Reverse Engineering Heterogeneous Beliefs

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March 22, 2016

Abstract

In a setting with heterogeneous agents and standard expected utility preferences, this paper reverse engineers a cross-section of beliefs that is consistent with an arbitrary stochastic discount factor process. A key feature of the construction is that no agent makes systematic errors and there is substantial trade across agents induced by belief differences. Applying our methods to the asset pricing model of Bansal and Yaron (2004), we find that the dispersion of reverse engineered beliefs tracks measures of cross-sectional forecast variance compugmted using panel data on professional forecasters. A lower bound on belief dispersion, which can be directly computed from data on returns and aggregate consumption, is also provided.

Key words: stochastic discount process, complete markets, reverse engineering

1 Two reverse engineering exercises

Let $\{M_t^* > 0\}_t$ be a stochastic discount factor process satisfying $\lim_{t\to+\infty} \mathbb{E}M_t^* = 0$. Let $R_{t,t+1}$ be a gross return on an asset. An empirically successful stochastic discount factor process satisfies

$$\mathbb{E}_t \left(\frac{M_{t+1}^*}{M_t^*} \right) R_{t,t+1} = 1 \tag{1}$$

for every observed one-period gross return $R_{t,t+1}$. Constantinides and Duffie (1996) started with a discount factor process satisfying (1) for an arbitrary list of observed returns and

*I thank Tom Sargent for discussions and encouraging me to pursue this idea.

then reverse engineered an economy with a competitive equilbrium implying that $\{M_t^*\}$ process. Their economy featured: (i) incomplete markets, (ii) special patterns of heterogeneity in stochastic processes of individual's equilibrium income processes, and (iii) rational expectations (common beliefs). In Constantinides and Duffie's economy, agent i's personal stochastic discount factor process $\{M_t^i\}$ satisfies

$$\mathbb{E}_t \left(\frac{M_{t+1}^i}{M_t^i} \right) R_{t,t+1} = 1 \tag{2}$$

for each traded asset. Constantinides and Duffie construct a a no-trade competitive equilibrium that generates a cross section of $\{M_t^i\}$ processes that satisfy both (1) for all $t \geq 0$ and (2) for all $t \geq 0$ and for all i. A key ingredient of their construction is the cross-section variance of idiosyncratic income growth (or equivalently consumption growth since there is no no-trade in equilibrium) processes that varies systematically with the growth rate of aggregate consumption.

Here we reverse engineer another economy that is also consistent with an arbitrary empirically successful stochastic discount factor. Our economy features (i) complete markets and (ii) heterogeneous beliefs. We construct a cross section of stochastic processes for likelihood ratios representing heterogeneous beliefs that move in ways that verify equation (1) for a given $\{M_t^*\}$ process.

The paper is organized as follows. Section 2 develops the theory and characterizes the belief heterogeneity in terms of a panel of likelihood ratios. Section 3 discusses the properties of these induced beliefs and obtains a lower bound on dispersion in beliefs. Lastly, in section 4 we study an application with the stochastic discount factor in Bansal and Yaron (2004) as the candidate M^*.

2 A heterogeneous belief economy with complete markets

2.1 Setup

Consider an exchange economy with a continuum of agents. Let $s^t = [s_t, s_{t-1}, \dots, s_0]$ be a history of a state variable s_t. Histories s^t are described by a probability distribution $P_t(s^t)$. We assume that the total endowment C_t and the candidate stochastic discount process M_t^* are both measurable with respect to s^t. Agents trade a complete set of Arrow securites.

Agents' subjective beliefs $P_t^i(s^t)$ are absolutely continuous with respect to P_t for all t. Let $z_{i,t} = \frac{P_t^i(s^t)}{P_t(s^t)}$. For each i and every $t \geq 0$, z_{it} satisfies $\mathbb{E} z_{i,t} = 1$ and $z_{i,t} > 0$.

The two fundamental welfare theorems prevail. For equal initial Pareto weights, an optimal allocation solves

$$\max_{\{c_{i,t} \geq 0\}_{i,t}} \int \mathbb{E} \left[\sum_t \beta^t \left(\frac{c_{i,t}^{1-\gamma}}{1-\gamma} \right) z_{i,t} \right] di, \gamma > 1,$$

subject to

$$\int c_{i,t} \leq C_t, \quad t \geq 0. \tag{3}$$

We maintain:

Assumption A: $\left[\Delta \log M_{t+1}^* + \gamma \Delta \log C_{t+1} - \log \beta \right] < 0$ with probability one.

