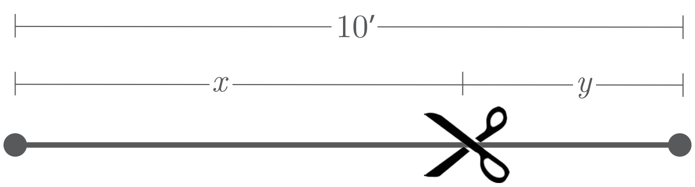

b) For what x-values does $g(x)$=0? $g(x)$=0 when $x$ =________ or $x$ =_________. These are called the **roots, zeros, or x-intercepts of *g*.**

c) The roots of $g$ are much easier to find than the roots of $f$ (Equation 1), why is this the case?

1.9 So far we've seen a function with 2 roots, $g$, and a function with no obvious roots, $f$ (Equation 1). Find and plot a highest power of two (quadratic) function with one obvious root. Call it $h(x)$. Or else.




1.10 You cut $x$ feet from a 10 foot rope, leaving $y$ feet of remaining rope.




a) Write an equation relating $x$ and $y$.

b) Solve for $y$ when $x$=7', $x$=10', and $x$=13'.

c) One of your answers from part b should be negative. Is this result meaninfgul? What does a negative answer tell you about your remaining rope? Does the specific value of your answer matter, or is knowing your answer is negative enough to reach a conclusion about the remaining rope?

1.11 You have exactly 10 minutes to get to class for your big test on lateral numbers! Your friend Gus (who's weirdly in to this sort of thing) says that class is exactly 10,000 feet away. Let's say you travel to class at a rate of $r$ feet/minute. You would like to get there a few minutes early. More specifically, let's say you will arrive $t$ minutes early.

a) Write an equation that relates $r$ and $t$.

b) Compute $t$ for $r$=1250 feet/minute, $r$=1000 feet/minute, and $r$=625 feet/minute.

c) One of your answers from part b should be negative. Is this result meaningful? What does a negative answer tell you about when you will arrive to class?

*Challenge*

1.12 Consider the function:

$$p(x) = x^4\text{-}10x^3\text{+}35x^2\text{-}50x\text{+}24$$

a) According to the fundamental theorem of algebra, how many roots should $p(x)$ have?

b) Find all the roots of $p(x)$. You may use technology if you would like, but it's not necessary to complete the problem.

# Imaginary Numbers Are Real

## Part 2: A Little History

Last time, we left off wondering how imaginary/lateral numbers could help us find the roots of our equation $x^2+1$, and further, how imaginary and negative numbers even became a part of modern mathematics after being avoided and ignored for a couple of thousand years, because, let's be honest here – they don't really make that much sense.[1]

But something happened in Europe around five centuries ago that would no longer allow mathematicians to ignore these numbers. An Italian mathematician, Scipione del Ferro, was trying to solve a problem not that different than ours.

At some point you've likely seen the quadratic formula. This formula is super useful because gives us the roots[2] of any equation with a highest power of two[3]- all you have to do is plug in $\mathrm{a}$, $\mathrm{b}$, and $\mathrm{c}$,and out pops the answer:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}, \tag{2}$$

where:

$$0 = ax^2 + bx + c.$$

Del Ferro was trying to find a formula like this for equations with a highest power of three - cubics. The general case is pretty tough:

$$0 = ax^3 + bx^2 + cx + d,$$

so Del Ferro first considered the case where the $x^2$ term is missing[4], and the last term is negative:

$$0 = ax^3 + cx - d.$$

In the 16th century, negative terms were way to sketchy to write, so del Ferro wrote his cubic as

$$x^3 + cx = d, \tag{3}$$

and required $\mathrm{c}$ and $\mathrm{d}$ to be positive.[5]

Now that we have our equation set up, the game here is to get $x$ by itself on one side, and all the constants[6] on the

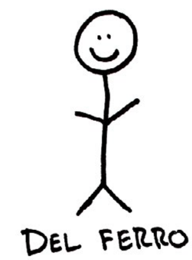



**Figure 5 | Highly-paid professional artist rendering of Scipione del Ferro.** 1465-1562.

other side. This is pretty easy in linear[7] equations, we can just add, multiply, divide, or subtract until we get $x$ alone.

Quadratics are a bit harder – you may have learned to do this in school – it requires some cleverness, and factoring by completing the square.

Del Ferro was trying to do the same thing for his cubic equation, and through some very clever substitution, he eventually found a solution: [8]

$$x = \sqrt[3]{\frac{d}{2} + \sqrt{\frac{d^2}{4} + \frac{c^3}{27}}} + \sqrt[3]{\frac{d}{2} - \sqrt{\frac{d^2}{4} + \frac{c^3}{27}}}. \tag{4}$$

Just like the quadratic formula, del Ferro's new formula allowed him to find the solution to cubic equations by simply plugging in values. Table 2 shows a summary of these types of polynomial equations and solutions.

Now, for some reason, the way mathematicians earned money in the 16th century was through challenging other mathematicians to what were basically "math duels" – so del Ferro kept his new formula a secret to use in his next duel.

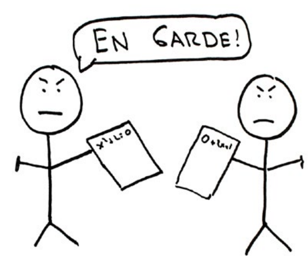



**Figure 6 | High-stakes math duel.** Because that's obviously what math is for. Dueling.

What happens next is a bit of a long story – here's the quick version. Del Ferro kept the formula secret until he was on his death bed, when he *finally* told his student

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1 Imaginary numbers: "Hey, we're having a little get together next week, we're hoping you can make it!" Mathematicians: "ehhh, we're a little busy doing *real* math."

2 Aka solutions, aka zeros – let's just say it lets' you find x!

3 Quadratic, hence the name…

4 Making the equation easier to solve, this is called a "depressed cubic"

5 Notice we lost the *"a"* here as well. We're allowed to do this by dividing through by a, and letting the "new" c be c/a and the "new" d be d/a. After all, they're just constants!

6 constants = a, b, c

7 highest power 1, this shown in more detail in the first row of Table 2.

8 If you think del Ferro wasn't that clever, try solving equation for x yourself. A full derivation is available at welchlabs.com/blog.

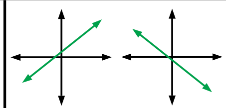
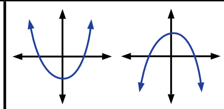
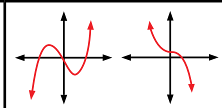
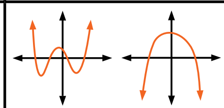
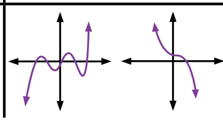
| Type of Equation | Highest Power | General Form | Graphs Look Like | General Solution | Solution Discovered |
|---|---|---|---|---|---|
| Linear | 1 | $ax + b = 0$ |  | $x = \frac{-b}{a}$ | A long time ago |
| Quadratic | 2 | $ax^2 + bx + c = 0$ |  | $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ | ~2000 BC |
| Cubic | 3 | $ax^3 + bx^2 + cx + d = 0$<br>$x^3 = cx + d$ * |  | $x = \sqrt[3]{\frac{d}{2} + \sqrt{\frac{d^2}{4} - \frac{c^3}{27}}} + \sqrt[3]{\frac{d}{2} - \sqrt{\frac{d^2}{4} - \frac{c^3}{27}}}$ | Early 1500s |
| Quartic | 4 | $ax^4 + bx^3 + cx^2 + dx + e = 0$ |  | [illegible] | 1540 AD |
| Quintic | 5 | $ax^5 + bx^4 + cx^3 + dx^2 + ex + f = 0$ |  | Proven not to exist in 1824! | |

**Table 2 | Polynomials and solutions.** *For cubic functions, the general solution shown is to this simplified case. Note that as we increase our highest power, polynomials become significantly harder to solve! The quartic case gets a little ridiculous - just one of the 4 solutions to the quartic equation is shown here - as you can see - it doesn't quite fit.

**Figure 7 | Niccolò Fontana Tartaglia.** 1500-1557. If you think your nickname sucks, it doesn't – this guy got his jaw sliced by a soldier as a kid, leading to a stammer for the rest of his life, and being called "stammerer" (Tartaglia) – even in the equations he helped develop.

Antionio Fior. Foir immediately thought he was invincible, or at least invincible in a math duel,[1] and challenged a way more skilled mathematician, Fontana Tartaglia, to a duel. Tartaglia had successfully solved similar cubics, but had thus far been unable to solve cubics of del Ferro's form. Suspecting Foir would be able to solve these tougher problems, Tartaglia freaked out before the math-off and figured out how to solve the equation at the last minute, and proceeded to completely dominate Fior.[2]

Tartaglia then went on to share the formula with the world! Not really, he kept it super secret so he could keep kicking butt in math duels.[3] That is, until a very talented mathematician named Girolamo Cardano heard about the formula, and pressured Tartaglia to share – he eventually went along, but only after Cardan swore to an oath of secrecy. Fortunately for us, after Cardan came across the surviving work of the original discoverer, del Ferro,[4] he figured that it wasn't such a big secret, and published the formula in his book Ars Magna. [5]

Cardan went on to improve on his borrowed formula, even making it work for cubics that included an $x^2$ term.[6] However, along the way Cardan came across a problem. In a slightly different version of the equation written as $x^3 = cx + d$, under certain values of c and d[7], the formula would break.

Let's take the innocent looking[8]

$$x^3 = 15x + 4, \qquad (5)$$

when we plug into Cardan's formula we get a result that involves the square root of negative numbers.[9]

$$x = \sqrt[3]{\frac{d}{2} + \sqrt{\frac{d^2}{4} - \frac{c^3}{27}}} + \sqrt[3]{\frac{d}{2} - \sqrt{\frac{d^2}{4} - \frac{c^3}{27}}}$$

$$x = \sqrt[3]{\frac{4}{2} + \sqrt{\frac{4^2}{4} - \frac{15^3}{27}}} + \sqrt[3]{\frac{4}{2} - \sqrt{\frac{4^2}{4} - \frac{15^3}{27}}}$$

$$x = \sqrt[3]{2 + \sqrt{4 - 125}} + \sqrt[3]{2 - \sqrt{4 - 125}}$$

$$x = \sqrt[3]{2 + \sqrt{-121}} + \sqrt[3]{2 - \sqrt{-121}}$$

**Figure 8 | Plugging into Cardan's formula.** When we try to use Cardan's formula to evaluate the simple cubic of Equation 5, we run into a small problem.

1 So not really invincible
2 This paragraph is a bit different in the accompanying video, where I got this detail wrong. I said that Tartaglia had falsely claimed to be able to solve these problems, the corrected story above is what actually happened.
3 Mwhahahaha
4 Remember him? ...from like...the last page...
5 Ars Magna = The Great Art (referring to algebra, instead of the lesser arithmetic). Fontana was not so happy about Cardan sharing his formula and accused him of plagiarism and such. Drama ensued.
6 Cardan did this through clever substitution. Given $f(x) = x^3 + bx^2 + cx + d$, substitute $x = x - b/3$.
7 $d^2/4 - c^3/27 < 0$
8 Cardan used this example in Ars Magna
9 Note that we're plugging into Cardan's modified version (shown in Table 2)

The square roots of a negative number created enough of problem to stop Cardan in his tracks. Square roots ask us to find a number, that when multiplied by itself, yield the number inside the root sign. The square root of nine is three because three times three is nine. Importantly, the square root of nine is also negative three, because negative three times negative three is also positive nine.

$$\sqrt{9} = 3, -3$$

But what about roots of negative numbers? What is the square root of negative nine? Positive three won't work, and neither will negative three, so we're stuck.

$$\sqrt{-9} = ?$$

Cardan was stuck too - he didn't know of any numbers, that when multiplied by themselves resulted in a negative. [1]

Now this certainly wasn't the first time the square root of a negative had shown up – usually mathematicians would interpret this as the problem's way of saying there are no solutions, and in many cases this is true.[2] However, in this case we *know* there is at least one solution, because of the way cubics are shaped.[3]

Regardless of their coefficients, cubic functions will always cross the x-axis at least once, meaning that our equation $x^3 = 15x+4$ will have at least one real solution.

So what we have here is a problem that must have an answer, a formula that has been proven to work. But when we put these together – and try to solve the problem with our formula – we quickly arrive at what appears to be impossible - the square roots of negative numbers.

Sometimes, when things break in math and science it means just that - they're broken – but there are other, more interesting situations in which broken mathematics give us the keys to unlock new insights. The way in which Cardan's' formula was broken turned out to be incredibly important to mathematics and science, and that's what we'll begin to discuss next time.

---

of del Ferro's original equation. Cardan's modification allows us to solve cases involving negative values of del Ferro's constant, c.

1 He actually did kinda know about these, but wasn't sure how to apply them here. See exercise 3.17.

2 See Exercises 2.11 and 2.14

3 in fact, one solution to Equation 5 is just 4. Check out Figure 13 for an example of how cubics are shaped.

# Exercises 2

*Discussion*

2.1 Why did del Ferro keep his discovery secret?

2.2 Why are problems like $\sqrt{-16}$ strange?

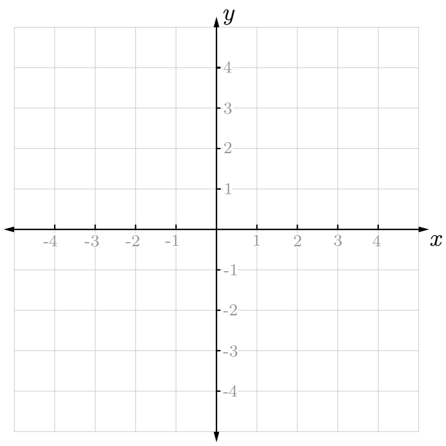
2.3 Why do you think math duels are way less popular today than in the $16^{th}$ century?

*Drill*

| | Plot | Solve by Factoring (If Possible) | Solve by Quadratic Formula (If Possible) |
|---|---|---|---|
| 2.4 | $f(x) = x^2 - x - 2$ | $0 = x^2 - x - 2$ | $0 = x^2 - x - 2$ |
| 2.5 | $g(x) = x^2 + 3x + 2$ | $0 = x^2 + 3x + 2$ | $0 = x^2 + 3x + 2$ |

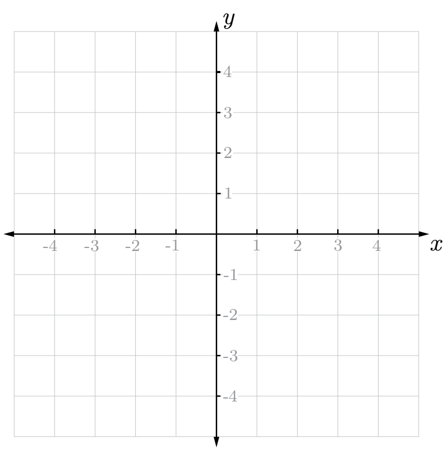
2.4 $f(x) = x^2 - x - 2$

| $x$ | $y$ |
|---|---|
| -2 | |
| -1 | |
| 0 | |
| 1 | |
| 2 | |
| 3 | |




2.5 $g(x) = x^2 + 3x + 2$

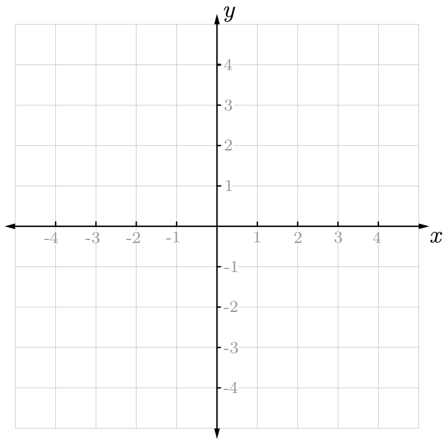
| $x$ | $y$ |
|---|---|
| -3 | |
| -2 | |
| -1 | |
| 0 | |
| 1 | |



| | Plot | Solve by Factoring (If Possible) | Solve by Quadratic Formula (If Possible) |
|---|---|---|---|
| 2.6 | $f(x) = x^2 - 2x + 1$ | $0 = x^2 - 2x + 1$ | $0 = x^2 - 2x + 1$ |
| 2.7 | $g(x) = x^2 + x - 4$ | $0 = x^2 + x - 4$ | $0 = x^2 + x - 4$ |
| 2.8 | $h(x) = x^2 + 1$ | $0 = x^2 + 1$ | $0 = x^2 + 1$ |

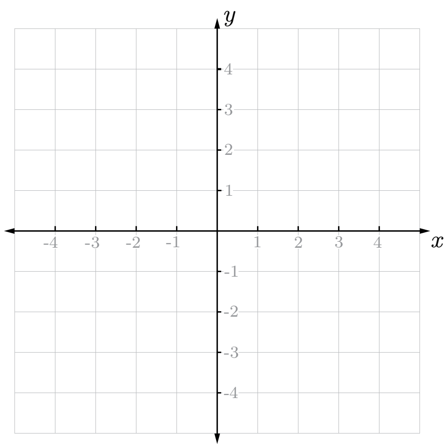
2.6

| $x$ | $y$ |
|---|---|
| -1 | |
| 0 | |
| 1 | |
| 2 | |
| 3 | |



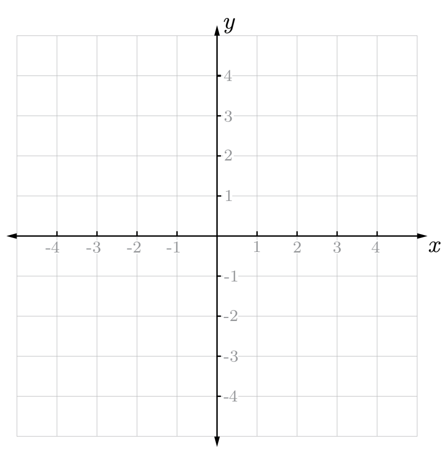

2.7

| $x$ | $y$ |
|---|---|
| -3 | |
| -2 | |
| -1 | |
| 0 | |
| 1 | |
| 2 | |




2.8

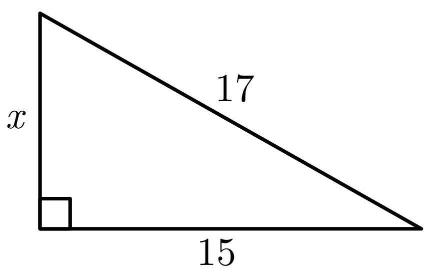
| $x$ | $y$ |
|---|---|
| -2 | |
| -1 | |
| 0 | |
| 1 | |
| 2 | |



*Critical Thinking*

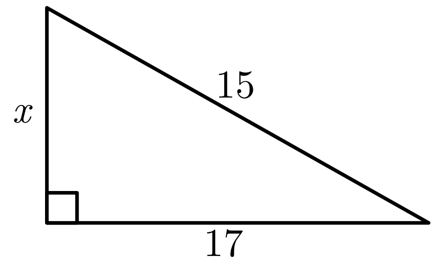
2.9 Why is the Quadratic Formula useful? Does it allow you to solve any types of problems that you couldn't otherwise? (Hint: compare exercise 2.7 to 2.4-2.6)

2.10 Solve for $x$:




2.11 Sometimes problems like 2.10 are less straightforward.

a) Solve for $x$:



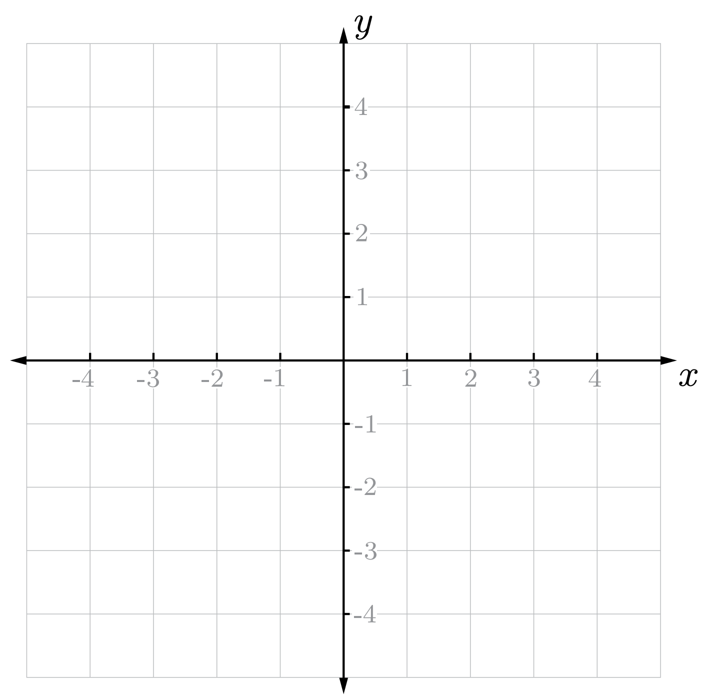

b) How did part a go? :) If you used the Pythagorean Theorem to solve for $x$, the result should have been the square root of a negative number. Sketchy. What's going on here?

2.12 Where do the parabola $y = x^2$-4 and the line y = $2x$-1 intersect?

2.13 Where do the parabola $y = x^2$-4 and the line y = $2x$-3 intersect?

2.14 a) Where do the parabola $y = x^2$-1 and the line y = $2x$-3 intersect?

b) Using the Quadratic Formula to solve part a should have resulted in the square root of negative number. What does this mean about the problem? Plot the parabola and line from part a below.



2.15 Solve the equation $x^3 = 8$ using del Ferro's formula (Equation 4).

2.16 Solve the equation $x^3 + 6x = 20$ using del Ferro's formula (Equation 4).

2.17 Solve the equation $x^3 = x + 2$ using Cardan's modified version of Del Ferro's formula (shown in the 3rd row of Table 2).

*Challenge*

2.18 Derive the Quadratic Formula from $\mathrm{a}x^2+\mathrm{b}x+\mathrm{c}=0$ by completing the square.

2.19 Derive del Ferro's formula from $x^3+\mathrm{c}x=\mathrm{d}$ by witchcraft.

2.20 What exact values of c and d result in the square roots of negative numbers in Cardan's formula (row 3 of, column 5 of Table 2)? Exactly how many real solutions must this type of cubic have?

# Imaginary Numbers Are Real

## Part 3: Cardan's Problem

We left off last time with Cardan and his broken formula for finding the roots of cubic functions. Cardan knew his problem had to have a solution – but didn't know what to do with the square roots of negative numbers that kept popping up in his equations.

Cardan came close to finding way to make his formula work, but got stuck in an algebraic loop, where a bunch of work would just lead him right back to where he started.

It took one more generation of mathematicians to get to the bottom of this. Cardan's student, Rafael Bombelli, made some incredible insights about what was really going on here.

Let's remember why Cardan was stuck – the square roots of negative numbers ask us to find a number, that when multiplied by itself, will yield a negative. Neither positive, nor negative numbers will work.

Bombelli's first big insight[1] was simply to accept that if positive numbers won't work, and negative numbers won't work, then maybe there's some other kind of number out there that will. Now, if there is some other type of number out there, a good follow up question is: what are we going to call it? After all, we need to use it in our equations.

Bombelli's approach was a very practical one.[2] Rather than dream up a new name and symbol[3] - Bombelli simply let the square root of negatives be their own thing. In the past, mathematicians would have thrown in the towel here and declared the problem "impossible", but Bombelli was able to press on simply by allowing the square root of negatives to exist.

Let's take a simple example of our new numbers – the square root of negative one. Now, for being a new kind of number, it doesn't look very exciting and kind of seems

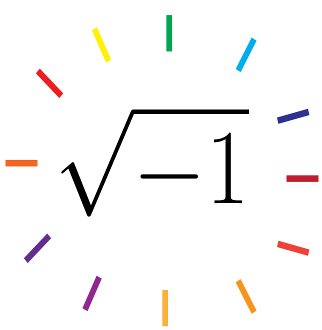



**Figure 10 | The Square Root of Minus One.** Here's the brand new number we need to fix our problems. It doesn't look like much, which is why our artist has added these bold, dramatic lines.

1 which wasn't totally new - Cardan kinda knew this
2 Although, not that imaginative.
3 Behold the great mystery number kweuasdktst whose symbol is ῷΩιἌ❧❁❃ and whos name must not be said.

**Figure 9 | Rafael Bombelli.** 1526-1572. Rafael was apparently pretty good at draining swamps, which is cool...I guess.

like our old numbers. But remember - it does have the exact special property we need - when we square it, the result is negative. Further, since this number is neither negative nor positive, it must be something new.

$$(\sqrt{-1})^2 = -1 \qquad (6)$$

Now if this all seems a bit fishy to you - like a slightly too convenient algebra trick, you're in good company.[4]

In fact, it's hard to introduce imaginary[5] numbers without them sounding like an arbitrary invention. However, before we dismiss the square root of minus one as some abstraction invented to torture students, let's review what we've learned thus far.

Cardan and Bombelli were genuinely stuck on a tough problem that they knew had a solution. What Bombelli was able to see, is that if he extended the existing number system, as had been done so many times before,[6] he could solve the problem.[7] Just as people needed fractions, zero, and negative numbers to solve new problems in past; to solve this problem, Bombelli now needed the square root of negative one to be its own, brand new, number.

Let's make sure we're clear about what it means for the square root of negative one to be its own number. If our new number is truly a discovery and *not* an invention, it should behave like the other numbers we already know about – it should follow the established rules of algebra and arithmetic. And it turns out it the square root of minus one does, for the most part.

Just as we can split apart the root of the product of 2 positive numbers,[8] we can also split apart our square

4 Cardan and Bombelli felt the same way!
5 Lateral
6 From integers to fractions to zero to negatives.
7 Remember that extending the number system before allowed us to solve problems we wouldn't have been able to otherwise, like x-3=1.
8 Like sqrt 2* sqrt 3 = sqrt 6, but be careful, sqrt -1 * sqrt -1 does not equal 1! (not all rules apply to imaginary numbers).

roots of negatives. The square root of minus 25 splits into the square root of 25 times the square root of negative one.

$$\sqrt{-25} = \sqrt{25} \cdot \sqrt{-1} \tag{7}$$

This process is important because it allows us to express the root of any negative using the square root of minus one. The square root of minus 25 becomes 5 times the square root of minus one.

$$\begin{aligned} \sqrt{-25} &= \sqrt{25} \cdot \sqrt{-1} \\ &= 5\sqrt{-1} \end{aligned} \tag{8}$$

We can use this process to expand the root of any negative number, and write it as some number we already know about, times the square root of minus one.

$$\begin{aligned} \sqrt{-16} &= \sqrt{16} \cdot \sqrt{-1} \\ &= 4\sqrt{-1} \\ \sqrt{-17} &= \sqrt{17} \cdot \sqrt{-1} \\ &= \sqrt{17}\sqrt{-1} \\ \sqrt{-18} &= \sqrt{18} \cdot \sqrt{-1} \\ &= 3\sqrt{2}\sqrt{-1} \end{aligned} \tag{9}$$

Let's quickly make sure our new numbers follow the same rules as other numbers. In algebra problems with $x$, only like terms can be added and subtracted. $2x+3x$ is $5x$, but $2+3x$ is just $2+3x$.[1] Likewise, $2\sqrt{-1}+3\sqrt{-1}=5\sqrt{-1}$, but $2 + 3\sqrt{-1}$ is just $2 + 3\sqrt{-1}$. Finally, unlike terms can be multiplied just as in algebra with $x$: 5 times $x$ is just 5 $x$, and 5 times $\sqrt{-1}$ is just $5\sqrt{-1}$.

Now, there are some cases where our new numbers behave a little strangely, but these can often be avoided by first separating out the square root of minus one. Table 3 shows some examples.

Now that we have a grasp on how our new numbers work, we can see how they fix one of our problems from last time. We now have a strategy for dealing with the roots of negatives. We can evaluate the square root of negative 9 we we're stuck on, and obtain 3 times $\sqrt{-1}$.

$$\begin{aligned} \sqrt{-9} &= \sqrt{-9 \cdot -1} \\ &= \sqrt{9} \cdot \sqrt{-1} \\ &= 3 \cdot \sqrt{-1} \end{aligned} \tag{10}$$

1 Because 2 and $3x$ are not like terms

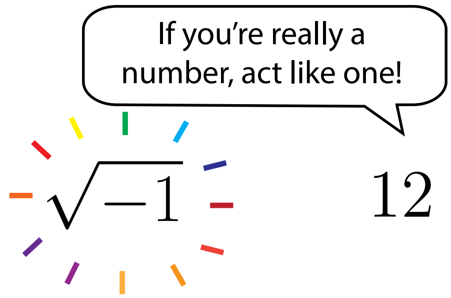



**Figure 11 | Suspicion.** Here we see twelve, an already established number, interacting with our new number for the first time. As we can see, it's not going well.

All this is important, but isn't enough to solve Cardan's problem – we still need to figure out how to deal with the cube roots of these numbers.[2] Bombelli was able to solve our problem through one more powerful insight here, and that's what we'll discuss next time.

| Algebra with $x$ | Algebra with $\sqrt{-1}$ |
|---|---|
| $\sqrt{2 \cdot 3} = \sqrt{2} \cdot \sqrt{3}$ | $\sqrt{-25} = \sqrt{25} \cdot \sqrt{-1} = 5\sqrt{-1}$ |
| $2x + 3x = 5x$ | $2\sqrt{-1} + 3\sqrt{-1} = 5\sqrt{-1}$ |
| $2 + 3x = 2 + 3x$ | $2 + 3\sqrt{-1} = 2 + 3\sqrt{-1}$ |
| $5 \cdot x = 5x$ | $5 \cdot \sqrt{-1} = 5\sqrt{-1}$ |
| $\sqrt{5} \cdot \sqrt{2} = \sqrt{10}$ | $\cancel{\sqrt{-5} \cdot \sqrt{-2} = \sqrt{-5 \cdot -2} = \sqrt{10}}$ |

**Table 3 | Algebra With the Square Root of Minus One.** The square root of minus one behaves like the numbers we already know about, for the most part. The one thing we are not allowed to do is shown in the bottom right square. Instead, we should: rewrite the expression as $\sqrt{-1}\sqrt{5} \cdot \sqrt{-1}\sqrt{2} = \sqrt{-1}\sqrt{-1}\sqrt{5}\sqrt{2} = -1 \cdot \sqrt{10} = -\sqrt{10}$

2 Cardan's formula in Figure 8 involve the cube root of the square root of negatives!

# Exercises 3

*Discussion*

3.1 Rafael Bombelli was born in 1526. For thousands of years before Bombelli's time, people used mathematics to solve all kinds of problems without any need for $\sqrt{-1}$. What do you think was so compelling about Cardan's problem that led Bombelli to consider using a completely new type of number to solve it?

3.2 Imagine you're working on a hard math problem. In fact, it's so hard, that no one on the planet has solved it yet. You work and work and work and get nowhere. Your friend Gus, who has been working on the same problem, excitedly tells you that he's found a solution! He shows you his work, and tells you that if works perfectly with one minor catch. To solve the problem he has allowed an entirely new kind of number to exist. A type of number that no one has used before. How would you respond to Gus?

*Drill*

Simplify the following. Helpful examples can be found in Table 3 and Equation 9.

3.3 $\sqrt{-16}$

3.4 $\sqrt{-51}$

3.5 $\sqrt{-4}$

3.6 $\sqrt{-5} + \sqrt{-5}$

3.7 $\sqrt{-5} + \sqrt{5}$

3.8 $16 + \sqrt{-16}$

3.9 $\sqrt{-4}\sqrt{4}$

3.10 $\sqrt{-512}$

3.11 $7/\sqrt{-7}$

3.12 $\sqrt{-4}\ \sqrt{-4}$

3.13 $1/\sqrt{-1}$

*Critical Thinking*

3.14 Why do all cubic functions have to have at least one root?

3.15 Find two numbers that multiply to 15 and add to 8.

3.16 Find two numbers that multiply to 11,187 and add to 212.

3.17 a) Find two numbers that multiply to 40 and add to 10.

b) How did part a go? This problem dates back to Cardan himself. In his book Ars Magna, Cardan gives the task of dividing 10 into parts whose product is 40, and calls this "manifestly impossible." You may have reached a similar conclusion. However, Cardan does push on in Ars Magna and gives the solution $5+\sqrt{-15}$, and $5-\sqrt{-15}$. Show that these numbers add to 10 and multiply to 40.

Interestingly, even though Cardan gave this solution, he didn't think very highly of it, saying: "So progressing arithmetic subtlety the end of which, as is said, is refined as it is useless."

*Challenge*

$$q(x) = 6x^4 - 5x^3 + 20x^2 - 20x - 16$$

3.18 One solution to $q(x)=0$ is $\sqrt{-4}$. Find all other solutions.

# Imaginary Numbers Are Real

## Part 4: Bombelli's Solution

Last time we decided to let $\sqrt{-1}$ be its own new type of number, hoping it would help us solve Cardan's problem. Doing this is helpful, but finding a solution requires one more insight from our friend Rafael Bombelli.

Bombelli knew that because of the way cubics are shaped, as shown in Figure 13, Equation 5 had to have a solution that didn't involve $\sqrt{-1}$, it had to be a plain old regular positive or negative number as he had seen before.

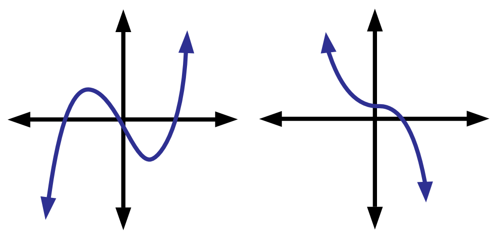


**Figure 13 | Some Cubics.** The end behavior of cubics mean they must have at least one real zero. More specifically, as we move to the left or right on our graph and follow our cubic curve, it must go up on one side and down on the other. Technically: as $x\to\infty$, $f(x)\to\infty$ and as $x\to-\infty$, $f(x)\to-\infty$, or as $x\to\infty$, $f(x)\to-\infty$ and as $x\to-\infty$, $f(x)\to\infty$.

His second big insight here was that for this to be the case, the root of minus one parts of each half of the equation must cancel out when added together! [1]

Bombelli used this idea to equate the two parts of the equation shown in Figure 12 to $a+b\sqrt{-1}$ and $a-b\sqrt{-1}$, where a and b are constants we need to find.[2]

$$\begin{aligned}\sqrt[3]{2+\sqrt{-121}} &= a+b\sqrt{-1}\\ \sqrt[3]{2-\sqrt{-121}} &= a-b\sqrt{-1}\end{aligned} \tag{11}$$

We can first eliminate that pesky cubed root by cubing both sides of Equation 11. The result is a particularly tough system of equations:

$$\begin{aligned}2 &= a(a^2-3b^2)\\ 11 &= b(3a^2-b^2)\end{aligned} \tag{12}$$

Bombelli was able to get around this through some clever guessing and checking. If we look at our original equation (Equation 5), and test a few integers[3] we eventually see that 4 is a solution! If we substitute 4 into our new equations, we can solve for a and b and obtain $a=2$ and $b=1$. These values make the two parts of Equation 11 equal to $2+\sqrt{-1}$and $2-\sqrt{-1}$.

1 They must be complex conjugates!

2 We're allowed to do this because $a+b\sqrt{-1}+a-b\sqrt{-1}=2a$

3 1...nope, 2...nope, 3...nope

$$x^3 = 15x+4, \tag{5}$$

$$x=\sqrt[3]{\frac{d}{2}+\sqrt{\frac{d^2}{4}+\frac{c^3}{27}}}+\sqrt[3]{\frac{d}{2}-\sqrt{\frac{d^2}{4}+\frac{c^3}{27}}}. \tag{4}$$

$$X=\sqrt[3]{\frac{d}{2}+\sqrt{\frac{d^2}{4}-\frac{c^3}{27}}}+\sqrt[3]{\frac{d}{2}-\sqrt{\frac{d^2}{4}-\frac{c^3}{27}}}$$

$$X=\sqrt[3]{\frac{4}{2}+\sqrt{\frac{4^2}{4}-\frac{15^3}{27}}}+\sqrt[3]{\frac{4}{2}-\sqrt{\frac{4^2}{4}-\frac{15^3}{27}}}$$

$$X=\sqrt[3]{2+\sqrt{4-125}}+\sqrt[3]{2-\sqrt{4-125}}$$

$$X=\sqrt[3]{2+\sqrt{-121}}+\sqrt[3]{2-\sqrt{-121}}$$

**Figure 12 | Reminder of Cardan's problem.** When we try to use Cardan's otherwise functional formula (Equation 4) to evaluate the simple cubic of equation 5, we run into a problem.

We can cube these to show that these are in fact equivalent to the left sides of Equation 11– and, more importantly, when we add the two parts, as Equation 4 tells us to do, we just get 4 – which we know is a solution to our original equation.[4] We have found the solution to Cardan's problem![5]

And what's really interesting is that our problem had nothing to do with the square root of minus one and neither did our answer – however, along the way, we found that by extending our number system to include the square root of minus one, we were able to find the solution.[6] And it turns out that extending the number system in this way is helpful in lots and lots of other problems as well.[7]

So what did Bombelli do to celebrate after discovering a number crucial to the future of science and mathematics?[8]

He actually did nothing. He discounted his discovery and basically said it was a hack.[9]

As ridiculous as that seems now, Bombelli drew pretty reasonable conclusion at the time. It just seemed a little

4 The $\sqrt{-1}$ parts cancel!

5 If this math seems a bit hacky, fear not, we'll learn a much more robust approach soon.

6 "The shortest path between two truths in the real domain passes through the complex domain." ---Jacques Hadamard

7 Lots of problems. Like lots and lots of problems. And not just math problems. Science problems. Engineering problems. Relationship problems. Ok, not that last one, but the rest are legit.

8 MATH RAVE?!?!

9 "The whole matter seemed to rest on sophistry rather than truth". - Rafael Bombelli. In case you aren't quite up to speed on your random words from the 17th century, Sophistry is the use of fallacious arguments, especially with the intention of deceiving.

too convenient – like this is a little trick devised just to solve problems like this.[1]

Squaring numbers, had, up until that point, largely been associated with what the operation is named for – squares. A square's area is equal to the length of its side, squared. So positive areas make sense – but what could a negative area be? What even is $\sqrt{-1}$?

Questions like these slowed down the development of imaginary numbers. It turns out there *is* a much deeper and richer meaning lurking below the surface, but it would take long after Bombelli's death to be revealed.

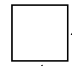
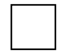
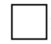
| Area | Associated Sqaure | Side Length |
|---|---|---|
| 16 |  4<br>4 | $\sqrt{16}=4$ |
| 9 |  3<br>3 | $\sqrt{9}= 3$ |
| 4 |  2<br>2 | $\sqrt{4}=2$ |
| 1 |  1<br>1 | $\sqrt{1}=1$ |
| -1 |  ?<br>? | $\sqrt{-1}=?$ |

**Table 4 | Sketchy Squares.** Historically, the operation of squaring was associated with what it was named for, squares. But what could a square with a minus area possibly look like?

1 This is how a lot of student seem to feel when they first meet $\sqrt{-1}$, and it's a completely legitimate reaction – if you feel this way about it – you're in good company.

# Exercises 4

*Discussion*

4.1 Why do you think Cardan and Bombelli mistrusted $\sqrt{-1}$?

4.2 The Greeks were arguably the most sophisticated mathematicians of the ancient world, and today are remembered for their many contributions to geometry. If Bombelli were to travel back in time, do you think he could convince the ancient Greeks to adopt $\sqrt{-1}$?

*Drill*

4.3 $\sqrt{-1}\cdot(1+\sqrt{-1})$

4.4 $(1-\sqrt{-1})\cdot(1+\sqrt{-1})$

4.5 $(2+3\sqrt{-1})\cdot(2+3\sqrt{-1})$

4.6 $(5-2\sqrt{-1})^2$

4.7 $(5\sqrt{-1}-2\sqrt{-1})^2$

Solve for a and b:

4.8 $2+\sqrt{-121} = a+b\sqrt{-1}$

4.9 $\sqrt{-4} = a+b\sqrt{-1}$

4.10 $\sqrt{4} = a+b\sqrt{-1}$

Find a solution by guessing and checking:

4.11 $a^2 + b^2 = 50$
$a - b^3 = 6$

4.12 $a(a^2 + b^2) = 0$
$a + b = 1$

4.13 $a(a^2 + b^2) = 2$
$a + b = 2$

*Critical Thinking*

In this section, we'll try to get a better feel for exactly how Bombelli solved Cardan's problem, $x^3=15x+4$.

4.14 How did Bombelli justify writing the equation below?

$$\begin{aligned} \sqrt[3]{2+\sqrt{-121}} &= a + b\sqrt{-1} \\ \sqrt[3]{2-\sqrt{-121}} &= a - b\sqrt{-1} \end{aligned} \tag{11}$$

4.15 Show that cubing the top line of Equation 11 results in:

$$2+\sqrt{-121} = a(a^2-3b^2) + b\sqrt{-1}(3a^2-b^2)$$

4.16 Show that:

$$\begin{aligned} 2 &= a(a^2-3b^2) \\ 11 &= b(3a^2-b^2) \end{aligned} \tag{12}$$

by equating the "normal" and "$\sqrt{-1}$" parts from exercise 4.15.

