Risk sharing with model misspecification in economies without aggregate fluctuations*

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1 Introduction

In a previous paper, Bhandari (2013a) studied a redistribution problem with heterogeneous beliefs, heterogeneous information and aggregate fluctuations. A Pareto planner assigned consumption to two uncertainty averse agents who differed in their initial priors over a common set of forecasting models of aggregate endowment. One of the agents received a privately observed taste shock. The analysis exploited two key features, presence of aggregate risk and the multiplicative nature of taste shocks. In this paper we study implications of model misspecification in environments with no aggregate risk and additive un-insurable idiosyncratic income risk. Absent other forms of heterogeneities, under complete markets, risk sharing scheme implies constant consumption for all the agents. In the paper we study the consequences of two forms of market incompleteness.

First we restrict agents such that they can only trade a risk-free bond. The key finding is that in contrast to the results of this paper, relative pessimism is diminishing in wealth, a result that is not dependent on the value of IES. Having a large amount of wealth in assets that yield non-contingent return, lowers the volatility of consumption and consequently concerns for misspecification for richer agents.

Next we study Pareto optimal risk sharing schemes under the restrictions that individual incomes are private information. Working with ex ante identical but finite agents, we impose an additional restriction that transfers to individuals only depend on the reports of histories of their incomes. This rules out trivially optimal allocations where the planner can impose the first best by using reports from one agent to punish possible misreports by the second agent.

Within the class of such restricted allocations, the efficient risk sharing scheme without the concerns for misspecification have a property that either one of the two agents is driven to immiseration. This comes from the dynamics of continuation values associated with efficient incentives. Since agents linearly aggregate continuation values a mean zero perturbation of continuation values (from a static risk sharing scheme) delivers the same ex ante value but relaxes incentive constraints. The insight in Atkeson Lucas (1992) suggests that such perturbations are always profitable and optimal incentives would imply that continuation values will spread. With enough bad shocks, some agent can drift towards immiseration. However, as we show in this paper, when there are concerns for uncertainty, agents with lower continuation values are relatively more pessimistic and consequently over-estimate the states when they have lower continuation values. The planner alters the risk sharing arrangement by reducing the amount by which continuation values are lowered. This generates a force away from immiseration.

2 Setup

1. **Agents** : There is a finite set $I = 2$ of infinitely lived agents. Each type i has a unit mass of identical individuals. Henceforth $i = 1, 2$ refers an arbitrary individual of type i.

2. **Technology** : For most of the analysis, I study an exchange economy [1]. There is a Lucas tree that yields a constant amount of aggregate endowment $\bar{y}$. This output is randomly split into the 2 agents with shares $\boldsymbol{s}_t \in \mathcal{S} \subset \Delta = \{\boldsymbol{s} \in \mathcal{R}_+^2 : \sum_i s_i = 1\}$. Every individual of type i has the same endowment : $y_{i,t} = \bar{y} s_{i,t}$.

3. **Information**: Individual incomes $y_{i,t}$ are privately observed. In section 3, we will study a bond economy that corresponds to assuming hidden savings as well.

4. **Models:** We allow for the approximating model to be specified as a prior over a set of models $\mathcal{M}$

   - M be the cardinality of the Model Space
   - $\mathcal{M} = \{\alpha_m\}_{m \leq M}$ is the Model Space

   Given a model we have $P(s^*|s,m) = \mathbb{P}_S[\alpha_m]$. We can have the model space evolving (like regime changes) as a Markov process $\{\mathcal{M}, \mathbb{P}_M, \pi_M^0\}$ [2]. With this setup we have the following state-space

$$m_{t+1}|m_t \sim \mathbb{P}_M$$

$$s_{t+1}|s_t, m_t \sim \mathbb{P}_S(\alpha_{m_t})$$

5. **Preferences** : Following Hansen and Sargent [2007] the preferences of the agent are described by 2 sets of objects,

   - **Ambiguity** $\forall i \in I$,
     (a) Approximating Models : $\langle P_M^i, P_S^i, \pi^i \rangle$
     (b) Entropy Penalty - θ_j^i where $j = 2$ captures the doubts about the parameter m and $j = 1$ about the s^* given s
   - **Time and Risk** :
     (a) Risk Aversion - γ^i
     (b) Subjective discount factor - δ^i

The agents can have potentially different preferences but I will mostly concentrate on the cases where the only differences in the agent is their endowment stream.

[1] In section 4 , I extend some results to a production economy with technology linear in labor and agents endowed with productivities. The main reason for the departure is to study the temporal properties of "tax" wedges.

[2] This allows me interpret type (II) ambiguity as concerns for unknown parameters when $P_M = \mathbb{I}$ and regimes otherwise.

3 Bond Economy

This section describes a bond economy without aggregate risk. The agents are identical with respect to the their endowment shocks and differ in the initial asset holdings. I first begin with a static (partial equilibrium) example explaining how the resolution of model uncertainty depends on the asset levels. In particular note that γ plays a minimal role here in driving the results.

3.1 Static Example

Consider an agent with a risky endowment y. The support for y is such that consumption is non-negative even for the worst income shock. The agent has risk free assets which pays b.

$$c(y, b) = y + b$$

$$V^R(b) = \min_m \mathbb{E}m[u(c) + \theta \log(m)]$$

such that $Em = 1$

Proposition 1. *For every b there exists a threshold $\bar{y}(b)$ such that* $\frac{\partial m(y,b)}{\partial b} > 0 iffy > \bar{y}(b)$

Since $\tilde{p}(y) = p(y)m^*(y, b)$ we have that the richer the agent over weights "sufficiently" good realizations of y. The intuition is that with higher b the relative fluctuations in y are not large enough to distort the distribution of y. Large assets provide the buffer for self-insurance and hence reduce concerns for model uncertainty.

Figure 1 shows an example with $\theta = 1$ and $\gamma = .5$ and a risky endowment y that has a density proportional to a standard normal. The solid line depicts the density under the approximating model. The two dotted lines show how the worst case density "shifts" we change the asset levels. In particular the dotted red line being closer to the benchmark illustrates the self-insurance effect of high assets. Figure 2 shows how the threshold $\bar{y}(b)$ flattens out as b increases. The shaded region are the realizations of y that are relatively over weighted by the marginally 'rich' agent.

3.2 Dynamic case

This section describes the bond economy without aggregate risk : $\mathcal{Y} = \{\bar{y}\}$ and normalizing aggregate supply of bonds to zero. The shocks s instead of affecting the tastes of Agent 2 affect the share of endowments. The beliefs of the agents are given by initial priors on a finite set of Markov models for s denoted by $\mathcal{M} = \{P_S(s^*|s, m)\}_m$ and $\pi_i^0 \in \Delta(\mathcal{M})$. The endowments of Agent 1 and Agent 2 are ys, $y(1 - s)$ respectively[3].

With the zero aggregate supply of bonds and common initial priors - π_0, the sufficient state variables for this economy are (B^1, s, π) :- The assets of Agent 1, current realization of the distributional shock and the common prior over the set of models $\mathcal{M}$

[3]Note that s is not an *idiosyncratic* shock as all agents of type i have the same endowment. This allows us to aggregate symmetric decisions over individual types of agent and keep track only of how wealth is distributed across types.

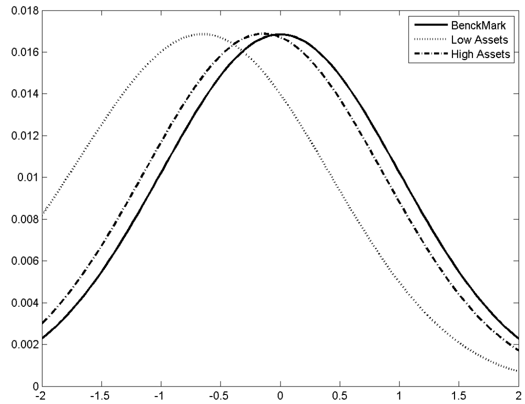


Figure 1: *This figure shows the reference and worst case models for two levels of assets in the static example with* $\theta = 1, \gamma = 0.5$

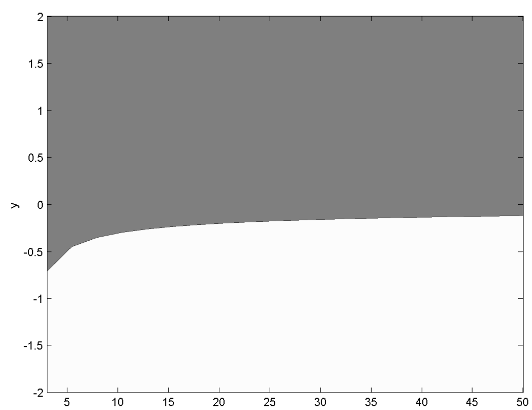


Figure 2: *This figure plots* $\bar{y}(b)$ *for the static example with* $\theta = 1, \gamma = 0.5$. *The shaded region are realizations of* y *for which* $m_b^*(y, b) > 0$

3.2.1 Agents problem

Given bond prices q, let $Q^i(b, B^1, s, \pi)$ be the value of Agent i with assets b and aggregate state (B^1, s, π)

$$\mathcal{Q}^1(b, B^1, s, \pi) = \max_{c,b^*} u(c) + \delta \mathbb{T}_{\theta_2} \mathbb{T}^1_{\theta_1,m,s} \mathcal{Q}^1(b^*, B^{1*}, s^*, \pi^*) \tag{1}$$

subject to

$$c + qb^* = ys + b \tag{2a}$$

$$\pi^* \propto \sum_m P_S(s^*|s, m)\pi(m) \tag{2b}$$

$$b^* \geq \underline{\mathrm{b}}^1(s) \tag{2c}$$

Where $\underline{\mathrm{b}}^1(s)$ is the natural debt limit for Agent 1 in state s

Similarly we can describe Agent 2's problem as

$$\mathcal{Q}^2(b, B^1, s, \pi) = \max_{c,b^*} \left[u(c) + \delta \mathbb{T}^2_{\theta_2} \mathbb{T}^1_{\theta_1,m,s} \mathcal{Q}^2(b, B^{1*}, s^*, \pi^*)\right] \tag{3}$$

subject to

$$c + qb^* = y(1 - s) + b \tag{4a}$$

$$\pi^* \propto \sum_m P_S(s^*|s, m)\pi(m) \tag{4b}$$

$$b^* \geq \underline{\mathrm{b}}_2(s) \tag{4c}$$

Where $\underline{\mathrm{b}}^2(s)$ is the natural debt limit for Agent 2 in state s.

Remark 1. *Along equilibrium paths market clearing will impose upper limits on asset positions of individual agents too*

3.2.2 Equilibrium

Given q, the interior solutions to these problems pin down the consumption savings decisions of both agents. Let $\mathcal{B}^i[b, B^1, \pi, s, q]$ be the savings of Agent i.

$$\mathcal{B}^1(b, B^1, s, \pi, q) : qu_c[ys + b - qb^*] - \delta \tilde{\mathbb{E}}^1_s Q^1_b(b^*, B^{1*}, s^*, \pi^*) = 0 \tag{5a}$$

$$\mathcal{B}^2(b, B^1, s, \pi, q) : qu_c[y(1 - s) + b - qb^*] - \delta \tilde{\mathbb{E}}^2_s Q^2_b(b^*, B^{1*}, s^*, \pi^*) = 0 \tag{5b}$$

The expectations are taken with respect to the worst case model averaged marginals $\sum_m \tilde{\pi}_i(m)\tilde{P}^i_S(s^*|s, m)$. Like before $\tilde{\pi}^i$ and $\tilde{P}^i_S$ can be computed using the value functions Q^i for each agent.

3.3 A Minimally Stochastic case

We first analyze the equilibrium under special dynamics for s which reduces the problem essentially to a 2 period version that can be quickly solved and study how wealth differences affect the worst case beliefs of agents. The simple economy is constructed under the following dynamics for $\{s_t\}_{t>0}$ given s_0

1. $s_1|s_0 \sim P_S[s^*|s, m]$
2. $s_{t+1} = s_t$ for $t \geq 1$

This is a *minimally stochastic case* which features an absorbing state for s_t from $t = 1$. The value of the agent can now be computed backwards - Let $\mathcal{Q}^{i*}$ be the value of Agent i from period 1 onwards and $\mathcal{Q}^{i0}$ denote the value after s_0 has been realized. The stationary environment after period 1 implies that

- $\mathcal{Q}^{1*}[b, B^1, s, \pi] = \frac{u[ys+(1-\delta)b]}{1-\delta}$ and $\mathcal{B}^{1*}[b, B^1, s, \pi] = b$
- $\mathcal{Q}^{2*}[b, B^1, s, \pi] = \frac{u[y(1-s)+(1-\delta)b]}{1-\delta}$ and $\mathcal{B}^{2*}[b, B^1, s, \pi] = b$
- $q(B^1, s, \pi) = \delta$

Now we can derive the objects in $t = 0$ using the above as terminal conditions. The following proposition states that there exist an inverse relationship between assets and the weights that agents give to states when they have low income.

Let $z_1(B^1, s, \pi) = \frac{\sum_m \tilde{\pi}^1(m)\tilde{P}^1(s|s_0,m)}{\sum_m \pi(m)P_S(s|s_0,m)}$ be Agent 1's (equilibrium) worst case likelihood ratio.