2.2 Equilibrium

Let $\{\lambda_t\}_{t=0}^{\infty}$ be a sequence of Lagrange multiplier on the feasibility constraints (3). First-order conditions with respect to $c_{i,t}$ imply $\beta^t z_{i,t} c_{i,t}^{-\gamma} = \lambda_t$. Integrating across i implies $(\beta^{-t}\lambda_t)^{-\frac{1}{\gamma}} \int z_{i,t}^{\frac{1}{\gamma}} di = C_t$, which in turn implies that consumption profiles satisfy

$$c_{i,t} = C_t \frac{z_{i,t}^{\frac{1}{\gamma}}}{\int z_{i,t}^{\frac{1}{\gamma}} di}. \tag{4}$$

In a competitive equilibrium, agent i's Euler equation

$$\mathbb{E}_t \beta \left(\frac{z_{i,t+1}}{z_{i,t}} \right) \left(\frac{c_{i,t+1}}{c_{i,t}} \right)^{-\gamma} R_{t,t+1} = 1 \tag{5}$$

is satisfied for every asset traded.

2.3 Construction of Beliefs

We want to reverse engineer a set of likelihood ratios $z_{i,t}$ that make equations (5) consistent with an empirically successful discount factor process $\{M_t^*\}$ that satisfies (1). For this, we require

$$\mathbb{E}_t\beta\left(\frac{z_{i,t+1}}{z_{i,t}}\right)\left(\frac{c_{i,t+1}}{c_{i,t}}\right)^{-\gamma}R_{t,t+1}=\mathbb{E}_t\left(\frac{\lambda_{t+1}}{\lambda_t}\right)R_{t,t+1}=\mathbb{E}_t\left(\frac{M^*_{t+1}}{M^*_t}\right)R_{t,t+1}.$$

A sufficient condition for these equations to hold is evidently $\frac{\lambda_{t+1}}{\lambda_t}=\frac{M^*_{t+1}}{M^*_t}$. Using equation (4), this is equivalent with

$$\beta\left(\frac{C_{t+1}}{C_t}\right)^{-\gamma}\left(\frac{\int z_{i,t+1}^{\frac{1}{\gamma}}di}{\int z_{i,t}^{\frac{1}{\gamma}}di}\right)^{\gamma}=\frac{M^*_{t+1}}{M^*_t}. \tag{6}$$

Therefore, to complete our reverse engineering exercise, we must construct a set of likelihood ratio processes $\{z_{i,t}\}$ such that $z_{i,t+1}>0$, $\mathbb{E}_t\frac{z_{i,t+1}}{z_{i,t}}=1$ and

$$\gamma\left[\log\left(\int z_{i,t+1}^{\frac{1}{\gamma}}di\right)-\log\left(\int z_{i,t}^{\frac{1}{\gamma}}di\right)\right]=\Delta\log M^*_{t+1}+\gamma\Delta\log C_{t+1}-\log\beta, \tag{7}$$

which we obtain by taking logs on both sides of equation (6).

To accomplish this, let

$$z_{i,0}=\exp\{n_0\}, \tag{8a}$$

$$\frac{z_{i,t+1}}{z_{i,t}}=\exp\left\{n_{i,t+1}\right\}, \tag{8b}$$

where $n_{i,t}=-\frac{\sigma_{t+1}^2}{2}+\sigma_{t+1}\epsilon_{t+1}^i$ and $\{\epsilon_{t+1}^i\}\sim N(0,1)$ and are i.i.d across agents i and t. Evidently, $z_{i,t}>0$ and

$$\mathbb{E}_t\frac{z_{i,t+1}}{z_{i,t}}=\mathbb{E}_t\left(\mathbb{E}\exp\{n_{i,t+1}\}|\sigma_{t+1}^2\right)=\mathbb{E}_t\exp\left(-\frac{\sigma_{t+1}^2}{2}+\frac{\sigma_{t+1}^2}{2}\right)=1,$$

so that $\frac{z_{i,t+1}}{z_{i,t}}$ and $z_{i,t}$ are indeed both likelihood ratios as required.