4.17 Back in exercise 4.15, we only cubed the top line of Equation 11. Cube both sides of the bottom line of 11, and turn the result into a system of equations as the did in Exercise 4.16. What do you notice?

4.18 Using Equation 12, and the fact that $x=4$ is a solution to our Cardan's problem, $x^3=15x+4$, derive that $a = 2$ and $b = 1$.

4.19 The solution we found to Equation 12, a = 2 and b = 1, mean that this must be true:

$$\sqrt[3]{2+\sqrt{-121}} = 2+\sqrt{-1}$$
$$\sqrt[3]{2-\sqrt{-121}} = 2-\sqrt{-1}$$

cube both sides to each equation to prove this.

4.20 Finally, show that using our solution above does result in a solution to Cardan's problem, $x^3{=}15x{+}4$.

4.21 So, what *is* the expression below (remarkably) equivalent to?

$$\sqrt[3]{2+\sqrt{-121}} + \sqrt[3]{2-\sqrt{-121}}$$

If you feel that Bombelli's method is a bit hacky, you're in good company. The sketchiness here really surrounds taking the cube root of our new types of numbers that involve √-1. Problems like this have confused lots of very smart mathematicians. In the next few sections, we'll find a nice way to think about these problems that makes this process much simpler and intuitive.[1]

*Challenge*

4.22 Solve for a and b:

$$\sqrt{\sqrt{-4}} = a + b\sqrt{-1}$$

4.23 Perhaps not surprisingly, Equation 12 has more than one solution. Show that other solutions to Equation 12 lead to more solutions to Cardan's original problem, $x^3{=}15x{+}4$. You may use technology. (Unless you're a complete bad-ass and don't need it)

1 Once we cover this exciting new ground, we'll revisit this problem in the Critical Thinking exercises in Section 8.

# Imaginary Numbers Are Real

## Part 5: Numbers are Two Dimensional

We left off with Bombelli's discovery that if he allowed the square root of minus one to be its own number, he could solve problems that had been stumping mathematicians for decades.

Despite the usefulness of his discovery, Bombelli and other mathematicians generally regarded it as a hack – after all, what could it possibly mean to take the square root of a negative number? Just like our friends zero and negative numbers before, the square root of negative one was generally regarded with suspicion because it didn't correspond to anything people could think of in the real world.

For this reason, the square root of minus one was given the terrible names imaginary or impossible. A century or so later Euler began using the symbol $i$ to indicate the square root of negative one, making the algebra less clunky.[1]

$$\sqrt{-1} = i \tag{13}$$

Unfortunately, the name imaginary stuck around, and that's still what we call these guys. In response, everything on the original number line gets the name real.[2]

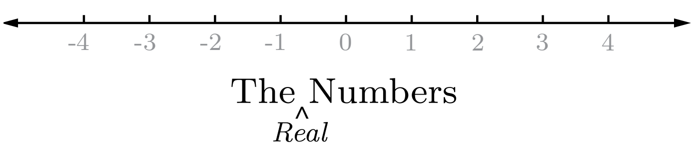



**Figure 15 | The numbers get re-branded.** Because everything that's not imaginary must be real, obviously...

When we put together a real and imaginary part, we get what we now call a complex number.

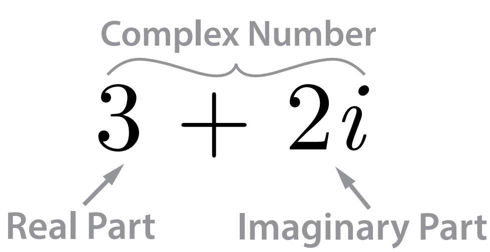



**Figure 16 | Complex Numbers.** When we put together a real and imaginary/lateral number, the result is what we'll call a complex number.

1 We'll start doing this too! I'm really sick of saying and writing root minus 1

2 Imagine how this must make lateral numbers feel. "Hey, we're going to call all the numbers, except for you, *real*."

**Figure 14 | Leonard Euler.** 1707-1783. Brilliant mathematician who got sick of writing $\sqrt{-1}$ all the time.

What is remarkable about this time period[3] is that although imaginary and complex numbers were used in calculations and derivations[4], the deeper meaning behind these numbers was left undiscovered for over 200 years after Bombelli's death.

Before we dive into this deeper meaning, let's think about $i$ algebraically for a moment.

If we raise $i$ to higher and higher powers, it doesn't get bigger as other numbers would. We know $i$ squared is -1 from the definition, and if we keep multiplying $i$ by itself, we see a pattern that repeats every four multiplications.[5] Over and over and over and over. Hold on to that fact for a few paragraphs.

| Real Numbers | Imaginary Numbers | Work |
|---|---|---|
| $2^1=2$ | $i^1=i$ | |
| $2^2=4$ | $i^2=-1$ | |
| $2^3=8$ | $i^3=-i$ | $i^3=i\cdot i^2=i\cdot -1$ |
| $2^4=16$ | $i^4=1$ | $i^4=i^2\cdot i^2=-1\cdot -1$ |
| $2^5=32$ | $i^5=i$ | $i^5=i^4\cdot i=1\cdot i$ |
| $2^6=64$ | $i^6=-1$ | $i^6=i^4\cdot i^2=1\cdot -1$ |
| $2^7=128$ | $i^7=-i$ | $i^7=i^4\cdot i^3=1\cdot -i$ |
| $2^8=256$ | $i^8=1$ | $i^8=i^4\cdot i^4=1\cdot 1$ |
| $2^9=512$ | $i^9=i$ | $i^9=i^4\cdot i^4\cdot i=1\cdot 1\cdot i$ |

**Table 5 | Patterns.** When we raise real numbers to higher and higher powers, they get bigger (for real numbers greater than 1). Interestingly, this is not the case with imaginary numbers. Instead, a pattern emerges.

3 ~1600-1800 AD

4 Pensively

5 It has a period of 4.

Let's return to our friend, the number line. Remember that all the numbers we know about[1] show up here, except imaginary numbers. They are nowhere to be found.

If we think back to our original problem with roots of negative numbers, we can visualize this using the number line. Remember the issue we had was finding a number, that when multiplied by itself, would yielded a negative.

To see this more clearly, we'll use arrows instead of dots to indicate numbers (Fig 17). Multiplying a positive by itself maintains direction on the number line – it stays positive. If we multiply by a negative, we flip directions, or rotate by 180°. Squaring a negative lands us in the positive numbers because we start on the left side with our first negative and rotate 180° when we multiply by the second negative. So there's no way to land on a negative number when squaring, a positive squared results in a positive and negative squared requires starting in negative territory, and when we multiply by the other negative we arrive back in the positive numbers.

So what we need is something in the middle. A number that when we multiply by it, only rotates 90°, not 180° as negatives do.

This is *exactly* what imaginary numbers do – $i$ squared is negative one, meaning that the first $i$ puts us 90° from the positive real numbers, and multiplying by $i$ rotates us 90° further, exactly where we wanted to be, firmly in negative number territory. [2]

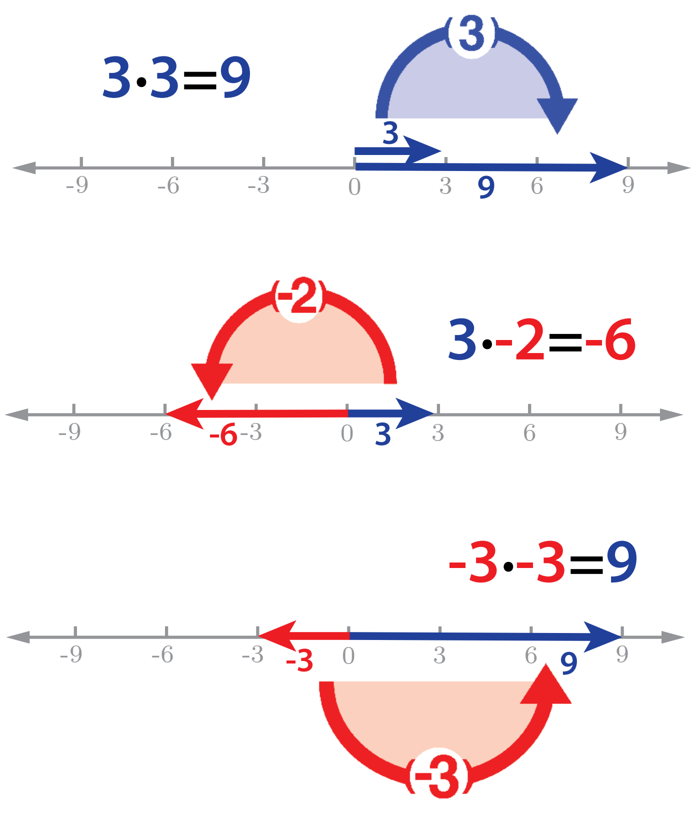



**Figure 17 | Multiplication using the number line.** Multiplying by a positive number maintains direction on the number line, while multiplying by a negative switches direction. Squaring can never result in a negative number, because multiplying a positive time a positive maintains direction, while multiplying a negative by a negative flips us back in the positive direction.

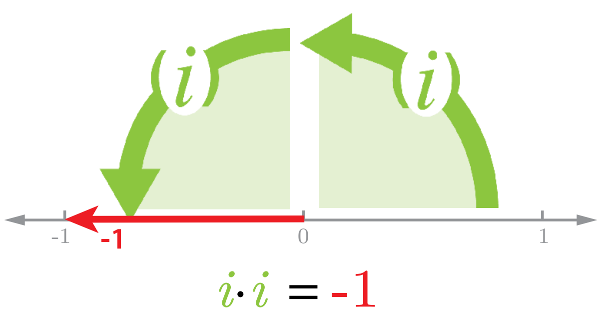



**Figure 18 | Multiplication by $i$.** One way to understand what multiplying by $i$ is as a rotation by 90°. Multiplying $i$ by itself moves us a total of 180°, exactly what we need to land on -1.

Back to that fact you're hanging on to from Table 5. Since multiplying by $i$ corresponds to one 90° rotation, if we place our imaginary axis at a right angle to our number line, our algebra will perfectly fit with our geometry (Fig 19).

If we start with the real number 1 and multiply by $i$, algebraically we get $i$ – which geometrically corresponds to a 90° rotation from 1 to $i$. Multiplying by $i$ again results in $i$ squared, which, by definition, is minus 1 which again matches a 90° rotation from $i$. As we keep raising $i$ to higher and higher powers, we keep rotating around, with our values repeating every fourth power, just as they did

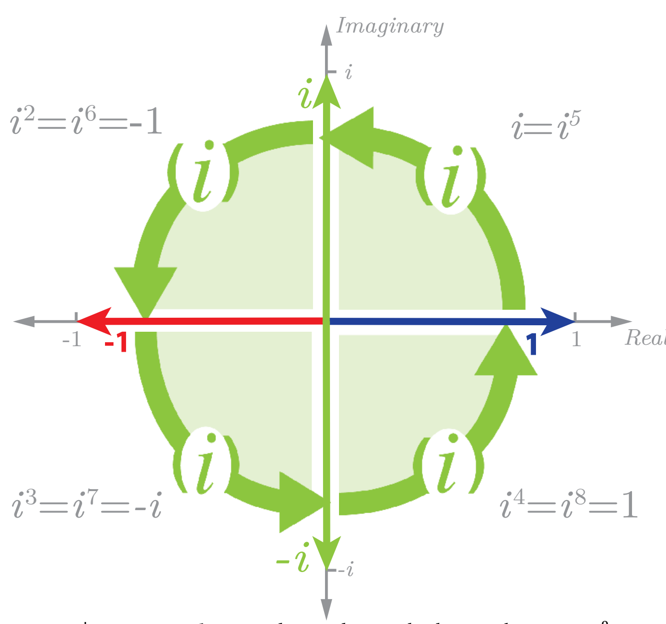



**Figure 19 | Pattern Matching.** Understanding multiplication by $i$ as a 90° rotation perfectly matches the behavior we see in Table 5. Crazy.

1 Integers, zeros, fractions, rational numbers, irrational numbers, transcendental numbers

2 Minus $i$ does the same thing!

algebraically.

So the insight here is that imaginary numbers do not exist apart from the real numbers, but right on top of them, hiding in a perpendicular dimension.

This is the deeper meaning beneath imaginary numbers. They aren't just some random extra number or hack - they are the natural extension of our number system from 1 dimension to 2.

Numbers are 2 dimensional.[1]

And what's even more remarkable, is that if we accept this – that numbers have a hidden dimension – we end up not only with more complete mathematics, but incredibly powerful tools for science and engineering.

Next time we'll show how and why thinking of numbers this way is useful.

1 If anyone tells you this is obvious or easy they're lying. Despite using $i$ in calculations, generations of very bright mathematicians missed this for over 200 years.

# Exercises 5

*Discussion*

5.1 Why might interpreting multiplication by a negative number as a 180° rotation make sense?

5.2 Why might it make sense for imaginary numbers to be placed at a right angle to real numbers?

5.3 Do you think numbers are two-dimensional? Why or why not?

*Drill*

5.4 $i^2$

5.5 $i^{20}$

5.6 $i^{21}$

5.7 $i^{22}$

5.8 $i^{23}$

5.9 $i^{1000}$

5.10 $i^{1001}$

5.11 $i^{1002}$

5.12 $i^{1003}$

5.13 $i^{-1}$

5.14 $i^{-20}$

*Critical Thinking*

5.15 As shown in Figure 18, multiplying $i$ by itself ($i^2 = -1$) can be interpreted as a 180° rotation from the positive real axis to the negative real axis. Since $(-i)^2$ also equals -1, what could multiplying by $-i$ represent?

5.16 True or false?

$$\sqrt{-1} < 2$$

The central idea of this section, that imaginary numbers should be placed at a right angle to the real numbers, is not obvious. We know this because it took over 200 years after Bombelli's death to discovered. This idea was finally unearthed by two non-mathematicians separately, Casper Wessel and Jean-Robert Argand, around 1800. However, like most good ideas in math and science, this one took quite some time to be accepted. In 1831, the mathematician Augustus deMorgan said of the topic:

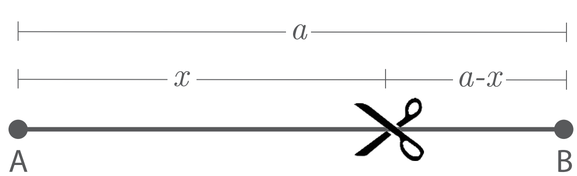
*"We have shown the symbol $\sqrt{-1}$ to be void of meaning, or rather self-contradictory and absurd."*

It's interesting to think about how exactly mathematicians struggled with and argued about these ideas during this period. A long standing practice in mathematics, (dating most notably back to the Greeks), is the idea of geometric proof. Speaking very roughly: if we can represent a mathematical idea visually, it must be true. Whether or not this is the right way to approach mathematics is an open question, but this way of thinking is certainly valuable, and fueled mathematical development for millennia. When $\sqrt{-1}$ began showing up in mathematics in the 17th century, many mathematicians attempted and failed to find a way to explain or understand $\sqrt{-1}$ visually/geometrically. This difficulty contributed to the mistrust of $\sqrt{-1}$.

Now that $\sqrt{-1}$ has a strong visual interpretation (existing at a right angle to the real numbers) and has become firmly rooted in mathematics, it's interesting to look back at historical attempts to show visually what the $\sqrt{-1}$ could possibly mean. One interesting example comes from an 1803 publication from Lazare Carnot:

*Given a line segment AB of length a, how can it be divided into two shorter segments so that the product of their lengths is equal to one-half the square of the original length?*

Let's temporarily assume that drawing pictures to represent mathematics is a good idea and draw a picture.

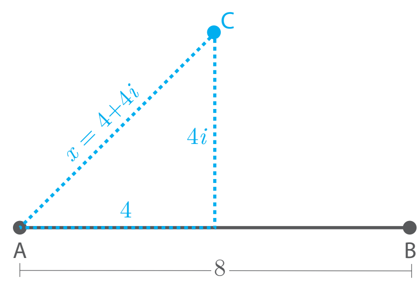



From Carnot's problems statement, we can write the formula:

$$x(a-x) = \frac{1}{2}a^2$$

5.17 Let's say $a = 8$ in Carnot's problem. Solve for $x$, the length of one of our segments.

How did Exercise 5.17 go? When Carnot solved this equation for $x$, his result was a complex number (hopefully your result was complex as well, specifically $4+4i$ and $4-4i$). Carnot interpreted the fact that his result was complex as meaning that the cut point we're looking for does not lie on $AB$, thus the problem was physically impossible. However, other mathematicians (such as the Frenchman Abbè Adrien-Quentin Buèe) interpreted this result differently. Roughly speaking, Buèe agreed that the problem was physically impossible, but argued that the results was nonetheless meaningful. More specifically, if we allow imaginary numbers to exist in the dimension perpendicular to the real axis, we obtain some type to picture like this:




5.18 What is the distance between A and C, shown above?

5.19 Does your answer to 5.18 make sense in terms of Carnot's original problem?

So, if we allow imaginary numbers to live perpendicularly to the real numbers, the answer we obtain does make some type of sense. Sure, there's still no way to cut the string the achieve what we want physically, but the results also are not meaningless. They tell us there is an answer, but we must move 4 units in the perpendicular (imaginary) direction to find it. Problems like this don't make a very strong case for $\sqrt{-1}$, which helps me understand why this matter was so contentions for so long. Only later, when our geometric interpretation $\sqrt{-1}$ was shown to be indispensable to math and science did these ideas become widely accepted.

# Imaginary Numbers Are Real

## Part 6: The Complex Plane

Last time we arrived at an incredibly powerful tool in mathematics, science, and engineering: the complex plane. The complex plane is an extension of the number line, where we include the imaginary dimension vertically. Just as we can plot xy coordinates on the xy plane, we can plot complex numbers on the complex plane.

This arrangement is why Gauss preferred the term lateral over imaginary and inverse instead of negative.[1] Gauss suggested we should call the numbers to the right of the origin are direct, numbers to the left inverse, and numbers up or down lateral.

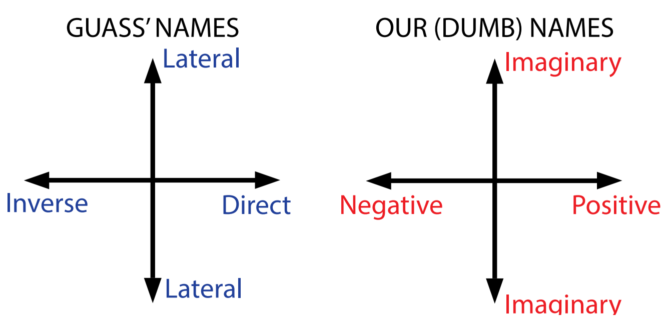



**Figure 20 | Gauss' names for numbers and ours.** "That this subject [imaginary numbers] has hitherto been surrounded by mysterious obscurity, is to be attributed largely to an ill adapted notation. If, for example, +1, -1, and the square root of -1 had been called direct, inverse and lateral units, instead of positive, negative and imaginary (or even impossible), such an obscurity would have been out of the question." -Gauss

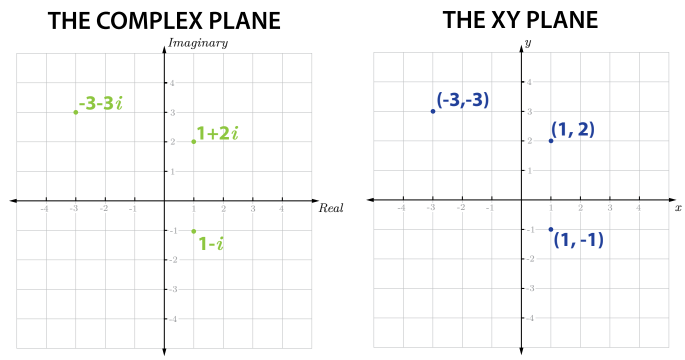



**Figure 20 | The complex plane and its xy cousin.** These planes have a lot in common. They both come outfitted for adventure with cartesian coordinate systems. However, the complex plane has has a special twist that makes it way hotter than its boring cousin.

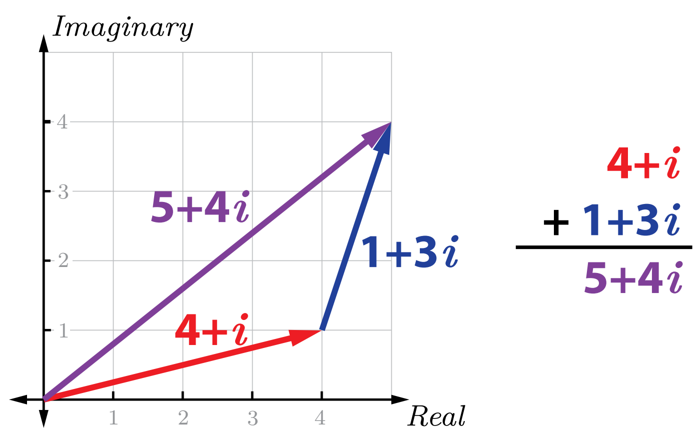



**Figure 21 | Adding complex numbers.** We add complex numbers by adding the real and imaginary parts seperately. This has a nice visual interpretation, adding complex numbers is exactly like adding vectors, we place the second arrow at the end of our first, and wherever it's tip lands is our result.

Now that we've seen the complex plane, let's discuss why it's so powerful. We've seen two dimensional planes before[2] where each axis represents a different quantity, in fact we started our whole series with one.

In a normal xy plane, there's no required connection between the dimensions, no rules about how they should relate to one another. On the complex plane however, we have the rules of algebra with complex numbers we discussed earlier. These rules impose a very specific and useful relationship between our two dimensions.

This first rule is how complex numbers add and subtract. The real and imaginary parts add independently, making complex numbers and the complex plane useful for problems involving movement in two dimensions. If we travel in one direction for a certain distance, and then in another direction, we can add the components of each part of our trip together to find the total distance we have traveled in each direction (Fig 21).[3]

So that's cool, but as you may already know, we can do the same exact thing with vectors. Where complex numbers really get interesting is through multiplication.

We can multiply complex numbers together by foiling, just as we do with binomials in algebra, with the minor complication that we know $i^2$ can be replaced with -1.

This is a perfectly valid algebraic solution to our problem

$$\begin{aligned}&(3+2i)(1+2i)\\&=3+6i+2i+4i^2\\&=3+8i-4\\&=-1+8i\end{aligned} \tag{14}$$

lem – but is only half the picture. There is another, equally valid, way to think about multiplying complex numbers. And it has everything to do with the complex plane.

Instead of just telling you what this interpretation is, it will be more way more fun to try to figure it out with

1 We first saw this idea from Gauss back in Part 1.
2 Like the x, y (Cartesian coordinate system) plane
3 AKA Displacements

some examples.[1] To discover this deeper meaning for yourself[2], all you need know is the following: how to multiply complex numbers algebraically as we just did, how to plot numbers on the complex plane, the Pythagorean theorem, and finally how to use arctangent to find angles. [3]

What's pretty cool here is that if you're able to figure out what we're after here – the interpretation of complex number multiplication using the complex plane – you'll have figured out a super useful bit of math that was unknown to the smartest mathematicians on the planet until only 2 centuries ago.

Next time we'll uncover this interpretation of complex multiplication using the complex plane – and we're going to do it with only 4 examples:

1. $(4+3i)\cdot i$
2. $(4+3i)\cdot 2i$
3. $(4+3i)\cdot(4+3i)$
4. $(2+i)\cdot(1+2i)$

Through considering each of these problems on the complex plane and looking at the patterns that naturally emerge, we'll arrive at the deeper meaning we're looking for.

Do try this at home, you'll find these problems nicely laid out for you in the *Critical Thinking* section of the following exercises. Even if you're already a boss at complex numbers or have no idea what I'm talking about – I promise it's a valuable process - and we'll sort our all the details next time. [4]

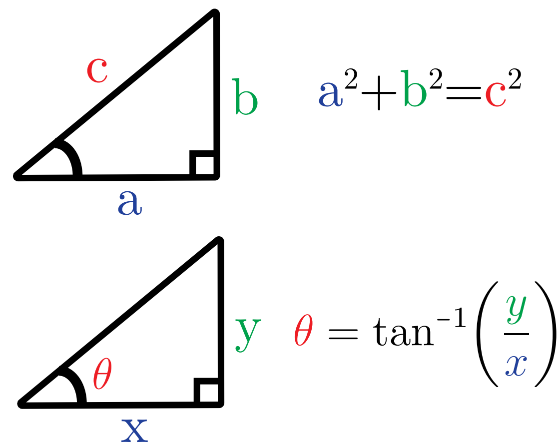



**Figure 22 | Tools you'll need.** These are the tools you'll need for your mission, should you choose to accept. You will also need to use complex multiplication as shown in Equation 14 and to plot points on the complex plane as shown in Figure 20.

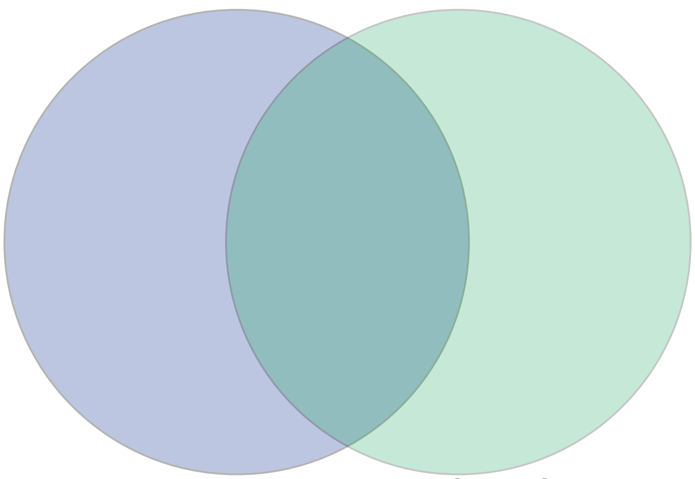
1 ...I promise.
2 Just as Wessel, Argand, and Gauss did a couple centuries ago… but Bombelli failed to (!).
3 …or you could just google. But I promise you'll learn less that way.
4 Do it. Seriously. Do it. Do the math problem. Sit down and do it. It will take like half an hour tops. And if you figure it out, you can tell all your friends your smarter than the 16th Century mathematician Gerolamo Cardano. And they'll definitely think you're really cool. For sure. Do it.

# Exercises 6

*Discussion*

6.1 Why did Guass prefer the names direct, inverse, and lateral over positive, negative, and imaginary?

6.2 What do the xy plane and the complex plane have in common? How are the different? Fill in your answers in the super cool Venn diagram below.




*Drill*

*Simplify:*

6.3 $(1+i)(1+i)$

6.4 $(1+i)(1\text{-}i)$

6.5 $(2+2i)(1\text{-}2i)$

6.6 $\text{-}i(3+5i)$

6.7 $(x+(1+i))((x+(1\text{-}i))$

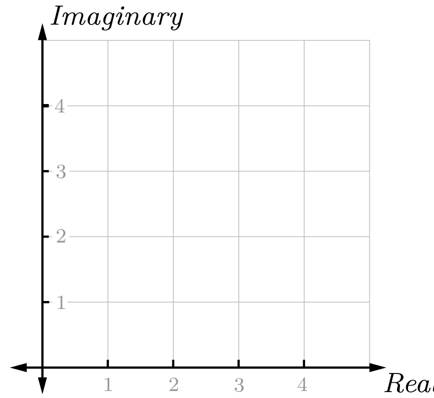
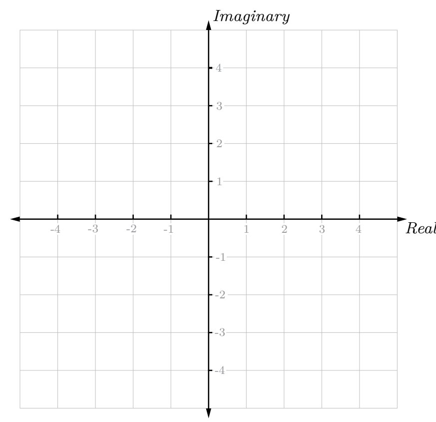
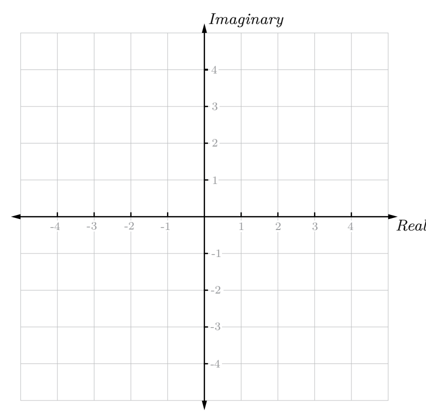
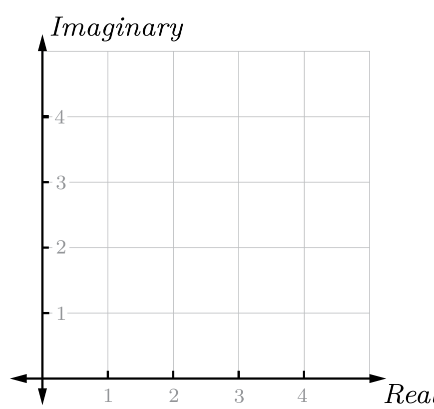
6.8 $(\text{-}2\text{-}i)^2$

*Simplify (your answer should not have an i in the denominator):*

6.9 $\dfrac{1}{1-i}$

6.10 $\dfrac{i}{1+2i}$

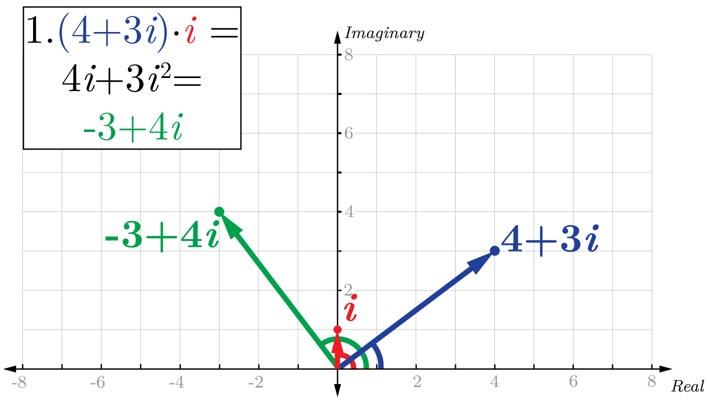
6.11 $\dfrac{\frac{1}{2}+\sqrt{3}i}{1-\sqrt{2}i}$

| | Problem | Solve Algabraically | Solve Graphically |
|---|---|---|---|
| 6.12 | $(2+i)\ +\ (1+2i)$ | |  |
| 6.13 | $(-3+i)\ +\ 2i$ | |  |
| 6.14 | $(2+i)\ -\ (1+2i)$<br>**Vaguely helpful hint: Try writing as an addition problem.* | |  |
| 6.15 | $2\cdot(2+i)$ | |  |

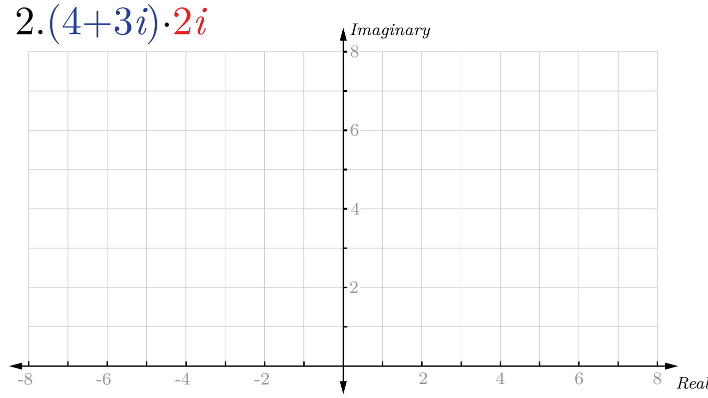
*Critical Thinking*

6.16 **What is the connection between complex multiplication and the complex plane?** To figure this out, we'll look at four examples. This is by far my favorite exercise in the entire workbook. It's cool because if you figure it out, you'll have discovered something that mathematicians missed for over 200 years.

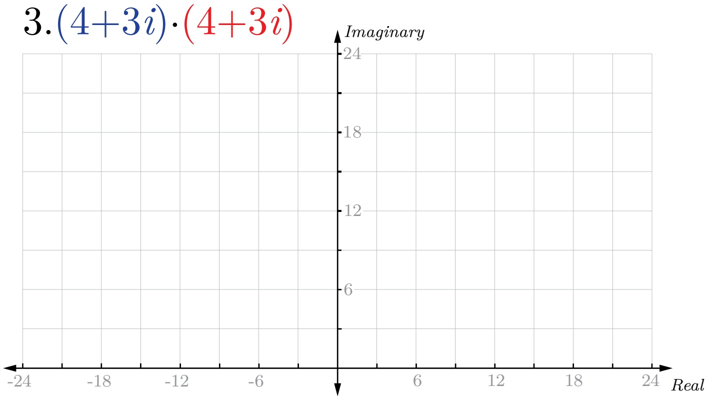
a) To get started, for each example compute the product, plot the two numbers we're multiplying together, and plot the result. The first example has been done for you.[1]



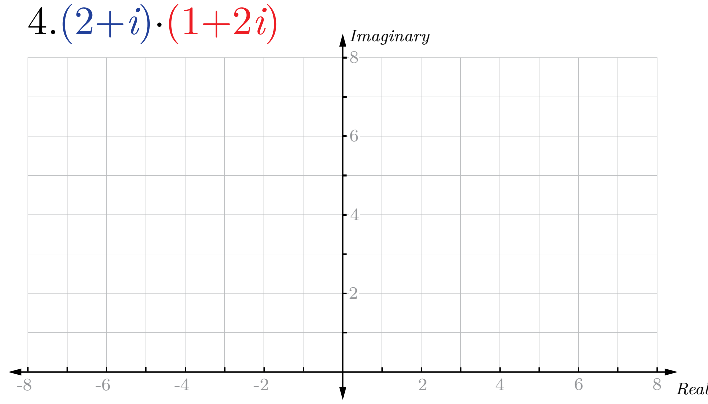

2.$(4+3i)\cdot 2i$




3.$(4+3i)\cdot(4+3i)$




4.$(2+i)\cdot(1+2i)$




b) Use your results from a to complete the table below.

| Problem | Result | Angle 1 | Angle 2 | Result Angle | Distance 1 | Distance 2 | Result Distance |
|---|---|---|---|---|---|---|---|
| $(4+3i)\cdot i$ | $-3+4i$ | 36.9° | 90° | 126.9° | 5 | 1 | 5 |
| $(4+3i)\cdot 2i$ | | | | | | | |
| $(4+3i)\cdot(4+3i)$ | | | | | | | |
| $(2+i)\cdot(1+2i)$ | | | | | | | |

c) What patterns do you see? What is the connection between complex multiplication and the complex plane?

1 Wow, that Stephen Welch is such a nice guy, doing a whole problem for you!

# Imaginary Numbers Are Real

## Part 7: Complex Multiplication

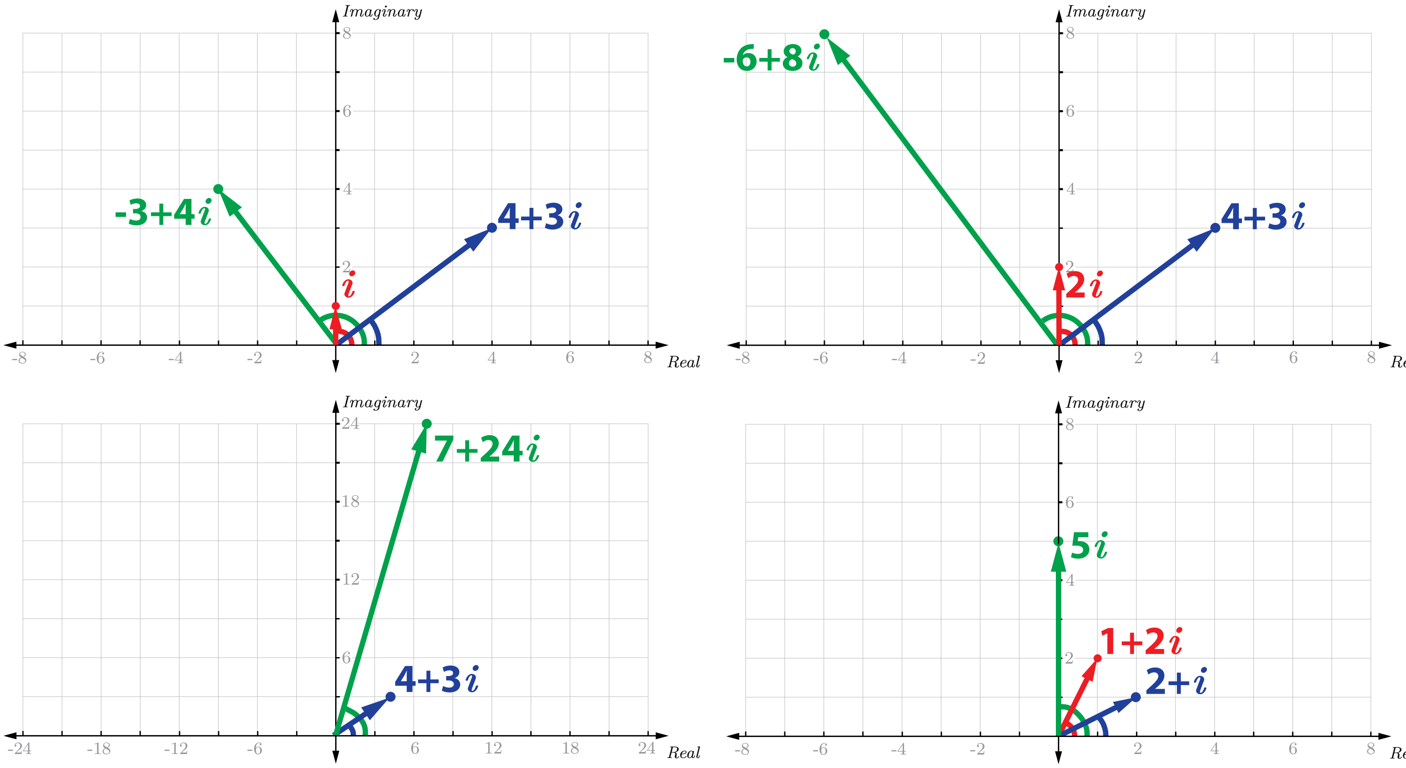


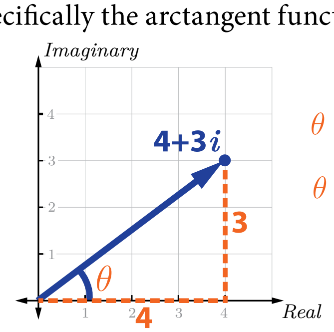

**Figure 23 | Our four examples from last time plotted on the complex plane.** We'll use these examples to figure out the connection between the complex plane and complex multiplication.

Last time we left off with a real math problem: what is the connection between complex multiplication and the complex plane?

To get to the bottom of this we'll use the four examples we mentioned last time. For each example, we'll plot the two numbers we're multiplying together in Figure 23. We'll also compute the result algebraically and add it to each plot.

Our job now is to look for patterns. Back in part five, we learned that $i$ has something to do with rotation on the complex plane. So a good thing to keep track of here will be the angle our complex numbers make with the real axis.

We can determine our angles using a little trigonometry, specifically the arctangent function.




$$\theta = \tan^{-1}\left(\frac{3}{4}\right)$$

$$\theta = 36.9^\circ$$

**Figure 24 | Arctangent.** We'll use the arctangent function to find the angle each of our complex numbers makes with the real axis.

For each example, we'll add the angle of each complex number to Table 5. Now let's look for a connection between our three angles.

| Problem | Angle 1 | Angle 2 | Result | Result Angle |
|---|---|---|---|---|
| $(4+3i)\cdot i$ | 36.9° | 90° | $-3+4i$ | 126.9° |
| $(4+3i)\cdot 2i$ | 36.9° | 90° | $-6+8i$ | 126.9° |
| $(4+3i)\cdot(4+3i)$ | 36.9° | 36.9° | $7+24i$ | 73.8° |
| $(2+i)\cdot(1+2i)$ | 26.6° | 63.4° | $5i$ | 90° |

**Table 6 | Patterns?** What is the connection between our angles for each example?

After a little pondering[1], we see that the angle of our result is exactly equal to the angles of the numbers we're multiplying, added together.

This is the first of half the connection we're looking for: when multiplying on the complex plane, the angle of

1 Hmmmm...

our result is equal to the sum of the angles of the numbers we're multiplying.

Let's now have a closer look at our first two examples. Notice that the angles are identical, but the resulting complex numbers are not. This means that just keeping track of angles alone is not enough to sufficiently describe complex multiplication in the complex plane – there is something else going on.

