

The Optimal Maturity of Government Debt

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Abstract

A Ramsey planner chooses a distorting tax on labor and manages a portfolio of bonds of different maturities in a representative agent economy with aggregate shocks. Covariances of bonds' returns with the primary deficit are key determinants of Ramsey portfolios. We estimate these moments in U.S. data and calibrate a model with a representative agents who has Epstein-Zin preferences that matches these moments. The implied optimal portfolio does not short any bond and allocates approximately equal portfolio shares to bonds of different maturities, slightly tilt towards longer maturities when the outstanding debt is large, and requires little re-balancing in response to aggregate shocks. These portfolio prescriptions differ from those of models often used in the business cycle literature. The differences are driven by counterfactual asset pricing implications of the standard models.

1 Introduction

An important insight of portfolio theory is that the composition of an optimal investment portfolio is determined by covariances between returns on the securities available to the investor and her income and expenditure shocks. An optimal portfolio uses fluctuations in returns to hedge investor's shocks and smooth her consumption stream. The choice of a maturity structure of government debt is a portfolio problem, so covariances of returns on debts of different maturities with shocks to government revenues and expenditures shapes an optimal maturity structure. Canonical macroeconomic models used to study an optimal maturity structure typically fail to describe observed behavior of asset prices. That renders doubtful the portfolio prescriptions of those models.

We modify a representative agent model in ways designed to generate empirically plausible asset prices, then proceed to deduce normative prescriptions for the optimal management of government debt. We begin by establishing that covariances between some returns on debts of different maturities with innovations to the present value of government revenues and expenditures determine an optimal choice of maturity structure. We then document key facts about these covariances in the U.S. during the post WWII period. Finally, we calibrate an Epstein-Zin representative agent economy to match these stylized fact and then proceed to compute an optimal maturity structure. As a by-product of our analysis, we develop numerical methods to characterize the portfolio problems with many securities and return structures.

Given the observed behavior of debt returns, we find that it is not possible for the government fully to hedge its expenditure and tax collection shocks. That makes the value of outstanding debt fluctuate with aggregate shocks. An optimal debt portfolio is long in both long-term and short-term debt, with an optimal mix depending on the value of outstanding debt. In our baseline calibration with one- and five-year bonds, the optimal share of 5 year bonds in the portfolio is about 45% when the debt to GDP ratio is about 30 percent; it increases to 55% when the debt to GDP ratio is about 160 percent. These ratios change little in response to typical business-cycle frequency shocks because an optimal portfolio requires little re-balancing across maturities.

These prescriptions differ from those of representative agent Ramsey models, such as Buera and Nicolini (2004) and Angeletos (2002). Those models assert that an optimal maturity structure of government debt can fully hedge government shocks,

but that it does so only by taking extremely long and short positions in debts of different maturities. Thus, a typical finding in those models is that the government should issue long-term debt valued at tens or even hundreds times GDP while simultaneously taking offsetting short positions of short-term debt of similar magnitudes. The optimal portfolio massively re-balances after each shock, again with a change in debt positions often of the order of tens of times GDP. Furthermore, the composition of an optimal portfolio is very sensitive to the assumed menu of traded maturities.

Our findings differ from these because we succeed in bringing equilibrium distributions of asset returns much closer to observed distributions. Observed returns on government debts are quite volatile. A substantial part of that volatility is unrelated to shocks in government revenues or expenditures over business cycle frequencies. As a result, holding large positions in any given maturity is risky for the government, and the government's ability to hedge its revenues and expenditures shocks is limited. To manage that risk, in our model the government should hold quantitatively similar positions in debts of different maturities and smoothly change the duration of the outstanding government debt as it accumulates or decreases total outstanding debt. In contrast, those earlier Ramsey models assume that debt returns are exclusively driven by the same aggregate shocks that affect government revenues and expenditures. That assumption generates fluctuations in returns on government debt that are small and also highly correlated with these shocks. That means that the government can fully hedge its shocks in the face of small fluctuations in returns only by using a lot of leverage, in particular, by taking huge short and long positions in debts of different maturities. Shocks to returns that are uncorrelated with a government's financial needs would make holding such positions extremely risky. But those shocks were not present in those earlier models.

Our paper builds on a large literature on optimal taxation and debt management over the business cycle, going at least to the work of Barro (1979) and Lucas and Stokey (1983). Buera and Nicolini (2004) and Angeletos (2002) are two benchmark specifications of those models to incomplete market settings with debt of different maturities. Faraglia et al. (2012) restrict a government's ability to re-balance its portfolio, while Debortoli et al. (2016 forthcoming) restricting a government's ability to commit. Lustig et al. (2008) allow for nominal bonds of several maturities but impose portfolio restrictions that prevent the government from going short in any maturity. Farhi (2010) allows the government to hold a risky asset in the form of

capital. All these papers use similar specifications of preferences and shocks that fail to match observed behavior of asset returns.

Our work builds on papers by Bohn (1990) and Bhandari et al. (2017a) who showed how insights from portfolio theory can be applied to extract prescriptions for optimal government security trades. We adapt general theoretical insights of those papers to study the optimal maturity structure of government debt.