Proposition 2. *There exists* $\bar{y}(b)$ *and* $\underline{B}^1_{-1,0}[s,\pi]$ *such that* $\lim_{b\to\underline{B}^1_{-1,0}} \mathcal{B}^{1,0}(b, \underline{B}^1_{-1,0}, s, \pi, q) = -\frac{ys_l}{1-\delta}$ *and* $\lim_{b\to\underline{B}^1_{-1,0}} \bar{y}[\mathcal{B}(b, \underline{B}^1_{-1,0}, s, \pi, q)] = ys_l$*. Further we have,*

$$\frac{\partial z_1(B^1, s, \pi)}{\partial B^1} > 0 \quad \textit{iff} \quad y_1(s) > \bar{y}(b)$$

as long as we have $ys_l < \bar{y}[\mathcal{B}(B^1, s, \pi, q)] < ys_h$

3.4 Long-run Dynamics of Heterogeneous Beliefs

I solve for the recursive competitive equilibrium using the algorithm detailed in Appendix 6.1. For the results in this section I use the following parameter values for technology and preferences. The results below depict two cases 1) Transient learning ($P_M = \mathbb{I}$) and 2) Permanent learning where the models switch with probability .1

	Description
Agggregate Income (y)	1.00
Agent 1 low share (s_l)	0.30
Agent 1 high share (s_h)	0.70
Probability of switching - Model 1 (1-α^S)	0.50
Probability of switching - Model 2 (1-α^S)	0.10
Risk aversion (γ)	0.50
Subjective discount factor(δ)	0.95
Ambiguity - Observable State (θ_1)	1.00
Ambiguity - Hidden State (θ_2)	1.00

With incomplete markets the history dependence encoded in state variable B^1 yields a non-degenerate ergodic distribution on wealth. Agent 1 accumulates (decumulates) assets in s_h (s_l) shocks. Figure 3 depict the change in the level of assets of Agent 1 : $\mathcal{B}^1 - B^1$ as a function of his initial wealth level. Since precautionary motives are strong when either Agent 1, Agent 2 has very low wealth, interest rates are low at extremes.
Figure 5 shows how relative pessimism diminishes with wealth. The top (bottom) row depicts the distorted transition probabilities ($\tilde{P}^1(s^*|s)$) for s^* conditional on $s = s_l$ ($s = s_h$). The dotted (solid) line is the probability of low income state for Agent 1 in the next period. The pessimistic twisting implies that as long the Agent is facing some risk, he will over estimate the persistence of the shock in bad times. However as his assets grow, the self-insurance mechanism starts kicking in. The non-contingent part of his income - interest on savings rises and the realized consumption risk is lower. This makes the dotted line line slope downwards towards the common reference probability as his wealth rises. Figure **??** depicts how Agent 1 re-asses his estimation of the hidden state m. The top (bottom) panel shows Agent 1's pessimistic twisting for the Bayes prior (at $\pi = .5$) as a function of his assets B^1. Note that the curves in both the panels are very different. In particular $\tilde{pi}^{,1}$ is not monotonic. Since the agent over-estimates the persistence when $s = s_l$, his worst case models are very similar. This has an unusual effect of minimizing the effect of errors in estimation of m. In other words since concerns for errors in the transition matrices offset concerns for errors in the parameters. However in the bottom panel the worst case models are sufficiently different. In this case, concerns for consumption risk decline with wealth and the distorted priors are closer to the Bayes estimate when Agent 1 has high wealth.
After computing the decision rules and the value functions using the algorithm above I simulate the economy for 25000 periods with the starting value $(B^1, \pi, s) = (0, .5, 1)$ [4]. Figure **??** and 8plots the ergodic distribution of assets and prices in the two cases. Let $\tilde{\pi}_{gap} = \frac{|\tilde{\pi}^1 - \tilde{\pi}^2|}{\tilde{\pi}}$ lastly, I plot in figure 9 the ergodic distribution of differences in model priors when learning is permanent.

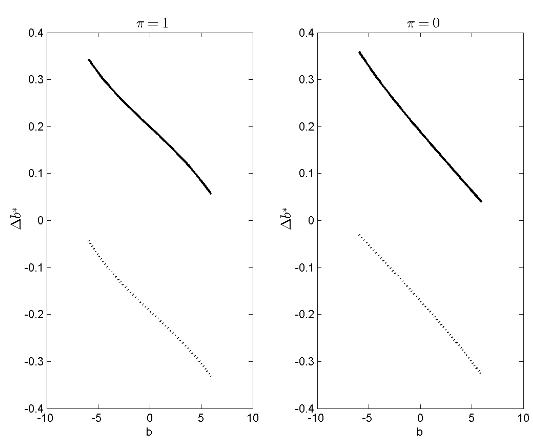


Figure 3

[4]The ergodic wealth distribution is invariant to the starting value

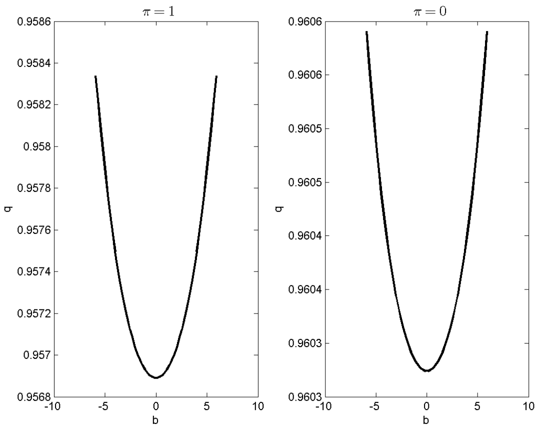


Figure 4: *The figure plots averaged bond prices as a function of Agent-1's level of assets.*

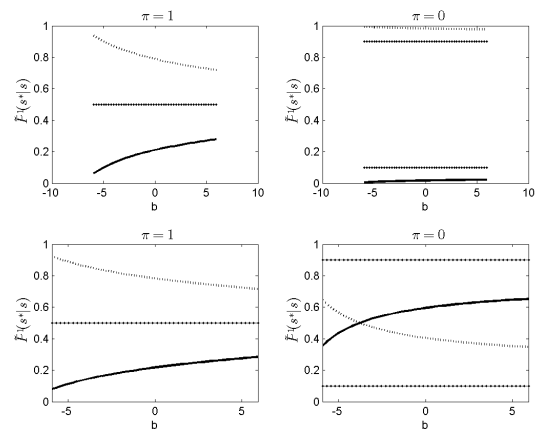


Figure 5: *The plots above depict $\tilde{P}^1(s^*|s,m)$. The left (right) column is the IID (NonIID) model and the dotted(solid) line refers to $s^* = s_l$ (s_h). The top (bottom) panels refer to $s = s_l(s_h)$*

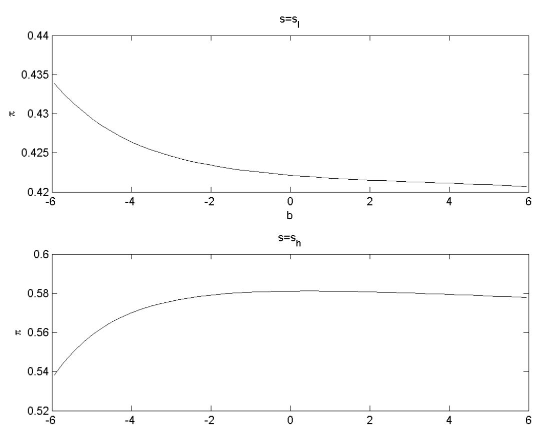
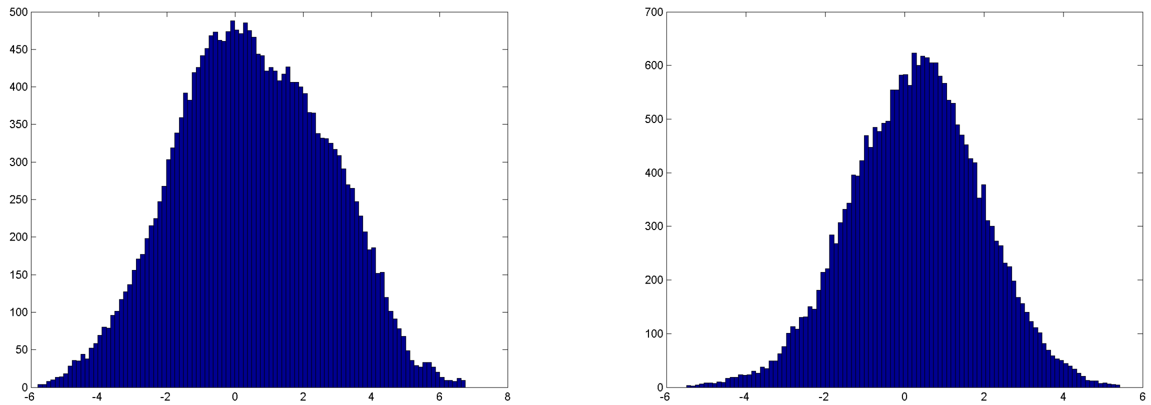
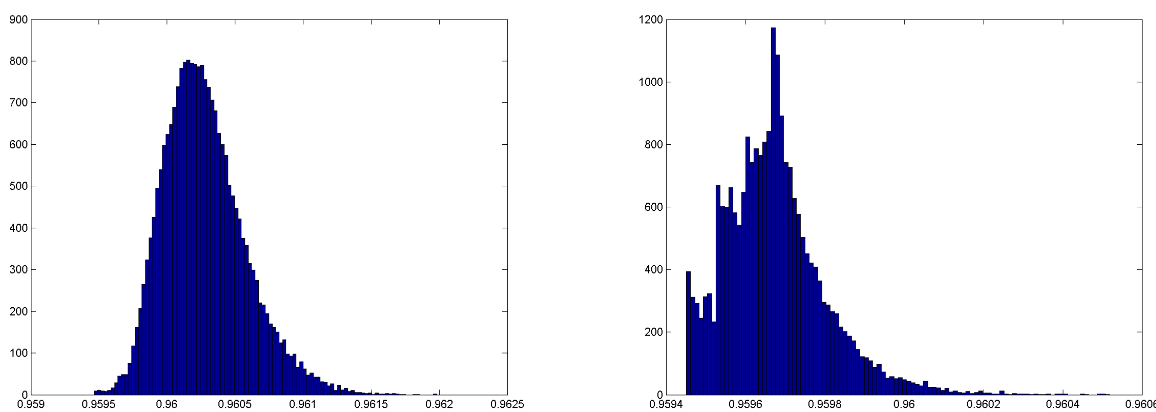


Figure 6: *This plots $\tilde{pi}^1$ given $\pi = \frac{1}{2}$. The top (bottom) panel is $s = s_l$ ($s = s_h$)*



(a) *Transient Learning* **(b)** *Permanent Learning*

Figure 7: *The figure plots the ergodic wealth distribution of B^1. The left panel is the transient learning case with IID model and the right panel refers to $P_M \neq \mathbb{I}$*



(a) *Transient Learning* **(b)** *Permanent Learning*

Figure 8: *The figure plots the ergodic wealth distribution of bond prices. The left panel is the transient learning case with IID model and the right panel refers to $P_M \neq \mathbb{I}$*

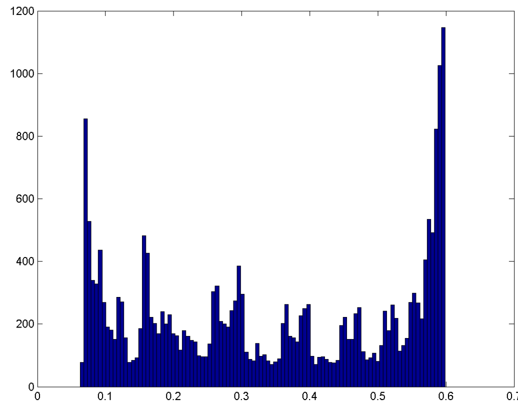


Figure 9: *This plots the ergodic distribution of* $\tilde{\pi}_{gap} = \frac{|\tilde{\pi}^1 - \tilde{\pi}^2|}{\tilde{\pi}}$

3.5 Efficient Allocations with Asymmetric Information

In this section I will study an example without aggregate risk and but further restrict the ability of the Planner to redistrubte consumption across agents and states. In particular individual incomes will be private information and the Planner has to design a risk sharing arrangement that mandates consumption to individuals depending only on his reported income. [5]

Within these class of allocations, presence of misspecification concerns affect the nature of incentive compatible schemes and the long run dynamics of consumption shares. On one hand, in absence of concerns for misspecification, the economy features an ever growing spread in consumption shares while activating these doubts keeps the long run inequality bounded.

I will characterize the (constrained) efficient allocations recursively by modifying problem **??** so that it accounts for the additional restrictions that capture incentive compatibility.

In the this section, I will describe the problem recursively for two agents with the following simplifying restrictions to the setup

1. $\mathcal{Y} = \{\bar{y}\}$and $\mathcal{S} = \{(s_l, s_h), (s_h, s_l)\} \quad s.t \quad s_l < s_h \quad s_l + s_h = 1$. Thus there is no aggregate risk and the individual incomes can take two values $y_i \in \{y_l = s_l\bar{y}\ , y_h = s_h\bar{y}\}$ for each of the two agents

2. $P^i(s_{t+1}|s^t) = P(s_{t+1})$: The agents have common priors about the distributional shock that is I.I.D over time

All agents are ex-ante identical and Planners choice can be described by a report contigent $\{T_t(y_i^t)\}_t$ that specifies a transfer to an individual at time t depending on the history of his reported income.

[5] I have two types, so knowing the income reports of one agents and the aggregate endowment, in principle one can figure out the income of the other agent. I ignore schemes that use this information and restrict consumption of an individual to be a function of only his idiosyncratic reports. This allows me to retain the forces similar to environments like Atkeson and Lucas (1994) with a parsimonious state space. Allowing richer mechanisms open up strategic concerns that I leave for future work.