So what *is* the difference between these examples? It looks like multiplying by $2i$ has pushed our result further from the origin than multiplying by $i$.

A good follow up question is "how much further?"

We can measure the distance between the origin and our complex numbers by forming right triangles and using the Pythagorean theorem.

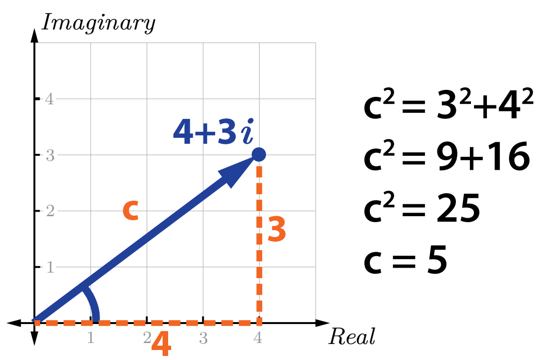



**Figure 25 | Pythagorean Theorem.** We'll use the Pythagorean Theorem to find the distance between each of our complex numbers and the origin.

Just as before, let's compute our measurement for each example and look for patterns We'll put the results in Table 7.

| Problem | Distance 1 | Distance 2 | Result | Result Distance |
|---|---|---|---|---|
| $(4+3i)\cdot i$ | 5 | 1 | $-3+4i$ | 5 |
| $(4+3i)\cdot 2i$ | 5 | 2 | $-6+8i$ | 10 |
| $(4+3i)\cdot(4+3i)$ | 5 | 5 | $7+24i$ | 25 |
| $(2+i)\cdot(1+2i)$ | $\sqrt{5}$ | $\sqrt{5}$ | $5i$ | 5 |

**Table 7 | More Patterns?** What is the connection between our distances for each example?

After some more pondering[1], we see that if we multiply the distances from the origin of the numbers we're multiplying, we obtain the distance from the origin of the result!

We now have the complete picture. When we multiply complex numbers on the complex plane, their angles from the real axis add, and their distances from the origin multiply.

This is the connection we were looking for between complex multiplication and the complex plane.

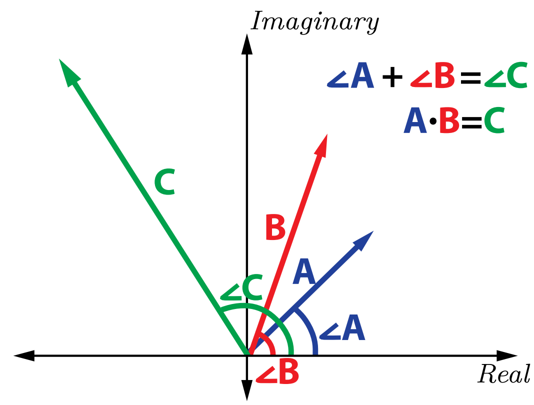



**Figure 26 | Our Result.** When we multiply two complex numbers, their angles add and distances to the origin multiply.

We now have completely separate, but completely equivalent interpretations of complex multiplication. To multiply two complex numbers together, we can follow the rules of algebra, OR, we can find each numbers distance from the origin and angle to the real axis on the complex plane and multiply and add each.

And what's really cool here is that although these approaches look and *are* totally different, but they *do* the same exact thing. What we're seeing here is the same underlying process from two separate vantage points. I really like this idea, because it reminds me that there's more to math than what we see on the page. There are deeper truths embedded in our universe, and math is one way of expressing them.

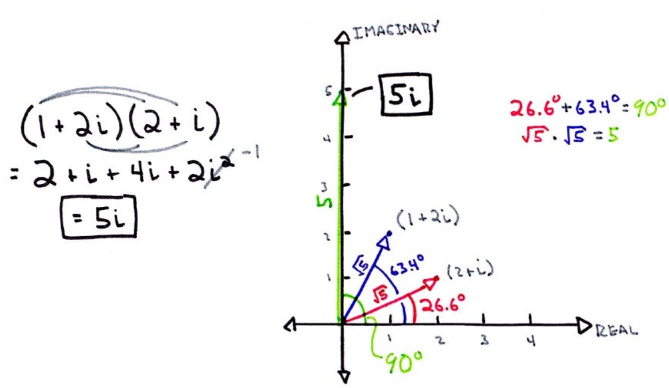



**Figure 27 | Two perfectly good ways to multiply complex numbers.** We can multiply complex numbers algebraically as shown on the left, or we can use the complex plane as shown on the right.

1 Also hmmmmm...

Now that we've made our discovery, let's formalize our results a bit. We found that the quantities we should keep track of when multiplying complex numbers on the complex plane are the distance from the origin and the angle from the real axis.

These quantities turn out to be so important, that we use them as another way to write complex numbers. Instead of writing complex numbers as the sum of their real and imaginary parts[1], we instead write them as their distance from the origin and the angle they make with the real axis. This is called polar form, and the distance from the origin gets a special name, magnitude.[2]

Multiplying complex numbers in polar form is super easy – we just multiply the magnitudes and add the angles. Division is pretty simple too, especially compared to dividing in rectangular form - to divide in polar form we divide our magnitudes, and subtract our angles.

Next time we'll show that this discovery is not only cool, but useful. We'll use the complex plane to make hard algebra problems easier, faster, and more intuitive.

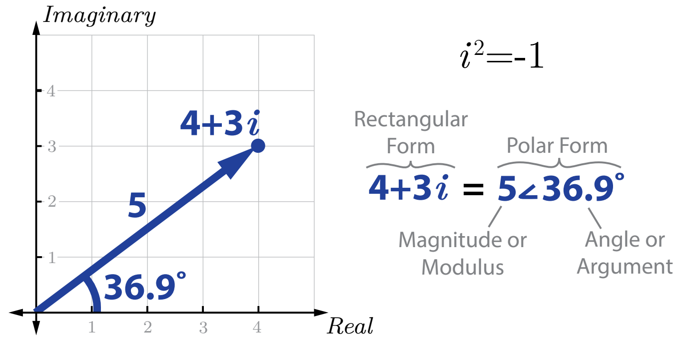



**Figure 28 | Two ways to write complex numbers.** We can write complex numbers in rectangular or polar form.

1 Rectangular form!

2 Magnitude also goes by the name modulus, and the angle is also called the argument.

# Exercises 7

*Discussion*

7.1 The main idea of this section is that there are two very different, but completely equivalent methods we can use to multiply complex numbers. How could such seemingly different approaches can yield the same exact results?

7.2 Can you think of an example where using the polar form of complex numbers would make life easier?

7.3 Why do you think it took over 200 years for the connection between complex multiplication and the complex plan to be discovered?

*Drill*

*Convert to polar form:*

7.4 1+$i$

7.5 2-3$i$

7.5 $\frac{1}{2} + \frac{\sqrt{3}}{2}i$

*Convert to rectangular form:*

7.6 $1\angle 45^\circ$

7.7 $5\angle 30^\circ$

7.8 $5\angle 150^\circ$

7.9 $3\angle 270^\circ$

*Critical Thinking*

*Solve these suckers:*

7.10 (1+$i$)(1+$i$)

7.11 $(\sqrt{2}\angle 45^\circ)(\sqrt{2}\angle 45^\circ)$

7.12 How are 7.10 and 7.11 related?

*Convert to polar form, solve, and convert your answer back to rectangular form. For extra credit draw a pretty picture that shows what's happening.*[1]

7.13 $(2+2i)(2+2i)$

7.14 $\left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)\left(\frac{\sqrt{3}}{2}+\frac{1}{2}i\right)$

7.15 $\frac{\left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)}{\left(\frac{\sqrt{3}}{2}+\frac{1}{2}i\right)}$

7.16 $(1+i)(1+i)(1+i)$

7.17 $(1+i)(1+i)(1+i)(1+i)$

7.18 $\frac{2i}{1+i}$

7.19 $\frac{-2}{i}$

1 Each problem is worth 1 extra credit point. 5 extra credit points my be converted into 1 cool point at the discretion of the point holder. Upon achieving 3 cool points, demand some type of prize from your teacher/professor/friends/family members. Be persistent.

*Challenge*

In school you may have been required to memorize some formulas like these:

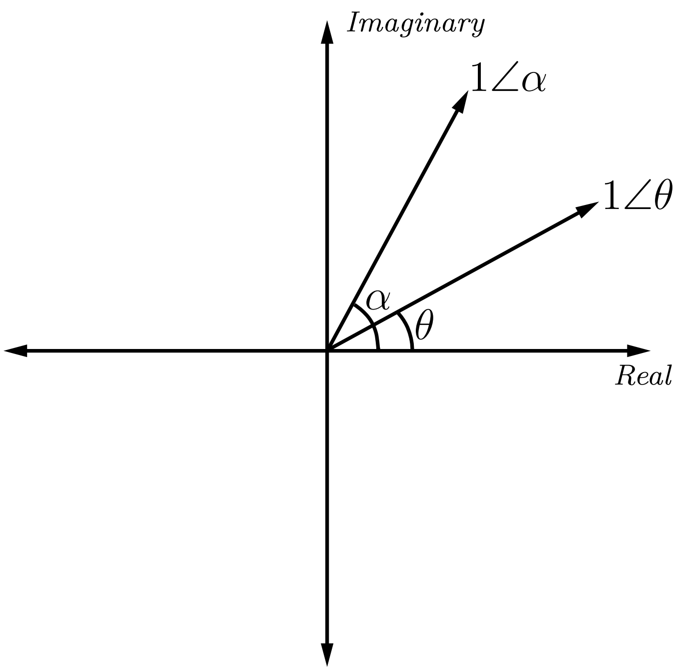
(1) $cos(\alpha + \theta) = cos(\theta)cos(\alpha) - sin(\theta)sin(\alpha)$
(2) $sin(\alpha + \theta) = sin(\theta)cos(\alpha) + cos(\theta)sin(\alpha)$
(3) $cos(2\theta) = cos^2(\theta) - sin^2(\theta)$
(4) $sin(2\theta) = 2sin(\theta)cos(\theta)$

Since memorization is basically the worst thing ever[1], let's try to find a way to never have to memorize these formulas again.

7.20 Let's begin by considering two complex numbers, each with a magnitude of 1:

$$1\angle\alpha \qquad 1\angle\theta$$

To make things a little more visual, let's say our two complex numbers look something like this on the complex plane:




In terms of $\alpha$ and $\theta$, compute the product:

$$(1\angle\alpha)(1\angle\theta)$$

and add it to the plot above in the generally correct location.

1 Memorization does not equal learning!

7.21 Convert to rectangular form using sin and cos:

a) $1\angle\alpha$

b) $1\angle\theta$

c) $1\angle(\alpha + \theta)$

7.22 Substitute your answers from 7.21 into[2]:

$$(1\angle\alpha)(1\angle\theta) = 1\angle(\alpha + \theta)$$

Expand and simplify your result.

7.23 Derive Equations 1 and 2 by equating the real and imaginary parts of each side of your result to 7.22.

7.24 Derive Equations 3 and 4 by considering the special case where $\alpha = \theta$.

2 This is hopefully the equation you obtained in 7.20!

# Imaginary Numbers Are Real

Part 8: Math Wizardry

Let's solve a simple equation:

$$x^3 = 1 \tag{15}$$

What value of $x$ would make this equation work?

If you said 1, great job, $1^3=1$. We've found one answer. Now, are there any more answers?

Way back in part one we introduced the Fundamental Theorem of Algebra, which says that a polynomial must have as many roots[1] as its highest power. We can rearrange our equation as $x^3$-1=0 to make it a more obvious polynomial, and since our highest power[2] is 3, this equation must have 3 solutions.

There is a way to find all three answers without using complex numbers along the way, but it involves perfect cube factoring and the quadratic formula – and takes like 7 steps.

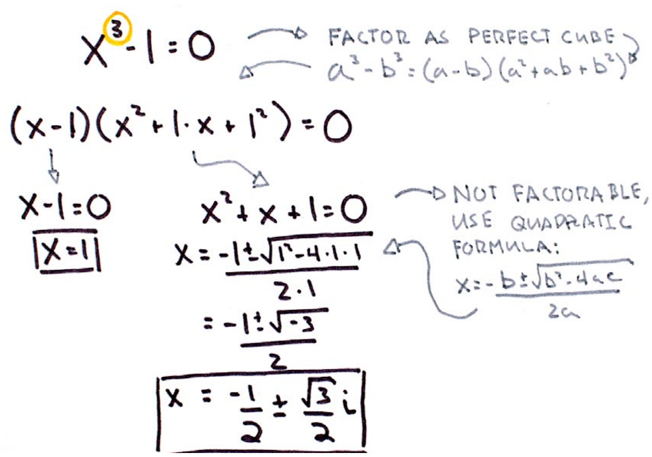



**Figure 29 | Solving Equation 15 by factoring.** We can solve Equation 15 by factoring, but doing it this way is hella long. And you have to remember how to factor the difference of perfect cubes. And the Quadratic Formula. Lame.

Instead, let's try to solve the problem visually using the complex plane. Our question, in words, is: what numbers, when multiplied by themselves 3 times, equal 1?

$$x^3 = x \cdot x \cdot x = 1$$

Let's think about this problem using the polar form of complex numbers we discussed last time. We can think about 1, in the complex domain, as a number with a magnitude of 1, and an angle of zero. Or 360.[3]

When we multiply numbers in the complex domain, their magnitudes multiply and their angles add. Our result should have a magnitude of 1, since 1 times 1 times

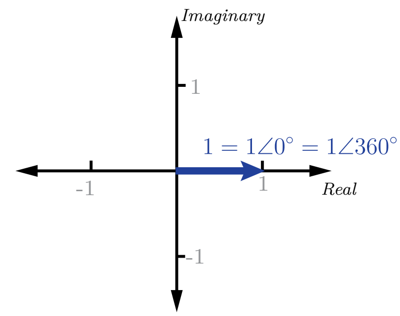



**Figure 30 | The number one.** The real number one shown on the complex plane.

1 is…1[4], if we give each our $x$'s a magnitude of 1,[5] our resulting magnitude will work out to 1 – easy. But what about our angles?

We know that when multiplying complex numbers, our angles add, so we need an angle, that when added together 3 times gives 0.[6] Or 360. 360 seems a little more reasonable, so what's the correct angle here?

Well, since we're dividing 360 into three even parts, the right answer here is 360 divided by 3, or 120°. When we put this together with the magnitude of 1, we have a second solution! It's the complex number with a magnitude of 1 and an angle of 120.

This answer makes a lot of sense on the complex plane:

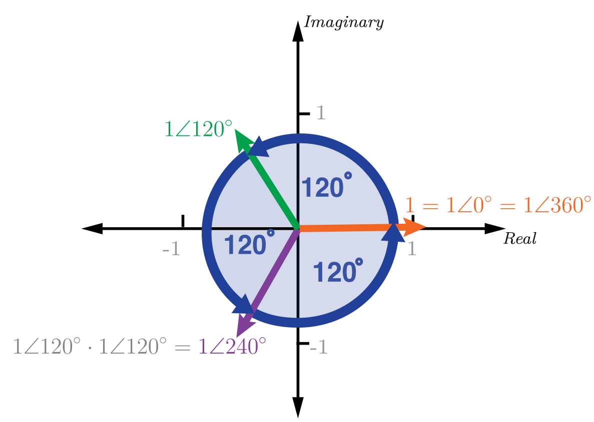



**Figure 31 | A solution!** Our second solution is shown in green, the complex number with and angle of 120 degrees and magnitude of 1. Here, we can see that multiplying this number by itself 3 times results in 1.

Multiplying our answer by itself once results in a magnitude of 1 at an angle of 240°, and multiplying by our answer again lands us exactly where we wanted to be – at a magnitude of one and an angle of 360° – also known

1 AKA Solutions. Aka Zeros.
2 AKA Degree
3 or -360, or -720, 720, or 1080, or if you're up to it, 360*n, where n an integer.
4 Or 1 to whatever power you want…
5 Because 1 = cubed root of 1. 1*1*1 =1!
6 This would actually yield our first answer, 1!

as...one.

So we've found an answer, the complex number with a magnitude of one and an angle of 120°. This may seem like a strange answer for algebra class, so let's put it back in Cartesian form.

Our good friend the unit circle can save us some time here. We would like to know the rectangular coordinates of a point on the unit circle at an angle of 120°.[1] According to the unit circle, our answer is -1/2 for the real part and $\sqrt{3}/2$ for the imaginary part.[2]

Ok, so now that we have an answer in rectangular form, let's try it! If we multiply out our result:

$$\begin{aligned}
x^3 &= 1\\
\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)^3 &= 1\\
\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right) &= 1\\
\left(\frac{1}{4}-\frac{\sqrt{3}}{4}i-\frac{\sqrt{3}}{4}i+\frac{3}{4}i^2\right)\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right) &= 1\\
\left(\frac{1}{4}-\frac{2\sqrt{3}}{4}i-\frac{3}{4}\right)\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right) &= 1\\
\left(-\frac{1}{2}-\frac{2\sqrt{3}}{4}i\right)\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right) &= 1\\
\left(\frac{1}{4}-\frac{\sqrt{3}}{4}i+\frac{\sqrt{3}}{4}i-\frac{3}{4}i^2\right) &= 1\\
\frac{1}{4}+\frac{3}{4} &= 1
\end{aligned}$$

Pretty cool, right? We we're able to solve a tough algebra problem visually using the complex plane, and get the same exact answer we obtained through factoring, as shown in Figure 29.

Finally, we've only found two answers so far, and the fundamental theorem of algebra demands that we have 3.

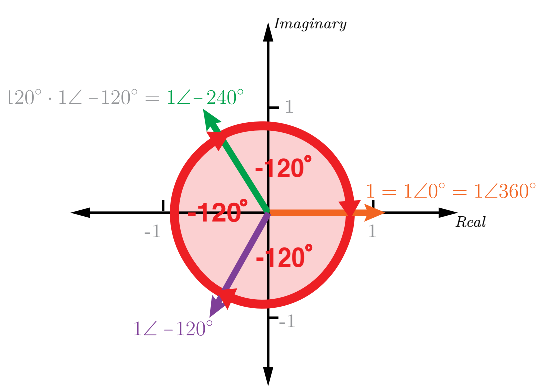



**Figure 33 | Our final solution to $x^3$=1.** We find our final solution, the complex number with a magnitude of 1 and angle of -120, by moving clockwise around our circle in three equal steps.

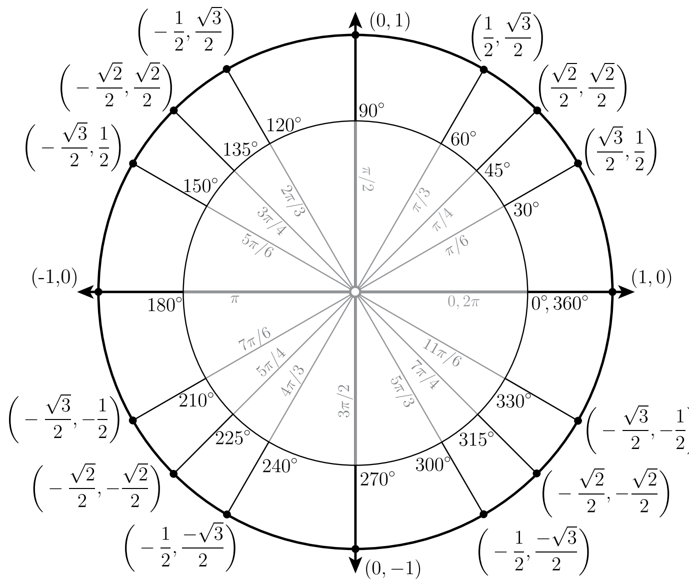



**Figure 32 | Your BFF the unit circle.** Yeah, it's kind of a big deal. Today it's going to help us convert between polar and rectangular coordinates.

We find our missing answer going the other way around our circle. If we start at negative 120 and multiply our number by itself 3 times, we also land at the purely real number one. So our missing answer is -1/2-($\sqrt{3}/2$)$i$.[3]

So we've found all the answers to our problem $x^3$=1, and shown how the complex plane allows us to find these visually. In this case, using complex numbers saved us *some* time over the algebraic approach – and for more complicated problems, the complex plane becomes even more useful. For example, what if we change the power in our original problem to say 8? That is $x^8$-1=0. We could try to factor this[4] – or just have a quick look at our complex plane and realize that, just like our last problem where we needed to divide the unit circle into 3 equal portions – we need to, in this case, divide our circle into 8 equal portions – so our solutions need to be at 45°, again along the unit circle. All 8 answers. Done.[5]

Next time we'll show how imaginary numbers are the missing puzzle piece that make algebra complete.

1 We know we're on the unit circle because our number has a magnitude of 1

2 You could also use a 30/60/90 special right triangle or sin and cosine… $x$ = 1*cos(120), $y$=1*sin(120).

3 Notice that our 2 complex roots are complex conjugates, this will always be the case when our polynomials have real coefficients.

4 (x^4-1)(x^4+1)= (x^2-1)(x^2+1)(x^4+1)= (x-1)(x+1)(x^2+1)(x^4+1)=0...

5 One solution would be $\sqrt{2}/2$+($\sqrt{2}/2$)$i$. This is called an n[th] root of unity problem and has lots of cool applications, like the Fourier Transform!

# Exercises 8

*Discussion*

8.1 Why must the equation $x^3=1$ have 3 solutions?

8.2 How does the complex plane make tough problems easier?

8.3 Why to you think the mathematician Jacques Hadamard wrote:

*"The shortest path between two truths in the real domain passes through the complex domain."*

-Jacques Hadamard, ~1945

*Drill*

*Answer in rectangular form:*

8.4 $(1+i)^6$

8.5 $(1 + i\sqrt{3})^6$

8.6 $f(x) = x^2 + 1$

8.7 $\sqrt{i}$

8.8 $\sqrt{\dfrac{\sqrt{3}+i}{2}}$

8.9 $\sqrt[3]{8i}$

*Critical Thinking*

Let's say we're given a given complex number with a magnitude[1] of 1, and an angle[2] of $\theta$. We can convert our number to rectangular form as follows:

$$1\angle\theta = cos(\theta) + isin(\theta)$$

8.10 Using a picture, show why the above equation is legit.

Now let's consider what happens when we raise our number to various powers. Write each answer in rectangular form, using $sin$, $cos$, and $\theta$.[3]

8.11 $(1\angle\theta)^2$

8.12 $(1\angle\theta)^{1/3}$

8.13 $(1\angle\theta)^n$

1 or modulus

2 or argument

3 Feel free to explain you answer with a picture for one extra credit point per problem.

Your answer to 8.13 should be something like:

$$(1\angle\theta)^n = cos(n\theta) + isin(n\theta)$$

This is known as de Moivre's theorem.[4] Note that the right side of the equation is sometimes abbreviated using $cis$: $cis(x) = cos(x) + isin(x)$. De Moivre's theorem can be expanded to numbers that have magnitudes other than 1 without too much headache. Let's say we're given some complex number with a magnitude or r and an angle of $\theta$. We can convert our number to rectangular form as follows

$$(r\angle\theta) = r(cos(\theta) + isin(\theta))$$

8.14 Using a picture, show why the above equation is legit.

Now let's consider what happens when we raise our new number to various powers. Write each answer in rectangular form, using $r$, $sin$, $cos$, and $\theta$.

8.15 $(r\angle\theta)^2$

8.16 $(r\angle\theta)^{1/3}$

8.17 $(r\angle\theta)^n$

4 Although it never shows up in his work...makes sense...

Way the heck back in Part 4 (Exercises 4.14-4.21), we discussed how Bombelli solved Cardan's problem by allowing imaginary numbers to exist. A key part of Bombelli's argument was this rather shocking fact:

$$\sqrt[3]{2+\sqrt{-121}}+\sqrt[3]{2-\sqrt{-121}}=4$$

Now, with the help of the 200+ years of mathematical advancement we've covered since Part 4, we'll show that the above equation is must be true without breaking a sweat! This is remarkable when we consider how much trouble this equation gave Cardan and Bombelli.

8.18 To begin showing that the above equation is true, convert these two parts of the equation to polar form. Round your answers to 4 decimal places.

a) $2 + \sqrt{-121}$

b) $2 - \sqrt{-121}$

8.19 Using your results from the previous page, compute the cube root of your answers from 8.18, and convert your results to rectangular form.

a) $\sqrt[3]{2+\sqrt{-121}}$

b) $\sqrt[3]{2-\sqrt{-121}}$

8.20 Add together your answers to 8.19 a and b. If your result is 4, nice job! Think about how impressed Cardan and Bombelli would by your math wizardry. If you're answer isn't 4, check out the solutions in the back of the book.

*Challenge*

The brilliant mathematician Gottfried Wilhelm Leibniz (1646-1716), co-father to calculus, is known to have struggled deeply with the topics we're covering here.

*"I do not understand how...a quantity could be real, when imaginary or impossible numbers are used to express it."*

*-Leibniz*

After his death, Leibniz's unpublished work revealed that he calculated some special cases of Cardan's formula again and again, presumably looking for deeper insights like the ones that have been handed to us. In solving the cubics $x^3{=}13x{+}12$ and $x^3{=}48x{+}72$ using Cardan's formula, Leibniz discovered that:

$$\sqrt[3]{6+\sqrt{-\frac{1225}{27}}}+\sqrt[3]{6-\sqrt{-\frac{1225}{27}}}=4$$

and:

$$\sqrt[3]{-36+\sqrt{-2800}}+\sqrt[3]{-36-\sqrt{-2800}}=6$$

8.21 Show that the two equations above are true.

# Imaginary Numbers Are Real

## Part 9: Closure

Before we finish up the series and solve our problem from part one, let's talk about how complex numbers are the missing puzzle piece that make algebra complete.

Back in part two we saw how the definition of what a number is has evolved over time, beginning with the natural numbers. The Egyptians figured out that these numbers were missing something[1], and it's pretty obvious to us today that the natural numbers are incomplete.

However, as we saw with complex numbers, it's not always obvious when our numbers are missing something. Fortunately, there is a more sophisticated way to determine if we have all the types of numbers we need – the mathematical idea of closure.

Let's play a game. I'll give you a set numbers, and an algebraic operation. I want you to tell me if any two numbers in the set, when combined with the operation, give a number not in the set.

Our first set is the natural numbers and our operation is addition.

| SET OF NUMBERS | OPERATION |
|---|---|
| Natural: 1, 2, 3... | + |

**Figure 34 | Are the natural numbers closed under addition?**

So the question is: are there any two natural numbers that when added together, produce something that is *not* a natural number?

After a little noodling, it should seem pretty reasonable that any two natural numbers, added together, result in another natural number. Mathematically, we can say that the set of natural numbers is *closed* under addition.

Next, let's try the set of natural numbers and the operation of subtraction.

| SET OF NUMBERS | OPERATION |
|---|---|
| Natural: 1, 2, 3... | - |

**Figure 35 | Are the natural numbers closed under subtraction?**

For some pairs of natural numbers, like 6 and 4, things work just fine – 6 minus by 4 is 2, which is a natural number. But what about 2 minus 6? This results in an answer that is nowhere to be found in our set of natural numbers, so our set is not closed under subtraction. We need to expand our set to include zero[2] and negative numbers for this to be the case.

1 Fractions!
2 Because, for example 3-3=0.

So the set of natural numbers is not closed under subtraction, but the set of all integers is. By expanding our number system, we can guarantee that any subtraction question we can ask will have an answer.

| NUMBERS | SYMBOL | EXAMPLES | CLOSED UNDER |
|---|---|---|---|
| Natural | $\mathbb{N}$ | 1,2,3... | + |
| Integers | $\mathbb{Z}$ | ...-2, -1, 0, 1, 2... | +, - |

**Table 8 | Closure under subtraction.** To ensure we can handle all subtraction problems, we must expand our number system to include integers.

As we include more algebraic operations, we must continue to expand our number system. Division requires us to expand our number system to include fractions – also known as rational numbers. Rational comes from the word ratio[3] – rational numbers are numbers can be expressed as the *ratio* of two integers.[4]

We can show the relationship between the numbers we've covered using another invention of Euler's.[5] Using an Euler diagram,[6] we can visually express the idea that one set includes another – all integers are rational numbers, because we can always express them as a ratio of two integers, but not all rational numbers are integers.

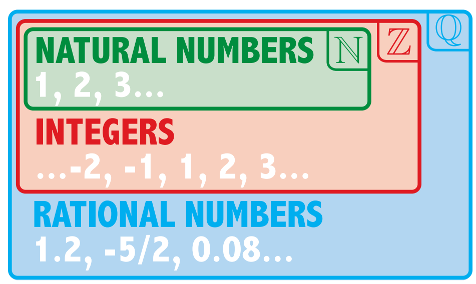



**Figure 36 | Relationship between natural numbers, integers, and rational numbers.** All natural numbers are integers, but not all integers are natural numbers.

Let's recap. So far we've made it to rational numbers, which includes numbers like 1, 0, -5.1, and -2/3. What operations are the rational numbers closed under?

Well, any two rational numbers added together yield another rational number, so we can say that rational numbers are closed under addition. We can say the same for subtraction, multiplication, and division.[7]

3 Which comes from the Greek word logos, which means word! Word.
4 Like 2/3
5 His other invention we're making use of is the notation $i = \sqrt{-1}$.
6 These are basically more flexible Venn diagrams.
7 With the notable exception of dividing by 0 – this is a new can of worms…. called calculus.

Now, what about powers and roots? Does a rational number raised to a rational power always yield a rational result?

It turns out that for problems like $(2/9)^2$ this is no problem – our result is rational.[1] Where we get into trouble is things like $(2)^{1/2}$. Raising something to the power of 1/2 is the same thing as taking the square root, so this is equivalent to $\sqrt{2}$. We'll save the full argument[2] for another day, but it turns out that $\sqrt{2}$ is not rational – there are no two integers, that when divided, equal exactly $\sqrt{2}$. Since numbers like this are not rational, we give them the name irrational.

There's one more class of numbers that are even cooler than irrationals - the *transcendental* numbers like π and *e*, we'll also save these for another day.

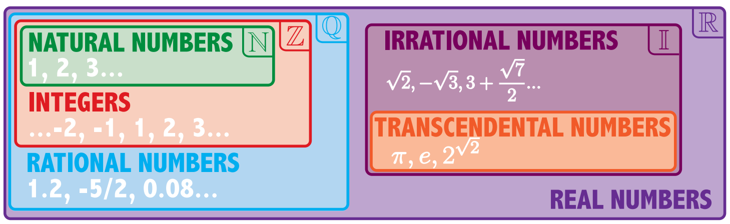



**Figure 37 | The real numbers.** The real numbers are an inclusive group! Real numbers include the natural numbers, integers, rational numbers, irrational numbers, and even transcendental numbers. But it still feels like we're missing something...

So, we've expanded our number system again to include irrationals, and all these numbers, taken together, form what we call the real numbers.

Let's play our game one more time. Our set is now the

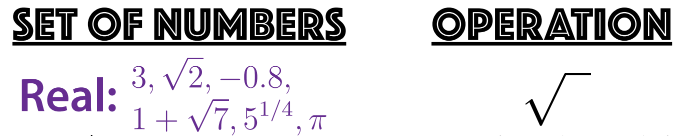



**Figure 38 | Are the real numbers closed under roots?** Time for one last round of everyone's favorite game.

real numbers, and our operation is taking roots. Do we have closure? Are there any real numbers, that when we take some root, yield a result that is not a real number?

The answer is, that despite all the types of numbers we've included along the way, we're still missing something. We can write an expression only using real numbers and roots – for example, $\sqrt{-9}$, that has no solution in the real numbers.

For this problem to have an answer, we must expand our number system once more to include imaginary numbers. And taking all our real numbers from before, together with imaginary numbers, we arrive at our broadest class of numbers – the complex numbers.

| NUMBERS | SYMBOL | EXAMPLES | CLOSED UNDER |
|---|---|---|---|
| Natural | ℕ | 1,2,3... | + x $()^x$ |
| Integers | ℤ | ...-2, -1, 0, 1, 2... | + - x |
| Rational | ℚ | .4, 1/2, -16 | + - x ÷ |
| Real | ℝ | .4, $\sqrt{2}$, 1/2, π, 51 | + - x ÷ |
| Complex | ℂ | $\sqrt{2}$, .5, π, $2i$ | + - x ÷ $\sqrt{\phantom{x}}$ $()^x$ |

**Table 9 | Closure.** Expanding our number systems to include closure under more algebraic operations.

1 $(2/9)^2=(4/81)$, which is hella-rational! 'Hella' is a super fun word I learned while at graduate school in northern California, I think it means something like "really". Use hella in a conversation for 1 extra credit point.
2 It's pretty cool, it means there are holes in the rational number line! See exercise 9.19.

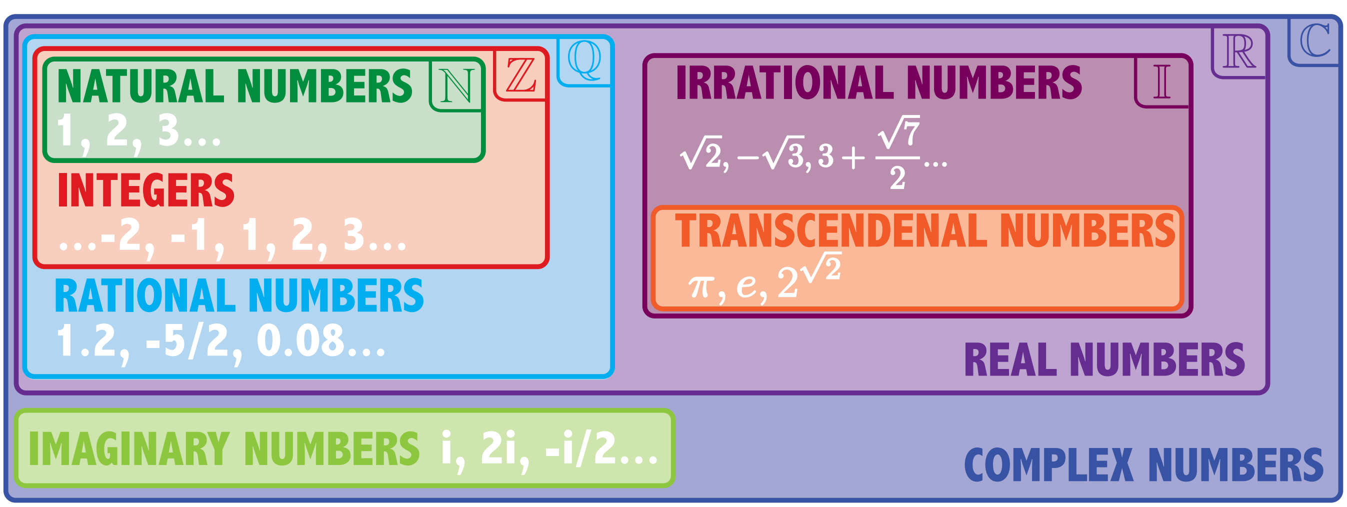



**Figure 39 | Finally.** All the numbers we need to answer any algebra question we can think of using addition, subtraction, multiplication, division, powers and roots. Complex numbers are closed under all algebraic operations.

Initially, mathematicians were concerned that even complex numbers were not sufficient – that problems like $\sqrt{-i}$ would result in an even more "complex number" – perhaps even a three dimensional number, instead of our two-dimensional complex numbers.

Fortunately, this turned out not to be the case. In fact, we can evaluate $\sqrt{-i}$, again using our good friend, the complex plane. Since $-i$ has a magnitude of 1 and an angle of -90°, we just need a number with a magnitude of 1 and an angle of -45°, according to the unit circle, $\sqrt{2}/2-(\sqrt{2}/2)i$.

So the square root of negative $i$ is just another complex number - there's no need for some wild new three-dimensional number. In fact, there's no operation in the world using addition, subtraction, multiplication, division, powers, and roots that the complex numbers can't handle.[1]

Imaginary numbers are the exact missing piece that make algebra complete.

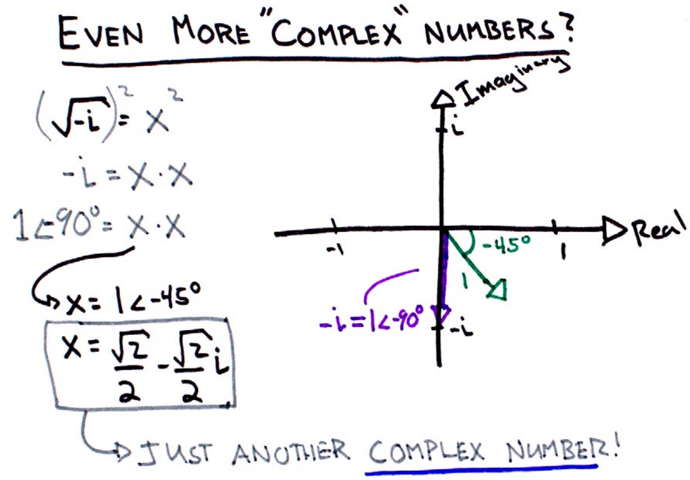



**Figure 40 | The Complex numbers are closed under the operations we care about.** Even crazy sounding problem like $\sqrt{-i}$ result in complex numbers. As long as we stick to multiplication, division, addition, subtraction, powers, and roots, there's no need for some crazy, "more complex" number.

1 Except dividing by zero – this leads us to calculus.

# Exercises 9

*Discussion*

9.1 What is mathematical closure?

9.2 Why might closure be important for scientists, engineers, and other people who use mathematics?

9.3 The real numbers seem pretty complete. What evidence do we have that indicates this is not true?

*Drill*

Time for a few more rounds of the closure game! Determine if the set is closed under the given operation. If the set is not closed, provide a counter-example.

9.4 Integers, multiplication.

9.5 Integers, division.

9.6 Real Numbers, division.

9.7 Irrational Numbers, addition.

9.8 Irrational Numbers, multiplication.

9.9 Transcendental Numbers, subtraction.

9.10 Imaginary Numbers, multiplication.

9.11 Integers, powers.

*Critical Thinking*

The idea of mathematical closure can be applied to sets beyond those we've discussed thus far. For example, let's consider the set of all even numbers (just in case you forgot, 0 is even).

9.12 Are the even numbers closed under addition? What about subtraction? For 1 extra credit point, prove that your answer it true for any two even numbers.

9.13 Are the even numbers closed under multiplication? What about division? For $\sqrt{2}$ extra credit points, prove that your answer is true for any two even numbers.

9.14 Under what algebraic operations are the odd numbers closed?

Consider the completely made-up operation:

$$\Lambda(\text{a,b}) = (\text{a} + \text{b}) \textit{ modulo } 4$$

Where modulo 4 means the remainder after dividing by four. For example, 9 *modulo* 4 = 1, because 4 goes into 9 twice, leaving a remainder of 1. Numbers less than 4 remain unchanged, for example, 3 *modulo* 4 = 3.

9.15 Show that $\Lambda(5, 3) = 0$, and $\Lambda(2, 1) = 3$.

9.16 Is the set $\{1, 2, 3, 4\}$ closed under $\Lambda$?

9.17 Name a set that would be closed under $\Lambda$.

*Challenge*

Using the kick-ass polar form of complex numbers we learned back in Part 7, solve the problem below and confirm that results belong to the set of Complex Numbers.

9.18 $(-i)^{1/6}$

9.19 $\sqrt{2}$ is irrational. This means that $\sqrt{2}$ cannot be expressed as a ratio of two integers. This fact didn't fit so well with the orderly world view of the ancient Greeks, and legend has it even lead to the murder of Hippasus of Metapontum. Let's prove this dangerous fact by contradiction. To do this, let's assume, as the Greeks did, that $\sqrt{2}$ is rational. If $\sqrt{2}$ is rational, then we can write $\sqrt{2} = m/n$, for integers $m$ and $n$, and let's say that $m$ and $n$ make our fraction in lowest terms - so they have no common factors.

a. Square both sides of our equation $\sqrt{2} = m/n$, and show how we can conclude from our result that $m$ must be even.

b. Since $m$ is even, let $m = 2k$, for some integer $k$. By plugging in $2k$ for $m$ into your result from part a, show that $n$ must also be even.

c. Why can $m$ and $n$ not both be even?

Since $m$ and $n$ cannot both be even, we've reached a contraction, indicating that our original assumption, that $\sqrt{2}$ is rational must be false, leaving us with the inevitable fact that $\sqrt{2}$ is irrational.

So…what the heck is going on in Figure 2?

We saw this shape in part one and then proceeded to not talk about it for like ten chapters. And at this point, I wouldn't really blame you if you thought this was some science fiction designed to get you excited about doing boring algebra problems.

But before we chalk this up to unnecessary special effects,[1] let's remember where this shape came from. We began our conversation in part one with an equation that appeared to have no solution: $x^2+1=0$.

After all the work we've done so far, you can probably to see how to find the answer algebraically: subtract one from both sides and take the square root, resulting in $x=i$ and $x=-i$, done.