Apply a Law of Large Numbers to conclude that $\int z_{i,t}^{\frac{1}{\gamma}}di\to\mathbb{E}_t z_{i,t}^{\frac{1}{\gamma}}$. Expressing $z_{i,t}=\exp\left\{\sum_0^t n_{i,s}\right\}$, compute

$$\mathbb{E}_t z_{i,t}^{\frac{1}{\gamma}}=\mathbb{E}_t\exp\left\{\frac{1}{\gamma}\sum_{s=0}^t n_{i,s}\right\}=\exp\left\{-\sum_{s=0}^t\frac{\sigma_s^2}{2\gamma}+\sum_{s=0}^t\frac{\sigma_s^2}{2\gamma^2}\right\}.$$

Substituting into equation (7) we obtain

$$-\frac{\sigma_{t+1}^2}{2\gamma} + \frac{\sigma_{t+1}^2}{2\gamma^2} = \frac{\Delta \log M_{t+1}^* + \gamma \Delta \log C_{t+1} - \log \beta}{\gamma}.$$

Solving for σ_{t+1}^2 gives

$$\sigma_{t+1}^2 = \frac{2\gamma}{1-\gamma} \left[\Delta \log M_{t+1}^* + \gamma \Delta \log C_{t+1} - \log \beta \right]. \tag{9}$$

For $\gamma > 1$, the left side of (9) is a variance that Assumption A makes non-negative. Equations (8) and (9) completely describe a panel of beliefs that supports the stochastic discount factor process $\{M_t^*\}$ in a competitive equilibrium with complete markets. The cross section distribution is a function of $\{\Delta M_t^*, \Delta C_t\}$.

3 Discussion

A trivial construction that can rationalize any stochastic discount factor, M^* with distorted beliefs would be to assign an identical wedge to all agents that adjusts rational expectation beliefs exactly to obtain M^*. However, this will generate predictable and common bias in beliefs across all agents and furthermore no motives for trade. Both these implications are odds with data. Panel data on return forecasts show that there is considerable heterogeneity beliefs and an extensive literature that tries to rationalize the volume of trade in financial markets. Our construction sidesteps both potential criticisms.

The one-period ahead likelihood ratios summarized by the (8) corresponds to biases that are i.i.d across all agents every period. In fact, ex-ante $\mathbb{E}_t \frac{z_{i,t+1}}{z_{i,t}} = 1$ and thus no agent is systematically biased. The predictable component is captured in how the distribution itself varies with aggregate shocks.

Moreover, the cross-sectional disagreements will make agents trade with each other. To see this assume that all agents have exactly the same endowment $y_{i,t} = Y_t$. In absence of heterogeneity in beliefs, presence of complete markets implies that their consumption is equal to their endowment and hence equal across each other. One measure of volume of trade due to differences in beliefs is the induced cross-sectional dispersion in consumption growth. We now relate it to our constructed stochastic process for σ_t.

From equation (4), the consumption share of agent i,

$$\delta_{it} \equiv \frac{c_{i,t}}{C_t} = \frac{z_{i,t}^{\frac{1}{\gamma}}}{\int z_{it}^{\frac{1}{\gamma}} di}$$

It is easy to see that see that the cross-sectional dispersion in growth rate of shares, $\log \delta_{i,t+1} - \log \delta_{i,t}$ is given by cross-sectional standard deviation of $\frac{1}{\gamma}\eta_{i,t+1} = \frac{\sigma_{t+1}}{\gamma}$. The implication is intuitive, for a given risk-aversion, more dispersion corresponds to more trade. Through equation (9), our theory links it the dynamics of aggregate returns and consumption.