Next I define a reporting strategy which is a collection of history contingent functions on the set of possible individual histories.

Definition 1. *A reporting strategy* $\boldsymbol{\sigma} = \{\sigma_t(y_i^t)\}_t$ *such that*

$$\sigma_t(y_i^t) = \tilde{y}_t$$

Definition 2. *A reporting strategy* σ^* *is said to be* truth-telling *if* $\sigma_t^*(y_i^t) = y_{i,t}$ *for all individual histories*

Let $\sigma^t(y_i^t) = \tilde{y}_i^t$ to be the collection of reports up to time t, given a $\boldsymbol{T}, \boldsymbol{\sigma}$ we can define the actual consumption as $\boldsymbol{c}(\boldsymbol{\sigma}) = \{T_t(\sigma^t(y_i^t)) + y_{i,t}\}_t$ and the value to any individual of $\{\boldsymbol{T}, \boldsymbol{\sigma}\}$ is given by

$$\mathcal{V}_{-1}[\boldsymbol{T}, \boldsymbol{\sigma}] = V_{-1}[\boldsymbol{c}(\boldsymbol{\sigma})]$$

The efficient allocation solves for $\boldsymbol{T}$ such that

$$\mathcal{V}_{-1}[\boldsymbol{T}, \boldsymbol{\sigma^*}] \tag{6}$$

subject to

$$\sum_i T(y_i^t) = 0 \quad \forall t \tag{7}$$

$$\boldsymbol{\sigma}^* \in \text{argmax}_\sigma \mathcal{V}_{-1}[\boldsymbol{T}, \sigma] \tag{8}$$

3.5.1 Recursive Formulation

As before we can characterize the efficient allocation recursively. It is useful to define the ex-ante version of Q - The maximum value to Agent 1 (Q^0) given a promised value v to Agent 2

$$\mathcal{Q}^0(v) = \max_{c_i(y_i), \bar{v}^*(y_2), Q(y_1)} \mathbb{T}_\theta \{(1-\delta)u[c_1(y_1)] + \delta Q(y_1)\} \tag{9}$$

Let $\Delta(y_i, \tilde{y}_i) = y_i - \tilde{y}_i$ be the short run gains misreporting

Incentive Constraints :

$$\mathbb{T}_\theta \{(1-\delta)u[c_2(y_2)] + \delta\bar{v}^*(y_2)\} \geq v^0 \quad \text{(`PK')} \tag{10a}$$

$$(1-\delta)u[c_1(y_1)] + \delta Q(y_1) \geq (1-\delta)u[c_1(\tilde{y}_1) + \Delta(y_1, \tilde{y}_1)] + \delta Q(\tilde{y}_1) \tag{10b}$$

$$(1-\delta)u[c_2(y_2)] + \delta\bar{v}^*(y_2) \geq (1-\delta)u[c_2(\tilde{y}_2) + \Delta(y_2, \tilde{y}_2)] + \bar{v}^*(\tilde{y}_2) \tag{10c}$$

Feasibility :

$$c_1(y_{1,l}) + c_2(y_{2,h}) = c_1(y_{1,h}) + c_2(y_{2,l}) = \bar{y} \tag{10d}$$

$$c_i(y_i) + \Delta(y_i, \tilde{y}_i) \geq 0 \tag{10e}$$

$$\mathcal{Q}^0(\bar{v}^*(\bar{y}-y_1)) \geq Q(y_1) \tag{10f}$$

$$v^{\max} \geq \bar{v}^*(y_2) \tag{10g}$$

Let $\underline{\mathcal{C}} = \{0, \bar{y}(s_h - s_l)\}$ and $\bar{\mathcal{C}} = \{\bar{y}(1 - s_h + s_l), \bar{y}\}$, we have $v_{\min}, v^{\max}$ satisfy the following

$$v_{\min} = \mathbb{T}_\theta \left\{u[\underline{\mathcal{C}}]\right\}$$
$$v_{\max} = \mathbb{T}_\theta \left\{u[\bar{\mathcal{C}}]\right\}$$

A couple of remarks regarding this recursive formulation. As in Atkeson and Lucas (1994) lemma 3.1-3.2, the temporary incentive constraints imposed in the recursive problem are sufficient for sequence problem.

The Bellman equation is ex-ante but the incentive constraints are ex-post. Although the concerns for model uncertainty manifest indirectly through the presence of Q^0 in the constraints, their key role will be in how the agents value various incentive compatible risk sharing arrangements. The Planner incorporates this link between different contracts and the associated endogenity in beliefs with respect to which these contracts are values. Given the solution of this problem, one can generate an allocation by iterating on the policy rules, Bellman equation using the history of reported shocks

Suppose we start time 0 with v_0, Agent 1's allocation can be obtained by

$$c_{1,t}(y_1^t) = c_1(y_{1,t}|v_{t-1})$$

where

$$v_{t-1} = \bar{v}^*(\bar{y} - y_{1,t-1}|v_{t-2})$$

and so on, where v_0 is the initial condition. Similarly we can get Agent 2's allocation

I assume agents start with equal Pareto wts., or $\mathcal{Q}^0(v_0) = v_0$ then $\boldsymbol{c}_1 = \boldsymbol{c}_2$

Let $\mu_s^i(y)$ denote the multiplier on incentive constraint for agent i in state s and λ on the promise-keeping constraint.

3.6 Optimal Contracts : Static

To build the key forces, consider the static case when $\delta = 0$. The following proposition summarizes the main features of the optimal allocation with concerns for misspecification when the agents essentially care for one period.

Proposition 3. *In the static case with $\delta = 0$, all incentive constraints bind. Further the optimal contract features has the following shape*

$$c_i[\lambda(v^0)] = c_i[\lambda(v^0), \tilde{s}] + \Delta^i(s, \tilde{s})$$

Let c_i^∞ be the corresponding contract when the agents are not concerned about misspecification($\theta \to \infty$). There exists a $\bar{\lambda}$ such that

$$c_2[\lambda, s] \geq (\leq) c_2^\infty[\lambda, s] \quad \lambda < \bar{\lambda} \quad y_2(s) \leq (\geq) y_2(\tilde{s})$$

$$c_1[\lambda, s] \geq (\leq) c_1^\infty[\lambda, s] \quad \lambda > \bar{\lambda} \quad y_1(s) \leq (\geq) y_1(\tilde{s})$$

Further $\bar{\lambda} = 1$ if shocks are equally likely

Note that this proposition compares the outcomes with benchmark keeping λ "fixed". This puts the consumption plans in both the economies on a comparable scale as the state variable - v is measured in 'utils' and is not invariant to changes in θ. This proposition states that agents with low (implied) Pareto weights are relatively (as compared to $\theta = \infty$) better off in the state when they are unlucky.

1. The IC constraint pins down the gap between consumption levels across states to be the amount agents can garner if they misreport. Since these constraints are ex-post in nature they hold independently of attitudes towards model misspecification

2. As compared to the benchmark, there are two forces which distort the optimal insurance scheme with concerns for misspecification.

   - Let s be such that $y^2(s) = y_l$ or Agent 2 is unlucky. His consumption diminishes as his relative Pareto weight λ becomes smaller

   $$\lambda \to 0 \implies \frac{c_2(s)}{y - c_1(s)} \to 0$$

   However the IC constraint restricts the consumption plan to be $[c_2(s), c_2(s)) + \Delta]$.

   - Since utility is concave, $u(c+\Delta) - u(c)$ diminishes with c for a fixed Δ. In terms of distorted beliefs this means that agents with low Pareto weights relatively over estimate the probability of the state in which they have low incomes. In particular

   $$\tilde{p}^2(s) > p(s) > \tilde{p}^1(s)$$

   - Optimal insurance requires the Planner to allocate higher consumption to agents who perceive the given state more likely. However this channel only appears with endogenous beliefs when agents fear model misspecification. So given everything else as $\lambda \to 0$ Agent 2's consumption in states s is higher than what he would get in the benchmark. Since the agents are otherwise symmetric, the converse is true when $\lambda \to \infty$.

With incomplete markets as above, when consumption goes to zero, the spread in consumption remains at Δ. Thus the prediction that relative pessimism is decreasing in average consumption shares is more robust to shapes of utility functions at zero. This also emphasizes why shutting of aggregate risk was helpful.

Figure (10) illustrates this in a numerical example with $\theta = 1, \gamma = .5, \frac{\Delta}{y} = .3$.

In the dynamic setting with $\delta > 0$, the Planner faces a choice of how to provide incentives : he can either distort menu of current consumption or promises to future consumption. In general it is optimal to use both of these instruments. However, concerns for misspecification makes the costs of fluctuating future promises sensitive to current inequality as measured in λ or dispersion in consumption shares.

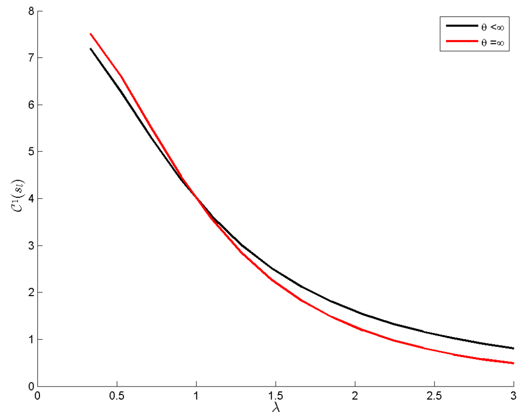


Figure 10: *The plot depicts consumption of Agent 1 in $t = 1$, state $s : y_1(s) = y_l$. The dotted line is the benchmark without concerns for model ambiguity.*

3.6.1 Optimal Contract : Dynamic

In this section I will analyze the dynamic contract with $\delta > 0$. Now the Planner chooses a contract : $\mathcal{U}(v^0)$ as a menu of report contingent consumption for both the agents and (ex-ante) promised values for Agent 2 [6],

$$\mathcal{U}(v^0) = \langle c_1(s), c_2(s)\bar{v}^*(s)\rangle$$

Before discussing the properties of the optimal contract, I will highlight a special class of dynamic incentive feasible contracts - *repeated static contracts.* These dynamic contracts are constructed by a sequence of static contracts discussed in the previous section. Given v^0, let $c^{ss}(v^0)$ solve the following equation

$$\exp\left\{-\frac{v^0}{\theta}\right\} = \left[\exp\left\{\frac{-u(y-c_1)}{\theta}\right\}P(s_l) + \exp\left\{\frac{-u(y-c_1-\Delta)}{\theta}\right\}P(s_h)\right]$$

The repeated static contracts are given by

$$\mathcal{U}(v^0) = \left\langle c^{ss}, c^{ss}+\Delta, \bar{y}-c^{ss}, \bar{y}-c^{ss}-\Delta, v^0, v^0\right\rangle$$

It is easy to see that they are incentive-feasible. The Planner loads all the incentives with across state consumption variation. These contracts are 'absorbing' in nature as the ex-ante promised values are constant and in general suboptimal except at $(v^0) \in \{v_{min}, v^{\max}\}$. The value of these contracts gives a reasonable lower bound to the value function. The upper bound can be constructed using the benchmark with no informational frictions.

$$\mathcal{Q}^{CM}(v^0) = u\left(y - u^{-1}\left[v^0\right]\right) \tag{11}$$

Thus the optimal value satisfies

$$\mathbb{T}_\theta u[c_1^{ss}] \leq \mathcal{Q}[v^0] \leq u\left(y - u^{-1}\left[v^0\right]\right)$$

[6] The promised values for Agent 1 are implicitly chosen through the Bellman equation

Let $\tilde{\Delta}_{i,c} = c_i(y_h) - c_i(y_l)$ and $\tilde{\Delta}_v = v(s_l) - v(s_h)$. The Planner ideally wants to provide insurance by giving more consumption in the low income states but this makes the agent misreport in periods of high income. The contract features incentive constraints which are slack in the periods when the agent has low income. This property is similar to that seen in environments similar to Thomas and Worral (XXX) where these are termed as downward binding incentive constraints.