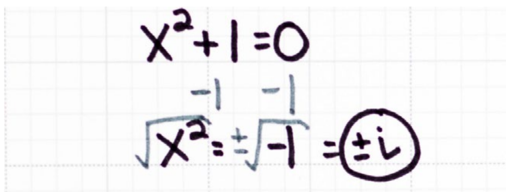



**Figure 41 | Solving Our Equation From Part One.** After all the work we've done so far, this isn't so bad.

So that's cool, but to really make sense of our shape from part one we need to dig a little deeper, and talk about functions of complex variables.

The kinds of functions most of us are used to, functions of real numbers[2], have inputs and outputs that can each be visualized using a single dimension. This means that all our $x$ values fit on a single number line, and so do our $y$ values. It seems pretty reasonable then[3], that if we want to figure out how $x$ and $y$ are related, we should put

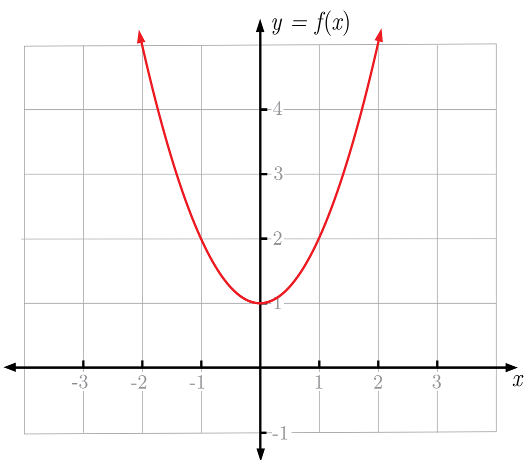



**Figure 42 | It's easy to take the cartesian coordinate system for granted.** Placing our $x$ and $y$ number lines at a right angle to each other like this is not obvoius!

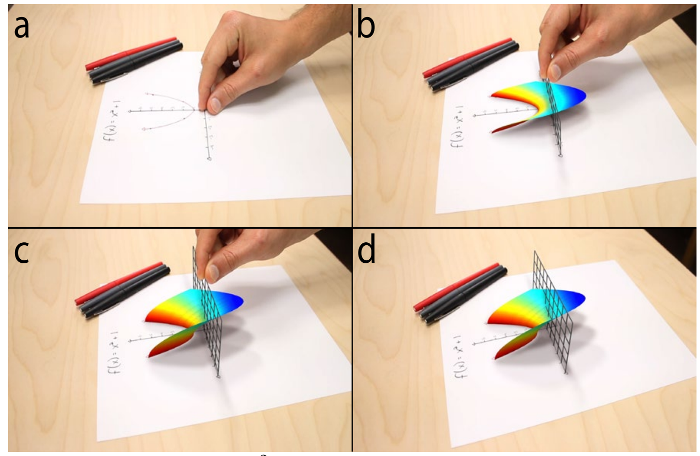



**Figure 2 | Graph of $f(x) = x^2+1$ where $x$ includes imaginary numbers.** Panels a-d show "pulling" the function out of the page.

our $x$ number line facing one way on a piece of paper, and put our $y$ number line one the same piece of paper, just facing the other way. This forms a two-dimensional grid known as the Cartesian coordinate system.

Apparently invented by Rene Descartes in the 16th century after watching flies crawl around, the Cartesian coordinate system is a super powerful tool for understanding the relationship between two variables.

The Cartesian coordinate system is powerful because it allows us to take abstract ideas, like functions, and turn then into something our brains can grasp much more intuitively – shapes. By giving each point on the plane its very own coordinates, Descartes was able to bring together the two largest areas of mathematics at the time: algebra and geometry. This greatly aided early efforts at classifying functions by Isaac Newton and others, and today the Cartesian coordinate systems shows up everywhere; helping us do all kinds of things, like spot trends in data.

So that's all fun and wonderful – but the Cartesian coordinate system does come with a disclaimer.

It only works in 2 dimensions.[4]

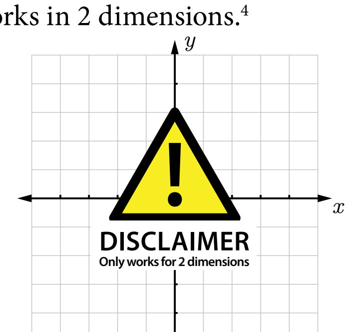



**Figure 43 | Limits of the Cartesian Coordinate System.** It only works in two dimensions :(

1 Transformers, anyone?
2 AKA Real-valued functions
3 Although no one figured this our until Descartes and Fermat in the 17th century.
4 Well, also 3, we'll get to that.

This limitation becomes a real problem when we start to think about functions of complex variables.

These functions take in complex numbers for inputs, and for the most part, also output complex numbers. This means the numbers we put in and get out of our function no longer fit on number lines. We now need two complex planes to keep track of our numbers: one for our input, one for our output.

This raises an important question – if, when visualizing these functions, what we're really interested in is the connection between the input and the output, how do we visualize what's happening on both planes simultaneously?

We could try to fit our input and output planes together somehow, as we did with our number lines for real-valued functions, but we quickly run into a pretty serious issue. As you likely know, the universe we live in has[1] three spatial dimensions – so there's no way to fit the four spatial dimensions we need into a single structure that our brains can comprehend – we simply run out of dimensions.

Fortunately, there are some very clever ways to see the relationship between two complex variables – but before we can get to these, we need to think about the mathematics of complex functions.

Even though using separate planes for our input and output is not a perfect solution, this approach can still help us get started. Let's try it out with our original function, $f(x)=x^2+1$. Before we begin, let's make a quick variable name change make things easier down the road. We'll change the name of our input variable from $x$ to $z$,[2] and call our output variable $w$. Since $z$ and $w$ each have a real and imaginary part, let's go one step further and give these parts names – we'll let $z = x + iy$, so $x$ and $y$ represent the real and imaginary parts of $z$,[3] and we'll let $w = u + iv$.

$$\begin{aligned} w &= f(z) = z^2+1 \\ z &= x + iy \\ w &= u + iv \end{aligned} \tag{16}$$

Just as we can use tables to keep track of our inputs and outputs for real-valued functions, we can also make tables to keep track of the inputs and outputs of complex functions. However, we now need four columns to keep track of our four variables: $x$, $y$, $u$, and $v$.

We can now experiment with our function: $f(z)=z^2+1$. If we plug a complex number into our function, for example $z$ equals $1+i$, we can do a little algebra

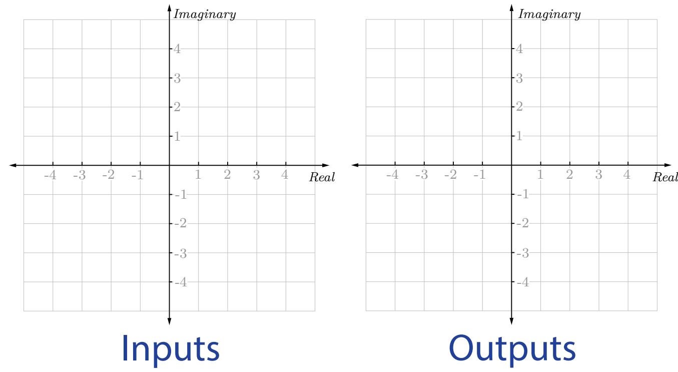



**Figure 44 | A Tale of Two Planes.** To keep track of the inputs and outputs of a function of complex variables, we need two complex planes.

and obtain our result:

$$\begin{aligned} f(1+i) &= (1+i)^2+1 \\ &= (1+i)(1+i)+1 \\ &= 1+i+i+i^2+1 \\ &= 1+2i-1+1 \\ &= 1+2i \end{aligned}$$

Plotting our inputs and outputs, we see that the point $1+i$ on our input $z$-plane was pushed, or mapped, by our function to $1+2i$ in our output $w$-plane.

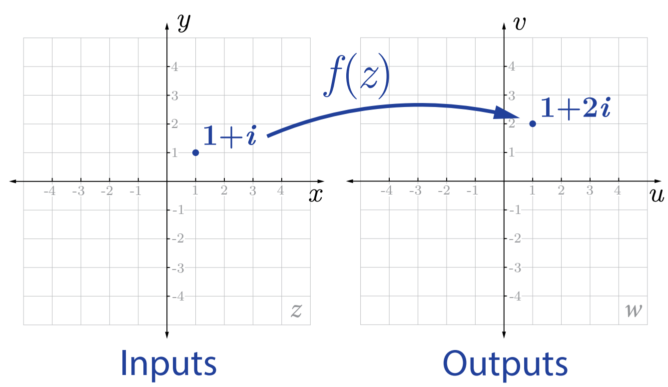



**Figure 45 | Mapping.** One way to visualize our complex function, $f(z)=z^2+1$, is by following how individual points are mapped from the input to output plane.

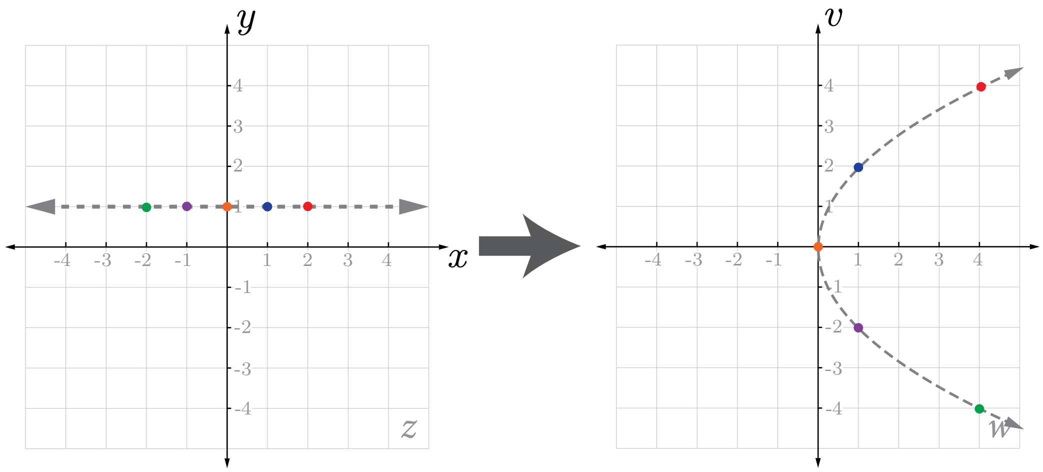
Let's plug in a few more points to see if we can find a pattern. If we test points along a straight line in our input space, we see that in our output space, our straight line is transformed into a curved line, as shown in Figure 46.

Now, as you can imagine, plugging in points like this can get pretty tedious. To speed things up, let's get a computer do it for us. And instead of having our computer just map certain points – let's have it map…all the points.[4]

We'll take advantage of the fact that the images are just collections of pixels that happen to be arranged on a grid. We'll use some code written in the programming language python to move every single pixel in an input

1 Probably.
2 Many resources use x to denote a real number, and z for a complex number. I have no idea why.
3 Respectively.

4 ALL THE POINTS!




| z | | w | |
|---|---|---|---|
| x | y | u | v |
| 1 | 1 | 1 | 2 |
| -2 | 1 | 4 | -4 |
| -1 | 1 | 1 | -2 |
| 0 | 1 | 0 | 0 |
| 2 | 1 | 4 | 4 |

**Figure 46 | Straight Lines Get All Curvy.** If we map points along a straight line in our input space, the result is a curved line in our ouput space!

image to its proper location in an output image.

To make this work, we'll assign each pixel in our input video a complex number that corresponds to its location on the complex plane. We can then let our code take care of the tedious work of moving each pixel to its new location, as dictated by our function, $z^2+1$.

Our code will move points exactly as we did by hand before - if we have a blue pixel at the $1+i$ location on our input graph, this blue pixel will be moved to the $1+2i$ location on our output graph, because $f(1+i)$ is equal to $1+2i$. [1]

We saw before that our function warped a straight horizontal line into a curvy one, so it should have some interesting effects on our video. We'll include some reference markers on top of our input and output planes to keep track of our numbers, but we won't transform these pixels.

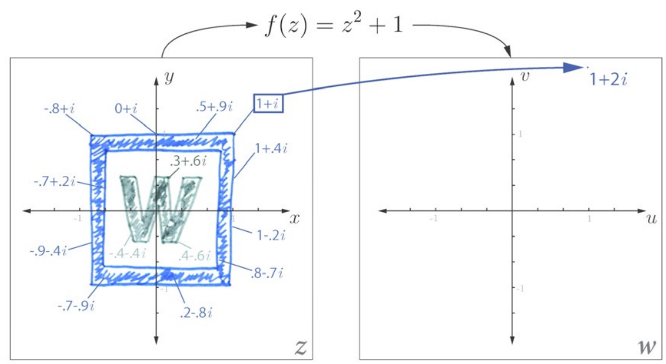



**Figure 47 | Every point gets a number.** To map all the points in the left image, we'll assign each pixel a complex number that corresponds to its location on the complex plane. Our code will then take care of the tedious work of mapping each pixel to its proper location on the output plane.

Alright, ready?

Let's start by drawing some simple lines in a grid. In our output plane we can see that our family of straight lines is turned into a family of curved lines. Cool, right?[2]

So we've found a pattern, but how is this pattern explained by our function $z^2+1$? And more importantly, how does this fit with everything else we've learned about complex numbers?

What else would be interesting to draw in our input space to test our mapping? What shape would *you* draw to learn more about what our function is doing?

Next time, more shapes.

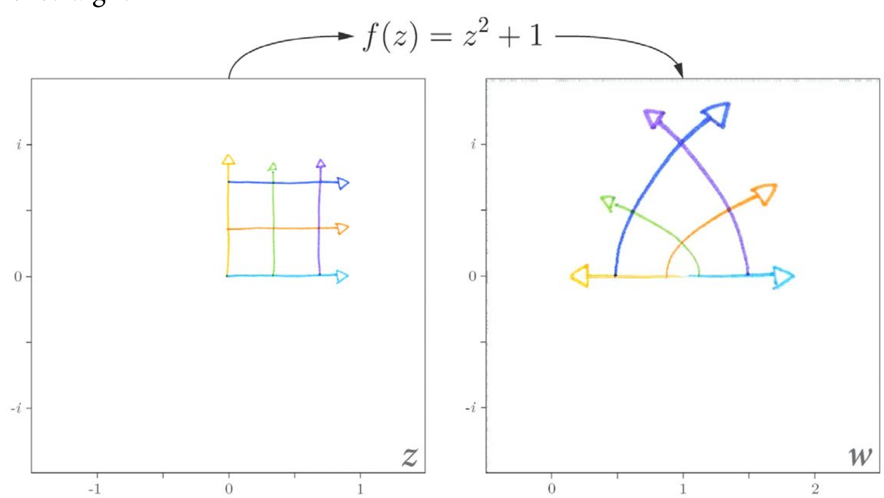



**Figure 48 | Mapping an Entire Image.** Here we've mapped all the points on our $z$-plane to the $w$-plane.

1 Python code available on github.

2 Notice that the angles between our lines have been preserved! This means our mapping is conformal.

# Exercises 10

*Discussion*

10.1 Why is the Cartesian Coordinate System so freaking cool?

10.2 Why are functions of complex variables so freaking hard to visualize?

10.3 What is the maximum number of dimensions the Cartesian Coordinate System will work in? Why?

10.4 What shape would you draw on the left plane of Figure 48 to better understand what our function, $f(z)=z^2+1$, is doing?

*Drill*

Let $g(z)=z^2+2z$. Calculate the following:

10.4 $g(1)$

10.5 $g(i)$

10.6 $g(1+i)$

10.7 $g(1-i)$

Let $h(z)=z^3+iz$. Calculate the following:

10.8 $h(1)$

10.9 $h(-i)$

10.10 $h(1-i)$

10.11 $h(2+2i)$

*Critical Thinking*

10.12 The Cartesian Coordinate System is so common today, it's easy to forget just how useful and non-obvious it is! Let's consider an example.

You and your friend Gus are planning a murder in a quiet beach town.You and Gus agree to hide the body in a cave located off Sunset Bay. However, there's a small problem. The cave can only be accessed at low tide. Being thorough pre-meditators, you and Gus devise a plan to predict exactly when low tides will occur, allowing you to schedule the perfect murder. Over the course of the days leading up to the Murder, Gus records the water depth at Sunset Bay every two hours. Check out Gus's data over there ----------------->

| Time (2 days before murder) | Water Depth (m) | Time (day before murder) | Water Depth (m) |
|---|---|---|---|
| 12:00 AM | 2.16 | 2:00 AM | Gus Fell Asleep |
| 2:00 AM | 1.95 | 4:00 AM | 0.29 |
| 4:00 AM | 1.35 | 6:00 AM | 0.77 |
| 6:00 AM | 0.59 | 8:00 AM | 1.33 |
| 8:00 AM | 0.30 | 10:00 AM | 1.81 |
| 10:00 AM | 0.20 | 12:00 PM | 2.26 |
| 12:00 PM | 0.77 | 2:00 PM | 1.85 |
| 2:00 PM | 1.34 | 4:00 PM | 1.27 |
| 4:00 PM | 1.89 | 6:00 PM | 0.51 |
| 6:00 PM | 2.07 | 8:00 PM | 0.07 |
| 8:00 PM | 1.96 | 10:00 PM | 0.17 |
| 10:00 PM | 1.43 | 12:00 PM | 0.60 |
| 12:00 AM | 0.66 | | |

a) The morning of the Murder is here! You and Gus pour over his notebook, trying to find the perfect time to commit the act and hide the body. Based on the data alone, (without drawing any pictures) when do you think You and Gus should hide the body?

b) Gus is NOT happy with your answer to part a. Let's use the Cartesian Coordinate System to convince Gus. Using the grid below, plot the water depth and time of day.

c) Based on your plot, when should you hide the body? Did you answer to part a and b match? Which method is more convincing? What advantages might using the Cartesian Coordinate System have?

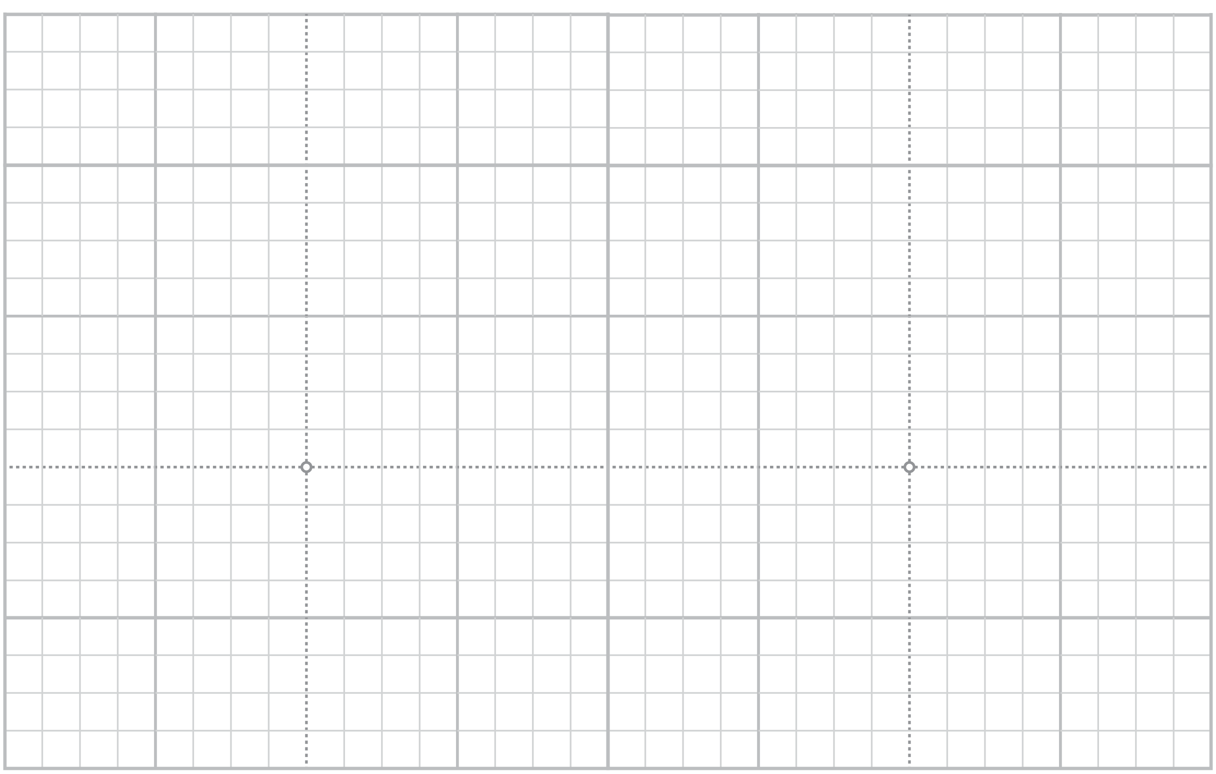


Figure 46 shows how one line was transformed from straight to curved by our function $f(z)=z^2+1$. In the exercises below, we'll experiment with mapping other shapes. For each exercise complete the table using $f(z)=z^2+1$, plot the input and output points, and speculate wildly about what exactly happened to the input shape.

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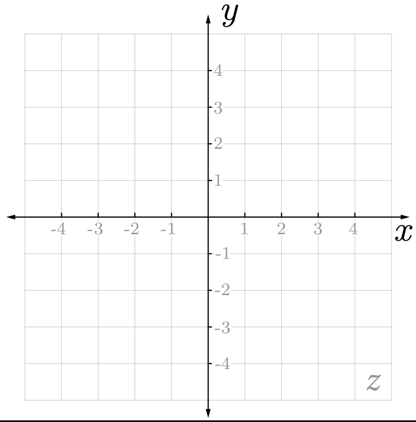
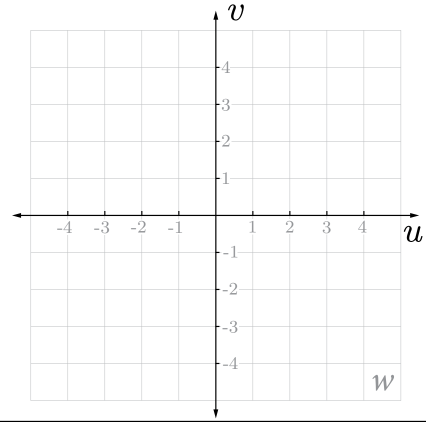
</td><td>
</td><td>
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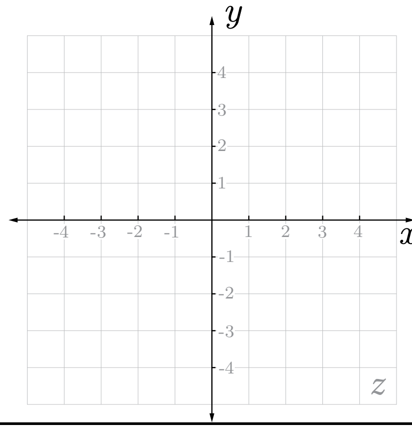
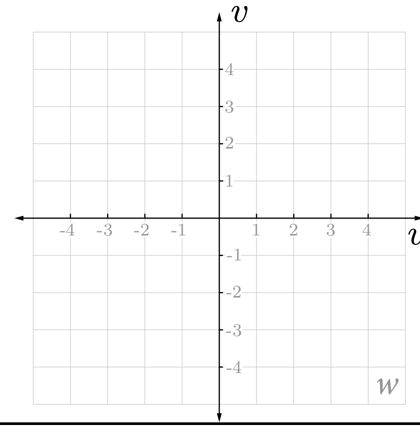
</td><td>
</td><td>
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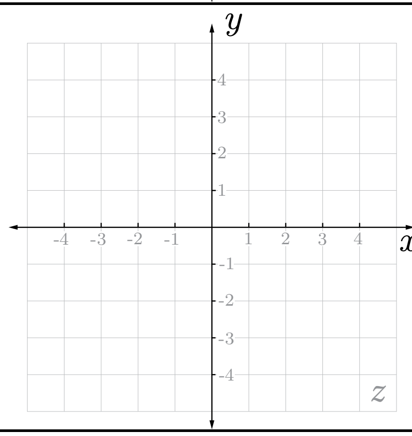
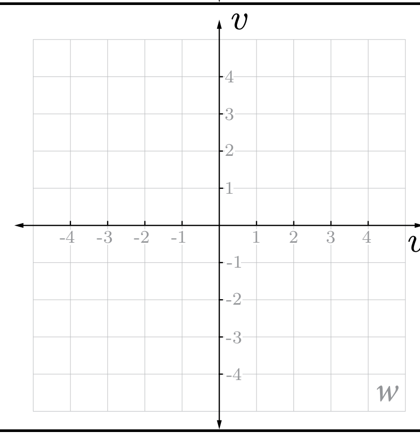
</td><td>
</td><td>
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<tr><td>10.16<br>MAKE YOUR OWN SHAPE!</td><td>
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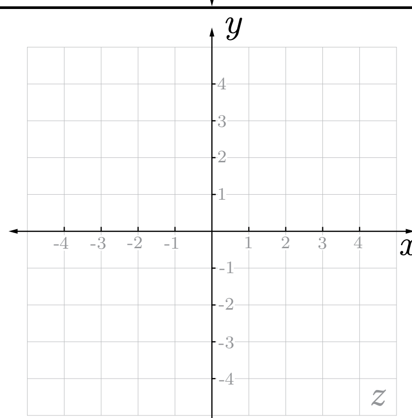
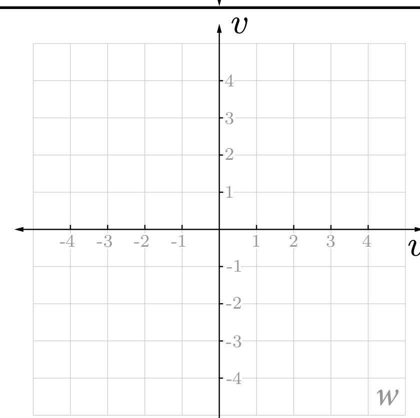
</td><td>
</td><td>
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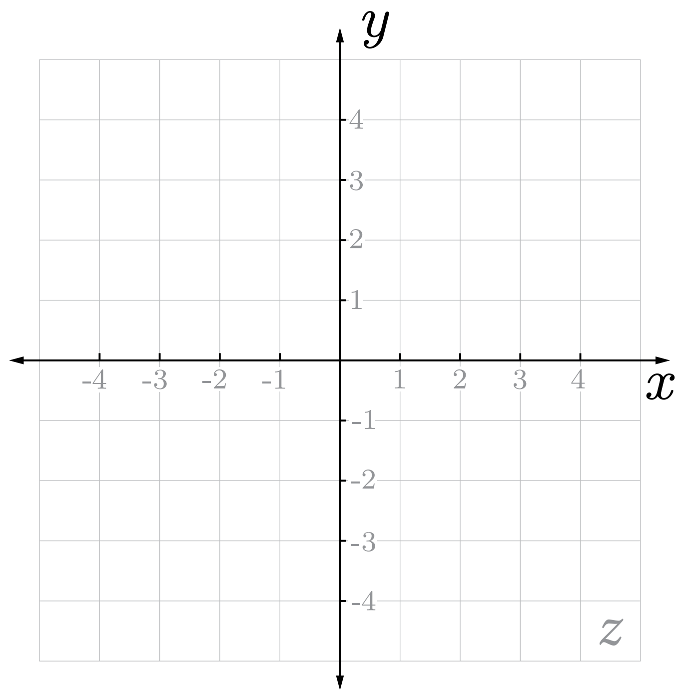
*Challenge*

$$
\begin{gathered}
w = f(z) = z^2 + 1 \\
z = x + iy \\
w = u + iv
\end{gathered}
\tag{16}
$$

10.17 a) Using Equation 16, draw curves/lines on the z-plane to the right where $u = 0$.

b)Using Equation 16, draw curves/lines on the z-plane to the right where $v = 0$.

c)What do your curves from parts a and b tell you about the zeros of $f$?



# Imaginary Numbers Are Real

## Part 11: Wandering in Four Dimensions

Last time, we left off trying to think of shapes to draw on our input plane that would help us better understand our function, $f(z)=z^2+1$.

Since $z^2$ means multiply $z$ by itself, and $z$ is a complex number – our function's behavior should have some connection to complex multiplication. Back in part seven we saw that one way to interpret complex multiplication is as a rotation and scaling of our input values – when we multiply two complex numbers together, their magnitudes multiply and their angles add.

So the $z^2$ part of our function should take our complex number, $z$, square its distance to the origin and double its angle. The plus one portion of our equation is a little less exciting - adding a *positive real* number will move all our points in the *positive real* direction, so to the right, in this case by one. Since this shift to the right doesn't affect the behavior that we're interested in, we'll leave it out of our equation for now.

Let's test the idea that our function will double the angle of its input values. What kind of shape should we draw to test this idea?

Ideally, we want to draw a shape that is made up points that are all at the same angle, to see if our function changes all points of the same angle in the same way.

So what kind of shape is made of points all at the same angle?

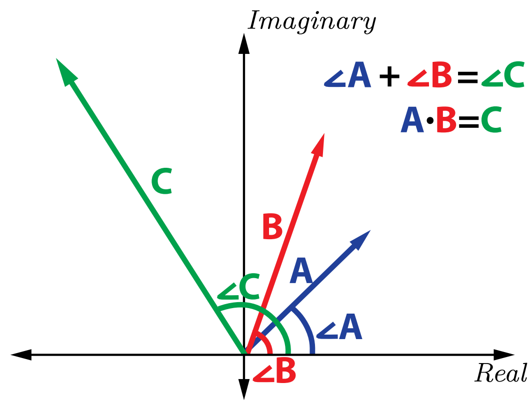



**Figure 49 | Complex Multiplication.** When we multiply two complex numbers, thier angles add and distances to the origin multiply.

This turns out to be a straight line through the origin. If we draw a few green lines through the origin as shown in the left pane of Figure 50, we see that the outputs are also straight lines, at what looks like double the angle!

So we've shown that our transformation doubles the angle of our input values – now what about the magnitude our inputs? We said earlier that when squaring a complex number, the magnitude of the complex number should also be squared. So the distance to the origin from our input points should be squared in our mapping.[1]

1 Just don't forget that numbers<1 get smaller when squared!

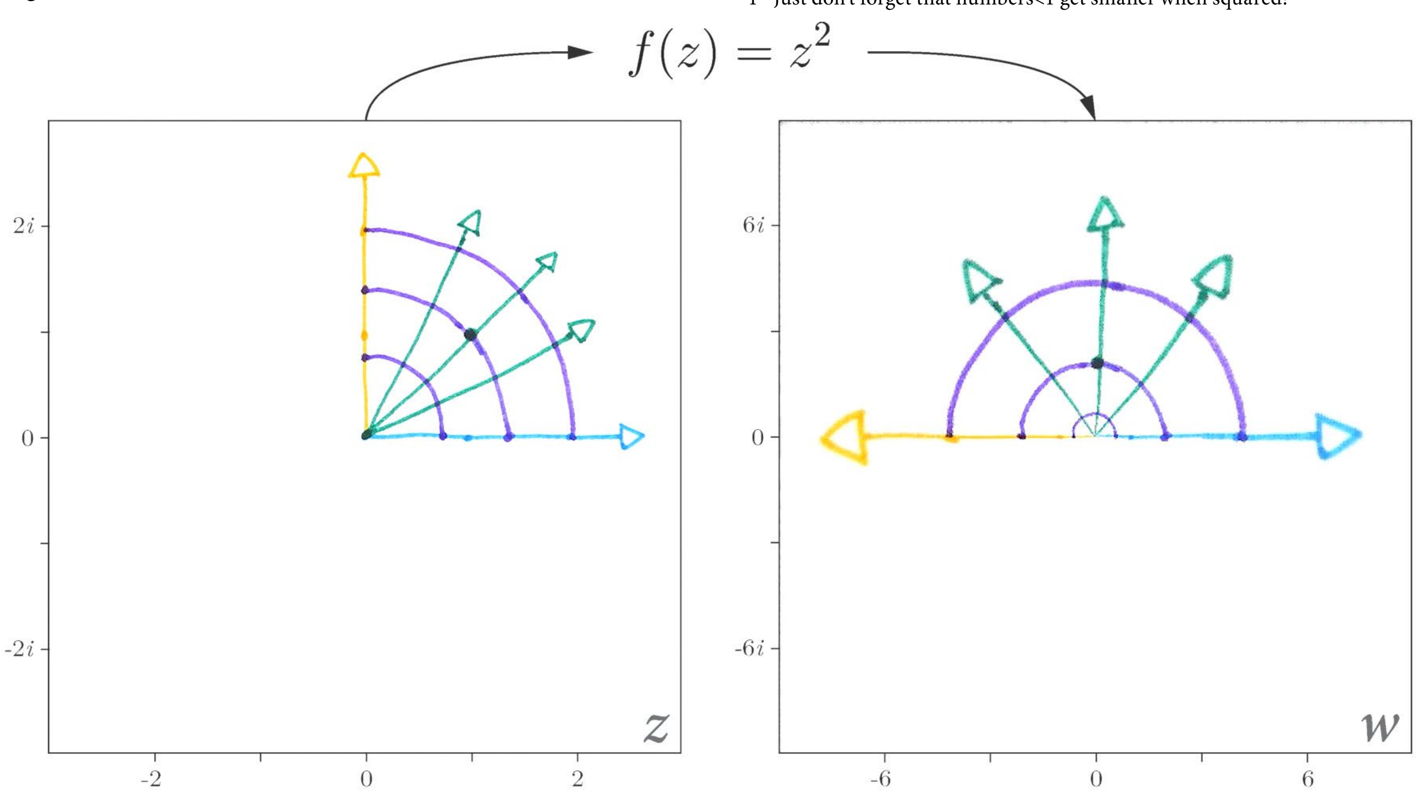



**Figure 50 | Mapping Pixels According to the Function $f(z)=z^2+1$.** By carefully choosing the shapes we draw on $z$, we can learn more about our complex function. Here we see that quarter circles are mapped to half circles with different radii, and lines through the origin are mapped to other lines through the origin with larger angles.

What kind of shape should we draw to test this idea? We want to test magnitude alone – so it would be nice to have a shape with a constant magnitude. What kind of shape has the same magnitude, or distance to the origin, everywhere?

This shape we need now is exactly a circle! If we add sections of a few circles to our picture as shown in Figure 50, we see that these result in new circle sections – but now at different distances to the origin. So as we expected, our circles are preserved, but their radii are changed.

So now we're really getting somewhere – by carefully choosing our input shape, we we're able to better understand exactly what our function does to complex numbers.

Wonderful.

But before we celebrate, let's keep a couple things in mind. For one, we're dealing with a really simple function. And secondly, even for this simple function – we can still run into trouble with our two complex plane setup.

For example, we know that our mapping doubles the angle of the input points. This is fine, until we use up too much of our input space. If we continue the circles we started earlier – once we arrive at 180 degrees[1], we begin to see a problem.

Our shapes have been expanded to fill up the entire output space, but we've only used half the input space!

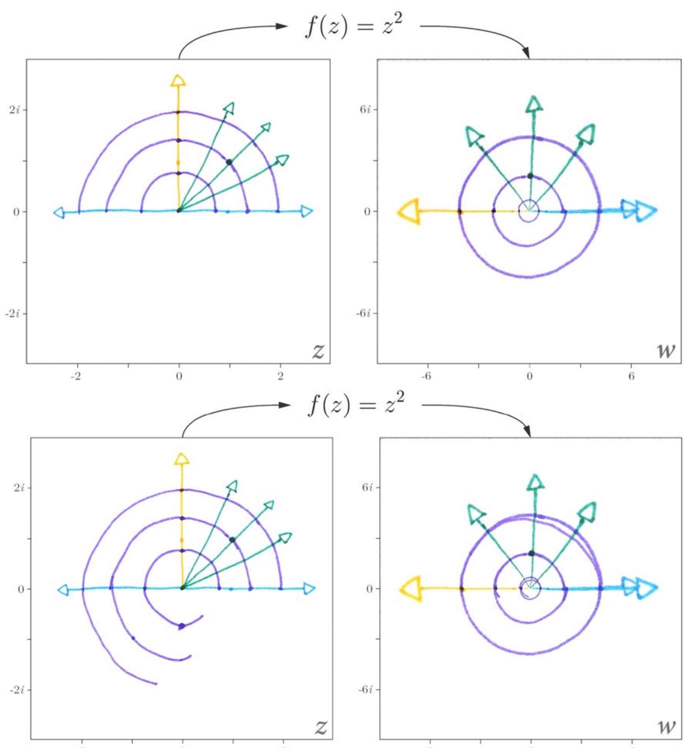



**Figure 51 | Out of Room!** If we continue our circles on the $z$-plane, our mapped circles have no where to go on $w$, except directly on top of our other points.

As we continue our circles, our new points have nowhere to go except directly on top of our old points. This makes sense algebraically - because of the way squaring works, points like $1+i$ and $-1-i$ will map to the same exact output value:

$$\begin{aligned} f(1+i) &= (1+i)^2 \\ &= (1+i)(1+i) \\ &= 1 + i + i + i^2 \\ &= 1 + 2i - 1 \\ &= 2i \end{aligned}$$

$$\begin{aligned} f(-1-i) &= (-1-i)^2 \\ &= (-1-i)(-1-i) \\ &= 1 + i + i + i^2 \\ &= 1 + 2i - 1 \\ &= 2i \end{aligned}$$

So it's not that our function is broken or anything – it's just that the technique we're using to visualize it can't really handle multiple values being mapped to the same location on the output plane – after all – which pixel should we display, the one at the $1+i$ location, or the one at the $-1-i$ location?

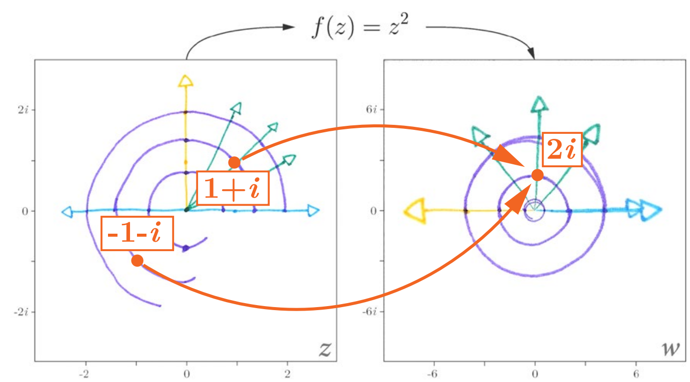



**Figure 52 | Double Trouble.** Two input points map to a single output point, meaning the bottom left part of our circle ends up directly on top of the upper left part. Not cool.

Mappings like this create problems in mathematics, although typically in the reverse direction. The reverse of a function, where our inputs become outputs and outputs become inputs, is called the inverse. The inverse of our function is pretty straight forward to find – we simply need to solve our equation for z: [2]

$$z = \pm\sqrt{w} \tag{17}$$

1 AKA pi.

2 Note, we're adding the plus or minus here for clarity, some resources omit it and take root(z) = +/- root(z), and some use the 1/2 power as an alternative to represent specific parts of the function.

We now we run into the real math problem.

Our inverse function represents the same exact connections as our forward function, just in the opposite direction. So our two inputs that mapped to the same output are now a single input that maps to two outputs. The $w$ value of $2i$ maps to both $z$=1+$i$ *and* $z$=-1-$i$. This mapping from a single input to multiple outputs is a big enough problem that our function's inverse to not even technically considered a function – the definition of a function requires that each input be mapped to one and only one output.

To make things nice and confusing, non-functions like this are called multi-functions. So our mapping is the same in either direction, but taken from $z$ to $w$ is considered a function, but from $w$ to $z$ is considered a multifunction.

Let's experiment with our multifunction. When we draw shapes on our $w$-plane, our shapes are duplicated and shrunk own onto our $z$-plane:

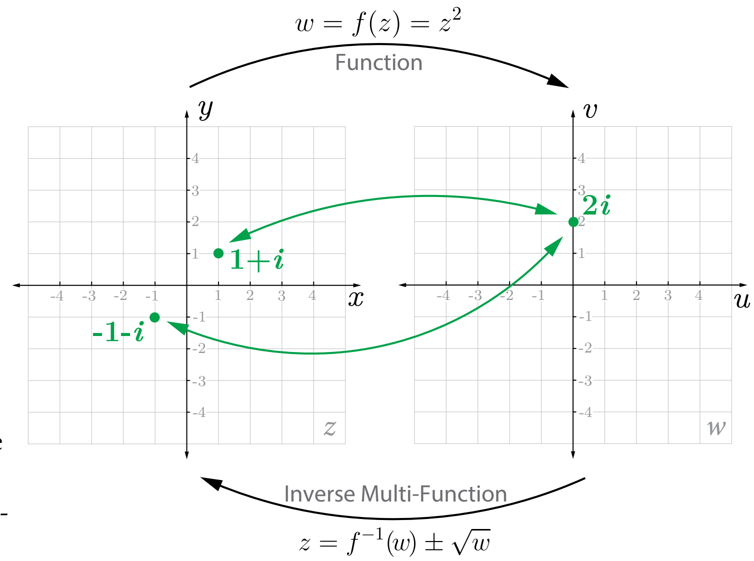



**Figure 53 | Our Function and Its Inverse.** We can think about our mapping in two ways, left to right or right to left.