We also build on a large empirical and theoretical literature about the behavior of bond returns. That literature has documented that returns on government bonds are quite volatile, not very correlated with typical macroeconomic shocks, and usually display rising yield curves. For example Campbell and Shiller (1991), Campbell (1995), Cochrane and Piazzesi (2005) and Ludvigson and Ng (2009). Piazzesi and Schneider (2006), Bansal and Shaliastovich (2013), Wachter (2006), and Albuquerque et al. (2016) constructed general equilibrium frameworks that are consistent with these facts. We extend the findings of this literature to study comovements between returns on government bonds and innovations to the present value of government revenue and expenditure shocks, objects that our theoretical framework asserts to determine an optimal debt composition. We also build on Albuquerque et al. (2016) approach to constructing a general equilibrium model that is consistent with the empirical asset pricing patterns.

To the best of our knowledge, we are the first to develop numerical methods applicable to incomplete market economies with Epstein-Zin preferences. Karantounias (2013) solved a Ramsey problem with Epstein-Zin preferences when markets are complete. We build on some of his insights, but the appearance of additional state variables in the Ramsey problem for our incomplete market economy requires a different computational approach than he was able to use. To that end, we extend the techniques developed in Evans (2014) and Bhandari et al. (2017b) by using perturbation theory to approximate a Ramsey plan sequentially around an endogenous sequence of current levels of government debt. We also build on insights of Guu and Judd (2001) and Devereux and Sutherland (2011) to approximate the dynamic portfolio problem of the Ramsey planner.

The rest of the paper is organized as follows. In Section 2 we obtain a convenient closed form approximation to an optimal government portfolio policy. This approximation shows the role that return covariances play in shaping an optimal debt composition. Section 3 presents the empirical facts about those covariances. Section

4 presents our general framework and numerical solution methods. Section 5 shows results.

2 Theory

We present a simple model that highlights the forces that shape an optimal government portfolio. Time is discrete, and the horizon is infinite. Shocks s_t driven by an exogenous Markov process take values in a finite set. Let $s^t = (s_0, s_1, \dots, s_t)$ denote a finite history. A representative agent consumes $c_t(s^t)$, works $l_t(s^t)$ hours, and pays a linear income tax rate $\tau_t(s^t)$. We will use the notation x_t to denote the random variable $x_t(s^t)$. The resource constraint is

$$c_t + g_t = \theta_t l_t. \quad (1)$$

A representative consumer and a government trade K securities that are in zero net supply. One unit of security k issued at time t pays a stream $\{p_{t+j}^k(1 - \delta^k)^j\}_{j=0}^{\infty}$ where $\delta_k \in (0, 1)$ and p_t^k is a measurable function of s^t . We use different payoff decay rates δ_k to represent bonds of different maturities and we use the random variables p_t^k to represent exogenous bond payouts in some examples. We let q_t^k denote the price of one unit of security k at time t . We let $-B_t^k$ denote the asset holdings of the government of security k at time t , so that a positive value of B_t^k denotes a net liability of the government. The government budget constraint is

$$g_t + \sum_k (p_t^k + (1 - \delta_k) q_t^k) B_{t-1}^k = \tau_t \theta_t l_t + \sum_k q_t^k B_t^k. \quad (2)$$

A one-period returns on security k is $R_t^k \equiv \frac{p_t^k + (1 - \delta^k) q_t^k}{q_{t-1}^k}$. Boldface letters denote a vector collecting information for all k securities, e.g., \mathbf{R}_t denotes the vector of returns R_t^1, \dots, R_t^K .

A representative agent ranks consumption and labor plans according to

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \eta_t \left[\frac{c_t^{1-\rho}}{1-\rho} - \theta_t^{1-\rho} \frac{l_t^{1+\phi}}{1+\phi} \right] \quad (3)$$

where η_t is the preference shock. The shock process follows

$$\begin{bmatrix} \ln \frac{\theta_t}{\theta_{t-1}} \\ \ln \frac{g_t}{\theta_t} \\ \mathbf{p}_t \\ \eta_t \end{bmatrix} = \begin{bmatrix} \bar{\theta} \\ \bar{g} \\ \mathbf{1} \\ 1 \end{bmatrix} + \sigma \begin{bmatrix} \epsilon_{\theta,t} \\ \epsilon_{g,t} \\ \boldsymbol{\epsilon}_{p,t} \\ \epsilon_{\eta,t} \end{bmatrix} \equiv \boldsymbol{\mu} + \sigma \mathbf{s}_t,$$

where σ is a positive scalar, $\boldsymbol{\mu}$ is a $(K + 3) \times 1$ vector, and we now specialize the Markov vector of shocks \mathbf{s}_t to have mean zero, be i.i.d. over time, and arbitrarily correlated contemporaneously.

We choose this specification of preference, shocks processes, and asset structure to simplify the analysis exposition in this section. We can readily extend the analysis to comprehend any finitely lived security extends, but that would require more notation. However, the basic insights conveyed by the simplified model of this section extend to more general preference specifications and shock processes, including ones that we consider in section 4 below.