Proposition 4. *With $\delta > 0$, incentive constraints are slack in the periods when agent i has low income.*

However, in this environment *both* constraints can be slack if the agent has low Pareto weight. Together with the constraints on bounds for consumption we have

- $\tilde{\Delta} \leq \Delta$ and $\tilde{\Delta}_v \geq 0$
- The IC and Feasibility constraints effectively partition the state space

$$\begin{aligned}
&c_1(s_l) = y - \Delta, \mu^1_{s_h} > 0, \mu^2_{s_l} = 0 && v \in \{v_{\min}, v_x\} \\
&\mu^1_{s_h} > 0, \mu^2_{s_l} = 0 && v \in \{v_x, v_y\} \\
&\mu^1_{s_h} > 0, \mu^2_{s_l} > 0 && v \in \{v_y, v_{yy}\} \\
&\mu^1_{s_h} = 0, \mu^2_{s_l} > 0 && v \in \{v_{yy}, v_{xx}\} \\
&c_1(s_h) = \Delta, \mu^1_{s_h} > 0, \mu^2_{s_l} = 0 && v \in \{v_{xx}, v^{\max}\}
\end{aligned} \tag{12}$$

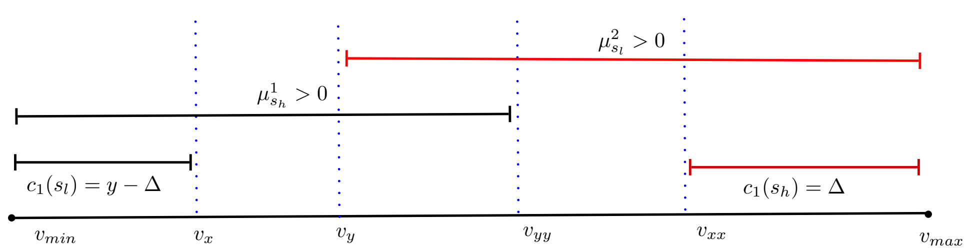


Figure 11: *This figure shows how the set of optimal contracts partition the endogenous state - ex-ante promised*

Figures 12 and 13 plot the optimal consumption menu and figure 14 shows gap in continuation values

At both the edges when the promised value to Agent 2 is either low enough or high enough, the bounds on consumption 10e are active. In particular for $v \leq v_x (v \geq v_{xx})$ we have $c_1(y_l) = y - \Delta(c_1(y_h) = \Delta)$. This gap is decreasing (increasing) in v in the aforementioned constrained regions. In the interior of $[v_x, v_{xx}]$, the gap in the consumption is low and stable. This region is where the Planner achieves maximal risk sharing. The additional lever whereby the (ex-ante) continuation values can be varied takes care of incentives and allows the Planner to reduce the spread in current consumption. Thus effectively the Planner uses the potential to vary consumption over time to reduce the spread between consumption over states. The Planner could always repeat the static contract with $\tilde{\Delta}_{i,c} = \Delta_i$ and no

spread in continuation values as discussed previously . The variation in continuation values allows the Planner to overcome short-run temptations to mis-report by implementing long-run (dynamic) punishments to the agent whose incentive constraint is binding. The recursive version of the problem chooses these punishment optimally. Like before the promised values can be mapped back to effective Pareto weights λ and we can interpret the variation in promised values as variation in Pareto weights.
Another point of distinction from the static case is that some incentive constraints are slack. Typically the constraint is slack for the agent with low Pareto weight and binds for the state when the Agent with high Pareto weight has a high income shock. In an subset of the region between v_x and v_y, the incentive constraints are binding for both the agents.

To summarize, the Planner manipulates two *wedges* to provide insurance respecting incentives : The gap in consumption menu $\tilde{\Delta}_{i,c}$ and the gap in continuation values $\tilde{\Delta}_v$. With IID shocks it can be shown that $\tilde{\Delta}_{i,c} \leq \Delta$ and $\tilde{\Delta}_v \geq 0$. Figures 12 and 13 plot the optimal consumption menu and figure 14 shows gap in continuation values

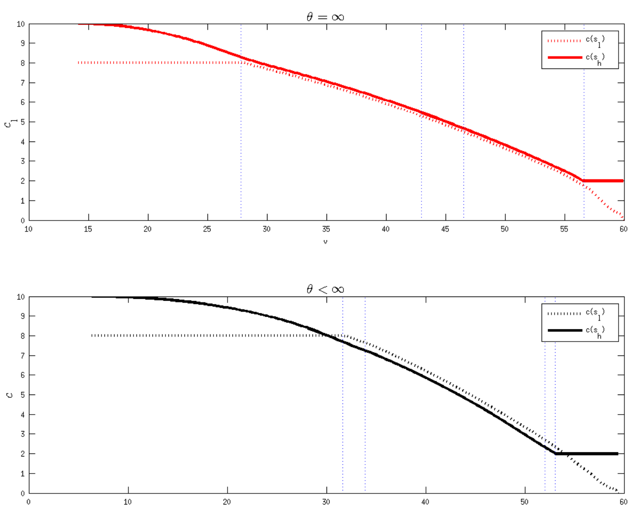


Figure 12: *This plots consumption levels as a function of the initial promised values*

Both the wedges - the across state wedge introduced by the consumption menu and the across time wedge introduced by variation in continuation values change with endogenous heterogeneous beliefs. Firstly agents over-estimate the probability of (respective) the low income state. The spread between values across states has a lower bound $u(y) - u(y - \Delta)$. This happens when both either of the bounds on consumption and incentive constraints are binding. In this region the agent whose incentive constraint is slack is relatively (with respect to the other agent) more pessimistic. Figure 16 plots the distorted beliefs as a function of the ex-ante promised value of Agent 2. Since the agent with lower Pareto weight typically has a slack IC, the relative pessimism is inversely related to Pareto weights. These differences in beliefs tilts the optimal risk sharing scheme in a way that the pessimistic agent gets more current and future consumption. Thus we have $\tilde{\Delta}_{1,c}$ lower than what would have been

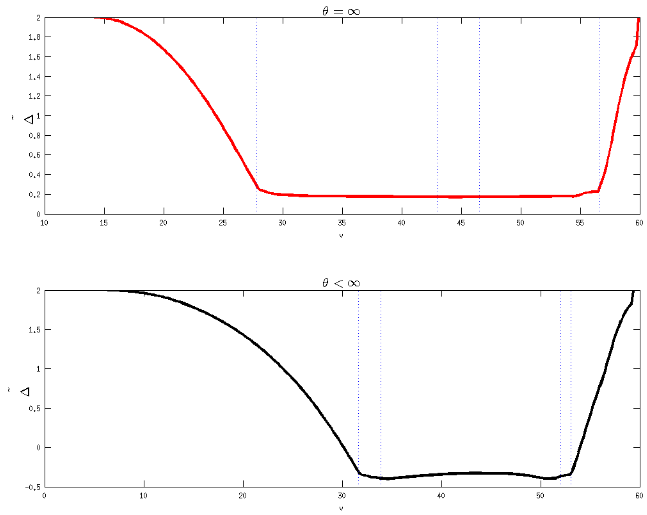


Figure 13: *This plots consumption gap for Agent 1* $\tilde{\Delta}_{1,c}$ *as a function of the initial promised values*

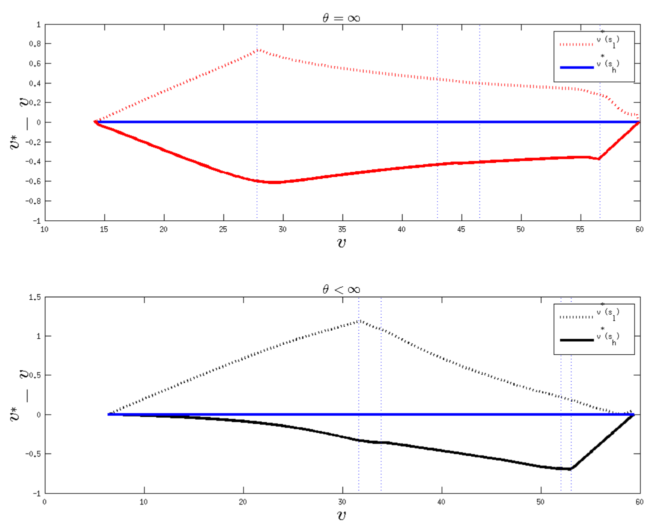


Figure 14: *This plots changes in (ex-ante) continuation values* $\bar{v}^*(z) - v$ *as a function of the initial promised values*

without concerns for misspecification[7]. Figure 15 plots the consumption gap for both the cases. As before, to make the contracts comparable, I align them on λ (multiplier on the promise keeping constraint) rather than promised values v^0. The figure clearly shows that $\tilde{\Delta}_{1,c}$ in the risk sharing phase is much lower for $\theta < \infty$.

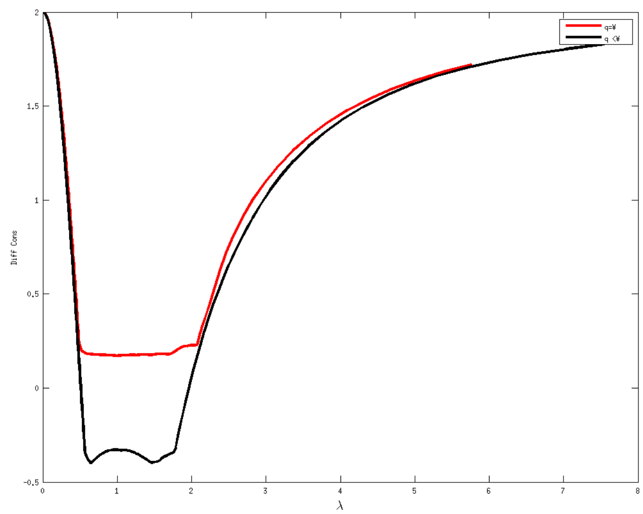


Figure 15: *This plots consumption gap $\tilde{\Delta}$ as a function of $\lambda(v)$*

The risk sharing arrangements shaped by these dynamic incentive considerations now interact with the agents concerns for misspecification and in turn affect the optimal risk sharing scheme. In the benchmark case (without concerns for model uncertainty) agents linearly aggregate their continuation utilities. Starting from from a repeated static contract, consider a mean zero perturbation to continuation values. With expected utility, this delivers the same value to the agent. However in an dynamic environment they can relax incentive constraints. The insight in Atkeson Lucas (1992) suggests that this perturbation is always profitable and optimal incentives imply continuation values will spread. With enough bad shocks, we can drift towards immiseration. An alternative way to interpret concerns for model uncertainty is *non-linear* aggregation of continuation values. Lower continuation values imply lower estimates of probabilities, the value from a mean zero perturbation is generally lower. This sets up the stage for possible survival forces

The long-run dynamics of this risk-sharing arrangement are captured by changes in ex-ante promised values $\tilde{\Delta}_v$ as a function of v. Without concerns for model uncertainty, the continuation values spread in a way that $\bar{v}^*(s_l) \geq v^0 \geq \bar{v}^*(s_h)$, with strict equality at the absorbing contract. The negative adjustment in $s : y_2(s) = y_l$ corresponds to a promise of lower future consumption to Agent 2 when his income is low and vice versa. The FOC conditions can be re-arranged to emphasize the two forces namely, *endogenous heterogeneity*

[7]It can also change signs: Δ_c is negative when cost of providing incentives through varying current consumption is lower than varying future consumption.

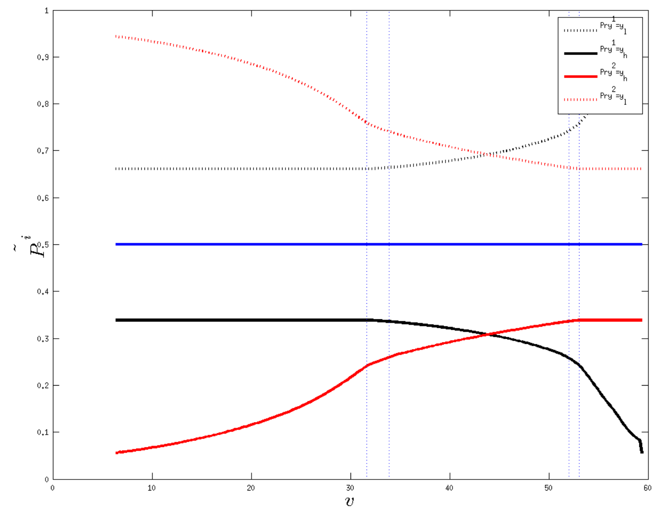


Figure 16: *This plots distorted beliefs for both Agents as a function of the initial promised values*

in beliefs and *optimal incentives* that interact in how Pareto weights move over time.

$$\lambda(s^t) = \lambda(s^{t-1}) \quad \underbrace{\frac{\tilde{P}^2(s_t)}{\tilde{P}^1(s_t)}}_{\text{Heterogeneous Beliefs}} \quad \underbrace{\left(\frac{1 + \mu^2(s_t) - \frac{\tilde{P}^2(s'_t)}{\tilde{P}^2(s_t)} \mu^2(s'_t)}{1 + \mu^1(s_t) - \frac{\tilde{P}^1(s'_t)}{\tilde{P}^1(s_t)} \mu^1(s'_t)} \right)}_{\text{Incentives}}$$

For low promised values for Agent 2 (i.e low λ), taking into account the slack incentive constraints, the one period ahead growth rate of λ_t is given by

$$\mathcal{G}_\lambda(y_{2,h}) = \frac{\tilde{P}^2(y_{2,h})}{\tilde{P}^1(y_{1,l})} \left(\frac{1}{1 - \frac{\tilde{P}^1(y_{1,h})}{\tilde{P}^1(y_{1,l}}\mu^1(y_{1,h})} \right) \tag{13}$$
$$\mathcal{G}_\lambda(y_{2,l}) = \frac{\tilde{P}^2(y_{2,l})}{\tilde{P}^1(y_{1,h})} \left(\frac{1}{1 + \mu^1(y_{1,h})} \right)$$

First consider the case when $\theta = \infty$, we have $\tilde{P}^i = P$ and since $\mu^1(s_h) > 0$,

$$\lim_{\lambda \to 0} \log\left(\mathcal{G}_\lambda(y_{2,h})\right) > 0 \tag{14}$$
$$\lim_{\lambda \to 0} \log\left(\mathcal{G}_\lambda(y_{2,l})\right) < 0$$

The negative growth rate in $s : y_2(s) = y_l$ or the states when Agent 2 is unlucky means that with enough bad shocks $v(s^t) \to v_{\min}$. This is the Atkeson Lucas (1994) force operating in our simple setup. Moving away from this benchmark and activating concerns for misspecifications (measured by $\theta < \infty$), there is a countervailing force to the incentives and if it is strong enough, it can potentially push the Agent 2 towards the center. For low θ's (or if utility is unbounded below), we can show that the agent with high Pareto weights has "bounded" pessimism, while the other agent puts arbitrarily high weights on the states he is unlucky

Proposition 5. *If* $\lim_{\lambda \to 0} \tilde{P}^2(y_{2,l}) = 1$ *then* $\lim_{\lambda \to 0} \log G_\lambda \geq 0$ *almost surely.*

Proof. As $\lambda \to 0$, Agent 1's distortions are bounded. Using the his IC and the bounds on consumption, the limiting probabilities are given by

$$\tilde{P}^1 \to \{\underline{p}(\Delta), \bar{p}(\Delta\} \tag{15}$$

where $\underline{p}(\Delta) = \frac{1}{1+\frac{P(y_h)}{P(y_l)}\exp\left\{\frac{u(y-\Delta)-u(y)}{\theta}\right\}}$. This is the distortion associated with the static contracts and we can rely on the concavity argument to see that when $\frac{\Delta}{\bar{y}}$ is low, it would make him distort his reference model rather moderately.