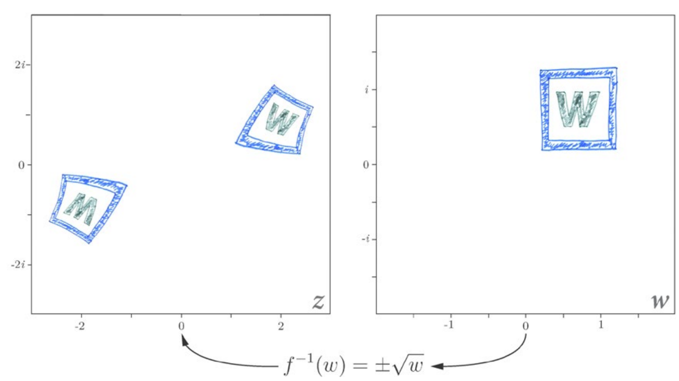



**Figure 54 | Double Trouble for Realsies.** When we draw on our $w$-plane, and map our pixels to our $z$-plane, we end up with 2 copies of our drawing!

Our shapes are copied because each point in $w$ is mapped to two points in $z$, and shrunk because the square root function takes the square root of the magnitude of our $w$ values and divides each angle by 2.

Let's experiment with one more type of shape – a path. We'll pick a starting point on our $w$ plane, wander around for while, and return to where we started. By following the paths on both our w and $z$-planes, we can get some idea of what happens as we wander around the 4-dimensional space of our complex multifunction.

Our blue path returns us right back to where we started on both the $z$ and $w$-plane. No surprise there. But if change our path a little, something weird happens.

Along our green path, our w path returns to where we started, but our $z$ paths…don't.

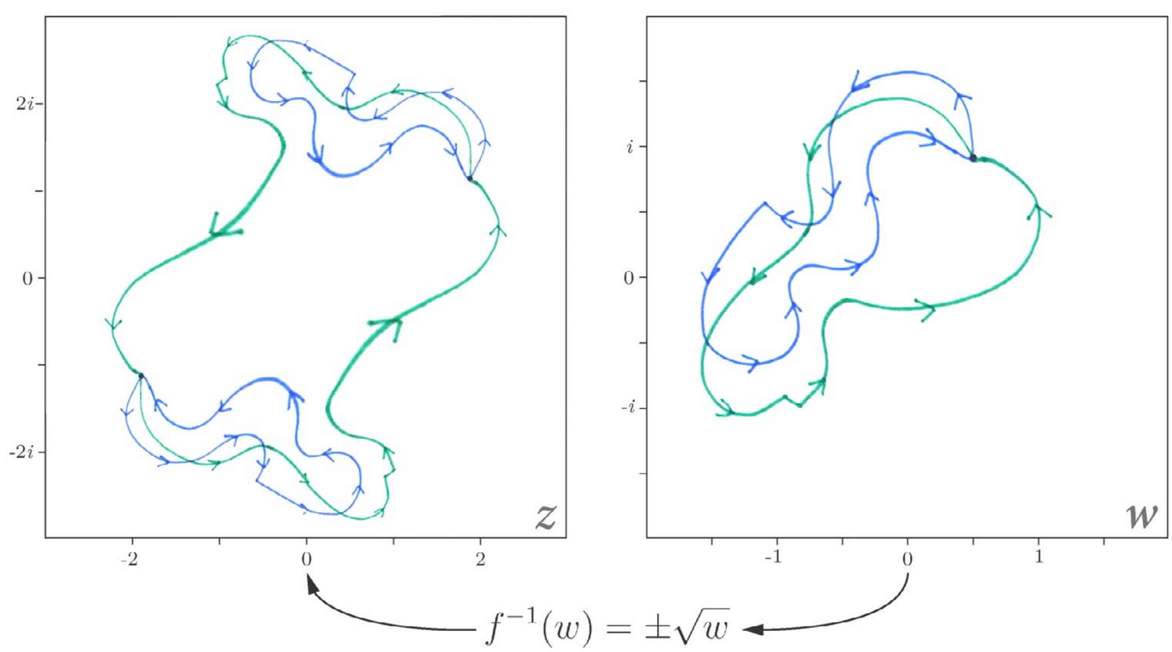



**Figure 55 | What the heck?** Our blue path ends up right where it started...but our green path jumps! What's going on here?

Somehow our $z$ values somehow jumped to a whole new part of the plane! Somehow we've wandered our way into a completely new part of our multifunction.

So it seems that some paths on $w$ lead us back to where we started, but others don't. What could be going on here? How is the complex landscape of our multifunction taking such similar paths in such different directions?

One reason I like math is that, for many problems,[1] someone much smarter than me has already given them some serious thought, and quite often found an elegant solution.

In this case, that person was one of Gauss' students - Bernhard Riemann. We'll discuss his solution next time.

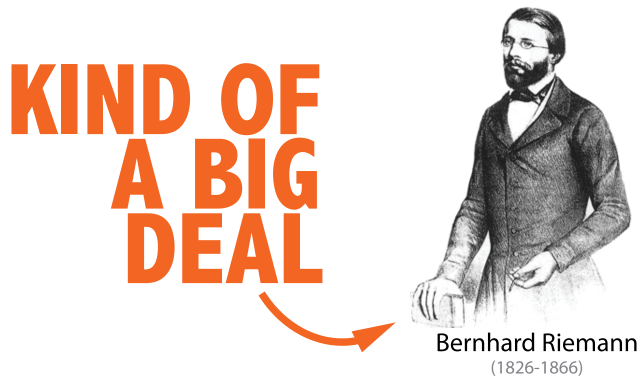



**Figure 56 | Bernhard Riemann.** The work of Berhnard Riemann continues to have a huge impact on modern mathematics. One of his ideas on prime numbers, known today as the Riemann Hypothesis, remains unsolved and is so important to mathematics that a correct solution will earn you a cool $1M from the Clay Mathematics Institute.

1 But certainly not ALL problems!

# Exercises 11

*Discussion*

11.1 Why did we choose to draw circles and lines through the origin in Figure 50?

11.2 In Figure 51 we run into a problem. Mutliple oints from our $z$-plane are mapped to the same location on our $w$-plane. Why does this happen?

11.3 In Figure 54, why do we end up with 2 copies of the shapes we draw on the $w$-plane?

11.4 Why do you think our paths behave so strangely in Figure 55?

*Drill*

Using the polar form of complex numbers we learned back in Parts 7 and 8, find all solutions to the the multifunction:

$$z = f^{-1}(w) \pm \sqrt{w}$$

11.5 $f^{-1}(-2i)$

11.6 $f^{-1}(-1)$

11.7 $f^{-1}\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)$

11.8 $f^{-1}(0)$

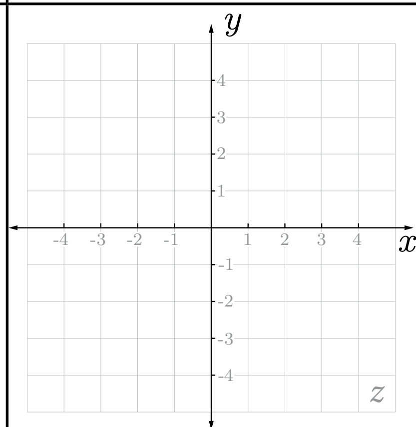
*Critical Thinking*

It's your turn to experiment with some paths! In each exercise below, compute the $x$ and $y$ values for each point according to the function $z = \pm\sqrt{w}$, plot the input and output points, and connect consecutive points with a line. You may use technology.

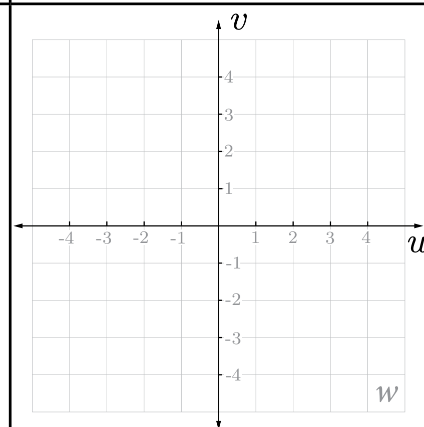
**Exercise 11.9**

| $x_1$ ($z_1$) | $y_1$ ($z_1$) | $x_2$ ($z_2$) | $y_2$ ($z_2$) | $u$ ($w$) | $v$ ($w$) |
|---|---|---|---|---|---|
| | | | | 4 | 0 |
| | | | | 0 | 2 |
| | | | | 0 | 4 |
| | | | | 2 | $2\sqrt{3}$ |
| | | | | 4 | 0 |

z-plane



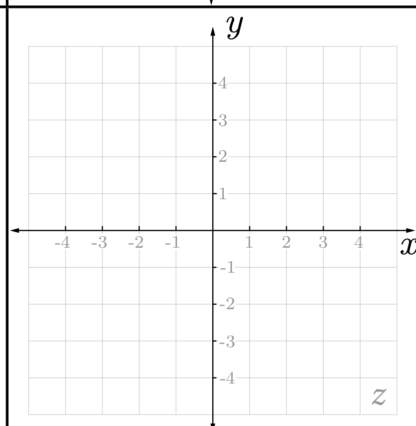

w-plane



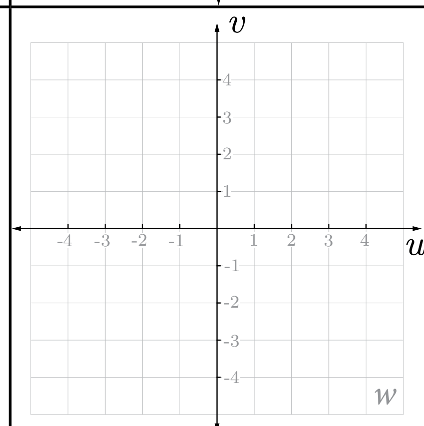

**Exercise 11.10**

| $x_1$ ($z_1$) | $y_1$ ($z_1$) | $x_2$ ($z_2$) | $y_2$ ($z_2$) | $u$ ($w$) | $v$ ($w$) |
|---|---|---|---|---|---|
| | | | | 4 | 0 |
| | | | | 0 | 2 |
| | | | | -4 | 0 |
| | | | | 0 | -2 |
| | | | | 4 | 0 |

z-plane



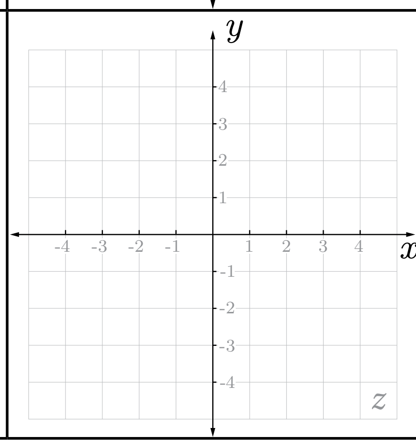

w-plane



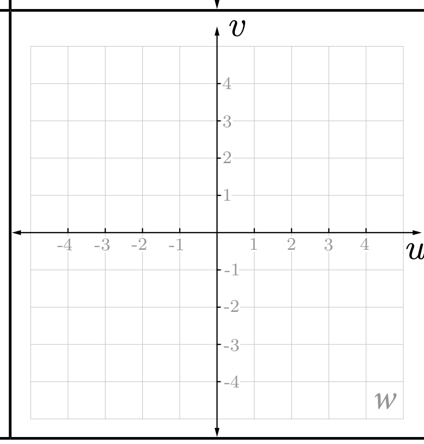

**Exercise 11.11** — MAKE YOUR OWN PATH!!

| $x_1$ ($z_1$) | $y_1$ ($z_1$) | $x_2$ ($z_2$) | $y_2$ ($z_2$) | $u$ ($w$) | $v$ ($w$) |
|---|---|---|---|---|---|
| | | | | | |
| | | | | | |
| | | | | | |
| | | | | | |
| | | | | | |

z-plane




w-plane




*Challenge*

11.12 Explain why our paths jump in Figure 55.

# Imaginary Numbers Are Real

## Part 12: Riemann's Solution

Last time we left off wondering why some paths on our $w$-plane led us to completely new values on our $z$-plane, while others didn't.

Gauss' student, Bernhard Riemann, made some powerful insights into problems like this in the mid-nineteenth century.

The first part of Riemann's contribution is the idea that, for functions like this, we need more than two complex planes to visualize our function. Since each point on our $w$-plane maps to two points on our $z$-plane, we can begin to resolve our ambiguity by adding a second $w$-plane, and letting each of our two points on $z$ map to its very own copy of the $w$-plane.

So that's fine, but it immediately raises an important question: how do we pick which $z$ values to map to each plane? A simple and effective approach here is to simply divide the $z$ plane into two halves – we'll let the right half map to our first $w$-plane, and our left half map to our second $w$-plane.[1] These restricted versions of our multifunction are called branches.

Let's draw a path again, but this time just on our first w-plane. Things look just fine until we cross the negative real axis, and our path on the $z$-plane suddenly jumps!

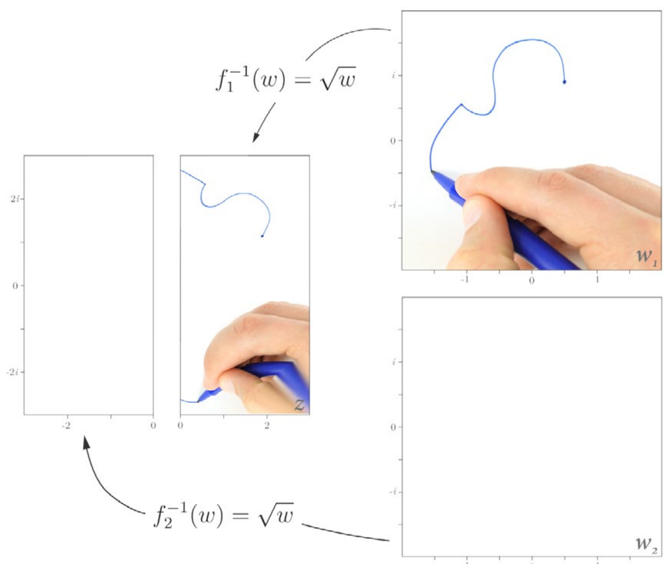



**Figure 58 | Crap!** Since we've decided to map our first $w$-plane to the right side of our z-plane, when we cross the negative real axis on $w$, our path on $z$ suddenly jumps!

This, of course, is what must happen – we've required points from our first w-plane to only map to the right side of our $z$-plane. Almost every[2] point on w has two possible solutions on $z$, and with our first branch, we've decided to always pick the one on the right.

So our path now jumps around on the $z$-plane, but

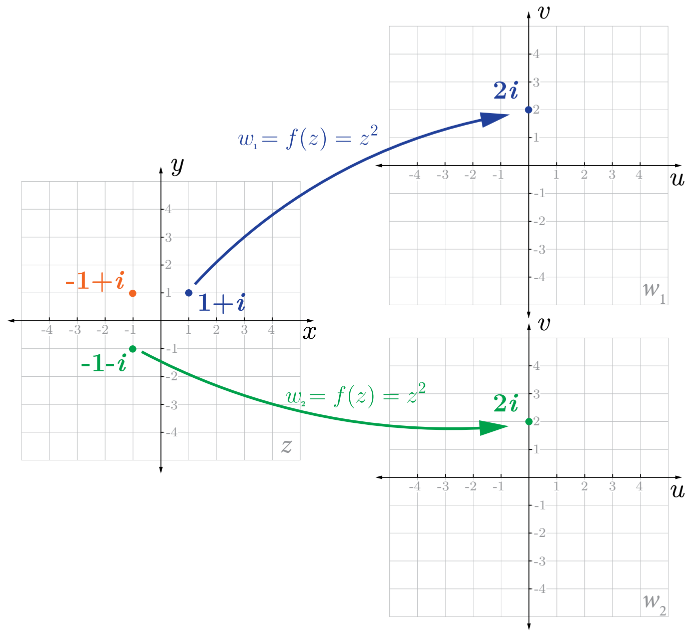



**Figure 57 | Riemann's Idea, Part I.** The first part of Riemann's idea is to allow each of our two solutions on $z$ to map to its very own copy of the $w$-plane. This does raise an issue though - which plane do we map other points, such as -1+$i$ to?

what's perhaps more disappointing here is that we haven't gained any insight into the interesting loop behavior we saw last time, in fact we can't even recreate it with these setup – no matter what kind of loop we draw, as shown in Figure 59, we always end up exactly where we started on both the $z$ and either $w$-plane, it seems that we've legalized this behavior out of existence.

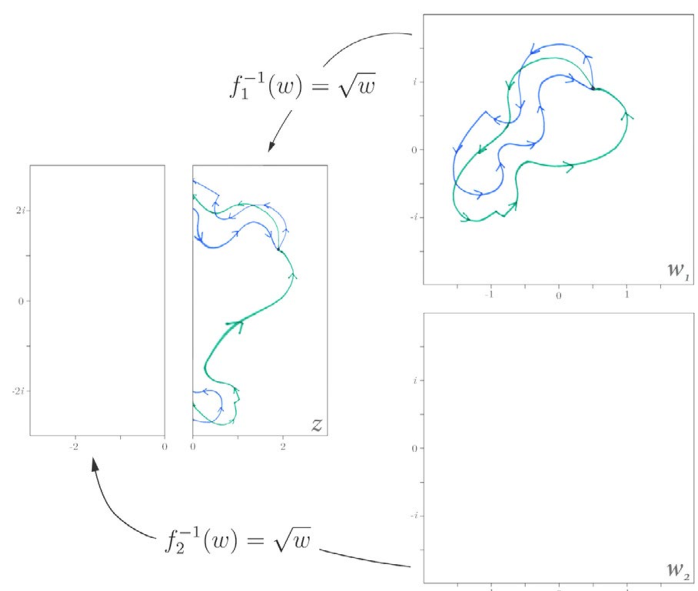



**Figure 59 | What Happened?** By splitting up our function like this, we no longer see the cool loop behavior in Figure 55. Lame.

Further, the fact that our function jumps across the $z$-plane means that our branches are discontinuous - a

1 These are called branches! We've made a "branch cut"
2 Which point(s) on w don't map to two points on z?

huge problem mathematically. Functions of complex variables are a big part of modern mathematics and science, and if our functions are jumping around like this, we can't do important things like take derivatives and integrals.[1]

So we've fixed the multi-valued problem by splitting our multifunction into branches, our function is now one-to-one[2], but in the process, we've introduced some serious issues - thus far Riemann's solution is not looking so great.

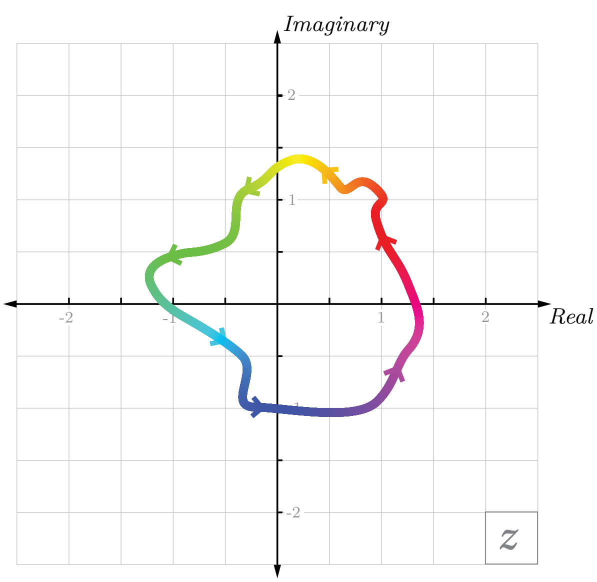
Fortunately, that was just part one, and part two is much cooler.

Let's consider our discontinuity problem in a bit more detail. We'll switch back to our forward function momentarily, and draw again on our $z$-plane.

1 In the language of calculus or real analysis, differentiable. Or in complex analysis, holomorphic or analytic.
2 Every input has exactly one unique output

**Figure 60 | Make your own Reimann Surface!**

The great mathematician Bernhard Riemann saw a way to fit together our two w-planes in such a way that our colored path would be made continuous. Your job is to recreate Riemann's idea by cutting out the two w-planes to the right, and positioning/cutting/taping them into a form that will make our curve continuous. This can be accomplished by making a single cut in each plane. If you succeed, you will have created a truly beautiful piece of mathematics, and the topic for our next section: **A Riemann Surface.**



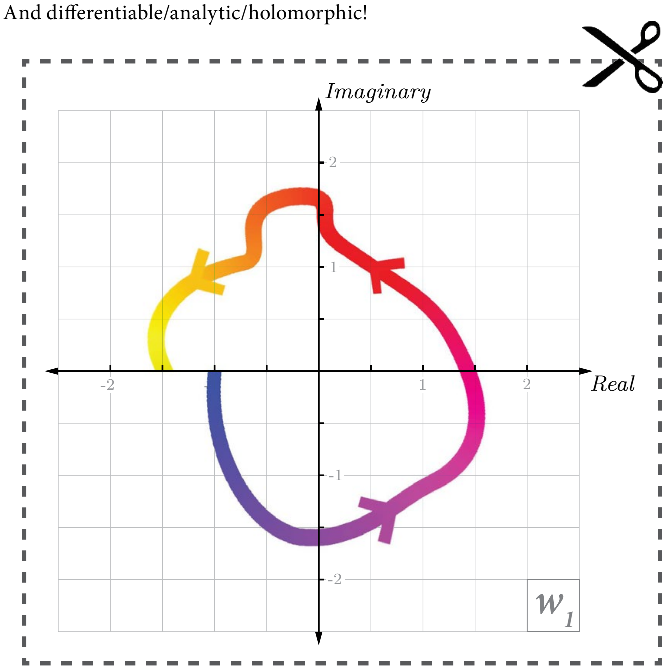

Let's pay careful attention to where our discontinuities show up. We'll follow the points along a single path, and to make sure we can tell the points of our path apart, we'll continuously change its color.

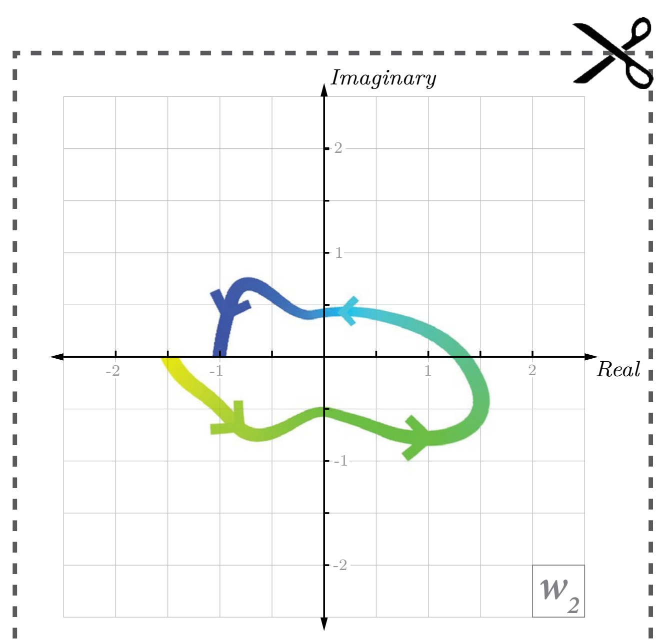
As we move from quadrant one to quadrant two on $z$ in Figure 60, we switch branches. We switch back to our first branch when moving from quadrant three to four. For our function to be continuous, we need to somehow connect our two $w$-planes at the exact points where our path jumps.

What Riemann saw here was a way to bring together our two complex planes in such a way that our multifunction would be perfectly continuous[3], while maintaining the nice one-to-one properties of our two $w$-plane solution.

To see Riemann's Solution, grab some scissors and tape, and check out the instructions below.

3 And differentiable/analytic/holomorphic!



# Exercises 12

*Discussion*

12.1 What's good about Riemann's two complex plane solution?

12.2 What's not so good about Riemann's two complex plane solution?

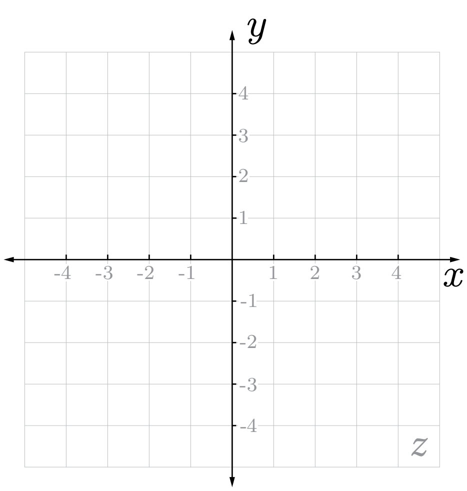
12.3 Why do our paths jump in Figures 58 and 59?

*Drill*

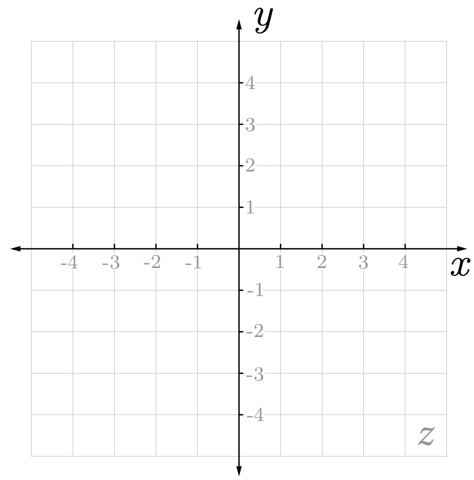
Find all plot solutions to the the multifunction:

$$z = f^{-1}(w) \pm \sqrt{w}$$

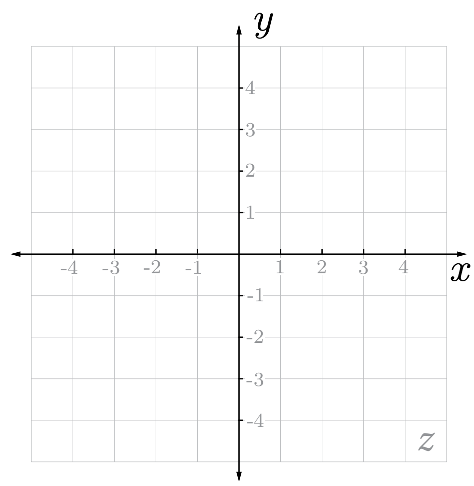
12.4 $f^{-1}(4)$



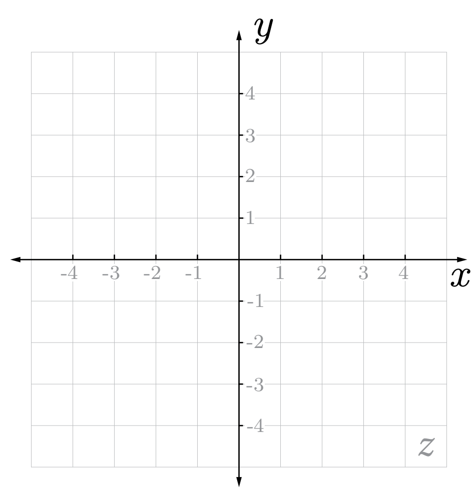

12.5 $f^{-1}(2i)$




12.6 $f^{-1}(-4)$




12.7 $f^{-1}(-2i)$

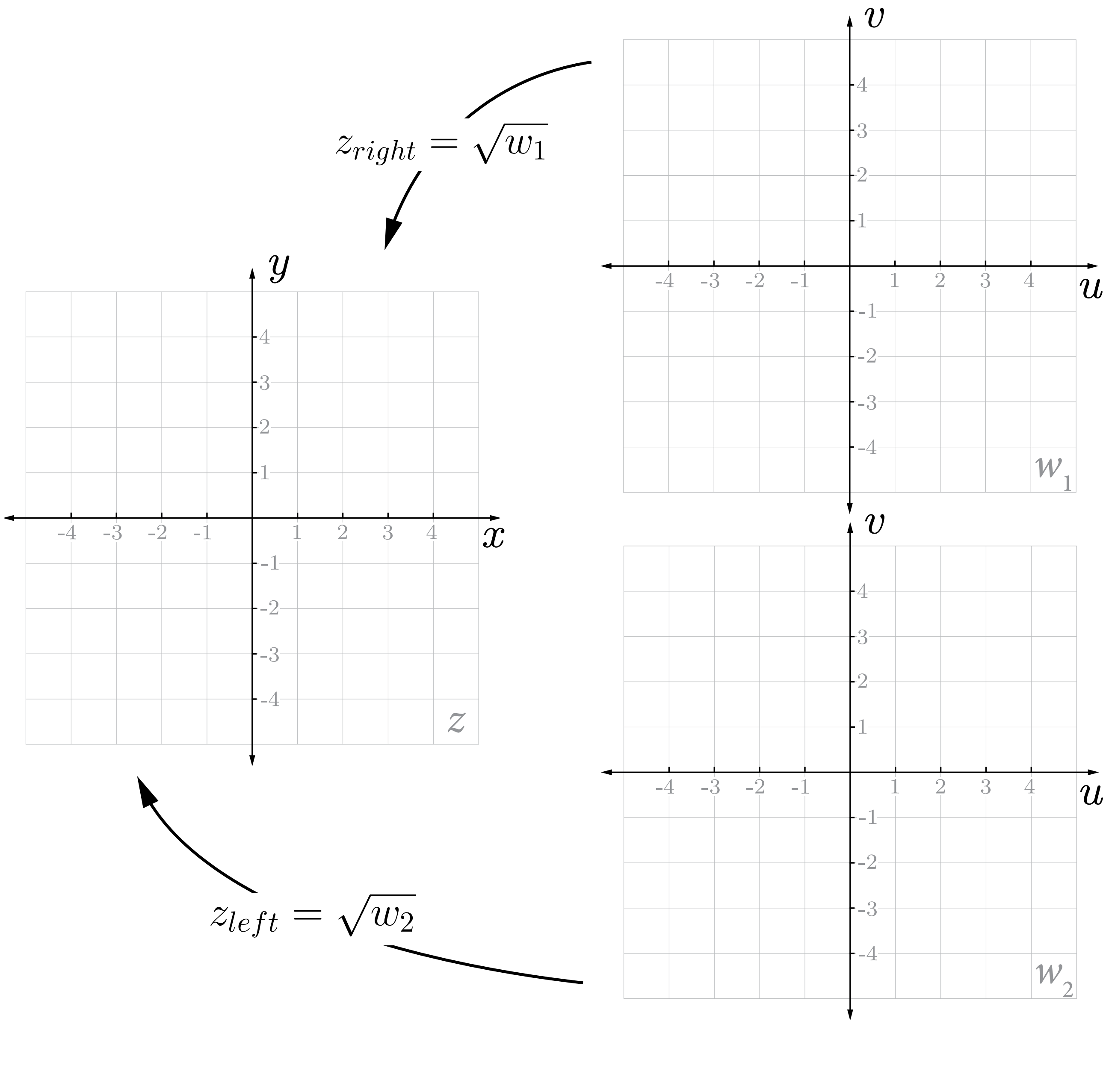


*Critical Thinking*

12.8 Let's try to get a better feel for what's happening in Figure 58. For each $w$ value in the tables below, compute the corresponding $z$ values. Each $w$ value will yeild two $z$ values, so choose the appropriate one based on which side of the plane (left of right) it's on. Plot all points, and connect consecutive points to form a path when possible. For points that end up on on the imaginary $z$ ($y$) axis (these could be interpreted as left or right side), keep both solutions. For one cool point, make your left and right paths different colors.

| $z_{\text{left}}$ | | $w_2$ | |
|---|---|---|---|
| $x_{\text{left}}$ | $y_{\text{left}}$ | $u$ | $v$ |
| | | 4 | 0 |
| | | 0 | 2 |
| | | 0 | 4 |
| | | 2 | $2\sqrt{3}$ |
| | | 4 | 0 |

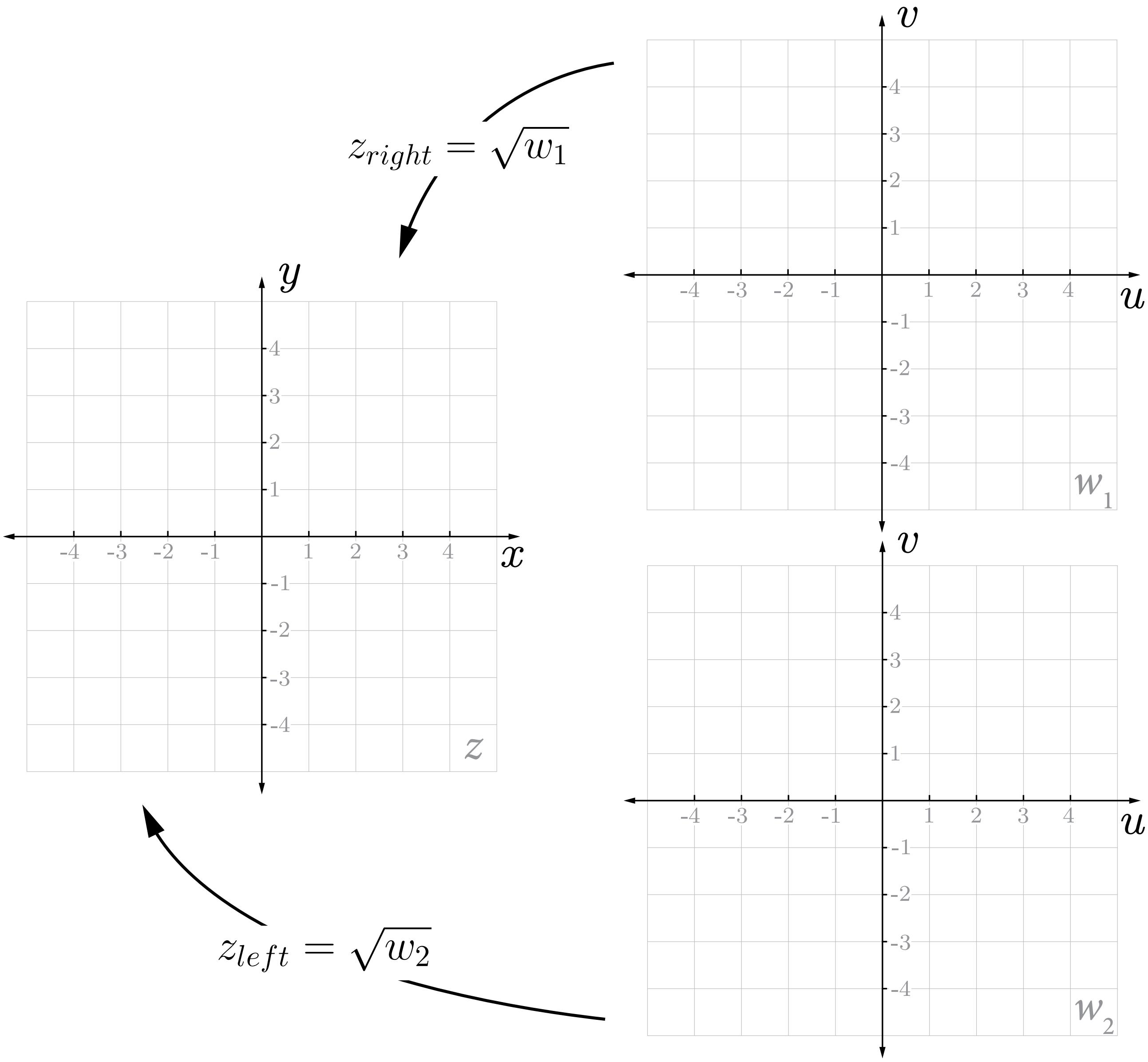
| $z_{\text{right}}$ | | $w_1$ | |
|---|---|---|---|
| $x_{\text{right}}$ | $y_{\text{right}}$ | $u$ | $v$ |
| | | 4 | 0 |
| | | 0 | 2 |
| | | 0 | 4 |
| | | 2 | $2\sqrt{3}$ |
| | | 4 | 0 |



12.9 For each $w$ value in the tables below, compute the corresponding $z$ values. Each $w$ value will yeild two $z$ values, so choose the appropriate one based on which side of the plane (left of right) it's on. Plot all points, and connect consecutive points to form a path when possible. For points that end up on on the imaginary $z$ ($y$) axis (these could be interpreted as left or right side), keep both solutions. For another cool point, make your left and right paths different colors.

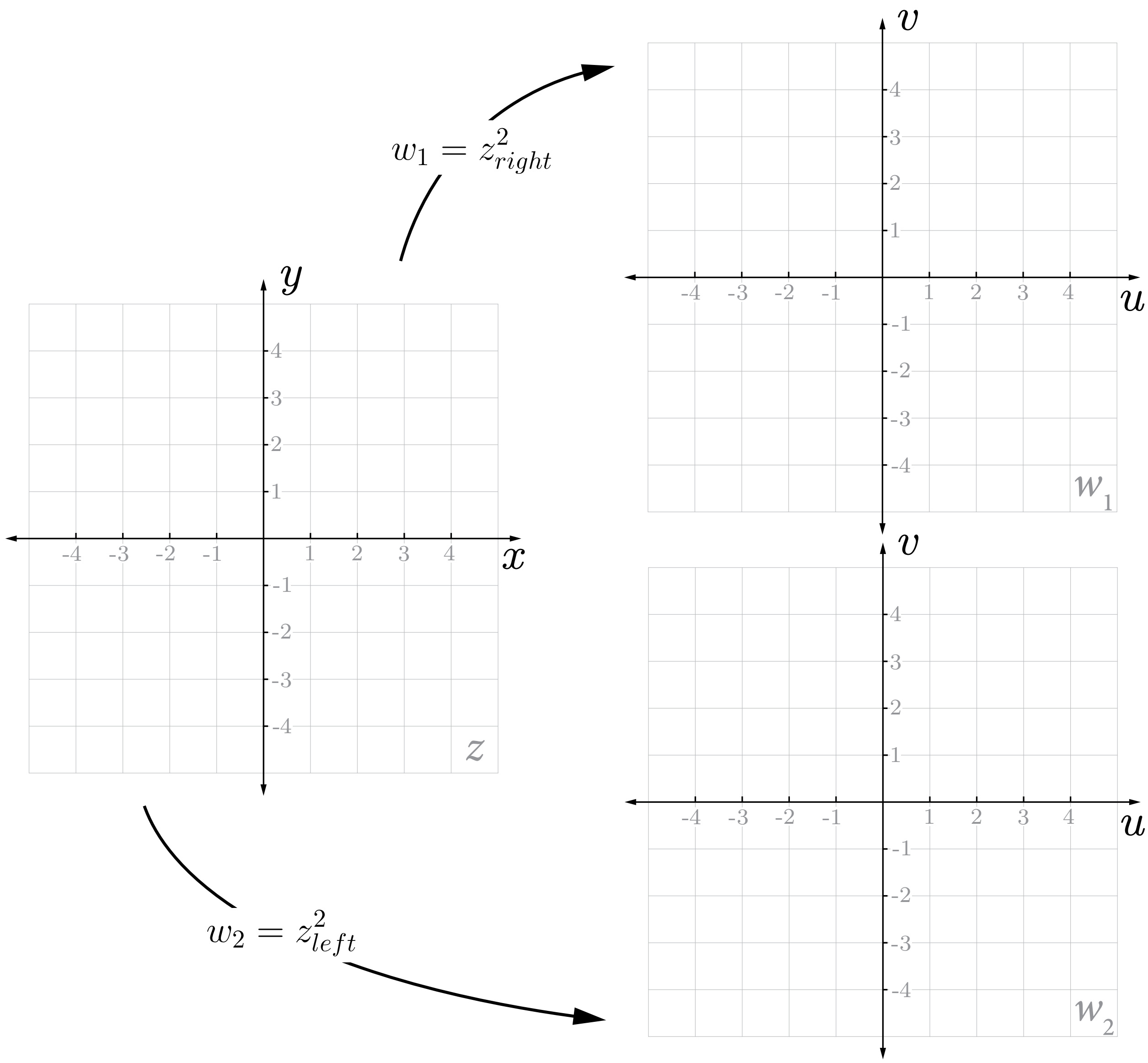
| $z_{\text{left}}$ | | $w_2$ | |
|---|---|---|---|
| $x_{\text{left}}$ | $y_{\text{left}}$ | $u$ | $v$ |
| | | 4 | 0 |
| | | 0 | 2 |
| | | -4 | 0 |
| | | 0 | -2 |
| | | 4 | 0 |

| $z_{\text{right}}$ | | $w_1$ | |
|---|---|---|---|
| $x_{\text{right}}$ | $y_{\text{right}}$ | $u$ | $v$ |
| | | 4 | 0 |
| | | 0 | 2 |
| | | -4 | 0 |
| | | 0 | -2 |
| | | 4 | 0 |



**12.10 Now Forwards!** Map each $z$ value in the table below to its corresponding $w$ value. Note that each $z$ value will map to *either* $w_1$ *or* $w_2$, with one exception: allow points along the imagary $z$ ($y$) axis to map to both $w_1$ *and* $w_2$ - this is a small hack (don't tell your mathematician friends) to make your graph easier to understand. Plot all points, and connect consecutive points to form a path when possible.

| $z$ | | $w_1$ | | $w_2$ | |
|---|---|---|---|---|---|
| $x$ | $y$ | $u$ | $v$ | $u$ | $v$ |
| 2 | 0 | | | | |
| 1 | 2 | | | | |
| 0 | 2 | | | | |
| -2 | 1 | | | | |
| -2 | 0 | | | | |
| -1 | -2 | | | | |
| 0 | -1 | | | | |
| 1 | -1 | | | | |
| 2 | 0 | | | | |



*Challenge*

12.11 In Figure 58, our first discontinuity on $z$ occurs when we first cross the negative real axis on $w$. Why does this happen at this specific location? Is this the result of how our function behaves, or of a choice we made?