We study the classical Ramsey problem of choosing the optimal sequences of taxes and debt portfolios $\{\tau_t, \mathbf{B}_t\}_t$ to finance the stochastic process $\{g_t\}_t$ so that the resulting competitive equilibrium maximizes (3). Since our economy is growing, it will be convenient to rescale several period t variables by the TFP shock θ_t so that we can work exclusively with random variables that are asymptotically stationary. Let $\hat{c}_t = c_t/\theta_t$, $\hat{g}_t = g_t/\theta_t$ and $\hat{B}_t^k \equiv q_t^k B_t^k/\theta_t$. Here \hat{B}_t^k corresponds to the market value of government holdings of security k normalized by θ_t . Then our constraints can be expressed

$$\hat{g}_t + e^{-g_{\theta,t}} \sum_k R_t^k \hat{B}_{t-1}^k = \tau_t l_t + \hat{B}_t, \quad (4)$$

$$\hat{B}_t = \sum_k \hat{B}_t^k, \quad (5)$$

$$\hat{c}_t + \hat{g}_t = l_t, \quad (6)$$

where $g_{\theta,t} = \bar{\theta} + \sigma \epsilon_{\theta,t}$. Finally, let the normalized primary deficit be defined as $\hat{X}_t \equiv \hat{g}_t - \tau_t l_t$.

2.1 Quasi-linear preferences

The analysis is particularly straightforward if we assume quasilinear preferences, $\rho = 0$, and no preference shocks, $\eta_t = 1$ for all t . In this case, R_t^k is governed by stochastic process that is determined solely by the exogenous stochastic process p_t^k . The representative agent's intra-temporal optimality condition is

$$l_t^\phi = 1 - \tau_t. \quad (7)$$

Therefore, $\{\hat{c}_t, l_t, \tau_t, \hat{B}_t, \hat{\mathbf{B}}_t\}_t$ is a competitive equilibrium if and only if it satisfies constraints (4) - (7). A recursive formulation of the Ramsey problem is associated with an optimal value function $V(\hat{B}_-)$ for the planner that solves a Bellman equation

$$V(\hat{B}_-) = \max_{\{l(s), \hat{B}(s)\}, \hat{\mathbf{B}}} \mathbb{E} e^{g_\theta(s)} \left\{ l(s) - \frac{l(s)^{1+\phi}}{1+\phi} + \beta V(\hat{B}(s)) \right\}$$

where the maximization is subject to

$$e^{-g_\theta(s)} \sum_k \hat{B}^k R^k(s) + l(s)^{1+\phi} = l(s) - \hat{g}(s) + \hat{B}(s),$$

$$\sum_k \hat{B}^k = \hat{B}_-,$$

and where we used the feasibility constraint to eliminate \hat{c} . Let $\tilde{B}(\hat{B}_-, s)$, $\tilde{\mathbf{B}}(\hat{B}_-)$, and $\tilde{l}(\hat{B}_-, s)$ be optimal policies for total debt, the portfolio, and the labor supply. The Ramsey plan can be recovered by setting the initial condition $\hat{B}_- = \hat{B}_{-1}$ and using policy rules

$$\tau_t = 1 - \tilde{l}(\hat{B}_{t-1}, s_t)^\phi, \hat{\mathbf{B}}_t = \tilde{\mathbf{B}}(\hat{B}_{t-1}), \text{ and } \hat{B}_t = \tilde{B}(\hat{B}_{t-1}, s_t)$$

for all $t \geq 0$.

This problem can be analyzed along the lines accomplished in Bhandari et al. (2017a). We construct second-order (small noise) Taylor expansions with respect to σ of policy functions that attain this Bellman equation, while simultaneously making the discount factor be small in the sense that $1 - \beta$ is of the same order as σ . This approach yields tractable linear approximations of policy functions from which we construct closed form characterizations of optimal portfolios. Let $X_{\tau,t} = g_t - \theta_t \tau l(\tau)$

be the primary deficit at a given tax rate τ , $PV_t(X_\tau)$ be a conditional mathematical expectation of the present discounted value of primary deficits,

$$PV_t(X_\tau) = \mathbb{E}_t \sum_j \beta^j X_{\tau,t+j},$$

and $\hat{P}V_t(X_\tau) \equiv PV_t(X_\tau)/\theta_{t-1}$. We use $\mathbb{C}[\mathbf{R}, \mathbf{R}]$ to denote the unconditional covariance matrix of the stochastic process for returns \mathbf{R}_t and $\mathbb{C}[\mathbf{R}, PV(X_\tau)]$ the vector of unconditional covariances of \mathbf{R}_t with $PV_t(X_\tau)$.