At the lower bound , if Agent 2's assessments of the states when he has low income shocks is arbitrarily close to unity, we can show that $\mu^1(y_{1,h})$ approaches $\frac{\underline{p}(\Delta)}{\bar{p}\Delta}$ This follows from the FOC with respect to $v^*(y_{2,h})$. A small increase in the continuation value for Agent 2 in these states $s : y_2(s) = y_l$ is equivalent to a $\lambda(s)$ fall in value for Agent 1. However this perturbation also relaxes his incentive constraint $s = s_h$. Thus equating the marginal gains and benefits we have $\tilde{P}^1(s_l)\lambda(s_l) = \lambda(s_l)\mu^1(s_h)\tilde{P}^1(s_h)$. This implies that $\mathcal{G}_\lambda \to 1$ and the continuation values rise almost surely. □

Towards low values of v^0, Agent 2 cares a lot about the states where he has a low endowment i.e $s = s_h$. Thus the Planner optimally reduces the downward adjustment to $\bar{v}^*[s_h]$ in these periods. This effectively speeds up the transition of the ex-ante promised values away from the absorbing contract. The figure 17 plots the growth rate of Lagrange multipliers for the two cases $\theta < \infty$ and $\theta = \infty$. We see in the bottom panel without concerns for misspecifications, the limiting growth rates are negative for low income states of Agent 2. However in the top panel they approach zero.

Thus endogenous beliefs pave a way of the necessary immiseration implied by efficiency. In related work Farhi and Werning [?] show that adding a paternalistic Planner (whose is more patient than the agents) can also give long run survival with efficiency in presence of asymmetric information

This proposition used the fact that $\tilde{P}^2(y_{2,l})$ went arbitrarily close to 1. With bounded utility, this is not necessarily true, hence whether or not the limiting force is strong enough to outweigh the incentives is a priori not clear and depends on the limiting value of $\mu^1(y_{1,h})$. So far, I don't have the precise characterization of the $\lim_{\lambda \to 0} \mu^1(y_{1,h})$. However numerical calculation show that for low enough θ, this growth rate is arbitrarily close to zero. Although exacerbating the concerns for misspecifications increases both the level of the multiplier and $\frac{P^2(y_{2,l})}{P^2(y_{1,h})}$, they seem to grow at different rates and the net effect in favor of heterogeneous beliefs for low enough θ. Figure 18 plots $\lim_\lambda \log \mathcal{G}_\lambda(y_{2,l})$ for a range of θ to emphasize this. To allow a "$\theta = \infty$" value the x-axis is modified in log scales. The long run drop in the

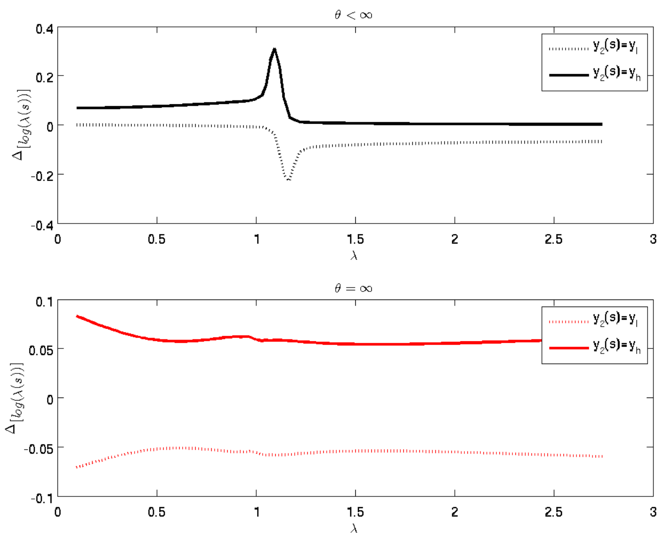


Figure 17: *This plots changes the (one-period ahead) growth rate of* λ^* *as a function of the* λ *. The dotted line refers to states when Agent 2 has low income. The top (bottom) panel is* $\theta < \infty (\theta = \infty)$

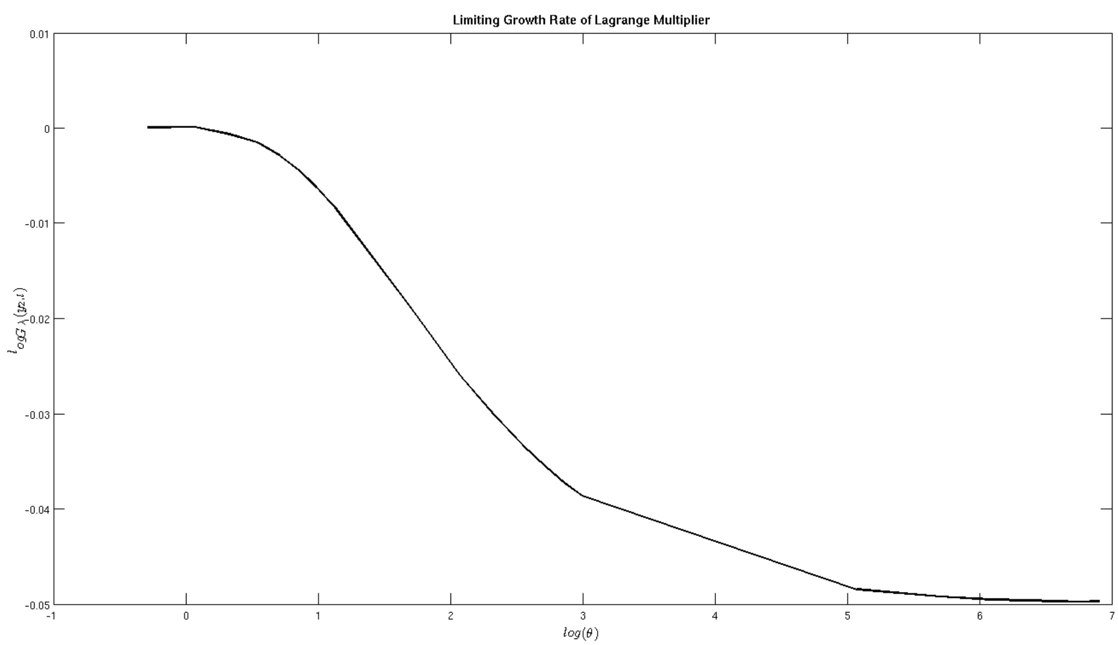


Figure 18: *This plots limiting growth rate* $\mathcal{G}_\lambda(y_{2,l})$ *as a function of* $\log(\theta)$

growth rate of Lagrange multiplier is negative around 5% and sharply drops to values close to zero around $\theta = 1$.

Naturally this raises the question of "what does $\theta = 1$ mean ?". Although the setup is far from any reasonable calibration exercise, one can make sense of value of θ, by mapping it to the familiar detection error probability. These were introduced in section **??** and basically refer to the threshold the agent puts on detection errors in finite samples done via a likelihood ratio test between his reference model and the worst case model that he builds in his mind accounting for the potential misspecifications. Computing these probabilities is a bit tricky in a setup with heterogeneous agents and history dependence in distortions. The way I do it here is as follows : Take a sample length T. Initialize the model with $\lambda_0 = 1$ which is roughly the mean of the ergodic distribution of Pareto weights. Draw N samples of length T from the reference IID model and compute the cases $M \leq N$ where the Agent would wrongly conclude that the sample came from his wort case beliefs. Let $r_1 = \frac{M}{N}$. Now do the reverse, i.e draw samples from the worst case distribution and test it against the approximating model to get r_2 similarly. In both cases the worst case likelihood is computed using the one-period ahead conditional likelihoods given the past history. The detection error probabilities naturally decline to zero with T. Keeping rest of the parameters unchanged, for a $T = 50$, θ of 1 maps to detection error probability of 15%.

Lastly I examine some long run properties of the model by simulating sample paths. In particular, asymptotically the ergodic distribution of values clusters in the region between $[v_y, v_{yy}]$. The right panel of figure 20 plots the ergodic distribution of v [8]. We see that with $\theta < \infty$, we have almost no mass on the constrained region. . Figure 19, plots the conditional exit times for both the cases - Prob$\{\bar{v}_{t+S} = v_{min} | v_t = v\}$ and Prob$\{\bar{v}_{t+S} = v^{max} | v_t = v\}$ for large S Figure 21 plots a typical sample path of Agent 1's consumption for the same sequence of shocks [9].

4 Skill shocks

So far, I studied a stylized two agent exchange economy to explore long run properties of Pareto optimal risk sharing outcomes. The key finding was uncovering how endogenous asymmetries for misspecifications concerns affected survival outcomes in alternative settings .In a stark contrast with benchmark where agents trust their priors , both settings 1) asymmetric priors - symmetric informations and 2) symmetric priors- asymmetric information feature survival forces.

This appendix replaces the endowment shocks with a technology linear in labor and shocks to skill distribution. The goal is use the model to study the patterns of the intra and inter temporal wedges embedded in constrained efficient allocations when agents fear model uncertainty. Related settings have been used in the NDPF literature for examining optimal taxation from first principles.

The main finding is that as compared to the benchmark, these wedges are higher and

[8]This is computed from a simulating a long sample using the law of motion $\bar{v}^*[z|v]$. The ergodic density is then obtained using the smooth kernel (Gaussian) on the last half of the sample

[9]The initial values are adjusted such that for both cases Agent 1 starts with the same consumption in period 1

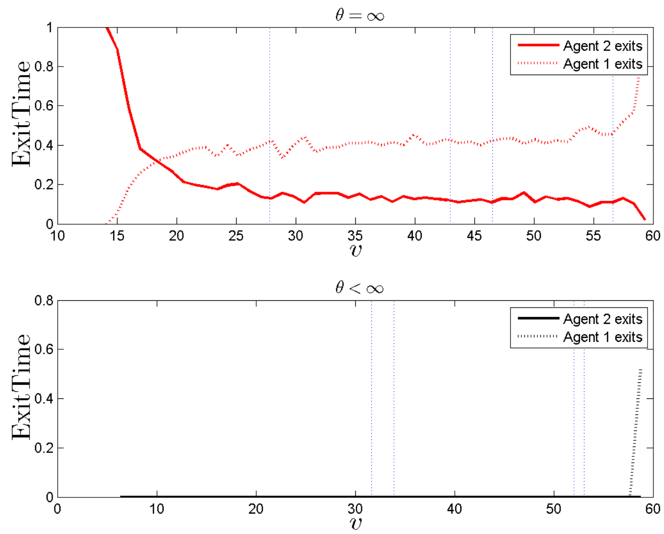


Figure 19: *This plots conditional exit probabilities as a function of initial promised values*

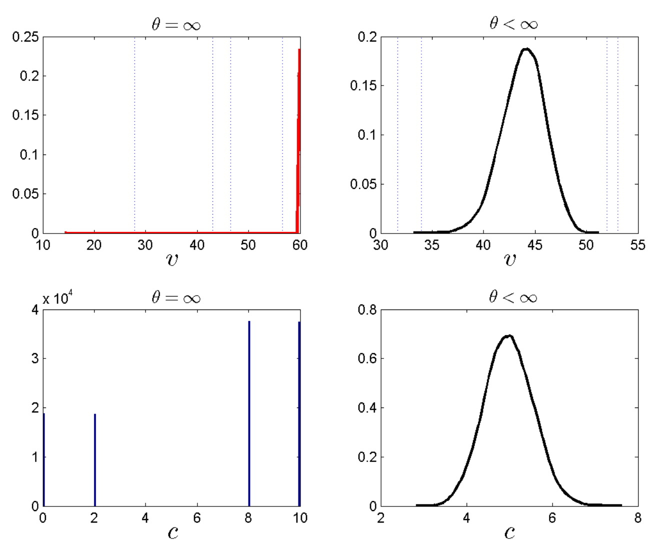


Figure 20: *This plots ergodic distribution of continuation values and consumption for Agent 1*

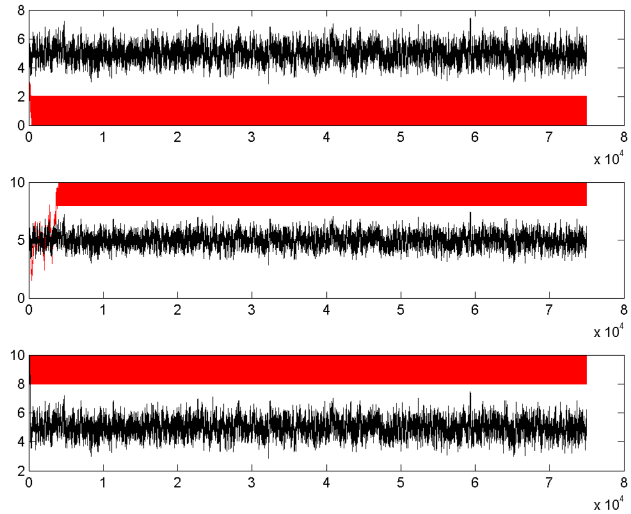


Figure 21: *This plots typical sample paths of consumption for Agent 1 with different initial conditions*

increase sharply with inequality. Thus long run equality is achieved by progressive (in wealth) labor and saving wedges.