# Imaginary Numbers Are Real

## Part 13: Riemann Surfaces

This is a Riemann Surface.

It's going to help us think in four dimensions.

We made it by cutting our planes at the discontinuities in our paths, and taping them together in a way that made our paths from one plane to the other continuous.

Riemann's big insight here was that the domain, the input values of our multifunction, should not be a flat, two dimensional plane. Our domain should instead be this – a curved surface[1] living in higher dimensional space. A Riemann surface.

What's incredible here is what the geometry[2] of our Riemann surface is going to allow us to do. Using our Riemann surface as the input space of our multifunction, we can fix literally all the problems we've encountered thus far: our function will one to one, continuous, and our Riemann surface will even help us elegantly explain the weird loop behavior we saw in part eleven.

Let's see how.

Riemann envisioned these surfaces as sheets covering the input plane, $w$.[3] Our Riemann surface is constructed from two copies of the complex plane, and the idea here is that each input value on w lies directly below its corresponding points on each layer of our Riemann surface.

If we follow a line straight up from the value $2i$ on our w-plane, we find two points that correspond to w equals $2i$. Our Riemann surface fixes our one-to-one problem just as our 2-complex plane solution did – each of our two solutions on z corresponds to its very own copy of the w-plane, these are called braches.

So our Riemann surface makes our mapping one-to-one, just as our two-complex plane approach did last time, but what about continuity? As we saw last time, a big problem with our two-complex plane solution is that it introduced discontinuities.

We constructed our surface in such a way that our colored path was continuous, but we encounter a weird self-intersection along the negative real axis - our planes have to pass right through eachother. Let's dig a little deeper. Remember the main idea here – using our Riemann surface as the input space, the domain, to our complex multifunction should give us clarity. So instead of drawing paths or shapes on our w-plane, we should really be drawing on our Riemann surface.

Drawing on three-dimensional surfaces can be a little challenging, so we'll make use of a tool that wasn't invented until over a century after Riemann's death – a computer!

Just as with our paper version, we'll start with our $w$-plane lying flat on the ground, and place our Riemann surface directly above.

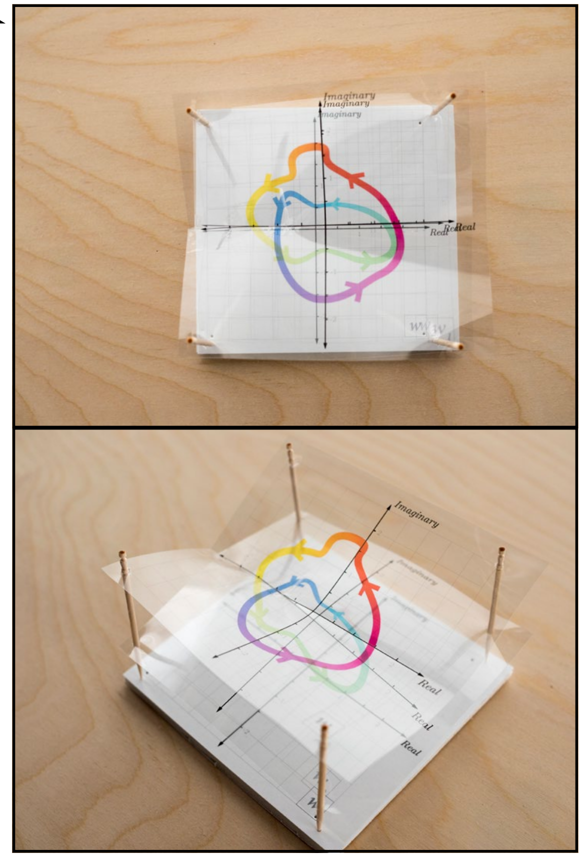



**Figure 61 | The Riemann Surface for Our Function** $z = f^{-1}(w) \pm \sqrt{w}$ **.** We made this surface by printing the paths from Figure 60 on transparencies, cutting the planes at their discontinuities, and taping the paths together.

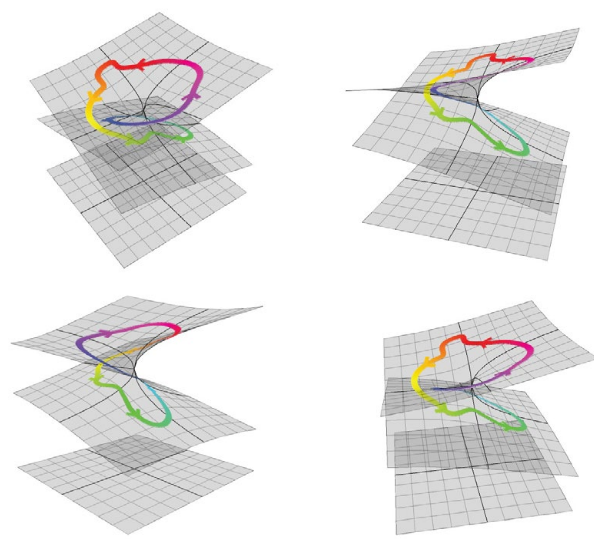


**Figure 62 | The Riemann Surface for Our Function** $z = f^{-1}(w) \pm \sqrt{w}$ **.** We made this surface using the programming language Python and the plotting library Plotly.

Before we start drawing paths all over our surface, let's make sure we know what we're looking at. We're trying to understand the complex function $w = z^2$, or taken in the other direction, $z$ equals plus or minus the square root of $w$. The mapping between w and z is the same both

1 Manifold!
2 Topology, really.
3 Or sphere!

directions. The visualization challenge here is that our mapping is four-dimensional, both $z$ and $w$ have real and imaginary parts – back in part 10 we called these $x$, $y$, $u$ and $v$.

What we're seeing when we visualize our Riemann surface is a two-dimensional surface in three-dimensional space. In this case, since we positioned our surface directly above the $w$-plane, two of our three dimensions correspond exactly to the real and imaginary parts of $w$, we called these $u$ and $v$. Part of Riemann's idea is that our third dimension should represent the $z$-value of our function. But as we know, $z$ is complex number – it has both a real and imaginary part, so there's no way to show both of these on a single axis.

What's often done when visualizing these surfaces, and what we're doing here, is to simply pick the real or imaginary part of $z$,[1] and use this value as the third dimension, the height, of our surface. In Figure 63, we're using $x$, the real part of $z$.

Doing this has a nice visual result: each point on our Riemann surface lives in 3d space at a location corresponding exactly to its $u$, $v$, and $x$ values. So each point on our surface represents a single solution to our equation, and 3 of the 4 values needed to describe the solution are represented by the points' location in 3D space.

This is a nice result, but we must remember that this is not the whole picture. There's another variable – the imaginary part of $z$, we called this $y$ – that is not included in our visualization at all.

This becomes important when trying to figure out if we've actually fixed our continuity problem. If we follow a path along a single branch of our Riemann surface, we

1 Or magnitude or angle!

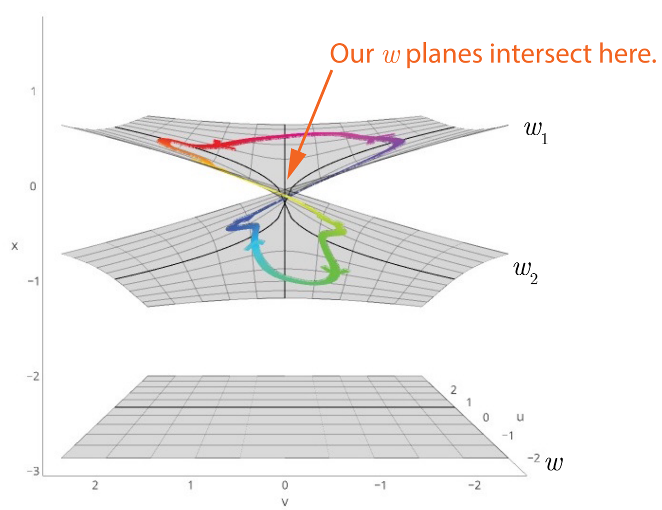



**Figure 64 | What to do about the intersection of our $w$ planes?** We would like to think that we've fixed our continuity problem, but as we follow a path around our surface, what should we do when we hit this point of intersection? Should we stay on the same plane or hop to the other one? How do we even figure this out?

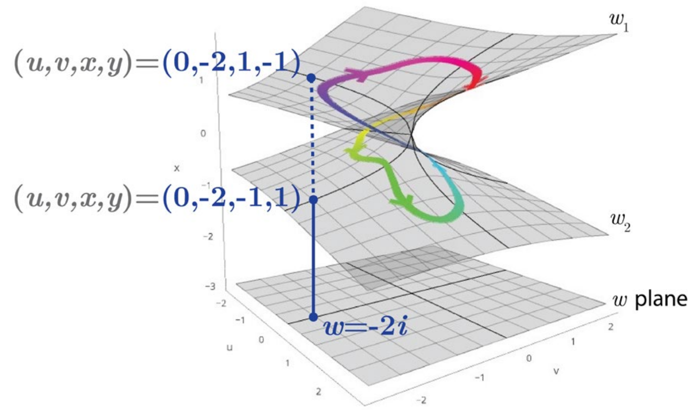



**Figure 63 | Each point on our surface is a solution to our function.** The point $w$=-2$i$, when plugged into our multifunction $z = f^{-1}(w) \pm \sqrt{w}$ yields two solutions, $z$=-1+$i$ and $z$=1-$i$. These solutions show up as the points $(u,v,x,y)$ = (0,-2,-1,1) and (0,-2,1,-1) on our Riemann Surface. 3 of our 4 coordinates show up as our points location in 3D space.

run into a bit of a problem when we hit our self-intersection, after all, should we stay on the same $w$-plane, or hope to the other one?

To answer this question let's try to figure out why our surface self-intersects in the first place. This intersection happens along the negative real axis on our $w$-plane. Let's consider a point on this axis: $w$ =-1. Plugging in -1 for $w$ yields two solutions:

$$z = f^{-1}(w) = \pm\sqrt{w}$$
$$z = \pm\sqrt{-1} = \pm i$$

Our two solutions, $z$ = +$i$, and -$i$ are clearly different, but have the same real part, zero. Since we're only visualizing the real part of $z$ – we have no way of *seeing* that these are in fact different points.

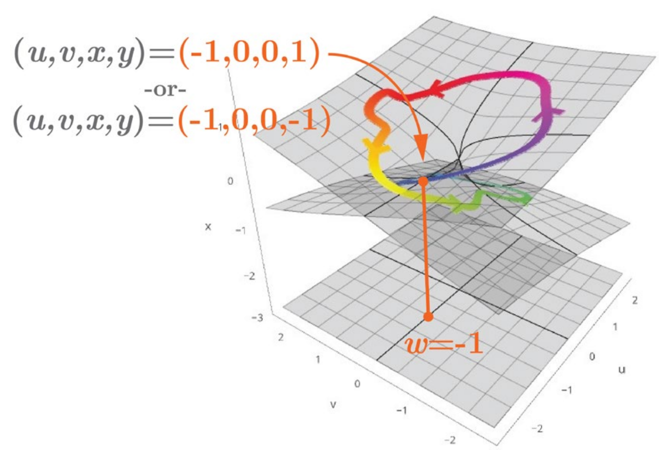



**Figure 65 | Tricky Tricky.** Along our self-intersection line, our two distinct solutions are collapsed into one becuase we don't have enough dimensions to show that out two points are, in fact, different.

This is the danger of visualizing high dimensional mathematical concepts. What we're really looking at here is a projection, a shadow, of our full four dimensional surface.

So our line of self-intersection actually…isn't. This is exactly like the two-dimensional shadows of 3-dimensional objects giving the appearance that the objects intersect.

There are inherent limitations to the types of structures we can visualize in the 3D space we inhabit.

However, there are some clever ways to get a feel for what's happening in our missing fourth dimension. One approach is to expand our visualization to include another dimensions of human perception,[1] such as color.

We'll color each point on our surface with a color that corresponds to the value of the 4th dimension of our function – in this case the imaginary part of $z$, $y$. To do this we need to decide which colors to match which numbers to – this is called a colormap.

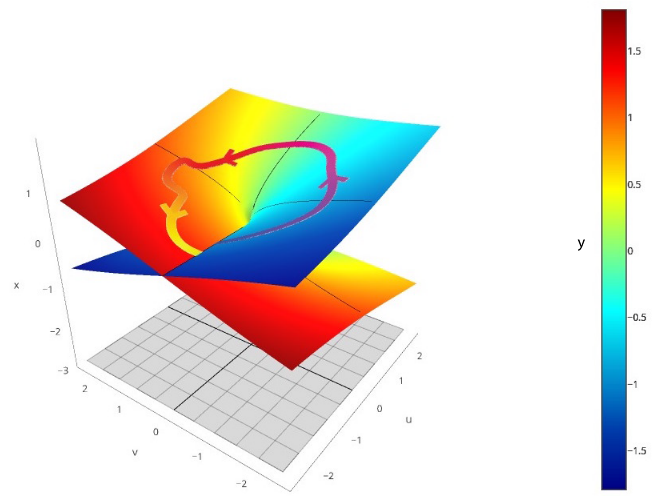



**Figure 67 | One way to visualize all 4 dimensions.** If we color our suface according to our missing value, y, we can get a better feel for what all 4 variables are doing at once. Since our planes are different colors at our line of intersection, this means that our surface actually doesn't self-intersect! The apparent self-intersection is a result of visualizing a 4D object in 3D space.

Our colors now give us a nice idea of what's happening in our missing 4th dimension, $y$. If we look at our line of self-intersection now, it's much more clear that since our colors[2] are different, our four-dimensional function actually doesn't intersect itself. The apparent self-intersection is just an artifact of our visualization technique.

So as we follow paths on our Riemann surface, the right thing to do with these self-intersection lines is to ignore them. If we now follow our path around our surface, we see that it's perfectly continuous, even at our weird self-intersection line.

Excellent.

So that's great, we have continuity – but there's still

**Figure 66 | Shadows are not the whole picture.** The shadows of the pens intresect, but the pens don't! Our Riemann surface is just a shadow of our full 4-dimensional function, complete with false intersections.

one missing piece of the puzzle – what about the weird behavior we saw in Part 11 where some paths ended up in new locations and others, didn't?

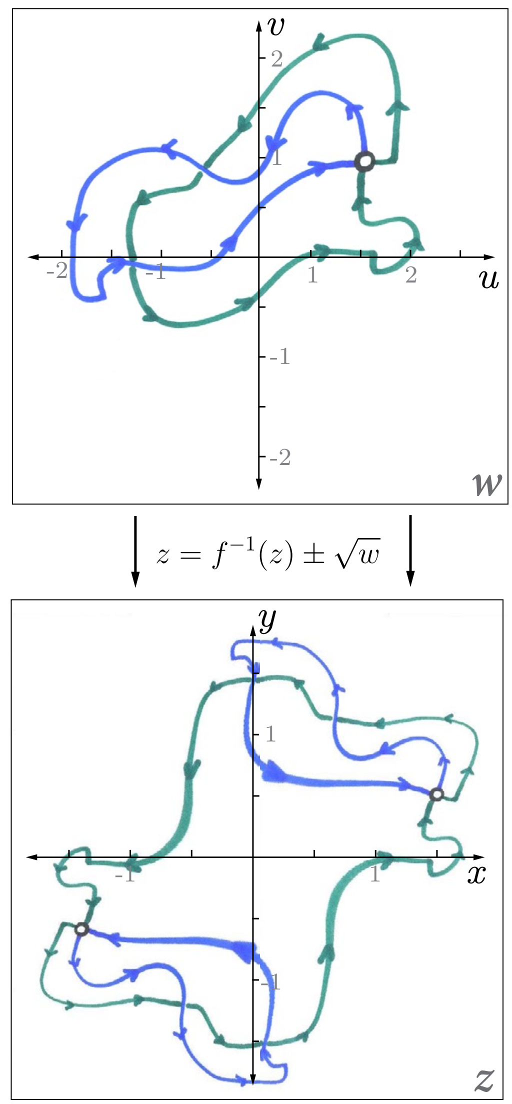



**Figure 68 | Another Pair of Suspicious Paths.** One more look at the suspicious types of paths we first saw back in Part 11. Out blue paths return to where they started, but our green paths don't! Hmmm...I wonder if there's a better way to think about this...

1 Sound?! ☺

2 Any y values

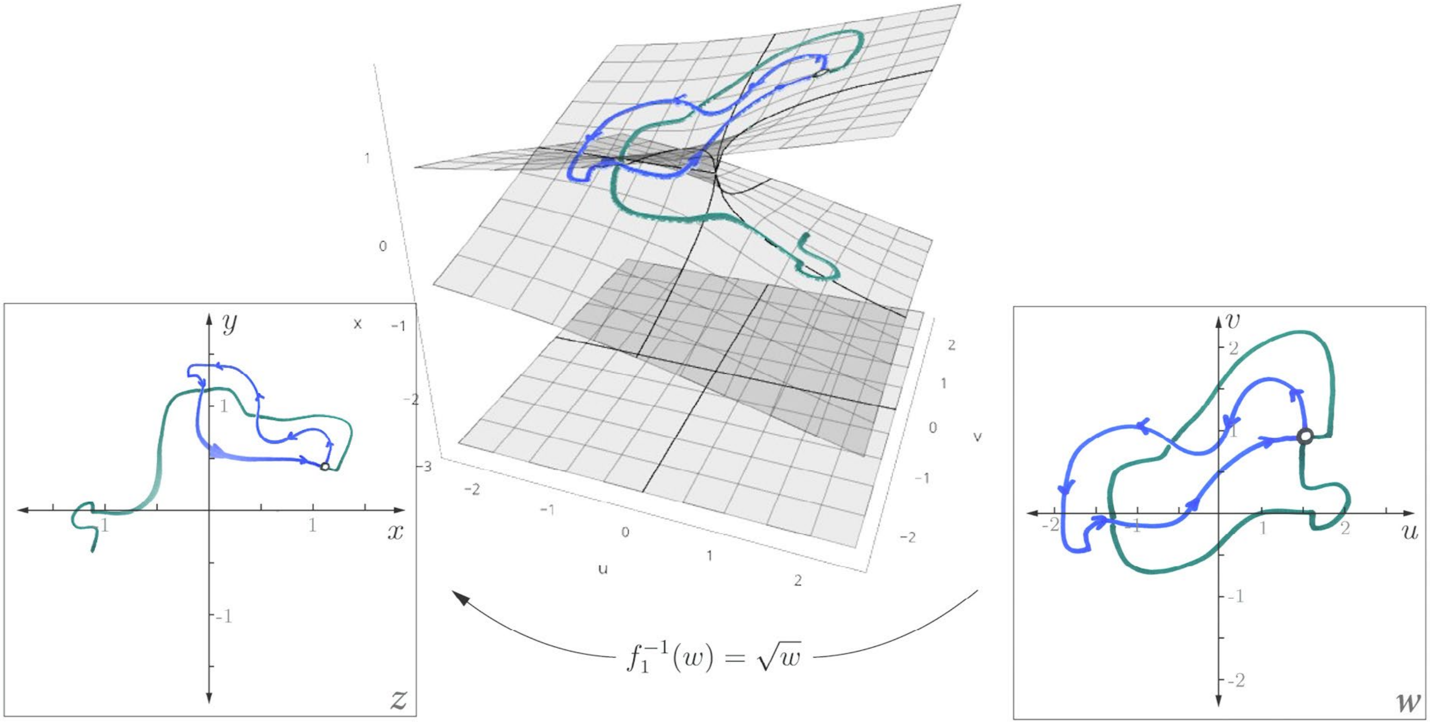



**Figure 69 | Riemann Surfaces are Cool AF.** Using our Riemann Surface, we can see exactly why our blue path returns to where it started, but our green path doesn't!

Let's recreate these suspicious paths, this time using our Riemann Surface to help us out.[1] To keep our surface from getting too crowded, we'll choose just one of our two paths to visualize first.

We'll draw the same exact paths on w and see how they show up on our Riemann surface as they are mapped to the $z$-plane.

So why does our green path start out in one location on our $z$-plane and end up in another?

*Simply because our green path leads us to the other layer of our surface.*

From the perspective of our $w$-plane, it appears that's we've returned exactly to our starting point, but we actually, haven't. The $w$-plane is just a projection, a shadow. In reality, our path has led us to a completely different branch of our function, with different $z$-values. [2]

Our Riemann surface allows us to clearly see that some paths on $w$ lead the other branch, and some don't. More specifically, paths that go around the central point on our Riemann surface end up on new branches. This point is called a branch point – branch points occur wherever the two branches of our function have the same exact value – and can tell us a great deal about how our complex function behaves. For us, this is the point $w = 0$.

So our Riemann surface not only fixes the troubles we ran into earlier, but beautifully explains the strange path behavior we saw! And this is just the beginning, Riemann surfaces are huge part of modern mathematics, and there's way more to say than we have time for here.

Alright, we're **finally** ready to answer the question – what the heck was happing way back in Figure 2?

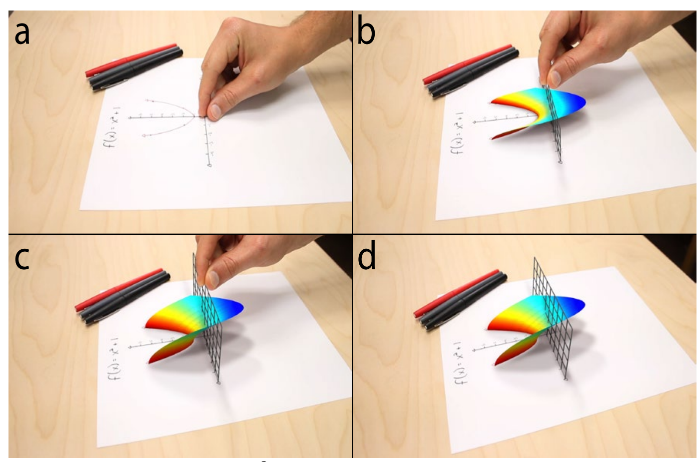



**Figure 2 | Graph of $f(x) = x^2+1$ where $x$ includes imaginary numbers.** Panels a-d show "pulling" the function out of the page.

Our entire discussion has centered around a single function, $f(z)=z^2+1$. And so far, we've looked at one way to visualize the Riemann surface for our function by plotting 3 of our 4 variables in 3D space.

The two-dimensional plot we started our discussion with way back in Figure 1, the one most of us see in math class, only shows 2 of our 4 variables: the real parts of $z$ and $w$. The surface in Figure 2 is the result of including one more variable: the imaginary part of $z$ as the vertical

1 But this time using a thicker marker to make things more clear.
2 The solution has undergone a monodromy along its path.

dimension of our visualization.

When we first saw this surface in part one, people asked a very good question - if, according to the fundamental theorem of algebra, our function is supposed to have exactly 2 roots, why does our surface appear to equal zero at way more than two locations?

This apparent contradiction has everything to do with the shortcomings of living in 3 dimensional space we've been discussing. When we visualize 4-dimensional functions in 3-dimensional space, we must remember that what we're really seeing is a projection, a shadow of the functions' full 4-dimensional form.

Let's have a closer look at our surface from Figure 2. In our opening shot, half of our surface was hidden behind our paper and the colors we used were chosen somewhat arbitrarily to roughly correspond to the surface height.

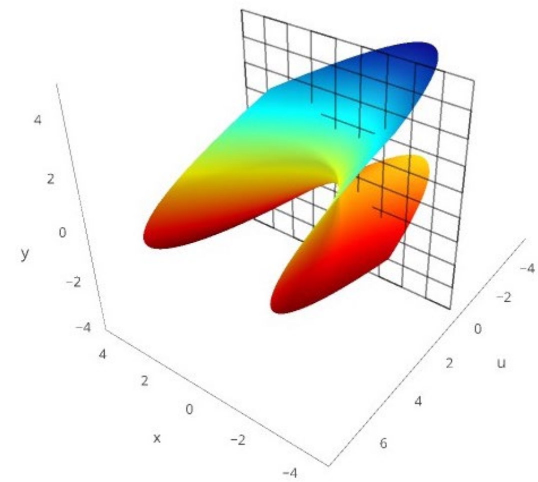



**Figure 70 | A Closer Look at Our Surface in Figure 2.** In Figure 2, half of our surface is hidden beneath the paper.

Now that we know a bit more about functions of complex variables, let's change the color of our surface to correspond to the fourth variable we were forced to leave out of our 3-dimensional visualization– $v$.

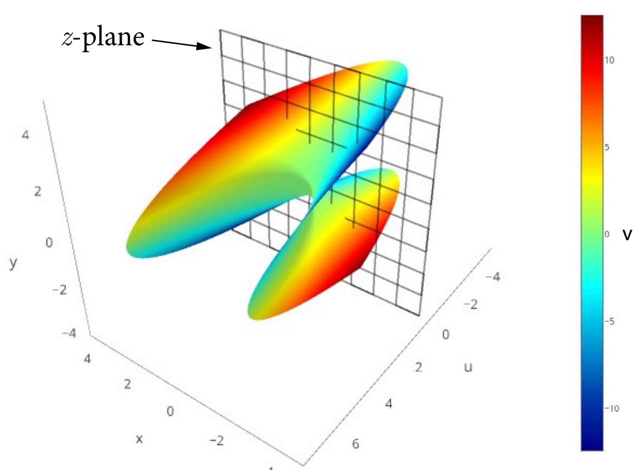



**Figure 71 | Colors!** To get a better feel for what's happening in all 4 dimensions, here we've colored our surface to correspond to hour missing variable, v.

Now that we have some idea of what all four variables are doing, let's look for the two roots predicted by Gauss. Remember, roots are where the output of our function equals zero. For this to be the case, both the real and lateral parts of our output variable $w$, $u$ and $v$, must be zero. Seeing where $u$ equals zero isn't too bad – this is where our surface intersects the $z$-plane.

Now, where does v, the imaginary part of our output variable $w$, equal zero? If we look at our colormap, this should be where our surface is green. It's difficult to see exactly which shade of green corresponds to zero,[1] so let's add an orange line to our surface where v equals exactly zero.

Now if we look closely, we see that both the real and imaginary parts of $w$ equal zero at exactly two points: $i$ and $-i$ on our complex $z$ plane, exactly as our algebra predicted in Figure 41!

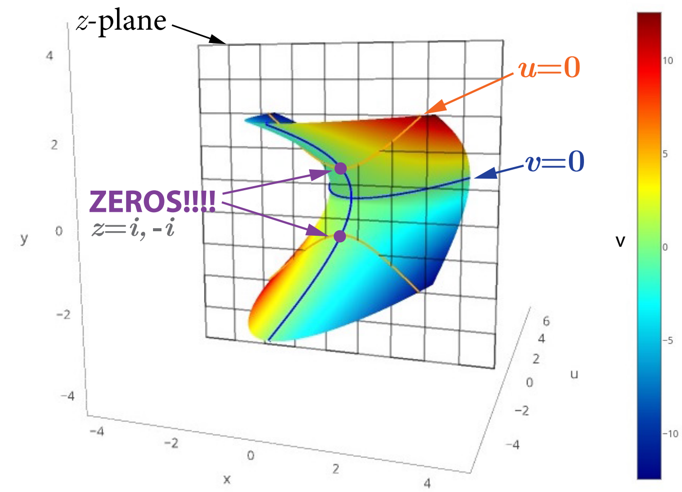



**Figure 72 | Finally.** The zeros we've been looking for since part 1! We've rotated around our surface from Figure 71 to get a better view. Our zeros occur where the real and imaginary parts of w, u and v, equal zero. This happens where our orange a blue lines intersect!

We have *finally* found our missing roots. So our friend Gauss was right all along. Our function *does* have exactly 2 roots.

Of course, finding these roots took some effort! We had to journey deep into mathematics, and ask ourselves what a number really is.[2] This led us to the strange, but necessary conclusion that numbers we should really be using in algebra are the two-dimensional complex numbers. This result dragged us down deeper into the four-dimensional world of complex functions and Riemann Surfaces. When we finally emerged, we saw that the algebra many of us learn in school is only a shadow of an elegant, powerful, and higher dimensional mathematics that has everything to do with the numbers that have been given the terrible name, imaginary. Thanks for reading.[3]

1 A limitation of using color instead of a spatial dimension!.

2 Man this feels like a movie montage.

3 You Rock!

# Exercises 13



*Discussion*

13.1 How did our Riemann Surface help us understand our wierd loop behavior?

13.2 Why was our line of self-intesection actually not?

*Critical Thinking*

13.3 We've shown 2 different surfaces in 3D space that represent our function $f(z)=z^2$ (Figure 62 and Figure 70). Are there other possible surfaces? If so, how many?

13.4 What kind of paths could we draw on our Riemann Surface that would return us *exactly* to where we started?

13.5 What would the Riemann Surface for $z = \sqrt[3]{w}$ look like?

Challenge

13.6 For 1,000,000 cool points, construct a Riemann Surface (in code or paper) for a function other than $f(z)=z^2$, and tweet a picture to @welchlabs.

## Exercises 1



*Discussion*

1.1 Why do you think most people who have lived on our planet would have been suspicious of negative numbers?

1.2 Why do you think negative numbers have been so widely accepted today, despite being somewhat sketchy?

1.3 Do you think negative numbers should be taught to elementary school students? Why or why not?

1.4 How would you explain negative numbers to a 5th grader?

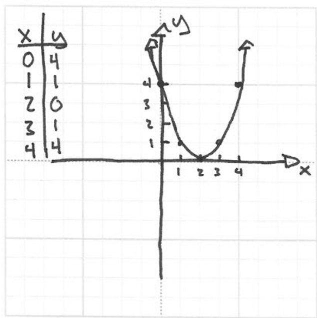
1.5 Which historical civilizations do you think embraced mathematics? Which didn't?

*Drill*

Plot each function.

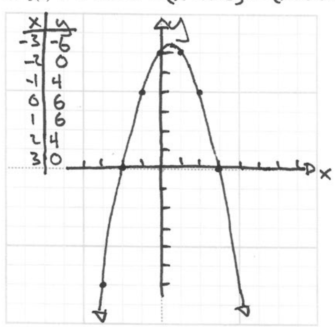
1.6 $f(x) = x^2$-4x+4 $= (x-2)^2$

| x | y |
|---|---|
| 0 | 4 |
| 1 | 1 |
| 2 | 0 |
| 3 | 1 |
| 4 | 4 |



1.7 $g(x) = -x^2+x+6 = -(x^2-x-6) = -(x-3)(x+2)$

| x | y |
|---|---|
| -3 | -6 |
| -2 | 0 |
| -1 | 4 |
| 0 | 6 |
| 1 | 6 |
| 2 | 4 |
| 3 | 0 |





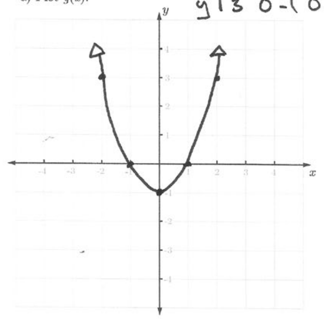
*Critical Thinking*

1.8 Consider the better-behaved sibling of Equation 1:

$$g(x) = x^2\text{-}1$$

| x | -2 | -1 | 0 | 1 | 2 |
|---|---|---|---|---|---|
| y | 3 | 0 | -1 | 0 | 3 |

a) Plot $g(x)$.



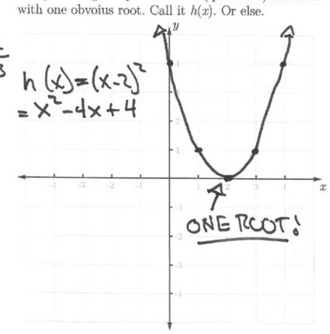
b) For what x-values does $g(x)$=0? $g(x)$=0 when $x$ = 1 or $x$ = -1. These are called the **roots, zeros, or x-intercepts of** $g$.

c) The roots of $g$ are much easier to find than the roots of $f$ (Equation 1), why is this the case?

BECAUSE g CROSSES THE X-AXIS.

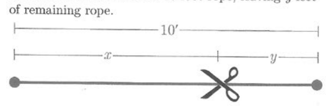
1.9 So far we've seen a function with 2 roots, $g$, and a function with no obvious roots, $f$ (Equation 1). Find and plot a highest power of two (quadratic) function with one obvoius root. Call it $h(x)$. Or else.

$h(x) = (x-2)^2$
$= x^2-4x+4$



ONE ROOT!

1.10 You cut $x$ feet from a 10 foot rope, leaving $y$ feet of remaining rope.



a) Write an equation relating $x$ and $y$.

X+y=10.

b) Solve for $y$ when $x$=7', $x$=10', and $x$=13'.

| x | y |
|---|---|
| 7 | 3 |
| 10 | 0 |
| 13 | -3 |

y=10-x

c) One of your answers from part b should be negative. Is this result meaninfgul? What does a negative answer tell you about your remaining rope? Does the specific value of your answer matter, or is knowing your answer is negative enough to reach a conclusion about the remaining rope?

THE VALUE -3 IS NOT VERY IMPORTANT, BUT THE FACT THAT ITS NEGATIVE MEANS WE'RE OUT OF ROPE!



1.11 You have exactly 10 minutes to get to class for your big test on lateral numbers! Your friend Gus (who's wierdly in to this sort of thing) says that class is exactly 10,000 feet away. Let's say you travel to class at a rate of $r$ feet/minute. You would like to get there a few minutes early. More specifically, let's say you will arrive $t$ minutes early.

a) Write an equation that relates $r$ and $t$.

D=R·T (DISTANCE = RATE·TIME)

TIME TRAVELING = 10,000/r

t = 10 - TRAVEL TIME => $\boxed{t = 10 - \frac{10{,}000}{r}}$

b) Compute $t$ for $r$=1250 feet/minute, $r$=1000 feet/minute, and $r$=625 feet/minute.

| r | t |
|---|---|
| 1250 | 2 |
| 1000 | 0 |
| 625 | -6 |

c) One of your answers from part b should be negative. Is this result meaninfgul? What does a negative answer tell you about when you will arrive to class?

t=-6, MEANING WE WILL ARRIVE 6 MINUTES LATE! UNLIKE EXERCISE 1.10, THE VALUE OF THE NEGATIVE NUMBER HERE MEANS SOMETHING.

*Challenge*

1.12 Consider the function:

$$p(x) = x^4\text{-}10x^3\text{+}35x^2\text{-}50x\text{+}24$$

a) According to the fundamental theorem of algebra, how many roots should $p(x)$ have?

4

b) Find all the roots of $p(x)$. You may use technology if you would like, but it's not neccessary to complete the problem.

USING RATIONAL ROOT THEOREM, ROOTS ~~SHOULD~~ COULD BE ±1, ±2, ±3, ±4, ±6, ±12, ±24

try 1: 1 | 1 -10 35 -50 24
1 -9 26 -24
1 -9 26 -24 0 THAT WAS LUCKY

$x^3-9x^2+26x-24=0$

try 2: 2 | 1 -9 26 -24
2 -14 24
1 -7 12 0 WOOHOO!

$x^2-7x+12=0$

$(x-4)(x-3)=0$

$x=4, 3$

$\hookrightarrow$ x=1, 2, 3, 4

## Exercises 2



*Discussion*

2.1 Why did del Ferro keep his discovery secret?

2.2 Why are problems like √-16 strange?

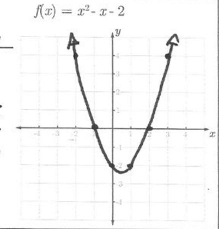
2.3 Why do you think math duels are way less popular today than in the 16th century?

*Drill*

| | Plot | Solve by Factoring (If Possible) | Solve by Quadratic Formula (If Possible) |
|---|---|---|---|
| 2.4 | $f(x) = x^2\text{-}\ x\text{-}\ 2$ | $0 = x^2\text{-}\ x\text{-}\ 2$ | $0 = x^2\text{-}\ x\text{-}\ 2$ |
| 2.5 | $g(x) = x^2 + 3x + 2$ | $0 = x^2 + 3x + 2$ | $0 = x^2 + 3x + 2$ |

2.4

| $x$ | $y$ |
|---|---|
| -2 | 4 |
| -1 | 0 |
| 0 | -2 |
| 1 | -2 |
| 2 | 0 |
| 3 | 4 |



$0 = (x-2)(x+1)$

$\Rightarrow \boxed{x = 2, -1}$

$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$

$x = \frac{1 \pm \sqrt{(1)^2-4\cdot1\cdot(-2)}}{2\cdot1}$

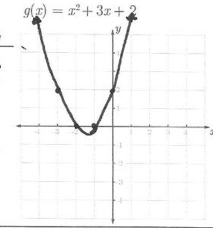
$x = \frac{1 \pm \sqrt{1+8}}{2}$

$x = \frac{1 \pm 3}{2} = \frac{4}{2}, \frac{-2}{2}$

$\boxed{= 2, -1}$

2.5

| $x$ | $y$ |
|---|---|
| -3 | 2 |
| -2 | 0 |
| -1 | 0 |
| 0 | 2 |
| 1 | 6 |



$0 = (x+2)(x+1)$

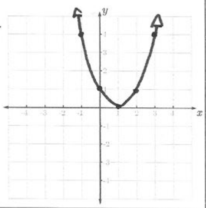
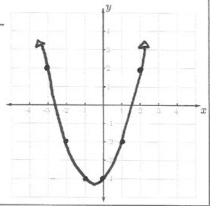
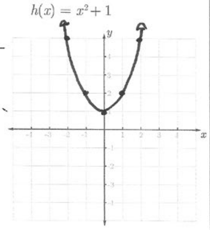
$\boxed{x = -1, -2}$

$x = \frac{-3 \pm \sqrt{3^2-4\cdot1\cdot2}}{2\cdot1}$

$x = \frac{-3 \pm \sqrt{9-8}}{2}$

$x = \frac{-3 \pm 1}{2} = \frac{-2}{2}, \frac{-4}{2}$

$\boxed{= -1, -2}$

| | Plot | Solve by Factoring (If Possible) | Solve by Quadratic Formula (If Possible) |
|---|---|---|---|
| 2.6 | $f(x) = x^2 - 2x + 1$<br>x: -1, 0, 1, 2, 3<br>y: 4, 1, 0, 1, 4<br> | $0 = x^2 - 2x + 1$<br>$0 = (x-1)(x-1)$<br>$\boxed{x = 1}$ | $0 = x^2 - 2x + 1$<br>$x = \frac{2 \pm \sqrt{(-2)^2 - 4 \cdot 1 \cdot 1}}{2(1)}$<br>$= \frac{2 \pm \sqrt{4-4}}{2}$<br>$= \frac{2 \pm 0}{2} = \boxed{1}$ |
| 2.7 | $g(x) = x^2 + x - 4$<br>x: -3, -2, -1, 0, 1, 2<br>y: 2, -2, -4, -4, -2, 2<br> | $0 = x^2 + x - 4$<br>~~0~~<br>NOT FACTORABLE! | $0 = x^2 + x - 4$<br>$x = \frac{-1 \pm \sqrt{1 - 4 \cdot 1 \cdot (-4)}}{2 \cdot 1}$<br>$= \frac{-1 \pm \sqrt{1 + 16}}{2}$<br>$= \frac{-1 \pm \sqrt{17}}{2}$ |
| 2.8 | $h(x) = x^2 + 1$<br>x: -2, -1, 0, 1, 2<br>y: 5, 2, 1, 2, 5<br> | $0 = x^2 + 1$<br>NOT FACTORABLE! | $0 = x^2 + 1$<br>$x = \frac{0 \pm \sqrt{0^2 - 4 \cdot 1 \cdot 1}}{2 \cdot 1}$<br>$= \frac{0 \pm \sqrt{-4}}{2}$<br>$= \frac{\pm\sqrt{-4}}{2}$ ← WE'LL LEARN ABOUT THIS SOON! |

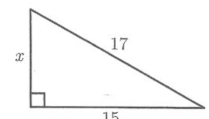


*Critical Thinking*

2.9 Why is the Quadratic Formula useful? Does it allow you to solve any types of problems that you couldn't otherwise? (Hint: compare exercise 2.7 to 2.4-2.6)

THE QUADRATIC FORMULA GIVES YOU THE ~~...~~ ROOTS, EVEN WHEN THE POLYNOMIAL IS NOT FACTORABLE.