3.1 Lower bound on belief heterogeneity

In this section we provide a lower bound on $\mathbb{E}_t\sigma_{t+1}^2$ that can be directly computed from observables using steps in Bansal and Lehmann (1997). Taking expectations on both sides of equation (9) we obtain

$$\mathbb{E}_t\sigma_{t+1}^2 = \left(\frac{2\gamma}{\gamma-1}\right)\left[-\mathbb{E}_t\Delta\log M_{t+1}^* + \log\beta - \gamma\mathbb{E}_t\Delta\log C_{t+1}\right]. \tag{10}$$

The Euler equation (1) can be written as

$$\frac{M_{t+1}^*}{M_t^*}R_{t,t+1} = 1 + \nu_{t+1},$$

where ν_{t+1} satisfies $\mathbb{E}_t\nu_{t+1} = 0$. Taking logs we have

$$\mathbb{E}_t\Delta\log M_{t+1}^* + \mathbb{E}_t\log R_{t,t+1} = \mathbb{E}_t\log(1+\nu_{t+1}) \leq 0.$$

Equivalently,

$$-\mathbb{E}_t\Delta\log M_{t+1}^* \geq \mathbb{E}_t\log R_{t,t+1}. \tag{11}$$

Since inequality (11) holds for any traded returns we can make it tight by considering the return that maximizes the right-hand side of inequality (11). Let $R^g \in \arg\max_{\tilde{R}\in\mathcal{R}_t}\mathbb{E}_t\log\tilde{R}$,

$$-\mathbb{E}_t\Delta\log M_{t+1}^* \geq \mathbb{E}_t\log R_{t,t+1}^g \tag{12}$$

The return R^g is also referred to as the return on the growth-optimal portfolio as it maximizes an investor's expected growth rate of his or her wealth.[1] Now using the bound (12) in equation (13) we obtain a lower bound on $E_t\sigma^2_{t+1}$:

$$\mathbb{E}_t\sigma^2_{t+1} \geq \left(\frac{2\gamma}{\gamma-1}\right)\left[\mathbb{E}_t \log R^g_{t,t+1} + \log\beta - \gamma\mathbb{E}_t \log\Delta C_{t+1}\right]. \tag{13}$$

Periods with high expected returns or low aggregate consumption growth are typically adverse aggregate states from the perspectives of the investors. Equation (13) requires the lower bound to be tighter in such states. The return R^g can be constructed by constructing a portfolio of observed returns. Roll (1973) and Fama and MacBeth (1974) suggest that a diversified market portfolio (like SP500 for U.S.) can be a good proxy for the growth-optimal portfolio. This makes recovering the lower bound from data operational.

4 Application

Just as Cogley (2002) and Violante et al. (2014) used the Constantinides and Duffie (1996) model to organize their empirical investigations, one can use our equations (8) and (9) to organize an empirical investigation of whether plausible amounts and evolution of belief heterogeneity can explain the observed risk-return tradeoffs captured by a successful discount factor satisfying (1). Here we study an example using a $\{M^*_t\}$ process used by Bansal and Yaron (2004) and Hansen et al. (2008).

The Bansal and Yaron (2004) model combines Epstein and Zin (1989) [EZ] preferences with a stochastic process for the logarithm of aggregate consumption that has a very persistent predictable component. With EZ preferences, the increment to the log of the stochastic discount factor process is

$$\Delta\log M^{EZ}_{t+1} = \log\beta - \rho\Delta\log C_{t+1} + (\rho-\alpha)[\log V_{t+1} - \log\mathcal{R}_t V_{t+1}], \tag{14}$$

where continuation values $\{V_t\}$ satisfy the recursion $V_t = \left[(1-\beta)C_t^{1-\rho} + \beta\mathbb{E}_t\ \left[V_{t+1}^{1-\alpha}\right]^{\frac{1-\rho}{1-\alpha}}\right]^{\frac{1}{1-\rho}}$, where ρ is the inverse of the intertemporal elasticity of substitution and α is a risk-aversion parameter. For this stochastic discount factor process, expression (9) for σ^2_{t+1} implies

$$\sigma^2_{t+1} = \frac{2\gamma}{1-\gamma}\left[(\gamma-\rho)\Delta\log C_{t+1} + (\rho-\alpha)\left(\log V_{t+1} - \log\mathcal{R}_t V_{t+1}\right)\right]. \tag{15}$$

[1] In theory it is also optimal if we assume that the investor has logarithmic preferences