4.1 Model

The environment is extends the basic setup in section 3.5 and I will outline the key differences. The two types of infinitely lived agents are now subject to stochastic skill shocks. The preferences are extended in a natural way by replacing $u(c)$ by $u(c, \frac{y}{\theta})$. I will focus on functional forms where utility from consumption and leisure is additively separable. The problem as before is to characterize constrained efficient allocations when skills and labor inputs are private information, however individual output produced is verifiable The skills of these agents are given by a stochastic process measurable with respect to s denoted by $\rho_i(s)$. I retain the IID-binary restriction on shocks which imply that skills are perfectly negatively correlated across agents.

We can modify the previous definitions in the obvious way to define feasibility, incentive compatibility and (constrained) efficiency. I proceed to the recursive characterization in the next section

4.2 Efficient Allocations

Extending the previous Bellman equation 9, let $Q(v^0)$ represent the maximum ex-ante discounted value that can be delivered to Agent 1 given a promise of v^0 to Agent 2.

$$Q(v^0) = \max_{c_1(s), c_2(s), y_1(s), y_2(s), v(s)} \mathbb{T}_\theta \left\{ (1-\delta) u \left(c_1(s), \frac{y_1(s)}{\rho_1(s)} \right) + \delta Q(v(s)) \right\} \tag{16}$$

subject to

$$\mathbb{T}_{\theta}\{(1-\delta)u\left(c_2(s),\frac{y_2(s)}{\rho_2(s)}\right)+\delta v(s')\}\geq v^0 \tag{17a}$$

$$c_1(s)+c_2(s)=y_1(s)+y_2(s) \tag{17b}$$

$$(1-\delta)u(c_1(s),\frac{y_1(s)}{\rho_1(s)})+\delta Q(v(s))\geq(1-\delta)u(c_1(s'),\frac{y_1(s')}{\rho_1(s)})+\delta Q(v(s'))\quad\forall s,s' \tag{17c}$$

$$(1-\delta)u(c_2(s),\frac{y_2(s)}{\rho_2(s)})+\delta v(s)\geq(1-\delta)u(c_2(s'),\frac{y_2(s')}{\rho_2(s)})+\delta v(s')\quad\forall s,s' \tag{17d}$$

The allocation achieves two goals in this setting - optimal consumption risk sharing and efficient labor allocation. Given a multiplier on the promise keeping constraint λ, the **first best** allocations are characterized by the following

$$\begin{aligned}
&\tilde{P}^1(s)u_c^1(s)=\lambda\tilde{P}^2(s)u_c^2(s) && \text{intra-temporal risk sharing}\\
&-\frac{u_l^1(s)}{u_c^1(s)\rho_1(s)}=\frac{u_l^2(s)}{u_c^2(s)\rho_2(s)}=1 && \text{production efficiency}\\
&\tilde{P}^1(s)\lambda^*(s)=\lambda\tilde{P}^2(s) && \text{inter-temporal risk sharing}\\
&c_1(s)+c_2(s)=y_1(s)+y_2(s) && \text{resource constraint}
\end{aligned}$$

For a given λ, subject to the resource constraints, the efficient allocation is characterized by a zero labor wedge and split of the output across the two agents that makes equalizes probability weighted marginal utilities of consumption today and in future . Production efficiency requires output to be relatively higher for agents that are more productive and compensating transfers to insure lower income in unproductive states. [10]

With incentive constraints the optimal allocation is characterized with two state dependent wedges for each agent: a *labor wedge* and *savings wedge* that summarize the differences from the first best.

[10] When agent have complete trust in their models, inter-temporal efficient risk allocation will imply that λ is constant. With doubts, since aggregate output is not constant presence of this small non diversifiable risk will induce fluctuations beliefs if the agents have unequal wealth. This channel is was completely shut down in section 3.5 with no aggregate risk.

$$\tilde{P}^1(s)u_c^1(s) = \lambda\tilde{P}^2(s)u_c^2(s)\left(\underbrace{\frac{1+\mu^2(s)-\frac{\tilde{P}^2(s')}{\tilde{P}^2(s)}\mu^2(s')}{1+\mu^1(s)-\frac{\tilde{P}^1(s')}{\tilde{P}^1(s)}\mu^1(s')}}_{\text{Relative savings wedge}}\right) \tag{18}$$

$$-\frac{u_l^i(s)}{u_c^i(s)\rho_i(s)} = \underbrace{\frac{1+\mu^i(s)-\frac{\tilde{P}^i(s')}{\tilde{P}^i(s)}\mu^i(s')}{1+\mu^i(s)-\frac{\tilde{P}^i(s')}{\tilde{P}^i(s)}\mu^i(s')\left[\frac{\rho_i(s)}{\rho_i(s')}\right]^{1+\gamma}}}_{\text{Agent i's labor wedge}} \tag{19}$$

$$\tilde{P}^1(s)\lambda^*(s) = \lambda\tilde{P}^2(s)\left(\underbrace{\frac{1+\mu^2(s)-\frac{\tilde{P}^2(s')}{\tilde{P}^2(s)}\mu^2(s')}{1+\mu^1(s)-\frac{\tilde{P}^1(s')}{\tilde{P}^1(s)}\mu^1(s')}}_{\text{Relative savings wedge}}\right) \tag{20}$$

$$c_1(s)+c_2(s) = y_1(s)+y_2(s) \tag{21}$$

$$\mu^i(s) \geq 0 \tag{22}$$

$$\mu^i(s)IC^i(s) = 0 \tag{23}$$

$$\tag{24}$$

As in the exchange economy, the incentive constraints are slack in states when the agents have low labor income

Lemma 1. $\mu^i(s) = 0$ *when* $\rho_i(s) < \rho_i(s')$

Proof. The proof is a standard perturbation argument. WLOG, if $\mu^1(s_l) > 0$, we have $IC^1(s_l) = 0$ and $IC^1(s_h) \geq 0$ for any feasible perturbation $\Delta c_1(s_l), \Delta y_1(s_l)$, we have

$$u_c[c_1(s_l)]\Delta c_1(s_l) + \frac{ul[\frac{y_1(s_l)}{\rho_1(s_l)}]}{\rho_1(s_l)}\Delta y_1(s_l) \geq u_c[c_1(s_l)]\Delta c_1(s_l) + \frac{ul[\frac{y_1(s_l)}{\rho_1(s_h)}]}{\rho_1(s_h)}\Delta y_1(s_l)$$

Thus we can find a perturbation that increases the utility of Agent 1 in state $s = 1$ and leave all other constraints satisfied. □

Before exploring how the wedges move over time, we note a few properties.

Lemma 2. *1. Positive labor wedge when productivity is low*

$$\tau_i^l(\rho_l) > 0$$

2. Zero labor wedge when productivity is high

$$\tau_i^l(\rho_h) = 0$$

3. *Positive savings wedge when productivity is low*

$$\tau_i^s(\rho_l) > 0$$

4. *Negative savings wedge when productivity is high*

$$\tau_i^s(\rho_h) < 0$$

5. *Zero mean property on savings wedges*

$$\tilde{\mathbb{E}}^i \tau_i^s = 0$$

6. *Monotonic relationship between savings and labor wedge*

$$\tau_i^l(\rho_l) = \frac{\tau_i^s(\rho_l)(\bar{e}-1)}{1+\tau_i^s(\rho_l)\bar{e}}$$

Where $\tau_i^l(\rho_l), \tau_i^s(s_l)$ *are the respective wedges implied by the optimal allocation and* $\bar{e} = \left(\frac{\rho_l}{\rho_h}\right)^{1+\gamma}$

Typically the incentive constraint when a particular agent is productive. The Planner ideally wants to insure the agent's bad draw by giving him higher consumption (and leisure) but is restricted by the possibility that this insurance would make him misreport his productivity in good times. This introduces a positive labor wedge and a negative savings wedge. These wedges are related to each other, in particular larger the savings wedge (in absolute value), larger is the labor wedge. These savings wedges differ in sign and are mean zero from the perspective of the distorted models of each agent.

Figures 22 depict how these wedges (for Agent 2) vary with λ, his implied Pareto weight. The red (black) line plots reflects the wedge in the when agents trust (do not trust) their reference models. At low λ, the incentive constraints are slack and the wedges are zero in both cases. However, the spread between the wedges increases as λ becomes large. Thus we have two general properties, firstly wedges increase λ and concerns for misspecification make them larger in magnitude.

The cost of incentives is largest when the Planner has a high Pareto weight on a particular agent, since his desired consumption is high. As the wedges reflect how restrictive the incentive constraints are , the magnitude of these wedges increases with the relative Pareto weight. A simple corollary of the above is that a utilitarian planner who cares about both the agents equally generates lowest average wedge.

Limited insurance implies that agents have low values in states when they are unproductive. The concerns for misspecification make them twist the reference model in a way that they over-estimate these states. Since in this setup, agents are symmetric, they distort in opposing directions generating heterogeneous beliefs from the lens of their worst case models. Concavity of the utility function imparts lower distortion when agents have high average consumption (or leisure). Thus relative pessimism (or the probability either agent assigns to the state he is unproductive, relative the other agent) is declining in Pareto

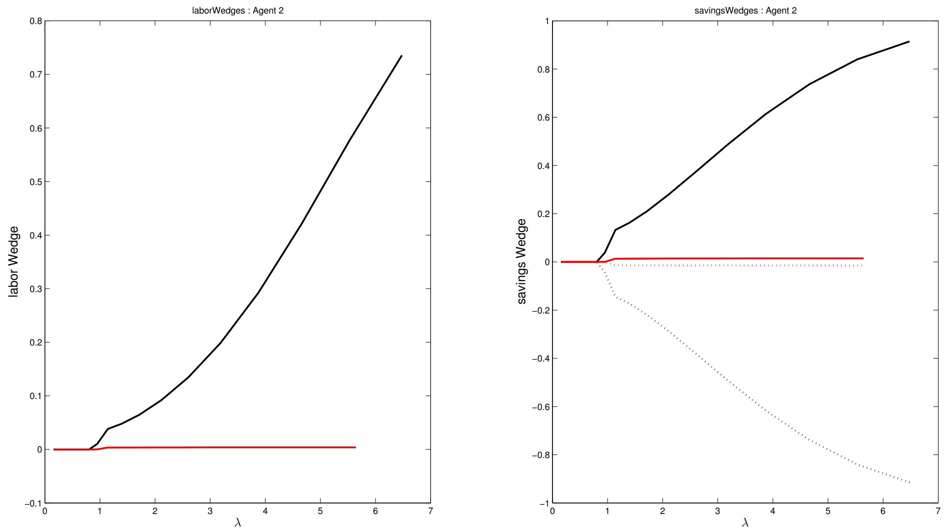


Figure 22: *The plot depicts the labor (left)and savings (right) wedges for Agent 2 as a function of λ. The red (black) line is case without (with) concerns for model uncertainty*

weights. This heterogeneity in beliefs, requires a tilt to the desired optimal insurance. However implementing this extra tilt to insurance is restricted by the same incentive constraints and thus misspecification concerns exacerbates the cost of incentives. The resulting distortions are higher.

The next issue is what happens to this economy in the long run. These large wedges at extreme values of λ hint at what would happen in this economy if we ever approached situations where inequality was high. Similar to the exchange economy we have a force towards drifts towards the center. The FOCs with respect to $v(s)$ help us elaborate this. The state variable ex-ante promised value - v is linked to λ by an Envelope condition. Equations 19 and 21 imply that λ is also the ratio of (past) marginal utilities and has the following law of motion

$$\lambda(s^t) = \lambda(s^{t-1}) \underbrace{\frac{\tilde{P}^2(s^t)}{\tilde{P}^1(s^t)}}_{\text{Heterogeneous Beliefs}} \underbrace{\frac{1-\tau_s^2(s^t)}{1-\tau_1^s(s^t)}}_{\text{Incentives}} \tag{25}$$

From an initial condition that yields $\lambda_0 = 1$, figure ?? shows how inequality spreads in the case where agents have no concerns for model uncertainty.

Asymptotically the long run wedges are active on on both the agents , are similar in magnitudes and display low volatility. In a way with time the Planner becomes utilitarian with about equal weights on both the agents. In the benchmark, as inequality increases, the wedges are active on a subset of agents (those with high Pareto weights) and display large volatility since the cost of incentive is high with higher inequality.