2.10 Solve for $x$:




$a^2 + b^2 = c^2$

$x^2 + 15^2 = 17^2$

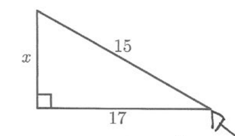
$x^2 + 225 = 289$

$x^2 = 64$

$\boxed{x = 8}$

2.11 Sometimes problems like 2.10 are less straightforward.

a) Solve for $x$:




$x^2 + 17^2 = 15^2$

$x^2 + 289 = 225$

$x^2 = -64$

$x = \pm\sqrt{-64}$ ???

b) How did part a go? :) If you used the Pythagorean Theorem to solve for $x$, the result should have been the square root of a negative number. Sketchy. What's going on here?

THIS TRIANGLE IS IMPOSSIBLE! THE Hypotenuse MUST BE THE LONGEST SIDE.



2.12 Where do the parabola $y = x^2$-4 and the line y = $2x$-1 intersect?

→ SET EQUATIONS EQUAL TO EACHOTHER TO FIND INTERSECTION:

$x^2 - 4 = 2x - 1$

$0 = x^2 - 2x - 3$

$0 = (x-3)(x+1)$

$x = 3, -1$

$y = 2x - 1 = 2(3) - 1 = 6 - 1 = 5$ | $y = 2(-1) - 1 = -2 - 1 = -3$

$\boxed{(3, 5)}$ | $\boxed{(-1, -3)}$

2.13 Where do the parabola $y = x^2$-4 and the line y = $2x$-3 intersect?

$x^2 - 4 = 2x - 3$

$x^2 - 2x - 1 = 0$

↳ NOT FACTORABLE.

$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$x = \frac{2 \pm \sqrt{(-2)^2 - 4 \cdot 1 \cdot (-1)}}{2 \cdot 1}$

$x = \frac{2 \pm \sqrt{4+4}}{2}$

$x = \frac{2 \pm \sqrt{8}}{2} = \frac{2 \pm 2\sqrt{2}}{2}$

$= 1 \pm \sqrt{2}$

$y = 2(1+\sqrt{2}) - 3$ | $y = 2(1-\sqrt{2}) - 3$

$y = 2 + 2\sqrt{2} - 3$ | $y = 2 - 2\sqrt{2} - 3$

$= -1 + 2\sqrt{2}$ | $= -1 - 2\sqrt{2}$

$(1+\sqrt{2}, -1+2\sqrt{2})$ | $(1-\sqrt{2}, -1-2\sqrt{2})$

2.14 a) Where do the parabola $y = x^2$-1 and the line y = $2x$-3 intersect?

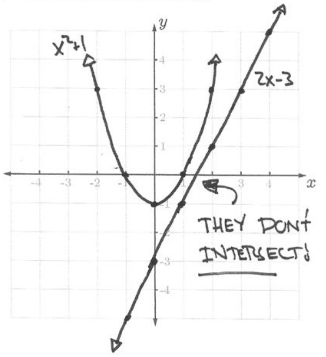
$x^2 - 1 = 2x - 3$

$x^2 - 2x + 2 = 0$ ⇒ NOT FACTURABL

$x = \frac{2 \pm \sqrt{2^2 - 4 \cdot 1 \cdot 2}}{2 \cdot 1}$

$x = \frac{2 \pm \sqrt{4-8}}{2} = \frac{2 \pm \sqrt{-4}}{2}$ ???

b) Using the Quadratic Formula to solve part a should have resulted in the square root of negative number. What does this mean about the problem? Plot the parabola and line from part a below.






2.15 Solve the equation $x^3 = 8$ using del Ferro's formula (Equation 4).

$$x = \sqrt[3]{\frac{d}{2} + \sqrt{\frac{d^2}{4} + \frac{c^3}{27}}} + \sqrt[3]{\frac{d}{2} - \sqrt{\frac{d^2}{4} + \frac{c^3}{27}}},$$

$$x^3 + cx = d.$$

$c = 0, \ d = 8$

$x = \sqrt[3]{\frac{8}{2} + \sqrt{\frac{64}{4} + \frac{0^3}{27}}} + \sqrt[3]{\frac{8}{2} - \sqrt{\frac{64}{4} + \frac{0}{27}}}$

$x = \sqrt[3]{4 + \sqrt{16}} + \sqrt[3]{4 - \sqrt{16}}$

$x = \sqrt[3]{8} + \sqrt[3]{0}$

$x = 2 + 0 \Rightarrow \boxed{x = 2}$

2.16 Solve the equation $x^3 + 6x = 20$ using del Ferro's formula (Equation 4).

$x = \sqrt[3]{\frac{20}{2} + \sqrt{\frac{400}{4} + \frac{216}{27}}} + \sqrt[3]{\frac{20}{2} - \sqrt{\frac{400}{4} + \frac{216}{27}}}$

$x = \sqrt[3]{10 + \sqrt{100 + 8}} + \sqrt[3]{10 - \sqrt{100 + 8}}$

$x = \sqrt[3]{10 + \sqrt{108}} + \sqrt[3]{10 - \sqrt{108}}$

↓

TECHNOLOGY :)

↓

2

2.17 Solve the equation $x^3 = x + 2$ using Cardan's modified version of Del Ferro's formula (shown in the 3rd row of Table 2).

$$x = \sqrt[3]{\frac{d}{2} + \sqrt{\frac{d^2}{4} - \frac{c^3}{27}}} + \sqrt[3]{\frac{d}{2} - \sqrt{\frac{d^2}{4} - \frac{c^3}{27}}},$$

$$x^3 = cx + d.$$

$x = \sqrt[3]{\frac{2}{2} + \sqrt{\frac{4}{4} - \frac{1}{27}}} + \sqrt[3]{\frac{2}{2} - \sqrt{\frac{4}{4} - 1/27}}$

$= \sqrt[3]{1 + \sqrt{26/27}} + \sqrt[3]{1 - \sqrt{26/27}}$

$\boxed{= 1.5213797}$

*Challenge*

2.18 Derive the Quadratic Formula from $ax^2+bx+c=0$ by completing the square.

2.19 Derive del Ferro's formula from $x^3+cx=d$ by witchcraft.

SHOWN IN VIDEO :)

2.20 What exact values of c and d result in the square roots of negative numbers in Cardan's formula (row 3 of, column 5 of Table 2)? Exactly how many real solutions must this type of cubic have?

$\frac{d^2}{4} - \frac{c^3}{27} < 0.$

↳ WHEN THIS IS TRUE, THERE WILL BE 3 REAL SOLUTIONS. SEE p. 20 OF PAUL NAHIN'S EXCELLENT BOOK: "AN IMAGINARY TALE: THE STORY OF $\sqrt{-1}$"

## Exercises 3



*Discussion*

3.1 Rafael Bombelli was born in 1526. For thousands of years before Bombelli's time, people used mathematics to solve all kinds of problems without any need for $\sqrt{-1}$. What do you think was so compelling about Cardan's problem that led Bombelli to consider using a completely new type of number to solve it?

3.2 Imagine you're working on a hard math problem. In fact, it's so hard, that no one on the planet has solved it yet. You work and work and work and get nowhere. Your friend Gus, who has been working on the same problem, excitedly tells you that he's found a solution! He shows you his work, and tells you that if works perfectly with one minor catch. To solve the problem he has allowed an entirely new kind of number to exist. A type of number that no one has used before. How would you respond to Gus?

*Drill*

Simplify the following. Helpful examples can be found in Table 3 and Equation 9.

3.3 $\sqrt{-16}$

$= \sqrt{16}\cdot\sqrt{-1} = \boxed{4\sqrt{-1}}$

3.4 $\sqrt{-51}$

$= \sqrt{51}\cdot\sqrt{-1} = \boxed{\sqrt{51}\sqrt{-1}}$

3.5 $\sqrt{-4}$

$= \sqrt{4}\ \sqrt{-1} = \boxed{2\sqrt{-1}}$

3.6 $\sqrt{-5} + \sqrt{-5}$

$\boxed{2\sqrt{-5}} = \boxed{2\sqrt{5}\sqrt{-1}}$

3.7 $\sqrt{-5} + \sqrt{5}$

$\sqrt{5}\sqrt{-1} + \sqrt{5} = \boxed{\sqrt{5}(\sqrt{-1}+1)}$

3.8 $16 + \sqrt{-16}$

$16 + \sqrt{16}\sqrt{-1}$

$\boxed{16 + 4\sqrt{-1}}$

3.9 $\sqrt{-4}\sqrt{4}$

$= 2\sqrt{4}\cdot\sqrt{-1}$

$= \boxed{4\sqrt{-1}}$

3.10 $\sqrt{-512}$

2 256

16 16

$= \sqrt{2\cdot 256}\cdot\sqrt{-1}$

$= \boxed{16\sqrt{2}\sqrt{-1}}$

3.11 $7/\sqrt{-7}$

$\frac{7}{\sqrt{-7}}\cdot\frac{\sqrt{-7}}{\sqrt{-7}} = \frac{7\sqrt{-7}}{\sqrt{7}\cdot\sqrt{-1}\cdot\sqrt{7}\cdot\sqrt{-1}} = \frac{7\sqrt{7}}{7\cdot -1} = \boxed{-\sqrt{7}}$

3.12 $\sqrt{-4}\ \sqrt{-4}$

$= \sqrt{4}\sqrt{-1}\sqrt{4}\sqrt{-1} = 2\cdot 2\cdot(\sqrt{-1})^2$

$\boxed{= -4}$

3.13 $1/\sqrt{-1}$

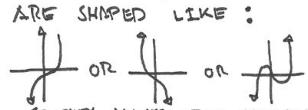
$\frac{1}{\sqrt{-1}}\cdot\frac{\sqrt{-1}}{\sqrt{-1}} = \frac{\sqrt{-1}}{-1} = -\sqrt{-1}$



*Critical Thinking*

3.14 Why do all cubic functions have to have at least one root?

BECAUSE ALL CUBICS ARE SHAPED LIKE:



SO THEY ALWAYS CROSS THE X-AXIS.

3.15 Find two numbers that multiply to 15 and add to 8.

$xy = 15 \quad x + y = 8$

$y = 8 - x$

$x(8-x) = 15$

$8x - x^2 = 15$

$0 = x^2 - 8x + 15$

$0 = (x-3)(x-5)$

$x = 3, 5$

3.16 Find two numbers that multiply to 11,187 and add to 212.

$xy = 11187 \quad x + y = 212$

$y = 212 - x$

$x(212-x) = 11187$

$212x - x^2 = 11187$ → FACTORABLE? WHO THE HECK KNOWS?

$0 = x^2 - 212x + 11187$

$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$

$= \frac{212 \pm \sqrt{212^2 - 4\cdot 1\cdot 11187}}{2\cdot 1} = \boxed{99, 113}$

3.17 a) Find two numbers that multiply to 40 and add to 10.

$xy = 40 \quad x + y = 10$

$x(10-x) = 40$

$10x - x^2 = 40$

$0 = x^2 - 10x + 40$ → NOT FACTORABLE.

$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$

$x = \frac{10 \pm \sqrt{100 - 4\cdot 1\cdot 40}}{2}$

$= \frac{10 \pm \sqrt{-60}}{2}$

$= 5 \pm \frac{2\sqrt{-15}}{2}$

$\boxed{= 5 \pm \sqrt{-15}}$

b) How did part a go? This problem dates back to Cardan himself. In his book Ars Magna, Cardan gives the task of dividing 10 into parts whose product is 40, and calls this "manifestly impossible." You may have reached a similar conclusion. However, Cardan does push on in Ars Magna and gives the solution $5+\sqrt{-15}$, and $5-\sqrt{-15}$. Show that these numbers add to 10 and multiply to 40.

$5 + \sqrt{-15}$
$+\ 5 - \sqrt{-15}$
$10$ ✓

$(5+\sqrt{-15})(5-\sqrt{-15})$

$25 + 5\sqrt{-15} - 5\sqrt{-15} - (\sqrt{-15})^2$

$25 - (-15)$

$\boxed{= 40}$ ✓

Interestingly, even though Cardan gave this solution, he didn't think very highly of it, saying: "So progressing arithmetic subtlety the end of which, as is said, is refined as it is useless."

*Challenge*

$q(x) = 6x^4 - 5x^3 + 20x^2 - 20x - 16$

3.18 One solution to $q(x)=0$ is $\sqrt{-4}$. Find all other solutions.

we'll see later that if $\sqrt{-4}$ is a solution, $-\sqrt{-4}$ must be one too

| | | | | | |
|---|---|---|---|---|---|
| $2\sqrt{-1}$ | 6 | -5 | 20 | -20 | -16 |
| | | $12\sqrt{-1}$ | $-10\sqrt{-1}$ −24 | $20-8\sqrt{-1}$ | 16 |
| $-2\sqrt{-1}$ | 6 | $-5+12\sqrt{-1}$ | $-10\sqrt{-1}$ −4 | $-8\sqrt{-1}$ | 0 ✓ |
| | | $-12\sqrt{-1}$ | $10\sqrt{-1}$ | $8\sqrt{-1}$ | |
| | 6 | -5 | -4 | 0 ✓ | |

$6x^2 - 5x - 4 = 0$

$6x^2 - 8x + 3x - 4 = 0$

$2x(3x-4) + 1(3x-4) = 0$

$(2x+1)(3x-4) = 0$

$x = -1/2,\ 4/3,\ 2\sqrt{-1},\ -2\sqrt{-1}$

## Exercises 4



*Discussion*

4.1 Why do you think Cardan and Bombelli mistrusted $\sqrt{-1}$?

4.2 The Greeks were arguably the most sophisticated mathematicians of the ancient world, and today are remembered for their many contributions to geometry. If Bombelli were to travel back in time, do you think he could convince the ancient Greeks to adopt $\sqrt{-1}$?

*Drill*

4.3 $\sqrt{-1}\cdot(1+\sqrt{-1})$

$= \sqrt{-1} + (\sqrt{-1})^2$

$= \boxed{\sqrt{-1} - 1}$

4.4 $(1-\sqrt{-1})\cdot(1+\sqrt{-1})$

$1 - \cancel{\sqrt{-1}} + \cancel{\sqrt{-1}} - (\sqrt{-1})^2$

$1 + 1 = \boxed{\cancel{0}}\ \boxed{2}$

JK!

4.5 $(2+3\sqrt{-1})\cdot(2+3\sqrt{-1})$

$4 + 6\sqrt{-1} + 6\sqrt{-1} + 9(\sqrt{-1})^2$

$4 + 12\sqrt{-1} - 9$

$\boxed{-5 + 12\sqrt{-1}}$

4.6 $(5-2\sqrt{-1})^2 = (5-2\sqrt{-1})(5-2\sqrt{-1})$

$= 25 - 10\sqrt{-1} - 10\sqrt{-1} + 4(\sqrt{-1})^2$

$= 25 - 20\sqrt{-1} - 4 = \boxed{21 - 20\sqrt{-1}}$

4.7 $(5\sqrt{-1}-2\sqrt{-1})^2$

$= (3\sqrt{-1})^2 = 9\cdot -1 = \boxed{-9}$

Solve for a and b:

4.8 $2+\sqrt{-121} = a+b\sqrt{-1}$

$\boxed{a=2}$

$\sqrt{-121} = b\sqrt{-1}$

$\sqrt{121}\cancel{\sqrt{-1}} = b\cancel{\sqrt{-1}}$

$\boxed{b = \sqrt{121} = 11}$

4.9 $\sqrt{-4} = a+b\sqrt{-1}$

$\boxed{a=0}$

$b\sqrt{-1} = \sqrt{-4}$

$b\cancel{\sqrt{-1}} = 2\cancel{\sqrt{-1}}$

$\boxed{b=2}$

4.10 $\sqrt{4} = a+b\sqrt{-1}$

$2 = a + b\sqrt{-1}$

$\boxed{b=0 \quad a=2}$

Find a solution by guessing and checking:

4.11 $a^2 + b^2 = 50$
$a - b^2 = 6$

*what perfect □'s add to Fifty?

$a = 7,\ b = 1$

4.12 $a(a^2 + b^2) = 0$
$a + b = 1$

$a = 0,\ b = 1$



4.13 $a(a^2 + b^2) = 2$
$a + b = 2$

$a = 1 \quad b = 1$

*Critical Thinking*

In this section, we'll try to get a better feel for exactly how Bombelli solved Cardan's problem, $x^3=15x+4$.

4.14 How did Bombelli justify writing the equation below?

$$\sqrt[3]{2+\sqrt{-121}} = a + b\sqrt{-1}$$
$$\sqrt[3]{2-\sqrt{-121}} = a - b\sqrt{-1} \qquad (11)$$

SINCE HE KNEW THE ANSWER WAS A REAL NUMBER, BOMBELLI GUESSED THAT THE "$\sqrt{-1}$" PARTS WOULD CANCEL WHEN ADDED.

4.15 Show that cubing the top line of Equation 11 results in:

$2+\sqrt{-121} = a(a^2-3b^2) + b\sqrt{-1}(3a^2-b^2)$

$(\sqrt[3]{2+\sqrt{-121}})^3 = (a+b\sqrt{-1})^3$

$2+\sqrt{-121} = (a^2 + 2ab\sqrt{-1} + b^2(\sqrt{-1})^2)(a+b\sqrt{-1})$

$= (a^2 + 2ab\sqrt{-1} - b^2)(a+b\sqrt{-1})$

$= a^3 + 2a^2b\sqrt{-1} - ab^2 + a^2b\sqrt{-1} + 2ab^2(\sqrt{-1})^2 - b^3\sqrt{-1}$

$= a^3 - 3ab^2 + 3a^2b\sqrt{-1} - b^3\sqrt{-1}$

$\boxed{= a(a^2-3b^2) + b\sqrt{-1}(3a^2-b^2)}$

4.16 Show that:

$$2 = a(a^2 - 3b^2)$$
$$11 = b(3a^2 - b^2) \qquad (12)$$

by equating the "normal" and "$\sqrt{-1}$" parts from exercise 4.15.

$= 11\sqrt{-1}$

$2 + \sqrt{-121} = a(a^2-3b^2) + \sqrt{-1}\,b(3a^2-b^2)$

4.17 Back in exercise 4.15, we only cubed the top line of Equation 11. Cube both sides of the bottom line of 11, and turn the result into a system of equations as the did in Exercise 4.16. What do you notice?

$(\sqrt[3]{2-\sqrt{-121}})^3 = (a-b\sqrt{-1})^3$

$2-\sqrt{-121} = (a^2 - ab\sqrt{-1} - ab\sqrt{-1} + b^2(\sqrt{-1})^2)(a-b\sqrt{-1})$

$= (a^2 - 2ab\sqrt{-1} - b^2)(a-b\sqrt{-1})$

$= a^3 - 2a^2b\sqrt{-1} - ab^2 - a^2b\sqrt{-1} + 2ab^2(\sqrt{-1})^2 + b^3\sqrt{-1}$

$= a^3 - 3ab^2 - 3a^2b\sqrt{-1} + b^3\sqrt{-1}$

$= a(a^2-3b^2) - \sqrt{-1}\cdot b(3a^2-b^2)$

$\boxed{2 = a(a^2-3b^2) \quad 11 = b(3a^2-b^2)}$

SAME EQUATIONS!

4.18 Using Equation 12, and the fact that $x=4$ is a solution to our Cardan's problem, $x^3=15x+4$, derive that $a = 2$ and $b = 1$.

FROM (4), $a + b\cancel{\sqrt{-1}} + a - b\cancel{\sqrt{-1}} = 4$

$2a = 4 \Rightarrow \boxed{a=2}$

$2 = a(a^2-3b^2)$

$2 = 2(2^2-3b^2)$

$1 = 4 - 3b^2$

$-3 = -3b^2$

$b^2 = 1 \Rightarrow \boxed{b = \pm 1}$

4.19 The solution we found to Equation 12, a = 2 and b = 1, mean that this must be true:

$$\sqrt[3]{2+\sqrt{-121}} = 2+\sqrt{-1}$$
$$\sqrt[3]{2-\sqrt{-121}} = 2-\sqrt{-1}$$

cube both sides to each equation to prove this.

$$\left(\sqrt[3]{2+\sqrt{-121}}\right)^3 = (2+\sqrt{-1})^3$$
$$2+\sqrt{-121} = (4+2\sqrt{-1}+2\sqrt{-1}+(\sqrt{-1})^2)\cdot(2+\sqrt{-1})$$
$$= (4+4\sqrt{-1}-1)(2+\sqrt{-1})$$
$$= (3+4\sqrt{-1})(2+\sqrt{-1})$$
$$= 6+3\sqrt{-1}+8\sqrt{-1}+4\cancel{(\sqrt{-1})^2}^{-1}$$
$$\boxed{2+11\sqrt{-1} = 2+11\sqrt{-1}} \checkmark$$

4.20 Finally, show that using our solution above does result in a solution to Cardan's problem, $x^3=15x+4$.

$$(2+\sqrt{-1})+(2-\sqrt{-1})$$
$$= 4,$$
$$4^3 = 15\cdot4+4$$
$$64 = 60+4 \checkmark$$

4.21 So, what *is* the expression below (remarkably) equivalent to?

$$\sqrt[3]{2+\sqrt{-121}}+\sqrt[3]{2-\sqrt{-121}}$$

4 (HOLY CRAP)

If you feel that Bombelli's method is a bit hacky, you're in good company. The sketchiness here really surrounds taking the cube root of our new types of numbers that involve $\sqrt{-1}$. Problems like this have confused lots of very smart mathematicians. In the next few workbook sections, we'll find a nice way to think about these problems that makes this process much simpler and intuitive.[1]

*Challenge*

4.22 Solve for a and b:

$$\left(\sqrt{\sqrt{-4}}\right)^2 = (a+b\sqrt{-1})^2$$
$$\sqrt{-4} = a^2+2ab\sqrt{-1}-b^2$$
$$0+2\sqrt{-1} = a^2-b^2+\sqrt{-1}\cdot2ab$$
$$0=a^2-b^2 \qquad \cancel{2}\cancel{\sqrt{-1}} = \cancel{\sqrt{-1}}\cdot\cancel{2}ab$$
$$a^2=b^2 \qquad 1=ab \Rightarrow b=1/a$$
$$a^2=(1/a)^2$$
$$a^4=1$$
$$\boxed{a=1 \Rightarrow b=1}$$

4.23 Perhaps not surprisingly, Equation 12 has more than one solution. Show that other solutions to Equation 12 lead to more solutions to Cardan's original problem, $x^3=15x+4$. You may use technology. (Unless you're a complete bad-ass and don't need it)

FROM WOLFRAM ALPHA:

$a=-1-\sqrt{3}/2 \quad b=\sqrt{3}-1/2 \Rightarrow -2-\sqrt{3}$

$a=\frac{\sqrt{3}}{2}-1 \quad b=-1/2-\sqrt{3} \Rightarrow \sqrt{3}-2$

1 Once we cover this exciting new ground, we'll revisit this problem in the Critical Thinking exercises in Section 8.

## Exercises 5



*Discussion*

5.1 Why might interpreting multiplication by a negative number as a 180° rotation make sense?

5.2 Why might it make sense for imaginary numbers to be placed at a right angle to real numbers?

5.3 Do you think numbers are two-dimensional? Why or why not?

*Drill*

5.4 $i^2$ $\quad -1$

5.5 $i^{20}$ $\quad 1$

5.6 $i^{21}$ $\quad i$

5.7 $i^{22}$ $\quad -1$

5.8 $i^{23}$ $\quad -i$

5.9 $i^{1000}$ $\quad 1$

5.10 $i^{1001}$ $\quad i$

5.11 $i^{1002}$ $\quad -1$

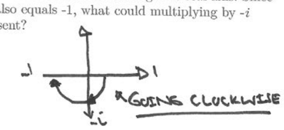
5.12 $i^{1003}$ $\quad -i$

5.13 $i^{-1}$ $= 1/i \Rightarrow \frac{1}{i}\cdot\frac{i}{i} = \frac{i}{i^2} = \frac{i}{-1} = -i$

5.14 $i^{-20}$ $= \frac{1}{i^{20}} = \frac{1}{1} = 1$

*Critical Thinking*

5.15 As shown in Figure 18, multiplying $i$ by itself ($i^2 = -1$) can be interpreted as a 180° rotation from the positive real axis to the negative real axis. Since $(-i)^2$ also equals -1, what could multiplying by $-i$ represent?




5.16 True or false?

$$\sqrt{-1} < 2$$

NEITHER! $\sqrt{-1}$ DOES NOT FIT ON THE NUMBERLINE. TRICK QUESTION SUCKA.



The central idea of this section, that imaginary numbers should be placed at a right angle to the real numbers, is not obvious. We know this because it took over 200 years after Bombelli's death to discovered. This idea was finally unearthed by two non-mathematicians separately, Casper Wessel and Jean-Robert Argand, around 1800. However, like most good ideas in math and science, this one took quite some time to be accepted. In 1831, the mathematician Augustus deMorgan said of the topic:

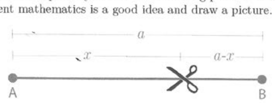
*"We have shown the symbol $\sqrt{-1}$ to be void of meaning, or rather self-contradictory and absurd."*

It's interesting to think about how exactly mathematicians struggled with and argued about these ideas during this period. A long standing practice in mathematics, (dating most notably back to the Greeks), is the idea of geometric proof. Speaking very roughly: if we can represent a mathematical idea visually, it must be true. Whether or not this is the right way to approach mathematics is an open question, but this way of thinking is certainly valuable, and fueled mathematical development for millennia. When $\sqrt{-1}$ began showing up in mathematics in the $17^{th}$ century, many mathematicians attempted and failed to find a way to explain or understand $\sqrt{-1}$ visually/geometrically. This difficulty contributed to the mistrust of $\sqrt{-1}$.

Now that $\sqrt{-1}$ has a strong visual interpretation (existing at a right angle to the real numbers) and has become firmly rooted in mathematics, it's interesting to look back at historical attempts to show visually what the $\sqrt{-1}$ could possibly mean. One interesting example comes from an 1803 publication from Lazare Carnot:

*Given a line segment AB of length a, how can it be divided into two shorter segments so that the product of their lengths is equal to one-half the square of the original length?*

Let's temporarily assume that drawing pictures to represent mathematics is a good idea and draw a picture.




From Carnot's problems statement, we can write the formula:

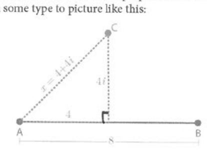
$$x(a-x) = \frac{1}{2}a^2$$

5.17 Let's say $a = 8$ in Carnot's problem. Solve for $x$, the length of one of our segments.

NOT FACT

$$x(8-x) = \frac{1}{2}8^2$$
$$8x-x^2 = 32 \Rightarrow 0 = x^2-8x+32$$
$$x = \frac{+8\pm\sqrt{(-8)^2-4\cdot1\cdot32}}{2\cdot1} = \frac{8\pm\sqrt{64-128}}{2}$$
$$= 4\pm\frac{\sqrt{-64}}{2} = 4\pm\frac{8\sqrt{-1}}{2} = 4\pm4i$$

How did Exercise 5.17 go? When Carnot solved this equation for $x$, his result was a complex number (hopefully your result was complex as well, specifically $4+4i$ and $4-4i$). Carnot interpreted the fact that his result was complex as meaning that the cut point we're looking for does not lie on $AB$, thus the problem was physically impossible. However, other mathematicians (such as the Frenchman Abbè Adrien-Quentin Buèe) interpreted this result differently. Roughly speaking, Buèe agreed that the problem was physically impossible, but argued that the results was nonetheless meaningful. More specifically, if we allow imaginary numbers to exist in the dimension perpendicular to the real axis, we obtain some type to picture like this:




5.18 What is the distance between A and C, shown above?

$$AC = \sqrt{4^2+4^2} = \sqrt{32} = \boxed{4\sqrt{2}}$$

5.19 Does your answer to 5.18 make sense in terms of Carnot's original problem? YES! (SORT OF). THE LENGTHS $4\sqrt{2}$, WHEN MULTIPLIED, YIELD EXACTLY HALF OF THE ORIGINAL LENGTH!

So, if we allow imaginary numbers to live perpendicularly to the real numbers, the answer we obtain does make some type of sense. Sure, there's still no way to cut the string the achieve what we want physically, but the results also are not meaningless. They tell us there is an answer, but we must move 4 units in the perpendicular (imaginary) direction to find it. Problems like this don't make a very strong case for $\sqrt{-1}$, which helps me understand why this matter was so contentions for so long. Only later, when our geometric interpretation $\sqrt{-1}$ was shown to be indispensable to math and science did these ideas become widely accepted.

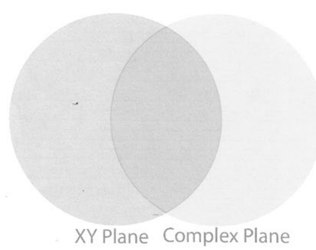
## Exercises 6



*Discussion*

6.1 Why did Guass prefer the names direct, inversé, and lateral over positive, negative, and imaginary?

6.2 What do the xy plane and the complex plane have in common? How are the different? Fill in your answers in the super cool Venn diagram below.




*Drill*

*Simplify:*

6.3 $(1+i)(1+i)$ $\quad 1+i+i+\cancel{i^2}^{-1}$ $\quad 2i$

6.4 $(1+i)(1-i)$ $\quad 1+i-i-\cancel{i^2}^{-1}$ $\quad 1+1 = 2$

6.5 $(2+2i)(1-2i)$

$$2-4i+2i-4i^2$$
$$2-2i-4\cdot-1$$
$$6-2i$$

6.6 $-i(3+5i)$

$$-3i-5\cancel{i^2}^{-1}$$
$$5-3i$$

6.7 $(x+(1+i))((x+(1-i))$

$$x^2+(1+i)x+(1-i)x+(1+i)(1-i)$$
$$x^2+x+\cancel{ix}+x-\cancel{ix}+1+\cancel{i}-\cancel{i}-\cancel{i^2}$$
$$\boxed{x^2+2x+2}$$

6.8 $(-2-i)^2$

$$= (-2-i)(-2-i)$$
$$= 4+2i+2i+\cancel{i^2}^{-1}$$
$$= 3+4i$$

*Simplify (your answer should not have an i in the denominator):*

6.9 $\frac{1}{1-i}\frac{(1+i)}{(1+i)} = \frac{1+i}{1+\cancel{i}-\cancel{i}-i^2} = \frac{1+i}{2}$

$$= \frac{1}{2}+\frac{i}{2}$$

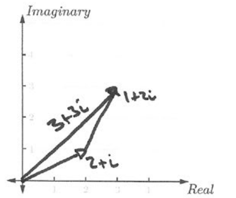
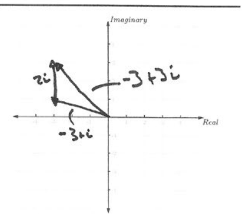
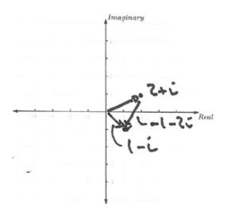
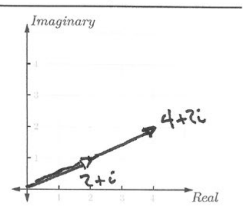
6.10 $\frac{i}{1+2i}\frac{(1-2i)}{(1-2i)} = \frac{i-2i^2}{1-\cancel{2i}+\cancel{2i}-4i^2}$

$$= \frac{i+2}{1+4} = \boxed{\frac{i+2}{5}} \text{ or } \boxed{\frac{2}{5}+\frac{i}{5}}$$

6.11 $\frac{\left(\frac{1}{2}+\sqrt{3}i\right)}{\left(1-\sqrt{2}i\right)}\frac{(1+\sqrt{2}i)}{(1+\sqrt{2}i)} = \frac{1/2+\sqrt{3}i+\frac{\sqrt{2}}{2}i+\sqrt{6}i^2}{1-\sqrt{2}i+\sqrt{2}i-2i^2}$

$$\boxed{= \frac{1/2-\sqrt{6}+(\sqrt{3}+\sqrt{2}/2)i}{3}}$$

Exercises 6 35

| | Problem | Solve Algabraically | Solve Graphically |
|---|---|---|---|
| 6.12 | $(2+i)+(1+2i)$ | $2+i$ <br> $+\ 1+2i$ <br> $3+3i$ |  |
| 6.13 | $(-3+i)+2i$ | $-3+i+2i$ <br> $=-3+3i$ |  |
| 6.14 | $(2+i)-(1+2i)$ <br> *Vaguely helpful hint: Try writing as an addition problem. | $=(2+i)+(-1-2i)$ <br> $=1-i$ |  |
| 6.15 | $2\cdot(2+i)$ | $4+2i$ |  |



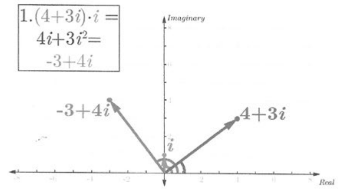
Exercises 6 36

*Critical Thinking*

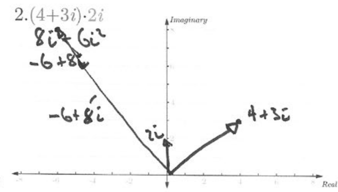
6.16 **What is the connection between complex multiplication and the complex plane?** To figure this out, we'll look at four examples. This is by far my favorite exercise in the entire workbook. It's cool because if you figure it out, you'll have discovered something that mathematicians missed for over 200 years.

a) To get started, for each example compute the product, plot the two numbers we're multiplying together, and plot the result. The first example has been done for you.[1]

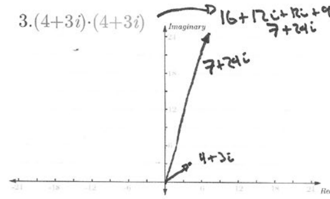
1. $(4+3i)\cdot i =$ $4i+3i^2=$ $-3+4i$



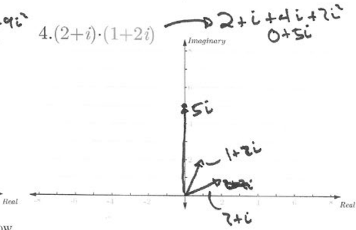

2. $(4+3i)\cdot 2i$ $8i+6i^2$ $-6+8i$




3. $(4+3i)\cdot(4+3i)$ → $16+12i+12i+9i^2$ $7+24i$




4. $(2+i)\cdot(1+2i)$ → $2+i+4i+2i^2$ $0+5i$




b) Use your results from a to complete the table below.

| Problem | Result | Angle 1 | Angle 2 | Result Angle | Distance 1 | Distance 2 | Result Distance |
|---|---|---|---|---|---|---|---|
| $(4+3i)\cdot i$ | $-3+4i$ | 36.9° | 90° | 126.9° | 5 | 1 | 5 |
| $(4+3i)\cdot 2i$ | $-6+8i$ | 36.9 | 90° | 126.9 | 5 | 2 | 10 |
| $(4+3i)\cdot(4+3i)$ | $7+24i$ | 36.9° | 36.9° | 73.8° | 5 | 5 | 25 |
| $(2+i)\cdot(1+2i)$ | $5i$ | 26.6° | 63.4° | 90° | $\sqrt{5}$ | $\sqrt{5}$ | 5 |

c) What patterns do you see? What is the connection between complex multiplication and the complex plane?

ANGLES ADD, DISTANCES MULTIPLY! SEE P. 37.

1 Wow, that Stephen Welch is such a nice guy, doing a whole problem for you!

## Exercises 7

40

*Discussion*

7.1 The main idea of this section is that there are two very different, but completely equivalent methods we can use to multiply complex numbers. How could such seemingly different approaches can yield the same exact results?

7.2 Can you think of an example where using the polar form of complex numbers would make life easier?

7.3 Why do you think it took over 200 years for the connection between complex multiplication and the complex plan to be discovered?

*Drill*

*Convert to polar form:*

7.4 $1+i$ $\quad \sqrt{1^2+1^2}=\sqrt{2}$ $\quad \theta=\tan^{-1}(1)=45°$ ⇒ $\boxed{\sqrt{2}\angle 45°}$

7.5 $2-3i$ $\quad \sqrt{2^2+3^2}=\sqrt{13}$ $\quad \theta=\tan^{-1}(-3/2)=-56.3°$ ⇒ $\boxed{\sqrt{13}\angle -56.3°}$

7.5 $\frac{1}{2}+\frac{\sqrt{3}}{2}i$ $\quad \sqrt{(1/2)^2+(\sqrt{3}/2)^2}=\sqrt{\frac{1}{4}+\frac{3}{4}}=\sqrt{1}=1$ $\quad \theta=\tan^{-1}\left(\frac{\sqrt{3}/2}{1/2}\right)=\tan^{-1}(\sqrt{3})=60°$ $\quad \boxed{1\angle 60°}$

*Convert to rectangular form:*

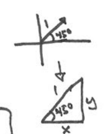
7.6 $1\angle 45°$

$\sin 45 = y/1$

$y=\sqrt{2}/2$

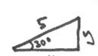
$\cos 45 = x/1$

$x=\sqrt{2}/2$



$\boxed{\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}i}$

7.7 $5\angle 30°$



$\sin 30 = y/5$

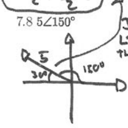
$y=5\sin 30 = 5/2$

$\cos 30 = x/5$

$x=5\cos 30 = 5\sqrt{3}/2$

$\boxed{\frac{5\sqrt{3}}{2}+\frac{5}{2}i}$

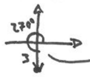
7.8 $5\angle 150°$



JUST LIKE THIS!

$\boxed{-\frac{5\sqrt{3}}{2}+\frac{5}{2}i}$

7.9 $3\angle 270°$



$-3i$

*Critical Thinking*

*Solve these suckers:*

7.10 $(1+i)(1+i)$ $\quad 1+i+i+i^2$ $\quad 1+2i-1=2i$

7.11 $(\sqrt{2}\angle 45°)(\sqrt{2}\angle 45°)$ $\quad \sqrt{2}\cdot\sqrt{2}=2$ $\quad 45°+45°=90°$ → $2\angle 90°$

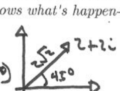
7.12 How are 7.10 and 7.11 related?

SAME PROBLEM, DIFFERENT FORMS.

Exercises 7 41

*Convert to polar form, solve, and convert your answer back to rectangular form. For extra credit draw a pretty picture that shows what's happening.*[1]

7.13 $(2+2i)(2+2i)$



$(2\sqrt{2}\angle 45°)(2\sqrt{2}\angle 45°)$

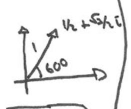
$\sqrt{2^2+2^2}=\sqrt{8}=2\sqrt{2}$

$\tan^{-1}\left(\frac{2}{2}\right)=45°$

$4(\sqrt{2})^2\angle 90°$

$8\angle 90°$ → $8i$

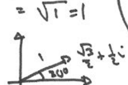
7.14 $\left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)\left(\frac{\sqrt{3}}{2}+\frac{1}{2}i\right)$



$(1\angle 60°)(1\angle 30°)$

$\sqrt{\left(\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}=\sqrt{1}=1$

$1\angle 90° = i$



$\tan^{-1}\left(\frac{1/2}{\sqrt{3}/2}\right)=30°$

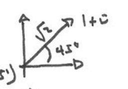
7.15 $\dfrac{\left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)}{\left(\frac{\sqrt{3}}{2}+\frac{1}{2}i\right)} = \dfrac{1\angle 60°}{1\angle 30°}$

$=\frac{1}{1}\angle 60-30$

$=1\angle 30°$

$=\sqrt{3}/2+1/2\, i$

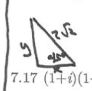
7.16 $(1+i)(1+i)(1+i)$



$(\sqrt{2}\angle 45°)(\sqrt{2}\angle 45°)\cdot(\sqrt{2}\angle 45°)$

$=\sqrt{2}\cdot\sqrt{2}\cdot\sqrt{2}\angle 45°+45°+45°$

$=2\sqrt{2}\angle 135°$



$x=2\sqrt{2}\cos 135°$

$=2$

$y=2$

$\boxed{-2+2i}$

7.17 $(1+i)(1+i)(1+i)(1+i)$

$(\sqrt{2}\angle 45°)^4 = 4\angle 180°$ ⇒ $\boxed{-4}$

7.18 $\dfrac{2i}{1+i} = \dfrac{2\angle 90°}{\sqrt{2}\angle 45°}$

$=\frac{2}{\sqrt{2}}\angle 90°-45°$

$=\frac{2\sqrt{2}}{2}\angle 45° = \boxed{\sqrt{2}\angle 45°}$

7.19 $\dfrac{-2}{i} = \dfrac{2\angle 180°}{1\angle 90°} = \frac{2}{1}\angle 180°-90°$

$\boxed{=2\angle 90°}$

1 Each problem is worth 1 extra credit point. 5 extra credit points my be converted into 1 cool point at the discretion of the point holder. Upon achieving 3 cool points, demand some type of prize from your teacher/professor/friends/family members. Be persistent.