Proposition 1. *The optimal debt portfolio $\mathbf{B}_{t-1} = \theta_{t-1} \hat{\mathbf{B}}_{t-1}$ satisfies*

$$\hat{\mathbf{B}}_{t-1} = -\mathbb{C}[\mathbf{R}, \mathbf{R}]^{-1} \mathbb{C}[\mathbf{R}, \hat{P}V(X_\tau)] + \frac{\mathbb{C}[\mathbf{R}, \mathbf{R}]^{-1} \mathbf{1}}{\mathbf{1}^\top \mathbb{C}[\mathbf{R}, \mathbf{R}]^{-1} \mathbf{1}} \left(\hat{B}_{t-1} + \mathbf{1}^\top \mathbb{C}[\mathbf{R}, \mathbf{R}]^{-1} \mathbb{C}[\mathbf{R}, \hat{P}V(X_\tau)] \right) + \mathcal{O}(\sigma, (1-\beta)) \quad (8)$$

where τ satisfies $B_{t-1} = \beta \mathbb{E}_{t-1}[PV_t(X_\tau)]$.

The easiest way to understand this proposition is to consider the variance minimization problem at $t-1$ given total debt B_{t-1}

$$\min_{\mathbf{B}} \text{Var}_{t-1} \left[\sum_k B^k R_t^k + PV_t(X_\tau) \right]$$

subject to

$$\sum_k B^k = B_{t-1},$$

where τ satisfies $B_{t-1} = \beta PV_t(X_\tau)$. Denote the solution to this variance minimization problem \mathbf{B}_{t-1}^* . The next proposition shows that the optimal portfolio characterized in Proposition 1 is approximately \mathbf{B}_{t-1}^* .

Proposition 2. *The optimal debt portfolio satisfies*

$$\mathbf{B}_{t-1} = \mathbf{B}_{t-1}^* + \mathcal{O}(\sigma, (1-\beta)).$$

Thus, Proposition 1 shows that the optimal Ramsey portfolio minimizes risk. The portfolio composition offsets fluctuations in the expected present discounted value of primary deficits that would occur at a constant tax rate. This expresses that the government would like to minimize the fluctuations in the deadweight loss of taxes by

making the tax rate as smooth intertemporally as possible. But shocks to government revenues and expenditures that move the expected present value of primary deficits make it impossible to satisfy the government's budget constraint at a constant tax rate. The optimal portfolio minimize the effect of shocks on deadweight losses.

An important insight that emerges from equation (8) is that co-movements of returns with the shocks to government expenditures and revenues are the critical objects that determines the optimal composition of the Ramsey portfolio. To illustrate this, we consider a simple example that will also prove helpful in later sections to understand the optimal portfolios that emerge in the models of Angeletos (2002) and Buera and Nicolini (2004) and how they differ from our results. Suppose that the government has access to two assets with returns R_t^1 and R_t^2 . For simplicity of exposition, assume $R_t^1 = \frac{1}{\beta}(1 + \alpha_1 \hat{P}V_t(X_\tau))$ and $R_t^2 = \frac{1}{\beta}(1 + \alpha_2 \hat{P}V_t(X_\tau) + \epsilon_t)$ where $\alpha_1 > \alpha_2 > 0$ and ϵ_t is orthogonal to $\hat{P}V_t(X_\tau)$.¹ From Proposition 1, the optimal portfolio is

$$\hat{\mathbf{B}}_{t-1} = \begin{bmatrix} -1/\alpha_1 \\ 0 \end{bmatrix} + \begin{bmatrix} \frac{\alpha_2^2 \sigma_{\hat{P}V}^2 - \alpha_1 \alpha_2 \sigma_{\hat{P}V}^2 + \sigma_\epsilon^2}{(\alpha_1 - \alpha_2)^2 \sigma_{\hat{P}V}^2 + \sigma_\epsilon^2} \\ \frac{\alpha_1^2 \sigma_{\hat{P}V}^2 - \alpha_1 \alpha_2 \sigma_{\hat{P}V}^2}{(\alpha_1 - \alpha_2)^2 \sigma_{\hat{P}V}^2} \end{bmatrix} (\hat{B}_{t-1} + 1/\alpha_1),$$

where σ_ϵ^2 is the variance of ϵ_t and $\sigma_{\hat{P}V}^2$ is the variance of $\hat{P}V_t(X_\tau)$. Consider two extreme cases. First, suppose that the returns are highly correlated with shocks to the primary surplus because $\sigma_\epsilon^2 \approx 0$. In this case, the optimal portfolio is given by

$$\hat{\mathbf{B}} \approx \begin{bmatrix} \frac{\alpha_2}{\alpha_1 - \alpha_2} \hat{B} - \frac{1}{\alpha_1 - \alpha_2} \\ \frac{\alpha_1}{\alpha_2 - \alpha_1} \hat{B} - \frac{1}{\alpha_2 - \alpha_1} \end{bmatrix}.$$

As $\alpha_1, \alpha_2 \rightarrow 0$ while maintaining $\alpha_1 > \alpha_2 > 0$, this portfolio diverges to $\begin{bmatrix} -\infty \\ \infty \end{bmatrix}$. This example captures key forces driving the Buera and Nicolini (2004) finding that the optimal government portfolio of the government issues huge amounts of long debt (tens or even hundreds times of GDP) and takes an offsetting short position in the short-term debt. In our example, securities 1 and 2 approximate a short-term and long-term bond. As we discuss in more details in Section 4, standard calibrations of Ramsey problems imply that marginal utility adjusted bond returns are highly