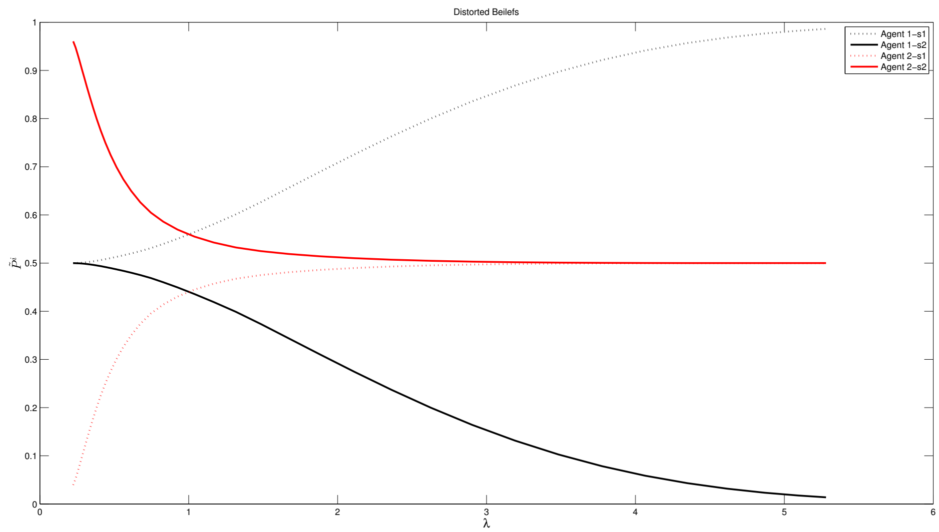
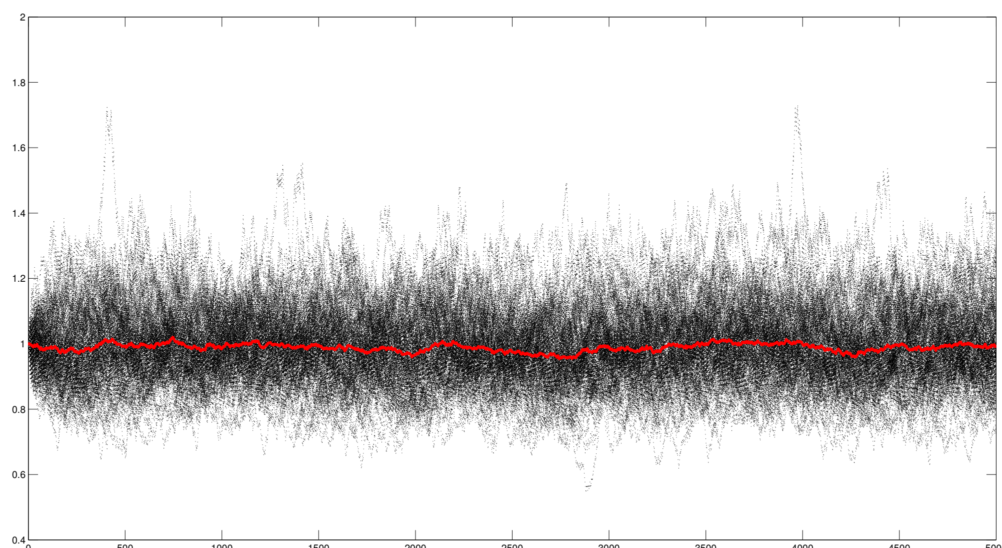
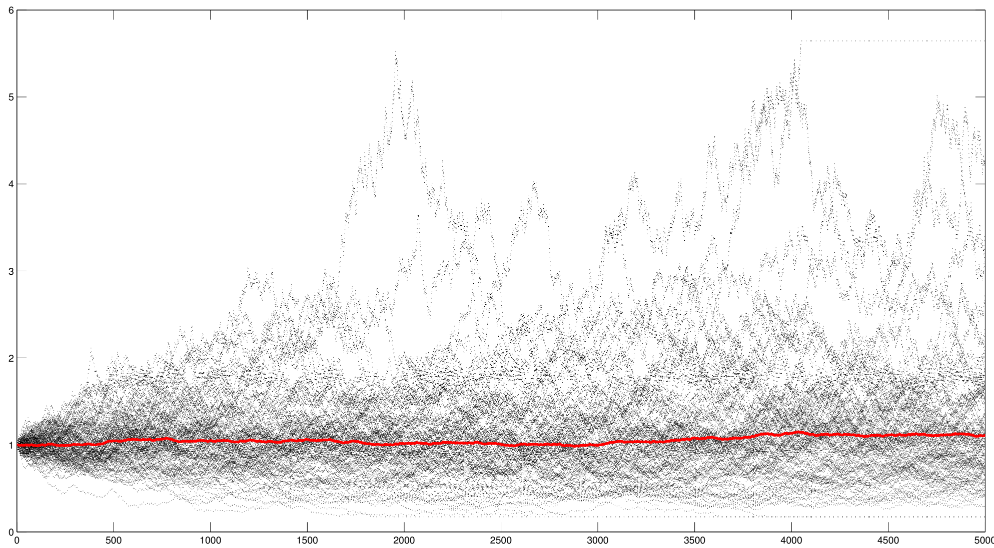


Figure 23: *The plot depicts the distorted beliefs as a function of λ. With low (implied) Pareto weights, Agent 2 puts relatively large weight on state $s = 2$, when he is unproductive*





5 Proofs

Proposition 2

Proof. We first derive some properties of how distortions to priors depend on wealth

Lemma 3. *Suppose consumption was given by $c(y,b) = y + b$ and z solves*

$$V^R(b) = \min_{z, Ez=1} \mathbb{E}z[u(c) + \theta \log(z)]$$

For every b there exists a threshold $\bar{y}(b)$ such that $\frac{\partial m(y,b)}{\partial b} > 0$ iff $y > \bar{y}(b)$

Proof. The choice for z^*

$$z^*(y,b) \propto \exp\left\{\frac{-(y+b)^{1-\gamma}}{\theta(1-\gamma)}\right\}$$

taking logs and differentiating with respect to b we have

$$\frac{\partial \log z^*(y,b)}{\partial b} = -\frac{(y+b)^{-\gamma}}{\theta} + \frac{\mathbb{E}\exp\left\{-\frac{(y+b)^{1-\gamma}}{(1-\gamma)\theta}\right\}(y+b)^{-\gamma}}{\theta_1 \mathbb{E}\exp\left\{-\frac{(y+b)^{1-\gamma}}{(1-\gamma)\theta}\right\}}$$

Define $\tilde{p}(y) = p(y)\frac{\exp\left\{-\frac{(y+b)^{1-\gamma}}{(1-\gamma)\theta}\right\}}{\mathbb{E}\exp\left\{-\frac{(y+b)^{1-\gamma}}{(1-\gamma)\theta}\right\}}$ we have

$$\frac{\partial d \log z^*(y,b)}{\partial b} = -\frac{(y+b)^{-\gamma} - \tilde{\mathbb{E}}(y+b)^{-\gamma}}{\theta}$$

Let $\bar{y}(b)$ be such that the numerator is zero

$$\bar{y}(b) = \left(\tilde{\mathbb{E}}(y+b)^{-\gamma}\right)^{-\frac{1}{\gamma}} - b$$

Since $y + b \geq 0$ as $y > \bar{y}(b)$ we have

$$\frac{\partial d \log z^*(y,b)}{\partial b} > 0$$

$\square$

Lemma 4. *There exists a* $\underline{B}^1_{-1,0}[s,\pi]$ *such that* $\lim_{b \to \underline{B}^1_{-1,0}} \mathcal{B}^{1,0}(b, \underline{B}^1_{-1,0}, s, \pi, q) = -\frac{ys_l}{1-\delta}$. *Further we also have*

$$\lim_{b \to \underline{B}^1_{-1,0}} \bar{y}[\mathcal{B}(b, \underline{B}^1_{-1,0}, s, \pi, q)] = ys_l$$

As $\mathcal{B}^{1,0}$ approaches $-\frac{ys_l}{1-\delta}$, marginal utility of consumption of Agent 1 in $s^* = s_l$ diverges to ∞. For an interior solution, the FOC would require his current consumption to go to zero as well. This means that $\underline{\mathrm{b}}_{-1,0}[s,\pi]$ will satisfy

$$\underline{\mathrm{b}}_{0,1} \approx q \frac{ys_l}{1-\delta} - ys$$

and from Agent 2's FOC along with market clearing we have that q is

$$q \approx \delta \frac{\tilde{\mathbb{E}} Q^{2*}_b(\frac{ys_l}{1-\delta}, -\frac{ys_l}{1-\delta}, s^*, \pi^*)}{u_c(y)}$$

This suggests $\underline{\mathrm{b}}_{-1,0}[s,\pi] = \delta \left(\frac{\tilde{\mathbb{E}} Q^{2*}_b(\frac{ys_l}{1-\delta}, -\frac{ys_l}{1-\delta}, s^*, \pi^*)}{u_c(y)} \right) \left(\frac{ys_l}{1-\delta}\right) - ys$

Following steps in lemma 3, the threshold for Agent 1's income to ensure that relative optimism rises with assets satisfies

$$\bar{y}[b^*] = \left(\check{\mathbb{E}}^1_s[ys^* + b^*(1-\delta)]^{-\gamma}\right)^{-\frac{1}{\gamma}} - (1-\delta)b^*$$

Note that the likelihood ratio $m(b^*, s^*) = \frac{\sum_m \tilde{\pi}^1(m) \tilde{P}^1_S(s^*|s,m)}{\sum_m \pi(m) P_Z(s^*|s,m)}$ The numerator can be simplified to

$$\left(\frac{\exp\{\frac{-u[ys^* + b^*(1-\delta)]}{\theta_1}\}}{\sum_m \exp\{\frac{-\delta \mathbb{T}^1_{\theta_1,m}[u(ys^* + b^*(1-\delta))]}{\theta_2}\}} \right) \sum_m \pi(m) P_S(s^*|s,m) F^1(m)$$

and $F^1(m) = \exp\left\{ \left(\frac{\theta_2 - \delta\theta_1}{\theta_1\theta_2}\right) \mathbb{T}^1_{\theta_1,m} \left[u(ys^* + b^*(1-\delta))\right] \right\}$

The derivative $\frac{\partial \log[z(b^*,s^*)]}{\partial b^*}$ is given by

$$-\left(\frac{1-\delta}{\theta_1}\right)[ys^* + b^*(1-\delta)]^{-\gamma} + (1-\delta)\sum_m \tilde{E}^1_{m,s}[ys^* + b^*(1-\delta)]^{-\gamma}\left(\frac{\delta}{\theta_2}\tilde{\pi}^1(m) + \left(-\frac{\delta}{\theta_2} + \frac{1}{\theta_1}\right)\hat{\pi}^*(m)\right)$$

where $\hat{\pi}^*(m) \propto \pi(m) P_S(s^*|s,m) F^1(m)$

Multiplying by $\frac{\theta_1}{1-\delta}$, we can define $\breve{\pi}^*$ as $\frac{\delta\theta_1}{\theta_2}\tilde{\pi}^1 + \left(1 - \frac{\delta\theta_1}{\theta_2}\right)\hat{\pi}^*$.

Now $\frac{\partial \log[z(b^*,s^*)]}{\partial b^*} \geq 0$ if and only if $ys_h \geq \bar{y}[b^*] \geq ys_l$

$$\bar{y}[b^*] = \left(\breve{\mathbb{E}}^1_s[ys^* + b^*(1-\delta)]^{-\gamma}\right)^{-\frac{1}{\gamma}} - (1-\delta)b^*$$

Now as $b \to \underline{\mathrm{B}}^1_{0,-1}[s,\pi]$ and $b^* = \mathcal{B}^{1,0}[b,s,\pi] \to -\frac{ys_l}{1-\delta}$

$$\bar{y}\left(\frac{-ys_l}{1-\delta}, s^*\right) = ys_l$$

As long as we have $ys_l < \bar{y}[\mathcal{B}(B^1, s, \pi, q)] < ys_h$ implies a negative association of assets levels and pessimism. □

Proposition 3

Proof. The FOC imply $c_1(s_l) = c(\lambda)$ such that

$$\frac{\tilde{P}^1(s_l)u_c[c] - \tilde{P}^2(s_l)\lambda u_c[y-c)]}{(1-\tilde{P}^1(s_l))u_c[c+\Delta] - \lambda(1-\tilde{P}^2(s_l))u_c[y-c-\Delta]} = -1 \tag{26}$$

Further $c_1^\infty(s_l) = c^\infty(\lambda)$ solves

$$\frac{u_c[c] - \lambda u_c[y-c]}{u_c[c+\Delta] - \lambda u_c[y-c-\Delta]} = -\frac{1-P(s_l)}{P(s_l)} \tag{27}$$

Now $\bar{\lambda}$ is such that $c_1^\infty(\bar{\lambda}) = c_1(\bar{\lambda})$

Using 26 and 27 $\bar{\lambda}$ will satisfy

$$\frac{\tilde{P}^1(s_l)u_c[c_1^\infty(\bar{\lambda})] - \tilde{P}^2(s_l)\lambda u_c[y - c_1^\infty(\bar{\lambda})]}{(1-\tilde{P}^1(s_l))u_c[c_1^\infty(\bar{\lambda})+\Delta] - \bar{\lambda}(1-\tilde{P}^2(s_l))u_c[y-c_1^\infty(\bar{\lambda})-\Delta]} = -1 \tag{28}$$

Let

$$F^1(c) = u_c(c) - u_c(c+\Delta) \tag{29}$$
$$F^2(c) = u_c(y-c-\Delta) - u_c(y-c) \tag{30}$$

This implies

$$(\tilde{P}^1(s_l) - P(s_l))F^1[c_1^\infty(\bar{\lambda})] + \bar{\lambda}(\tilde{P}^2(s_l) - P(s_l))F^2(c_1^\infty(\bar{\lambda})) = 0 \tag{31}$$

Define $\chi(\lambda)$ as the residual for the equation 31

$$(\tilde{P}^1(s_l) - P(s_l))F^1[c_1^\infty(\lambda)] + \lambda(\tilde{P}^2(s_l) - P(s_l))F^2[c_1^\infty(\lambda)] = \chi(\lambda) \tag{32}$$