*Challenge*

In school you may have been required to memorize some formulas like these:

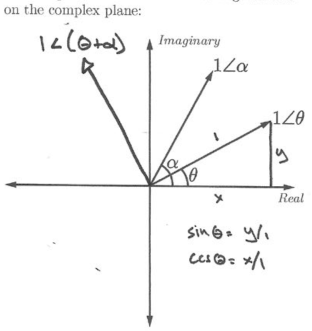
(1) $cos(\alpha+\theta) = cos(\theta)cos(\alpha) - sin(\theta)sin(\alpha)$
(2) $sin(\alpha+\theta) = sin(\theta)cos(\alpha) + cos(\theta)sin(\alpha)$
(3) $cos(2\theta) = cos^2(\theta) - sin^2(\theta)$
(4) $sin(2\theta) = 2sin(\theta)cos(\theta)$

Since memorization is basically the worst thing ever[1], let's try to find a way to never have to memorize these formulas again.

7.20 Let's begin by considering two complex numbers, each with a magnitude of 1:

$$1\angle\alpha \quad 1\angle\theta$$

To make things a little more visual, let's say our two complex numbers look something like this on the complex plane:




$\sin\theta = y/1$
$\cos\theta = x/1$

In terms of $\alpha$ and $\theta$, compute the product:

$$(1\angle\alpha)(1\angle\theta) = 1\angle(\alpha+\theta)$$

and add it to the plot above in the generally correct location.

1 Memorization does not equal learning!

7.21 Convert to rectangular form using sin and cos:

a) $1\angle\alpha = \cos\alpha + i\sin\alpha$

b) $1\angle\theta = \cos\theta + i\sin\theta$

c) $1\angle(\alpha+\theta) = \cos(\alpha+\theta) + i\sin(\alpha+\theta)$

7.22 Substitute your answers from 7.21 into[2]:

$$(1\angle\alpha)(1\angle\theta) = 1\angle(\alpha+\theta)$$

Expand and simplify your result.

$(\cos\alpha + i\sin\alpha)(\cos\theta + i\sin\theta)$
$\cos\alpha\cos\theta - \sin\alpha\sin\theta$
$+ i\sin\alpha\cos\theta + i\sin\theta\cos\alpha$
$= \cos(\alpha+\theta) + i\sin(\alpha+\theta)$

7.23 Derive Equations 1 and 2 by equating the real and imaginary parts of each side of your result to 7.22.

$\cos(\alpha+\theta) = \cos\alpha\cos\theta - \sin\alpha\sin\theta$
$\sin(\alpha+\theta) = \sin\theta\cos\alpha + \sin\alpha\cos\theta$ ✓

7.24 Derive Equations 3 and 4 by considering the special case where $\alpha = \theta$.

$\cos(\theta+\theta) = \cos(2\theta) = \cos^2\theta - \sin^2\theta$
$\sin(2\theta) = 2\sin\theta\cos\theta$

2 This is hopefully the equation you obtained in 7.20!



*Discussion*

8.1 Why must the equation $x^3=1$ have 3 solutions?

8.2 How does the complex plane make tough problems easier?

8.3 Why to you think the mathematician Jacques Hadamard wrote:

*"The shortest path between two truths in the real domain passes through the complex domain."*

-Jacques Hadamard, ~1945

*Drill*

*Answer in rectangular form:*

8.4 $(1+i)^6$

$1+i = \sqrt{2}\angle 45°$
$(\sqrt{2}\angle 45°)^6$
$= (\sqrt{2})^6 \angle 45\cdot 6$
$= 8\angle 270°$ $-8i$

8.5 $(1+i\sqrt{3})^6$

$(2\angle 60°)^6$
$2^6 \angle 6\cdot 60°$
$64\angle 360°$
$= 64\angle 0° = \boxed{64}$

$\theta = \tan^{-1}(\sqrt{3}) = 60°$
$\sqrt{(\sqrt{3})^2+1} = \sqrt{4} = 2$

8.6 $\dfrac{(1+i\sqrt{3})^3}{(1-i)^2} = \dfrac{(2\angle 60°)^3}{(\sqrt{2}\angle -45°)^2}$

$= \dfrac{2^3\angle 3\cdot 60°}{(\sqrt{2})^2\angle -45\cdot 2} = \dfrac{8\angle 180°}{2\angle -90°}$
$= 4\angle 270° = -4i$

8.7 $\sqrt{i}$

$i = 1\angle 90°$
$\sqrt{1\angle 90°} = \boxed{1\angle 45°}$
$= \sqrt{1}\angle \frac{90°}{2} \rightarrow \boxed{\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i}$

8.8 $\sqrt{\dfrac{\sqrt{3}+i}{2}} = \sqrt{\dfrac{\sqrt{3}}{2} + \dfrac{1}{2}i}$

$= \sqrt{1\angle 30°} = \sqrt{1}\angle 30/2$
$= 1\angle 15°$
$\boxed{= \cos 15° + i\sin 15°}$

8.9 $\sqrt[3]{8i}$

$\rightarrow \sqrt[3]{8\angle 90°}$
$= \sqrt[3]{8}\angle 90°/3 = \boxed{2\angle 30°}$

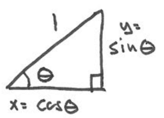


*Critical Thinking*

Let's say we're given a given complex number with a magnitude[1] of 1, and an angle[2] of $\theta$. We can convert our number to rectangular form as follows:

$$1\angle\theta = cos(\theta) + isin(\theta)$$

8.10 Using a picture, show why the above equation is legit.




Now let's consider what happens when we raise our number to various powers. Write each answer in rectangular form, using *sin*, *cos*, and $\theta$.[3]

8.11 $(1\angle\theta)^2$ $= (1\angle\theta)\cdot(1\angle\theta)$
$= 1\angle 2\theta$
$= \cos(2\theta) + i\sin(2\theta)$

8.12 $(1\angle\theta)^{1/3}$ $= \sqrt[3]{1\angle\theta}$
$= \sqrt[3]{1}\angle\theta/3$
$= \cos\left(\frac{\theta}{3}\right) + i\sin\left(\frac{\theta}{3}\right)$

8.13 $(1\angle\theta)^n$ $= 1^n\angle n\theta$
$= \cos(n\theta) + i\sin(n\theta)$

1 or modulus
2 or argument
3 Feel free to explain you answer with a picture for one extra credit point per problem.

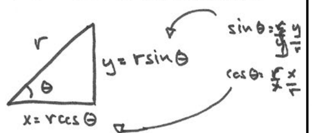
Your answer to 8.13 should be something like:

$$(1\angle\theta)^n = cos(n\theta) + isin(n\theta)$$

This is known as de Moivre's theorem.[4] Note that the right side of the equation is sometimes abbreviated using *cis*: $cis(x) = cos(x) + isin(x)$. De Moivre's theorem can be expanded to numbers that have magnitudes other than 1 without too much headache. Let's say we're given some complex number with a magnitude or r and an angle of $\theta$. We can convert our number to rectangular form as follows

$$(r\angle\theta) = r(cos(\theta) + isin(\theta))$$

8.14 Using a picture, show why the above equation is legit.




Now let's consider what happens when we raise our new number to various powers. Write each answer in rectangular form, using *r*, *sin*, *cos*, and $\theta$.

8.15 $(r\angle\theta)^2$ $= r^2\angle 2\theta$
$= r^2(\cos 2\theta + i\sin 2\theta)$

8.16 $(r\angle\theta)^{1/3}$ $= \sqrt[3]{r}\angle\theta/3$
$= \sqrt[3]{r}\left(\cos\frac{\theta}{3} + i\sin\frac{\theta}{3}\right)$

8.17 $(r\angle\theta)^n$ $= r^n\angle n\theta$
$= r^n(\cos(n\theta) + i\sin(n\theta))$

4 Although it never shows up in his work...makes sense...



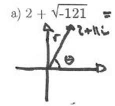
Way the heck back in Part 4 (Exercises 4.14-4.21), we discussed how Bombelli solved Cardan's problem by allowing imaginary numbers to exist. A key part of Bombelli's argument was this rather shocking fact:

$$\sqrt[3]{2+\sqrt{-121}} + \sqrt[3]{2-\sqrt{-121}} = 4$$

Now, with the help of the 200+ years of mathematical advancement we've covered since Part 4, we'll show that the above equation is must be true without breaking a sweat! This is remarkable when we consider how much trouble this equation gave Cardan and Bombelli.

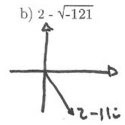
8.18 To begin showing that the above equation is true, convert these two parts of the equation to polar form. Round your answers to 4 decimal places.

a) $2+\sqrt{-121}$ $= 2 + 11\sqrt{-1} = 2 + 11i$




$r = \sqrt{2^2+11^2}$ $\quad\theta = \tan^{-1}\left(\frac{11}{2}\right)$
$r = \sqrt{125} = 5\sqrt{5}$ $\quad\theta = 79.6952°$
$\boxed{5\sqrt{5}\angle 79.6952° \text{ or } 11.1803\angle 79.6952°}$

b) $2-\sqrt{-121}$




$\boxed{5\sqrt{5}\angle -79.6952° \text{ or } 11.1803\angle -79.6952°}$

8.19 Using your results from the previous page, compute the cube root of your answers from 8.18, and convert your results to rectangular form.

a) $\sqrt[3]{2+\sqrt{-121}}$ $= \sqrt[3]{5\sqrt{5}\angle 79.6952°}$
$= \sqrt[3]{5\sqrt{5}}\angle 79.6952/3$
$= 2.2361\angle 26.5651°$
$= 2.2361(\cos(26.5651) + i\sin(26.5651))$ $= 2+i$

b) $\sqrt[3]{2-\sqrt{-121}}$

$\rightarrow 2.2361(\cos(-26.5651) + i\sin(-26.5651))$
$= 2-i$

8.20 Add together your answers to 8.19 a and b. If your result is 4, nice job! Think about how impressed Cardan and Bombelli would by your math wizardry. If you're answer isn't 4, check out the solutions in the back of the book.

$2+i$
$+\ 2-i$
$4$ WOOHOO!

*Challenge*

The brilliant mathematician Gottfried Wilhelm Leibniz (1646-1716), co-father to calculus, is known to have struggled deeply with the topics we're covering here.

*"I do not understand how...a quantity could be real, when imaginary or impossible numbers are used to express it."*

-Leibniz

After his death, Leibniz's unpublished work revealed that he calculated some special cases of Cardan's formula again and again, presumably looking for deeper insights like the ones that have been handed to us. In solving the cubics $x^3=13x+12$ and $x^3=48x+72$ using Cardan's formula, Leibniz discovered that:

$$\sqrt[3]{6+\sqrt{-\frac{1225}{27}}} + \sqrt[3]{6-\sqrt{-\frac{1225}{27}}} = 4$$

and:

$$\sqrt[3]{-36+\sqrt{-2800}} + \sqrt[3]{-36-\sqrt{-2800}} = 6$$

8.21 Show that the two equations above are true.

$6+\sqrt{-1225/27} = 9.0206\angle 48.3062°$
$\sqrt[3]{9.0206\angle 48.3062} = 2.0817\angle 16.1021°$
$= 2 + 0.5774i$

$2+0.5774i$
$+2-0.5774i$
$4$

PART TWO IS LEFT TO THE READER... BECAUSE I HAVE NO ROOM.

## Exercises 9

51

*Discussion*

9.1 What is mathematical closure?

9.2 Why might closure be important for scientists, engineers, and other people who use mathematics?

9.3 The real numbers seem pretty complete. What evidence do we have that indicates this is not true?

*Drill*

Time for a few more rounds of the closure game! Determine if the set is closed under the given operation. If the set is not closed, provide a counter-example.

9.4 Integers, multiplication.

CLOSED.

9.5 Integers, division.

NOT CLOSED

$\frac{2}{3} \neq$ INTEGER.

9.6 Real Numbers, division.

CLOSED

9.7 Irrational Numbers, addition.

$\sqrt{3} + \sqrt{3} = 2\sqrt{3}$

CLOSED.

9.8 Irrational Numbers, multiplication.

NOT CLOSED:

$\sqrt{3} \cdot \sqrt{3} = 3$

NOT IRRATIONAL

9.9 Transcendental Numbers, subtraction.

NOT CLOSED:

$\pi - \pi = 0$

9.10 Imaginary Numbers, multiplication.

NOT CLOSED:

$i \cdot i = -1$

9.11 Integers, powers.

CLOSED.

---

Exercises 9

52

*Critical Thinking*

The idea of mathematical closure can be applied to sets beyond those we've discussed thus far. For example, let's consider the set of all even numbers (just in case you forgot, 0 is even).

9.12 Are the even numbers closed under addition? What about subtraction? For 1 extra credit point, prove that your answer it true for any two even numbers.

CLOSED FOR + AND −.

$2k + 2n = 2(k+n)$

even numbers — INTEGERS — MUST ALSO BE EVEN!

$2k - 2n = 2(k-n)$

9.13 Are the even numbers closed under multiplication? What about division? For $\sqrt{2}$ extra credit points, prove that your answer is true for any two even numbers.

$(2k)(2n) = 4kn = 2(2kn)$

even numbers — INTEGERS — MUST BE EVEN!

$\boxed{\text{CLOSED UNDER MULTIPLICATION}}$

$\frac{2k}{2n} = \frac{k}{n}$ may not be even!

$\boxed{\text{NOT CLOSED UNDER } \div}$

9.14 Under what algebraic operations are the odd numbers closed?

$3+3=6$ ×

$3-1=2$ ×

$3 \cdot 3 = 9$ → maybe!

$6 \div 3 = 2$ ×

ODD NUMBERS

Test: $(2k+1)(2n+1)$

$= 4kn + 2k + 2n + 1$

even → even + 1 = odd!

CLOSED UNDER MULTIPLICATION

Consider the completely made-up operation:

$$\Lambda(a,b) = (a+b) \textit{ modulo } 4$$

Where modulo 4 means the remainder after dividing by four. For example, 9 *modulo* 4 = 1, because 4 goes into 9 twice, leaving a remainder of 1. Numbers less than 4 remain unchanged, for example, 3 *modulo* 4 = 3.

9.15 Show that $\Lambda(5, 3) = 0$, and $\Lambda(2, 1) = 3$.

$\Lambda(5,3) = 8 \bmod 4$
$= 0$

$\Lambda(2,1) = 3 \bmod 4$
$= 3$

9.16 Is the set {1, 2, 3, 4} closed under $\Lambda$?

$\Lambda(1,3) = 4 \bmod 4 = 0$

NOT IN {1, 2, 3, 4}

NO!

9.17 Name a set that would be closed under $\Lambda$.

{0, 1, 2, 3}

---

Exercises 9

53

*Challenge*

Using the kick-ass polar form of complex numbers we learned back in Part 7, solve the problem below and confirm that results belong to the set of Complex Numbers.

9.18 $(-i)^{1/6}$

$-i = 1\angle 270°$

$(1\angle 270°)^{1/6} = \sqrt[6]{1} \angle 270/6$

$= 1 \angle 45°$

$= \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} i$

COMPLEX!

9.19 $\sqrt{2}$ is irrational. This means that $\sqrt{2}$ cannot be expressed as a ratio of two integers. This fact didn't fit so well with the orderly world view of the ancient Greeks, and legend has it even lead to the murder of Hippasus of Metapontum. Let's prove this dangerous fact by contradiction. To do this, let's assume, as the Greeks did, that $\sqrt{2}$ is rational. If $\sqrt{2}$ is rational, then we can write $\sqrt{2} = m/n$, for integers $m$ and $n$, and let's say that $m$ and $n$ make our fraction in lowest terms - so they have no common factors.

a. Square both sides of our equation $\sqrt{2} = m/n$, and show how we can conclude from our result that $m$ must be even.

$(\sqrt{2})^2 = \left(\frac{m}{n}\right)^2$

$2 = \frac{m^2}{n^2}$

$2n^2 = m^2$

2 times some integer $n^2$ must be even!

so m is even!

b. Since $m$ is even, let $m = 2k$, for some integer $k$. By plugging in $2k$ for $m$ into your result from part a, show that $n$ must also be even.

$2n^2 = (2k)^2$

$2n^2 = 4k^2$

$n^2 = 2k^2$

MUST BE EVEN, Because 2 TIMES SOME INTEGER $k^2$ IS EVEN!

c. Why can $m$ and $n$ not both be even?

BECAUSE OUR FRACTION m/n WOULD NOT BE IN LOWEST TERMS!

Since $m$ and $n$ cannot both be even, we've reached a contraction, indicating that our original assumption, that $\sqrt{2}$ is rational must be false, leaving us with the inevitable fact that $\sqrt{2}$ is irrational.

## Exercises 10



*Discussion*

10.1 Why is the Cartesian Coordinate System so freaking cool?

10.2 Why are functions of complex variables so freaking hard to visualize?

10.3 What is the maximum number of dimensions the Cartesian Coordinate System will work in? Why?

10.4 What shape would you draw on the left plane of Figure 48 to better understand what our function, $f(z)=z^2+1$, is doing?

*Drill*

Let $g(z)=z^2+2z$. Calculate the following:

10.4 $g(1)$

$g(1) = 1^2 + 2 \cdot 1$
$= 1+2 = 3$

10.5 $g(i)$

$g(i) = i^2 + 2i$
$\boxed{= -1+2i}$

10.6 $g(1+i)$ $= (1+i)^2 + 2(1+i)$
$1 + 2i + \cancel{i^2} \; 2 + 2i$
$2 + 4i$

10.7 $g(1-i)$ $= (1-i)^2 + 2(1-i)$
$1 - 2i + i^2 + 2 - 2i$
$-2i + 2 - 2i$
$2 - 4i$

Let $h(z)=z^3+iz$. Calculate the following:

10.8 $h(1)$ $= 1^3 + i \cdot 1 = 1+i$

10.9 $h(-i)$ $= (-i)^3 + i \cdot (-i)$
$= -i^2 \cdot i + -i^2$
$= i + 1 = 1+i$

10.10 $h(1-i)$
$= (1-i)^3 + i(1-i)$
$= (1-2i+i^2)(1-i) + i - i^2$
$= (-2i)(1-i) + i - 1 = -2i + 2i^2 + i - 1$
$= -i - 2 - 1 = -i-3$

10.11 $h(2+2i)$
$= (2+2i)^3 + i \cdot (2+2i)$
$= (2+2i)(4+8i+4i^2) + 2i + 2i^2$
$= 8 + 16i + 8i^2 + \cancel{8i} + 16i^2 + 8i^3 + 2i + 2i$
$= 8 + 26i - 8 - 16 - 8i - 2i$
$= -16 + 16i$



*Critical Thinking*

10.12 The Cartesian Coordinate System is so common today, it's easy to forget just how useful and non-obvious it is! Let's consider an example.

You and your friend Gus are planning a murder in a quiet beach town. You and Gus agree to hide the body in a cave located off Sunset Bay. However, there's a small problem. The cave can only be accessed at low tide. Being thorough premeditators, you and Gus devise a plan to predict exactly when low tides will occur, allowing you to schedule the perfect murder. Over the course of the days leading up to the Murder, Gus records the water depth at Sunset Bay every two hours. Check out Gus's data over there ---------------->

| Time (2 days before murder) | Water Depth (m) | Time (day before murder) | Water Depth (m) |
|---|---|---|---|
| 12:00 AM | 2.16 | 2:00 AM | Gus Fell Asleep |
| 2:00 AM | 1.95 | 4:00 AM | 0.29 |
| 4:00 AM | 1.35 | 6:00 AM | 0.77 |
| 6:00 AM | 0.59 | 8:00 AM | 1.33 |
| 8:00 AM | 0.30 | 10:00 AM | 1.81 |
| 10:00 AM | 0.20 | 12:00 PM | 2.26 |
| 12:00 PM | 0.77 | 2:00 PM | 1.85 |
| 2:00 PM | 1.34 | 4:00 PM | 1.27 |
| 4:00 PM | 1.89 | 6:00 PM | 0.51 |
| 6:00 PM | 2.07 | 8:00 PM | 0.07 |
| 8:00 PM | 1.96 | 10:00 PM | 0.17 |
| 10:00 PM | 1.43 | 12:00 PM | 0.60 |
| 12:00 AM | 0.66 | | |

a) The morning of the Murder is here! You and Gus pour over his notebook, trying to find the perfect time to commit the act and hide the body. Based on the data alone, (without drawing any pictures) when do you think You and Gus should hide the body?

LOW TIDES @ 10a, 4am(ish), 8pm, repeat every ~16-18h, so murder around noon-2pm.

b) Gus is NOT happy with your answer to part a. Let's use the Cartesian Coordinate System to convince Gus. Using the grid below, plot the water depth and time of day.

c) Based on your plot, when should you hide the body? Did you answer to part a and b match? Which method is more convincing? What advantages might using the Cartesian Coordinate System have?

→ Hide Body @ 1pm.
→ parts a+b are close, b = more precise
→ Cartesian Coordinate Sys lets you see the gts!

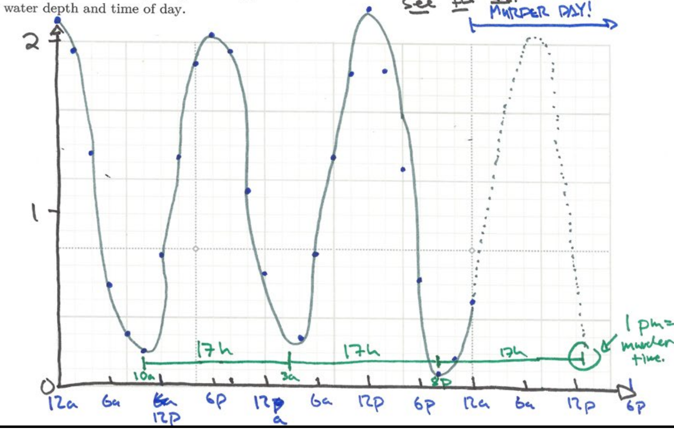
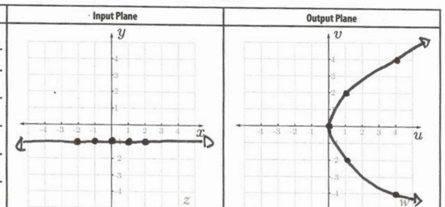





Figure 46 shows how one line was transformed from straight to curved by our function $f(z)=z^2+1$. In the exercises below, we'll experiment with mapping other shapes. For each exercise complete the table using $f(z)=z^2+1$, plot the input and output points, and speculate wildly about what exactly happened input shape.

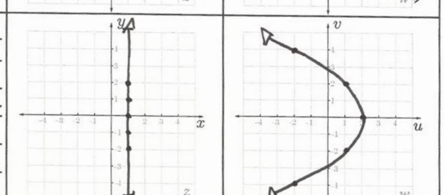
**10.13**

| $x$ | $y$ | $u$ | $v$ |
|---|---|---|---|
| -2 | -1 | 4 | 4 |
| -1 | -1 | 1 | 2 |
| 0 | -1 | 0 | 0 |
| 1 | -1 | 1 | -2 |
| 2 | -1 | 4 | -4 |



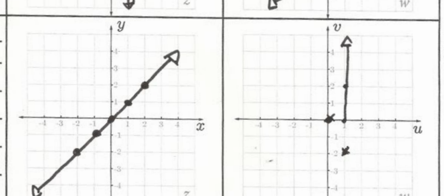
**10.14**

| $x$ | $y$ | $u$ | $v$ |
|---|---|---|---|
| 1 | -2 | -2 | -4 |
| 1 | -1 | 1 | -2 |
| 1 | 0 | 2 | 0 |
| 1 | 1 | 1 | 2 |
| 1 | 2 | -2 | 4 |



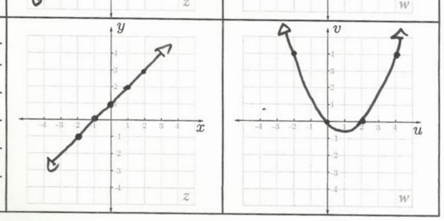
**10.15**

| $x$ | $y$ | $u$ | $v$ |
|---|---|---|---|
| -2 | -2 | 1 | 8 |
| -1 | -1 | 1 | 2 |
| 0 | 0 | 1 | 0 |
| 1 | 1 | 1 | 2 |
| 2 | 2 | 1 | 8 |



**10.16** MAKE YOUR OWN SHAPE!

| $x$ | $y$ | $u$ | $v$ |
|---|---|---|---|
| -2 | -1 | 4 | 4 |
| -1 | 0 | 2 | 0 |
| 0 | 1 | 0 | 0 |
| 1 | 2 | -2 | 4 |
| 2 | 3 | -4 | 12 |





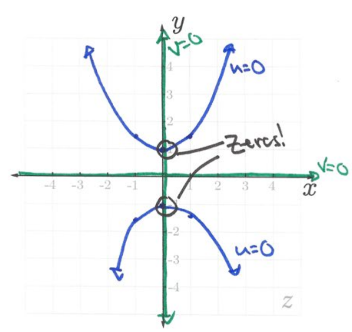
*Challenge*

$$w = f(z) = z^2+1$$
$$z = x + iy \quad (16)$$
$$w = u + iv$$

10.17 a) Using Equation 16, draw curves/lines on the z-plane to the right where $u = 0$.

b) Using Equation 16, draw curves/lines on the z-plane to the right where $v = 0$.

c) What do your curves from parts a and b tell you about the zeros of $f$?




$(u+iv) = (x+iy)^2 + 1$
$u+iv = x^2 + 2xyi - y^2 + 1$
$u = x^2 - y^2 + 1 \quad v = 2xy$

a) $u=0$: $0 = x^2 - y^2 + 1 \Rightarrow 1 = y^2 - x^2$ ← hyperbola!

| $x$ | 0 | ±1 |
|---|---|---|
| $y$ | ±1 | $\pm\sqrt{2}$ |

b) $v=0$: $0 = 2xy \Rightarrow$ true when $x=0$ OR $y=0$.

c) Zeros are where both $u$ and $v = 0$, so at $z = \pm i$!

## Exercises 11

65

*Discussion*

11.1 Why did we choose to draw circles and lines through the origin in Figure 50?

11.2 In Figure 51 we run into a problem. Mutliple oints from our $z$-plane are mapped to the same location on our $w$-plane. Why does this happen?

11.3 In Figure 54, why do we end up with 2 copies of the shapes we draw on the $w$-plane?

11.4 Why do you think our paths behave so strangely in Figure 55?

*Drill*

Using the polar form of complex numbers we learned back in Parts 7 and 8, find all solutions to the the multifunction:

$$z = f^{-1}(w) \pm \sqrt{w}$$

11.5 $f^{-1}(-2i)$ $= \pm\sqrt{2\angle 270^\circ}$
$= \pm\sqrt{2}\angle 135^\circ$
$= \pm(-1 \pm i) = \boxed{-1+i,\ 1-i}$

11.6 $f^{-1}(-1)$ $\pm\sqrt{1\angle 180^\circ}$
$= \pm 1\angle 90^\circ = \pm i$

11.7 $f^{-1}\left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right) = \pm\sqrt{1\angle 60^\circ}$
$= \pm 1\angle 30^\circ$
$= \pm\left(\sqrt{3}/2 + 1/2\, i\right)$
$= \sqrt{3}/2 + 1/2\, i,$
$-\sqrt{3}/2 - 1/2\, i$

11.8 $f^{-1}(0)$ $= \pm\sqrt{0} = \boxed{0}$

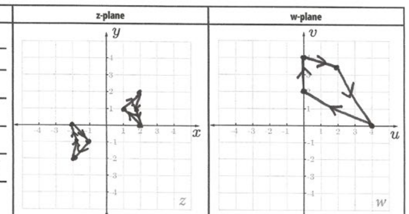
Exercises 11 66

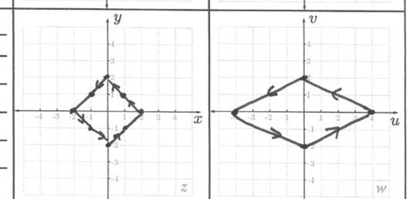
*Critical Thinking*

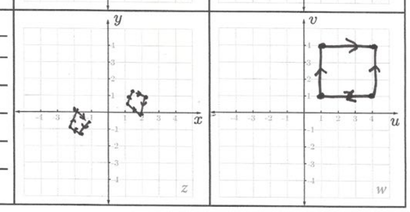
It's your turn to experiment with some paths! In each exercise below, compute the $x$ and $y$ values for each point according to the function $z = \pm\sqrt{w}$, plot the input and output points, and connect consecutive points with a line. You may use technology.

| Exercise | $z_1$ | | $z_2$ | | $w$ | |
|---|---|---|---|---|---|---|
| | $x_1$ | $y_1$ | $x_2$ | $y_2$ | $u$ | $v$ |
| 11.9 | -2 | 0 | 2 | 0 | 4 | 0 |
| | -1 | -1 | 1 | 1 | 0 | 2 |
| | -2 | -2 | 2 | 2 | 0 | 4 |
| | $-\sqrt{3}$ | -1 | $\sqrt{3}$ | 1 | 2 | $2\sqrt{3}$ |
| | -2 | 0 | 2 | 0 | 4 | 0 |
| 11.10 | -2 | 0 | 2 | 0 | 4 | 0 |
| | -1 | -1 | 1 | 1 | 0 | 2 |
| | 0 | -2 | 0 | 2 | -4 | 0 |
| | -1 | 1 | 1 | -1 | 0 | -2 |
| | -2 | 0 | 2 | 0 | 4 | 0 |
| 11.11 MAKE YOUR OWN PATH!! | 1.1 | 0.46 | -1.1 | -.46 | 1 | 1 |
| | 1.6 | 1.25 | -1.6 | -1.25 | 1 | 4 |
| | 2.2 | 0.9 | -2.2 | -0.9 | 4 | 4 |
| | 2.0 | 0.25 | -2.0 | -0.25 | 4 | 1 |
| | | | | | 1 | 1 |

z-plane / w-plane

11.9


11.10


11.11


*Challenge*

11.12 Explain why our paths jump in Figure 55.

SEE PARTS 12-13 :)

## Exercises 12

70

*Discussion*

12.1 What's good about Riemann's two complex plane solution?

12.2 What's not so good about Riemann's two complex plane solution?

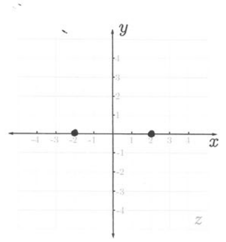
12.3 Why do our paths jump in Figures 58 and 59?

*Drill*

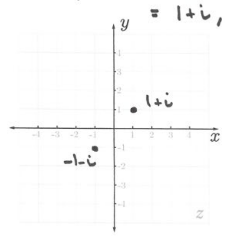
Find all plot solutions to the the multifunction:

$$z = f^{-1}(w) \pm \sqrt{w}$$

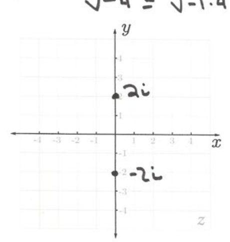
12.4 $f^{-1}(4)$ $\sqrt{4} = \pm 2$



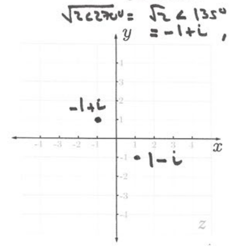
12.5 $f^{-1}(2i)$ $2i = 2\angle 90^\circ$
$\sqrt{2\angle 90^\circ} = \sqrt{2}\angle 45^\circ$
$= 1+i,\ -1-i$




12.6 $f^{-1}(-4)$ $\sqrt{-4} = \sqrt{-1\cdot 4} = \pm 2i$




12.7 $f^{-1}(-2i)$ $-2i = 2\angle 270^\circ$
$\sqrt{2\angle 270^\circ} = \sqrt{2}\angle 135^\circ$
$= -1+i,\ 1-i$



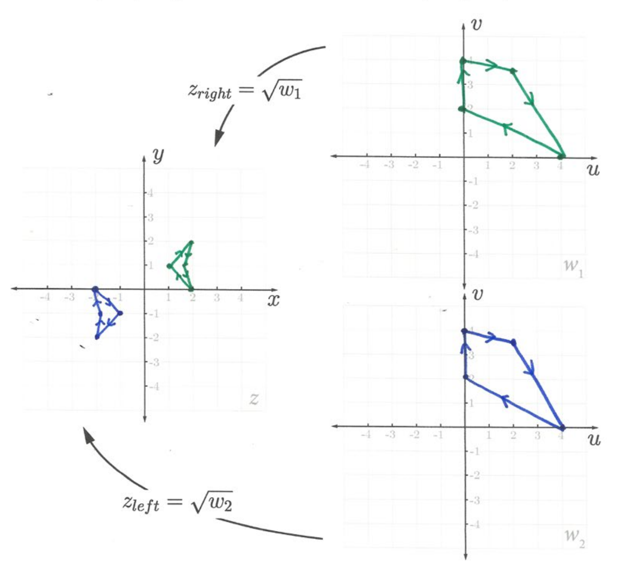

Part 12 Exercises 71

*Critical Thinking*

12.8 Let's try to get a better feel for what's happening in Figure 58. For each $w$ value in the tables below, compute the corresponding $z$ values. Each $w$ value will yeild two $z$ values, so choose the appropriate one based on which side of the plane (left of right) it's on. Plot all points, and connect consecutive points to form a path when possible. For points that end up on on the imaginary $z$ ($y$) axis (these could be interpreted as left or right side), keep both solutions. For one cool point, make your left and right right paths different colors.

| $z_{left}$ | | $w_2$ | |
|---|---|---|---|
| $x_{left}$ | $y_{left}$ | $u$ | $v$ |
| -2 | 0 | 4 | 0 |
| -1 | -1 | 0 | 2 |
| -2 | -2 | 0 | 4 |
| $-\sqrt{3}$ | -1 | 2 | $2\sqrt{3}$ |
| -2 | 0 | 4 | 0 |

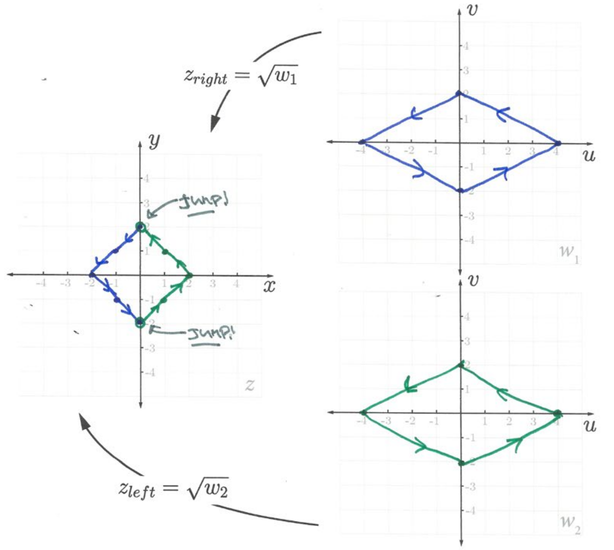
| $z_{right}$ | | $w_1$ | |
|---|---|---|---|
| $x_{right}$ | $y_{right}$ | $u$ | $v$ |
| 2 | 0 | 4 | 0 |
| 1 | 1 | 0 | 2 |
| 2 | 2 | 0 | 4 |
| $\sqrt{3}$ | 1 | 2 | $2\sqrt{3}$ |
| 2 | 0 | 4 | 0 |



12.9 For each $w$ value in the tables below, compute the corresponding $z$ values. Each $w$ value will yeild two $z$ values, so choose the appropriate one based on which side of the plane (left of right) it's on. Plot all points, and connect consecutive points to form a path when possible. For points that end up on on the imaginary $z$ ($y$) axis (these could be interpreted as left or right side), keep both solutions. For another cool point, make your left and right right paths different colors.

| $z_{left}$ | | $w_2$ | |
|---|---|---|---|
| $x_{left}$ | $y_{left}$ | $u$ | $v$ |
| -2 | 0 | 4 | 0 |
| -1 | -1 | 0 | 2 |
| 0 | ±2 | -4 | 0 |
| -1 | 1 | 0 | -2 |
| -2 | 0 | 4 | 0 |

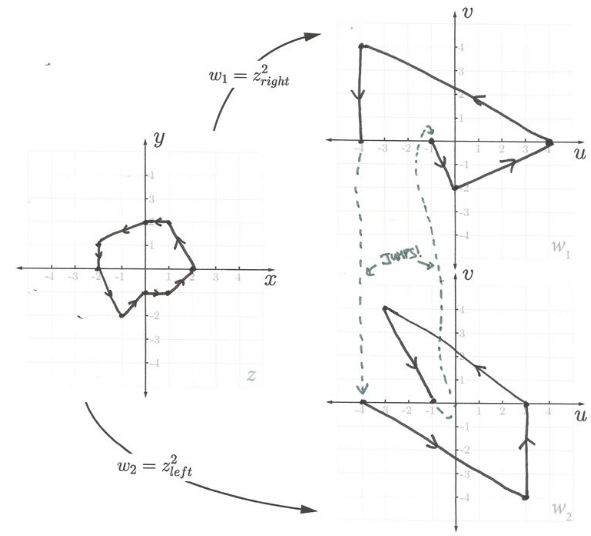
| $z_{right}$ | | $w_1$ | |
|---|---|---|---|
| $x_{right}$ | $y_{right}$ | $u$ | $v$ |
| 2 | 0 | 4 | 0 |
| 1 | 1 | 0 | 2 |
| 0 | ±2 | -4 | 0 |
| 1 | -1 | 0 | -2 |
| 2 | 0 | 4 | 0 |






**12.10 Now Forwards!** Map each $z$ value in the table below to its corresponding $w$ value. Note that each $z$ value will map to *either* $w_1$ *or* $w_2$, with one exception: allow points along the imagary $z$ ($y$) axis to map to both $w_1$ *and* $w_2$ - this is a small hack (don't tell your mathematician friends) to make your graph easier to understand. Plot all points, and connect consecutive points to form a path when possible.

| $z$ | | $w_1$ | | $w_2$ | | |
|---|---|---|---|---|---|---|
| $x$ | $y$ | $u$ | $v$ | $u$ | $v$ | |
| 2 | 0 | 4 | 0 | | | $w_1$ |
| 1 | 2 | -3 | 4 | | | |
| 0 | 2 | -4 | ~~2~~ 0 | -4 | ~~-4~~ 0 | |
| -2 | 1 | | | 3 | -4 | $w_2$ |
| -2 | 0 | | | 4 | 0 | |
| -1 | -2 | | | -3 | 4 | |
| 0 | -1 | -1 | 0 | -1 | 0 | |
| 1 | -1 | 0 | -2 | | | $w_1$ |
| 2 | 0 | 4 | 0 | | | |

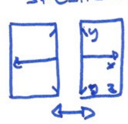





*Challenge*

12.11 In Figure 58, our first discontinuity on $z$ occurs when we first cross the negative real axis on $w$. Why does this happen at this specific location? Is this the result of how our function behaves, or of a choice we made?

THIS IS A DIRECT RESULT OF SPLITTING OUR Z-plane INTO HALVES:



→ THIS INTRODUCES OUR DISCONTINUITY ALONG THE Y-AXIS, LET'S CONSIDER AN EXAMPLE POINT ON THE Y-AXIS: $z = i$. THIS POINT IS MAPPED TO $w = z^2 = i^2 = \boxed{-1}$. SO WHEN WE CROSS THE NEGATIVE REAL AXIS ON W, WE CROSS THE CUT WE MADE ON Z. TO LEARN MORE ABOUT THIS, GOOGLE "BRANCH CUTS".

## Exercises 13



*Discussion*

13.1 How did our Riemann Surface help us understand our wierd loop behavior?

13.2 Why was our line of self-intesection actually not?

*Critical Thinking*

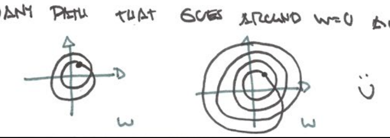
13.3 We've shown 2 different surfaces in 3D space that represent our function $f(z)=z^2$ (Figure 62 and Figure 70). Are there other possible surfaces? If so, how many?

YES! THERE ARE 4 DIMENSIONS: x, y, u, v. WE'VE SHOWN X, U, Y AND U, V, X SO FAR. There's 2 MORE: X, Y, V AND Y, U, V.

13.4 What kind of paths could we draw on our Riemann Surface that would return us *exactly* to where we started?

① ANY PATH THAT DOESN'T GO AROUND W=0

② ANY PATH THAT GOES AROUND W=0 AN EVEN # OF TIMES.




13.5 What would the Riemann Surface for $z = \sqrt[3]{w}$ look like?

3 LAYERS INSTEAD OF 2.

*Challenge*

13.6 For 1,000,000 cool points, construct a Riemann Surface (in code or paper) for a function other than $f(z)=z^2$, and tweet a picture to @welchlabs.

DO IT!

# About the Author

Stephen Welch is a huge jerk. Just look at that smug face. He thinks that, just like him, you should not only be good at math, but enjoy doing it. He has the audacity to think that math and science are perhaps the most beautiful discoveries we as humans have made, and they should be savored accordingly. Needless to say, we must stop this monster. Help show Stephen what a big jerk he is by watching and liking his You Tube videos and buying his books. That will show him.[1]

**Figure -1 | The jerk who wrote this workbook.** When he's not making videos, Stephen Welch spends his enormous You Tube profits on his professional Elephant Polo team.

1 Stupid Jerk.