¹This is the case if $R_t^1 = \frac{1}{\beta}(1 + a_1 \epsilon_{g,t} + a_2 \epsilon_{\theta,t})$ and $R_t^2 = \frac{1}{\beta}(1 + \alpha(a_1 \epsilon_{g,t} + a_2 \epsilon_{\theta,t}) + \epsilon_t)$.

correlated with government shocks ($\sigma_\epsilon^2 \approx 0$), marginal utility adjusted returns on long-term bonds move less in response to a temporary adverse shock by less than do marginal utility adjusted returns on short-term bonds ($\alpha_1 > \alpha_2 > 0$), but that bond returns overall are very smooth given the size of standard business cycle shocks (α_1 is close to zero). Given such a returns process, the government can hedge all of its shocks to primary deficits, but to do so requires taking a huge positive position in long debt and a negative position in the short debt. Such highly levered holding of the long bond amplifies its small negative co-movement with shocks to the primary government deficit and provides substantial insurance.

Alternatively, consider a case when σ_ϵ^2 is large, so that much of the variability of bond returns is not driven by the shocks to the primary deficit. As $\sigma_\epsilon^2 \rightarrow \infty$, the optimal portfolio converges to $\begin{bmatrix} \hat{B}_{t-1} \\ 0 \end{bmatrix}$. Fluctuations in returns that are orthogonal to the shocks to the primary government deficit make holding such security risky and the optimal portfolio in the economies with high σ_ϵ^2 looks very different from those with low σ_ϵ^2 . Thus, correlations of returns with the primary government deficit is a critical determinant of an optimal portfolio.

2.2 Recursive Ramsey problem with isoelastic preferences

Forces described in subsection 2.1 also operate in economies with a risk-averse representative agent. Consider now general isoelastic preferences (3). The implementability constraint can be written

$$\begin{aligned} & e^{-g\theta,t} \left(\sum_k \hat{c}_t^{-\rho} \eta_t \left[\frac{p_t^k + (1 - \delta^k)q_t^k}{\beta \mathbb{E}_{t-1} \eta_t e^{-\rho g\theta,t} \hat{c}_t^{-\rho} [p_t^k + (1 - \delta^k)q_t^k]} \right] \hat{c}_{t-1}^{-\rho} \eta_{t-1} \hat{B}_{t-1}^k \right) + \eta_t l_t^{1+\phi} \\ & = \hat{c}_t^{-\rho} \eta_t \hat{B}_t + \hat{c}_t^{1-\rho} \eta_t. \end{aligned}$$

Substitute $\mathcal{B}_t^k = c_t^{-\rho} \eta_t \hat{B}_t^k$, $\hat{\mathcal{B}}_t = c_t^{-\rho} \eta_t \hat{B}_t$, $\mathcal{R}_{t-1,t}^k = \hat{c}_t^{-\rho} \eta_t \left[\frac{p_t^k + (1 - \delta^k)q_t^k}{\beta \mathbb{E}_{t-1} e^{-\rho g\theta,t} \eta_t \hat{c}_t^{-\rho} [p_t^k + (1 - \delta^k)q_t^k]} \right] c_{t-1}^{-\rho}$ and $\mathcal{X}_t = \eta_t l_t^{1+\phi} - \hat{c}_t^{1-\rho} \eta_t$ to write these conditions as

$$e^{-g\theta,t} \left(\sum_k \mathcal{R}_{t-1,t}^k \hat{\mathcal{B}}_{t+1}^k \right) + \mathcal{X}_t = \hat{\mathcal{B}}_t,$$

$$\sum_k \hat{\mathcal{B}}_t^k = \hat{\mathcal{B}}_t,$$

$$\tilde{q}_t^k = \beta \mathbb{E}_t \eta_{t+1} e^{-\rho g_{\theta, t+1}} \hat{c}_{t+1}^{-\rho} [p_{t+1}^k + (1 - \delta^k) q_{t+1}^k],$$

$$\mathcal{R}_{t-1, t}^k = \frac{U_{c, t} p_t^k + (1 - \delta^k) \tilde{q}_t^k}{\tilde{q}_{t-1}^k}.$$

The Ramsey problem is recursive in state variables $(\hat{\mathcal{B}}_{t-1}, \tilde{\mathbf{q}}_{t-1})$, where $\tilde{\mathbf{q}}_t = \{\tilde{q}_t^k\}_k$. We can simplify it further if we observe that given a tax rate τ , consumption, $\hat{c}_\tau(s)$, and labor, $l_\tau(s)$, in any state s are pinned down by

$$\hat{c}(s) + g(s) = l(s),$$

$$\hat{c}(s)^{-\rho} (1 - \tau(s)) = l(s)^\phi.$$

Thus, the Ramsey problem can be written recursively as

$$\max_{\tilde{\mathbf{q}}(s), \hat{\mathcal{B}}(s), \tau(s), \hat{\mathcal{B}}_-} V(\hat{\mathcal{B}}_-, \tilde{\mathbf{q}}_-) = \mathbb{E} e^{[(1-\rho)g_\theta(s)]} \left\{ \eta(s) \left(\frac{\hat{c}_\tau(s)(s)^{1-\rho}}{1-\rho} - \frac{l_\tau(s)(s)^{1+\phi}}{1+\phi} \right) + \beta V(\hat{\mathcal{B}}(s), \tilde{\mathbf{q}}(s)) \right\}$$