The Bansal and Yaron dynamics for consumption growth are

$$\Delta \log C_{t+1} = \mu_c + g_t + \sigma_c \epsilon_{t+1}$$

$$g_t = A g_{t-1} + B \eta_t,$$

where $\eta_t \sim \mathcal{N}(0,1)$. We set $\rho = 1$ and deduce

$$\log V_{t+1} - \log \mathcal{R}_t V_{t+1} = \sigma_c \epsilon_{t+1} + \left[\frac{\beta B}{1-\beta A}\right] \eta_{t+1} - \left(\frac{1-\alpha}{2}\right) \left[\frac{\beta B}{1-\beta A} + \sigma_c\right]^2 .$$

As the consumption growth persistence parameter A approaches 1 from below, the term $\left[\frac{\beta}{1-\beta A}\right]$ serves to amplify risk prices. The expression for σ^2_{t+1} simplifies to

$$\sigma^2_{t+1} = \frac{2\gamma}{1-\gamma} \left((\gamma-1)(\mu_c + g_t) - 0.5(1-\alpha)^2 \left[\frac{\beta B}{1-\beta A} + \sigma_c\right]^2 + (\gamma-\alpha)\sigma_c \epsilon_{t+1} + (1-\alpha)\frac{\beta B}{1-\beta A}\eta_{t+1} \right) \tag{16}$$

Figure 1 plots the conditional mean $\mathbb{E}_t \sigma^2_{t+1}$ as a function of γ (left panel) and α (right panel) for three values of $g_t = \{-B, 0, B\}$. This is obtained by setting $\epsilon_{t+1} = \eta_{t+1} = 0$ in equation (16). We set $\beta = 0.998, \mu_c = 0.0015, \sigma_c = 0.0078, A = 0.979, B = 0.00034$ as in Bansal and Yaron (2004). We find that $\mathbb{E}_t \sigma^2_{t+1}$ varies inversely with g_t (with a unit elasticity). Next, keeping α fixed higher values of γ require a lower dispersion in likelihood ratios and on the other hand fixing γ and increasing α makes the stochastic discount factor more volatile and consequently requires a higher σ.

The American Association of Individual Investors Investor Sentiment Survey measures the percentage of individual investors who are bullish, neutral, or bearish on the stock market for the next six months. The survey is administered weekly to members of the American Association of Individual Investors. Assuming the investors who report neutral "agree" with each other, we construct a time-series of dispersion in investor expectations by adding up the percentage of bearish and bullish investors between 1987 when the survey first started and December 2015. We next compare the patterns in this empirical measure of dispersion of beliefs with the prediction of our theory, we plot it against $\mathbb{E}_t \sigma^2_{t+1}$.

To extract the values for $\mathbb{E}_t \sigma^2_{t+1}$, we first filter out a process for g_t from data on $\Delta \log C_t$ (which is measured as annualized growth rates in per-capital non durable consumption) and then use equation (16), setting $\epsilon_{t+1} = \eta_{t+1} = 0$. Figure 2 plots the two series[2]. We see