Note that,

$$\lim_{c\to 0} F^1(c) = \lim_{c\to 0}\left[\frac{u_c(c) - u_c(c+\Delta)}{c}c\right] = -u_{c,c}[\Delta][\lim_{c\to 0} c] = 0$$

$$\lim_{c \to y-\Delta} F^1(c) = [u_c(y-\Delta) - u_c(y)] \approx -\Delta u_{c,c}[y] > 0$$

$$\lim_{c \to y-\Delta} F^2(c) = 0$$

Since $\frac{\partial c_1^\infty(\lambda)}{\partial \lambda} < 0$, we have

$$\text{sgn}\chi(\infty) = \text{sgn}(\tilde{P}^2(s_l) - P(s_l))$$
$$\text{sgn}\chi(0) = \text{sgn}(\tilde{P}^1(s_l) - P(s_l))$$

Since $\tilde{P}^1(s_l) > P(s_l) > \tilde{P}^2(s_l)$ and the opposite for $s = s_h$, there exists a $\bar{\lambda}$ such that

$$\chi(\bar{\lambda}) = 0$$

Define $c^M = \frac{y-\Delta}{2}$. We have

$$c^M = y - c^M - \Delta$$

$c = c^M$ and $P(s_l) = \frac{1}{2}$ imply that

1. $\bar{c}(1) = c^M$
2. $F^1(c^M) = F^2(c^M)$
3. $\tilde{P}^1(s_l) + \tilde{P}^2(s_l) = 1$

From eq 31 we have that $\bar{\lambda}$ is pinned down by

$$(\tilde{P}^1(s_l) - P(s_l))F^1[\bar{c}(\bar{\lambda})] + \bar{\lambda}(\tilde{P}^2(s_l) - P(s_l))F^2(\bar{c}(\bar{\lambda})) = 0$$

$$F(c^M)(\tilde{P}^1(s_l) + \tilde{P}^2(s_l) - 1) = 0$$

$$\tilde{P}^1(s_l) = \frac{P(s_l)\exp\left\{-\frac{u[c]}{\theta}\right\}}{P(s_l)\exp\left\{-\frac{u[c]}{\theta}\right\} + (1-P(s_l))\exp\left\{-\frac{u[y-c]}{\theta}\right\}}$$

$$\tilde{P}^2(s_l) = \frac{P(s_l)\exp\left\{-\frac{u[y-c]}{\theta}\right\}}{P(s_l)\exp\left\{-\frac{u[y-c]}{\theta}\right\} + (1-P(s_l))\exp\left\{-\frac{u[y-c-\Delta]}{\theta}\right\}}$$

At $c = c^M$ and $P(s_l) = \frac{1}{2}$ we have

$$\tilde{P}^1(s_l) + \tilde{P}^2(s_l) = \frac{\exp\left\{-\frac{u[c^M]}{\theta}\right\} + \exp\left\{-\frac{u[y-c^M]}{\theta}\right\}}{\exp\left\{-\frac{u[y-c^M]}{\theta}\right\} + \exp\left\{-\frac{u[c^M]}{\theta}\right\}} \tag{33}$$

or

$$\tilde{P}^1(s_l) + \tilde{P}^2(s_l) = 1$$

□

Proposition 4

Proof.

Lemma 5. *Let $\tilde{\Delta}_c = c(s_h) - c(s_l)$ and $\tilde{\Delta}_v = v(s_l) - v(s_h)$the optimal contracts has*

$$\tilde{\Delta} \leq \Delta \& \tilde{\Delta}_v \geq 0$$

Proof. Adding $IC^1_{s_l}$ and $IC^1_{s_h}$ we have

$$u[c_1(s_h)] + u[c_1(s_l)] \geq u[c_1(s_l) + \Delta] + u[c_1(s_h) - \Delta]$$

Suppose $\tilde{\Delta}_c > \Delta$, the spread between the menu $\{c_1(s_l), c_1(s_h)\}$ on the LHS is greater than $\{c_1(s_l) + \Delta, c_1(s_h) - \Delta\}$. Since u is concave we have a contradiction.

Next suppose $\bar{v}^*(s_l) < \bar{v}^*(s_h)$, the incentive constraint of agent 2 in state s_l implies that

$$u[c_2(s_h) + \Delta] - u[c_2(s_l)] \leq \delta[\tilde{\Delta}_v] < 0$$

Using the resource constraint and monotonicity of u, this yields the contradiction

$$\tilde{\Delta}_c > \Delta$$

□

Lemma 6. *$\nexists v^0$ such that $\mu_i(, v^0, s) > 0$ for all i, s.*

Proof. Ignoring the non-negativity constraints, the first order conditions for problem 9 are as follows

$$\tilde{P}^1(s)u^1_c[c_1(s)] = \lambda \tilde{P}^2(s) u^2_c[c_2(s)] \left(\underbrace{\frac{1 + \mu^2(s) - \frac{\tilde{P}^2(s')}{\tilde{P}^2(s)} \frac{u_c[c_2(s) + \Delta_2(s,s')]}{u_c[c_2(s)]} \mu^2(s')}{1 + \mu^1(s) - \frac{\tilde{P}^1(s')}{\tilde{P}^1(s)} \frac{u_c[c_1(s) + \Delta_1(s,s')]}{u_c[c_1(s)]} \mu^1(s')}}_{\text{static wedge}} \right)$$

$$\tilde{P}^1(s)\lambda^*(s) = \lambda \tilde{P}^2(s) \left(\underbrace{\frac{1 + \mu^2(s) - \frac{\tilde{P}^2(s')}{\tilde{P}^2(s)} \mu^2(s')}{1 + \mu^1(s) - \frac{\tilde{P}^1(s')}{\tilde{P}^1(s)} \mu^1(s')}}_{\text{dynamic wedge}} \right)$$

$$c_1(s) + c_2(s) = \bar{y}$$
$$\mu^i(s) \geq 0$$
$$\mu^i(s) IC^i(s) = 0$$

Suppose there exist. Since all incentive constraints bind, we have

$$u\left(c_1(s_l)\right) - u\left(c_1(s_l) + \Delta\right) = u\left(c_1(s_h) - \Delta\right) - u\left(c_1(s_h)\right) \tag{34}$$
$$u\left(\bar{y} - c_1(s_l)\right) - u\left(\bar{y} - c_1(s_l) + \Delta\right) = u\left(\bar{y} - c_1(s_h) - \Delta\right) - u\left(\bar{y} c_1(s_h)\right) \tag{35}$$

These are two equations in two unknowns. I will argue that all roots to the above system lie on the subspace $\mathcal{N}(x, y : x - y = \Delta)$. For a given $c_1(s_l)$, the LHS of both the equations is fixed. Since u is concave , the RHS is a (strictly) monotonic function of $c_1(s_h)$. Consider equation **??** for instance, $u_c(c_1(s_h) - \Delta) > u_c(c_1(s_h))$ for all $c_1(s_h)$ as long as $\Delta > 0$. Thus for a given $c_1(s_l)$ there exists a unique $c_1(s_h) = c_1(s_l) + \Delta$ that solves the equation. Going back to the IC, we have $v(s_l) = v(s_h)$. Thus the only feasible contract is the static contract. This provides a converse the observation in Proposition 3 which stated all constraints bind in the static contract with $\delta = 0$.

Now the FOC provide the connection between the static wedge and the dynamic wedge under the optimal contract. First consider $s = s_l$

$$\frac{\tilde{P}^1(s_l)}{\tilde{P}^2(s_l)} \left(\frac{u_c(c_1(s_l)))}{\lambda u_c(y - c_1(s_l)))} \right) \left(\frac{1 + \mu^2(s_l) - \frac{\tilde{P}^2(s_h)}{\tilde{P}^2(s_l)} \frac{u_c[c_2(s_l) + \Delta_2(s_l, s_h)]}{u_c[c_2(s_l)]} \mu^2(s_h)}{1 + \mu^1(s_l) - \frac{\tilde{P}^1(s_h)}{\tilde{P}^1(s_l)} \frac{u_c[c_1(s_l) + \Delta_1(s_l, s_h)]}{u_c[c_1(s_l)]} \mu^1(s_h)} \right) < \left(\frac{1 + \mu^2(s_l) - \frac{\tilde{P}^2(s_h)}{\tilde{P}^2(s_h)} \mu^2(s_h)}{1 + \mu^1(s_l) - \frac{\tilde{P}^1(s_h)}{\tilde{P}^1(s_l)} \mu^1(s_h)} \right) =$$

or

$$\lambda > \left(\frac{u_c(c_1(s_l)))}{u_c(y - c_1(s_l)))} \right)$$

Now at the static contract, $c_1(s_h) = c_1(s_l) + \Delta$, further using concavity of u

we have

$$\lambda > \left(\frac{u_c(c_1(s_l)))}{u_c(y - c_1(s_l)))} \right) > \left(\frac{u_c(c_1(s_l) + \Delta))}{\lambda u_c(y - c_1(s_l) - \Delta))} \right)$$

With similar steps, we can order the wedges for $s = s_h$ and obtain

$$\lambda < \left(\frac{u_c(c_1(s_h)))}{u_c(c_2(s_h))} \right)$$

This yields the contradiction.

□

Lemma 7. *We need to only consider $\mu^1(s) > 0, \mu^2(s') > 0$ for $s \neq s'$*

Proof. Since the previous lemma 6 that the ICs for any agent cannot bind in both states holds for all v^0. We can rule out the cases where there is switching i.e from $\mu^i(s) > 0$ to $\mu_i(s') > 0$ as we change v^0. If there was then for some v^0, the IC constraints would bind for both states contradicting the previous lemma. Further $\mu_1(s) > 0, \mu_1(s) > 0$ for some s is also not possible. The agents are symmetric and reverse the role with respect to shocks as we go from a low v^0 to high v^0. If there was a s such that ICs for both agents were binding, then there would exist a λ such that it would bind for the other shock too. Thus again contradicting the previous lemma. The only possibilities then remain are $\mu^1(s) > 0, \mu^2(s') > 0$ for $s \neq s'$ □

Suppose this was not the case then by virtue of lemma 7 , they would bind in the states both the agents have high income and be slack in the other periods. This implies that

$$\frac{1 - \mu^2(s_h) \frac{\tilde{P}^2(s_h)}{\tilde{P}(s_l)}}{1 + \mu^1(s_l)} < 1$$

and similarly

$$\frac{1+\mu^2(s_h)}{1-\mu^1(s_l)\frac{\tilde{P}^1(s_l)}{\tilde{P}^1(s_h)}} > 1$$

This implies that

$$\frac{\tilde{P}^1(s_l)\lambda[s_l]}{\tilde{P}^2(s_l)\lambda[s_h]} > \frac{\tilde{P}^1(s_h)}{\tilde{P}^2(s_h)}$$

However , from lemma 5 we know that

$$v[s_l] \geq v[s_h]$$

Further pessimistic twisting implies that

$$\frac{\tilde{P}^1(s_l)}{\tilde{P}^2(s_l)} \geq 1 \geq \frac{\tilde{P}^1(s_h)}{\tilde{P}^2(s_h)}$$

Thus we have a contradiction.

□

6 Numerical algorithm and diagnostics

6.1 Bond Economy

1. Define a grid on b, π
2. Guess some prices $q_s^k(b,\pi)$ and a non-negative function for the consumption policy rule $\mathcal{C}_s^{k(j)}(b,\pi)[i]$ where $s = s_l, s_h$
3. Obtain the policy for savings using the budget constraint

$$\mathcal{B}_s^{k(j)}(b,\pi)[1] = \frac{b + ys - \mathcal{C}_s^{k(j)}[b,\pi][1]}{q_s^k(b,\pi)}$$

$$\mathcal{B}_s^{k(j)}(b,\pi)[2] = \frac{-b + ys - \mathcal{C}_s^{k(j)}[b,\pi][2]}{q_s^k(b,\pi)}$$

4. Get an approximation for $Q^j[i]$ by iterating on

$$Q_s^{k(j+1)}(b,\pi)[i] = \mathbb{T}^2_{\theta,2}\left[u(\mathcal{C}_s^{k(j)}[b,\pi][i]) + \delta \mathbb{T}^1_{\theta,m} Q_{s^*}^{k(j)}(\mathcal{B}_s^{k(j)}(b,\pi),\pi^*)[i]\right]$$

5. Now use the FOC to update $\mathcal{C}^{k(j+1)}[i]$

$$\mathcal{C}_s^{k(j+1)}[b,\pi][1] = \tilde{\mathbb{E}}_s\left(\frac{\delta\mathcal{C}^{k(j)}(\mathcal{B}_s^{k(j)},\pi^*)^{-\gamma}[1]}{q_s^k(b,\pi)}\right)^{-\frac{1}{\gamma}}$$

$$\mathcal{C}_s^{k(j+1)}[b,\pi][2] = \tilde{\mathbb{E}}_s \left(\frac{\delta \mathcal{C}^{k(j)}(\mathcal{B}_s^{k(j)},\pi^*)^{-\gamma}[2]}{q_s^k(b,\pi)} \right)^{-\frac{1}{\gamma}}$$

Where $\tilde{\mathbb{E}}$ is computed using $\tilde{\pi}[i]$,$\tilde{P}[i]$ using $Q^j[i]$

6. Now update q^{k+1}

$$q_s^{k+1}(b,\pi) = q_s^k(b,\pi) + \upsilon \left[\mathcal{B}_s^k(b,\pi)[1] + \mathcal{B}_s^k(b,\pi)[2]\right]$$

6.2 Dynamic Private Information

[TBA]