subject to

$$\mathcal{R}_{\tau(s)}^k(s) = \frac{[\eta(s) \hat{c}_\tau(s)(s)^{-\rho} p^k(s) + (1 - \delta^k) \tilde{q}^k(s)]}{\tilde{q}_-^k},$$

$$\tilde{q}_-^k = \beta \mathbb{E} [e^{-\rho g_\theta(s)} (\eta(s) \hat{c}_\tau(s)(s)^{-\rho} p^k(s) + (1 - \delta^k) \tilde{q}^k(s))], \quad (9)$$

$$\sum_k e^{-g_\theta(s)} \mathcal{R}_{\tau(s)}^k(s) \hat{\mathcal{B}}^k = -\mathcal{X}_{\tau(s)}(s) + \hat{\mathcal{B}}_- \quad \text{for } s, s_-, \quad (10)$$

$$\hat{\mathcal{B}}_- = \sum_k \hat{\mathcal{B}}^k.$$

Using the same approximation techniques as in subsection 2.1, we obtain

Proposition 3. *The optimal debt portfolio $\mathcal{B}_{t-1} = \theta_{t-1} \hat{\mathcal{B}}_{t-1}$ is*

$$\begin{aligned} \hat{\mathcal{B}}_{t-1} &= -\mathbb{C}[\mathcal{R}_\tau, \mathcal{R}_\tau]^{-1} \mathbb{C}[\mathcal{R}_\tau, PV(\mathcal{X}_\tau)] \\ &\quad + \frac{\mathbb{C}[\mathcal{R}_\tau, \mathcal{R}_\tau]^{-1} \mathbf{1}}{\mathbf{1}^\top \mathbb{C}[\mathcal{R}_\tau, \mathcal{R}_\tau]^{-1} \mathbf{1}} \left(\hat{\mathcal{B}}_{t-1} + \mathbf{1}^\top \mathbb{C}[\mathcal{R}_\tau, \mathcal{R}_\tau]^{-1} \mathbb{C}[\mathcal{R}_\tau, P\hat{V}(\mathcal{X}_\tau)] \right) + \mathcal{O}(\sigma, (1 - \beta)) \end{aligned} \quad (11)$$

where τ satisfies $\mathcal{B}_{t-1} = \beta \mathbb{E}_{t-1}(PV_t(\mathcal{X}_\tau))$.

Just like the optimal portfolio in the quasi-linear economy, the optimal portfolio $\theta_{t-1}\hat{\mathcal{B}}_{t-1}$ minimizes the conditional variance of the present value of the effective shocks, $\text{Var}_{t-1} [\sum_k \mathcal{B}^k \mathcal{R}_t^k + PV_t(\mathcal{X}_\tau)]$ subject to $\sum_k \mathcal{B}^k = \mathcal{B}_{t-1}$. We use the adjective “effective” to indicate that we have adjusted all variables by marginal utilities of consumption to recognize that the marginal cost of public funds now varies with the state of the economy. The cost of raising revenues is proportional to the marginal utility of consumption, necessitating the adjustment. Observe, however, that the main insights of Section 2.1 continue to hold when consumers are risk-averse: the optimal portfolio of the government minimizes fluctuations in the present value of primary deficits and portfolio’s composition depends critically on the covariances of securities returns with shocks to the deficits.

3 Bond returns data

We document patterns of holding period returns on US government bonds of several maturities. As discussed in the previous section, the covariance properties of the returns with shocks that drive fiscal needs are key in determining the optimal portfolio. We will use these moments as inputs to the quantitative analysis of a calibrated economy in section 4.

Our baseline results are compiled using bond prices data from the Fama-Bliss discount bond series in Center for Research in Security Prices (CRSP) database. This series records prices for artificial discount bonds with one to five years to maturity at a monthly frequency from 1952. Let q_t^n be the price of a bond of maturity n years at beginning of the year t and P_t be the price level.² We define the annual real holding period return as

$$R_{t,t+1}^n = \left(\frac{q_{t+1}^{n-1}}{q_t^n} \right) \left(\frac{P_t}{P_{t+1}} \right).$$

We plot returns in figure I. We show an estimated covariance matrix of returns with output growth and expenditure/output in Table II. We see that (a) returns are correlated across maturities and (b) mean returns as well as the volatility of returns are increasing in the length of maturity and (c) returns are not correlated with output growth. These patterns are also documented in Campbell (1995).