[2]Both series are standardized to have mean zero and standard deviation one

Average belief dispersion

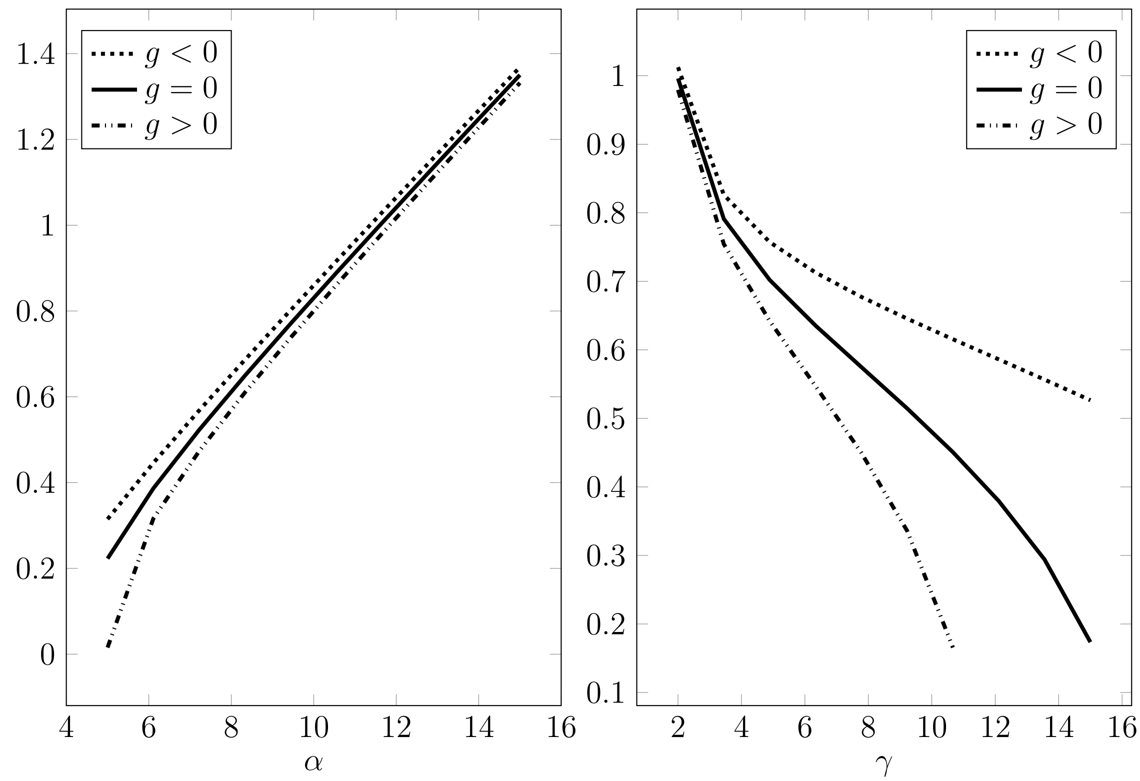

Figure 1: $\sqrt{\mathbb{E}_t \sigma_{t+1}^2}$ using the Bansal and Yaron stochastic discount factor for $g_t \in \{-2B, 0, 2B\}$ with respect to α (left) and γ (right).