We next use a simple regression of returns on output growth, primary deficit/output

²We use the GDP deflator series from NIPA for P_t

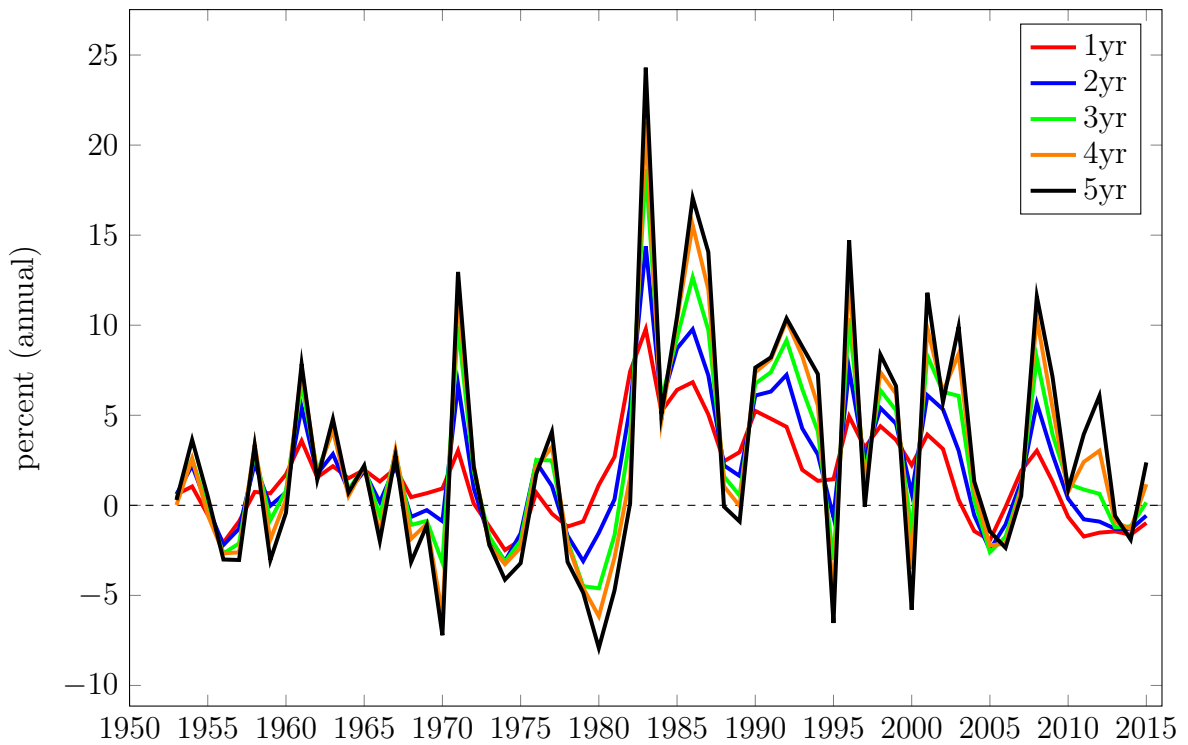


Figure I: Time series for annualized holding period real returns on bonds using Fama-Bliss data

Maturity	mean	1 yr	2 yr	3 yr	4 yr	5 yr	exp /output	output growth
1 yr	1.64	6.79	8.41	9.68	10.42	10.99	-0.84	0.22
2 yr	2.09	8.41	12.51	15.93	18.28	20.19	-2.08	0.21
3 yr	2.49	9.68	15.93	21.46	25.46	28.76	-3.25	0.34
4 yr	2.70	10.42	18.28	25.46	30.99	35.51	-4.09	0.20
5 yr	2.86	10.99	20.19	28.76	35.51	41.33	-5.10	0.21
exp/output	16.99	0.22	0.21	0.34	0.20	0.21	-0.43	5.32
output growth	3.14	-0.72	-0.13	0.44	1.01	1.58	0.33	-1.87

Table I: Descriptive statistics: means and covariance matrix for annualized holding period real returns on bonds using Fama-Bliss data

	1 yr	2 yr	3 yr	4 yr	5 yr
constant	5.54 (1.96)	4.66 (2.66)	3.54 (3.43)	2.63 (4.1)	1.58 (4.61)
output growth	-0.09 (0.16)	-0.07 (0.22)	0.00 (0.29)	0.00 (0.34)	0.03 (0.39)
deficits/ output	-0.31 (0.17)	-0.09 (0.22)	0.12 (0.3)	0.30 (0.34)	0.05 (0.40)
inflation	-0.20 (0.16)	-0.47 (0.21)	-0.71 (0.27)	-0.9 (0.32)	-1.11 (0.36)
R^2	0.07	0.08	0.11	0.13	0.16

Table II: OLS estimates for regression (12). Standard errors are in parenthesis.

and inflation to measure the part of the variation in returns that is unrelated to the fiscal needs.³

$$\log R_{t,t+1}^n = \alpha_0 + \alpha_y \text{output growth} + \alpha_g \text{deficits/output} + \alpha_\pi \text{inflation} + \epsilon_{t+1} \quad (12)$$

The OLS estimates are summarized in table II. We see that across maturities R^2 are in the range of 7% – 15% which shows that the orthogonal component in returns is large. The estimated coefficients on output growth are all statistically insignificant and except for the one year maturity, the coefficients on deficits/output are also insignificant. These results are largely consistent with predictability regressions in Fama and Bliss (1987), Cochrane and Piazzesi (2005), and Ludvigson and Ng (2009). For example, Ludvigson and Ng (2009) use dynamic latent factor methods to extract “macro factors” from a large data set of macro variables and use them to predict returns. In specifications in which they use only the macro factors, they find that about 75-80% of the variation in returns is orthogonal to macro factors. We read the findings in table II and related evidence as suggesting that variations in returns are largely driven by predictable movements discount rates that are uncorrelated to macroeconomic aggregates like output growth and deficits at business cycle frequencies.