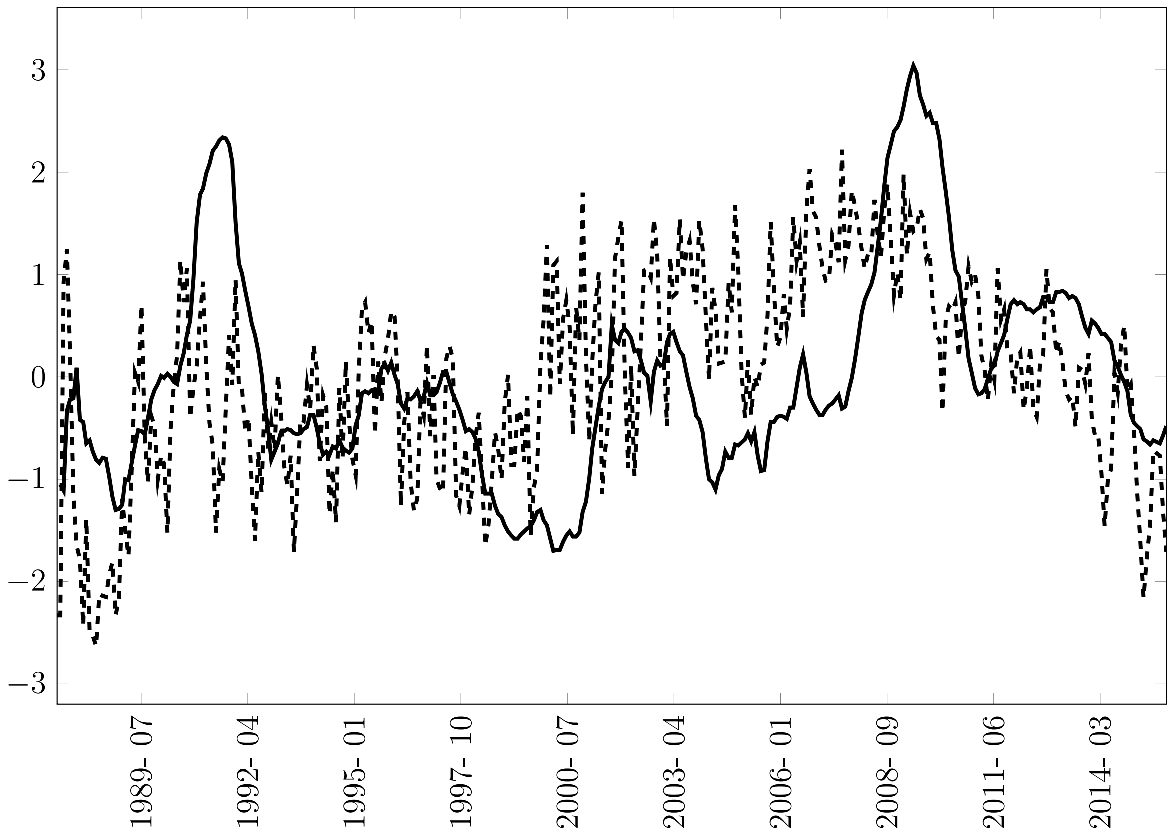

Figure 2: The dashed line is 1-percentage of investors that report “neutral” from American Association of Individual Investors, standardized to have mean zero and unit standard deviation. The bold line is the $\sqrt{E_t \sigma_{t+1}^2}$ for the Bansal and Yaron model using g_t that is filtered from per-capita non-durable consumption data.

that the model implied measure of dispersion picks up the key variation in the empirical measure of dispersion in beliefs. The two series are correlated with a statistically significant coefficient of 0.31.

5 Concluding remarks

This paper provides a framework where heterogeneity in beliefs can account for dynamics of asset prices. It helps that our characterization of belief heterogeneity is in terms of likelihood ratio process $\{z_{i,t}\}$ that also occurs, for example, in the model specification detection error statistics featured in Anderson et al. (2003) and Piazzesi et al. (2015). These approaches allows us confront models with a larger set of moment conditions than those typically used to estimate preference parameters like risk aversion. For instance, in the application to Bansal and Yaron (2004), we show how to map a collection of likelihood ratios into simpler statistics like cross-sectional variance of return forecasts. The next step is to use panel data from actual return forecasts to see if the magnitudes implied by asset pricing models used in the literature are in line with what we find. One avenue is using the Institutional Brokers' Estimate System (I/B/E/S) database that maintains a firm level earnings and price forecasts for a panel of analysts. This can be used to compute the empirical counterparts of forecast dispersions for the aggregate S&P 500 or other portfolios.

References

Anderson, Evan W., Lars Peter Hansen, and Thomas J. Sargent. 2003. A Quartet of Semigroups for Model Specification, Robustness, Prices of Risk, and Model Detection. *Journal of the European Economic Association* 1 (1):68–123.

Bansal, Ravi and Bruce N. Lehmann. 1997. Growth-Optimal Portfolio Restrictions On Asset Pricing Models. *Macroeconomic Dynamics* 1 (02):333–354.

Bansal, Ravi and Amir Yaron. 2004. Risks for the long run: A potential resolution of asset pricing puzzles. *The Journal of Finance* 59 (4):1481–1509.

Cogley, Timothy. 2002. Idiosyncratic risk and the equity premium: evidence from the consumer expenditure survey. *Journal of Monetary Economics* 49 (2):309–334.

Constantinides, George M and Darrell Duffie. 1996. Asset Pricing with Heterogeneous Consumers. *Journal of Political Economy* 104 (2):219–40.

Epstein, Larry G and Stanley E Zin. 1989. Substitution, Risk Aversion, and the Temporal Behavior of Consumption and Asset Returns: A Theoretical Framework. *Econometrica* 57 (4):937–69.

Fama, Eugene F and James D MacBeth. 1974. Long-Term Growth in a Short-Term Market. *The Journal of Finance* 29 (3):857–885.

Hansen, Lars Peter, John C Heaton, and Nan Li. 2008. Consumption strikes back? Measuring long-run risk. *Journal of Political Economy* 116 (2):260–302.

Piazzesi, Monika, Juliana Salomao, and Martin Schneider. 2015. Trend and Cycle in Bond Premia .

Roll, Richard. 1973. Evidence on "Growth-Optimum" Model. *The Journal of Finance* 28 (3):551–566.

Violante, Gianluca, Jonathan Heathcote, and Kjetil Storesletten. 2014. Consumption and Labor Supply with Partial Insurance: An Analytical Framework. *American Economic Review* 104 (7):2075–2112.