4 Quantitative analysis

A common choice in the real business cycle literature is to use separable expected utility preferences with CRRA form. However, studies of the “equity premium puzzle”

³We include inflation in the regression because in the current version of the paper we study real bonds and abstract from monetary policy. In this way we hope to get a conservative estimate of how large is the component of returns that is orthogonal to the government’s fiscal or monetary needs.

indicate that for reasonable parameter choices such preferences do not generate the observed high excess returns on risky assets or the volatility of these returns across time or the smooth risk free rates (see, for example, Hansen and Singleton (1982); Mehra and Prescott (1985); Hansen and Jagannathan (1991)). The pricing kernel affiliated with a representative agent having CRRA preferences is a smooth function of aggregate consumption growth that displays relatively smooth fluctuations at business cycle frequencies. As we showed in Propositions 1 and 3, an optimal government portfolio is determined by covariances of government bond returns. Thus, it is important to adopt a preference specification that implies realistic asset pricing properties.

We adopt Albuquerque et al.'s (2016) specification of recursive Epstein-Zin preferences augmented with the discount factor shocks. As Albuquerque et al. (2016) show, such a model matches a range of empirical facts about returns on bonds and equities such as the excess returns for the long bonds, an upward sloping term structure, variations in the pricing kernel and bond returns that have realistic covariances with aggregate output.⁴ While Albuquerque et al. (2016) consider an endowment economy, we further extend this specification by introducing the disutility of labor along the lines of Karantounias (2013). Specifically, we use the following recursion to value stochastic streams of consumption and leisure:

$$V_{t-1} = \mathbb{E}_{t-1} [W_t^{1-\gamma}]^{\frac{1}{1-\gamma}},$$

$$W_t \equiv \left((1-\beta)\eta_t \left[c_t^{1-\rho} - (1-\rho)\theta_t^{1-\rho} \frac{n_t^{1+\phi}}{1+\phi} \right] + \beta V_t^{1-\rho} \right)^{\frac{1}{1-\rho}}.$$

Here ρ measures the inverse of the intertemporal elasticity of substitution and γ measures risk aversion. A special case of this specification that sets $\rho = \gamma$ recovers the isoelastic form that we used in Section 2.2.

Defining $U(c, l) = \eta \left[\frac{c^{1-\rho}}{1-\rho} - \theta^{1-\rho} \frac{n^{1+\phi}}{1+\phi} \right]$, the agent's first-order necessary conditions

⁴Piazzesi and Schneider (2006) and Bansal and Shaliastovich (2013) construct and estimate models having a representative agent with Epstein and Zin (1989) preferences and also stochastic volatility and correlated shocks to inflation and consumption growth; Wachter (2006) specifies preferences with habit persistence in the style of Campbell and Cochrane (1999). We suspect that models like these that are calibrated to match the covariance matrix of returns would imply quantitatively similar optimal government portfolios to those we have constructed here. We plan to investigate this hunch in the future.

imply that

$$U_{n,t} = (1 - \tau_t)\theta_t U_{c,t},$$

and that assets are priced with by a stochastic discount factor

$$S_{t+1} \equiv \beta m_{t+1}^{\frac{\rho-\gamma}{1-\gamma}} \frac{U_{c,t+1}}{U_{c,t}},$$

where $m_{t+1} = \frac{V_{t+1}^{1-\gamma}}{\mathbb{E}_t V_{t+1}^{1-\gamma}}$. We assume that the government has access to K consols with non stochastic $p_t^k = 1$ payoff components and payoff decay rates $\{\delta^k\}_k$. Thus, the return on security k is given $R_t^k = \frac{1+(1-\delta^k)q_t^k}{q_{t-1}^k}$ and the household's Euler equations take the form

$$\mathbb{E}_{t-1} S_t R_t^k = 1 \quad \forall k.$$

Following earlier sections, we construct the normalized variables $\hat{c}_t = \frac{c_t}{\theta_t}$, $\hat{g}_t = \frac{g_t}{\theta_t}$, $\hat{W}_t = \frac{W_t}{\theta_t}$, $\hat{V}_t = \frac{V_t}{\theta_t}$, $\hat{B}_t = \frac{B_t}{\theta_t}$, and $\hat{B}_t^k = \frac{B_t^k}{\theta_t}$. After multiplying the budget constraint by $U_{c,t}$ we obtain the implementability constraint

$$\sum_k \frac{e^{-g_{\theta,t}} \hat{\mathcal{B}}_{t-1}^k \eta_t (\hat{c}_t^{-\rho} + (1 - \delta^k) \tilde{q}_t^k)}{\beta \mathbb{E}_{t-1} \left[m_t^{\frac{\rho-\gamma}{1-\gamma}} e^{-\rho g_{\theta,t}} (\hat{c}_t^{-\rho} + (1 - \delta^k) \tilde{q}_t^k) \right]} = \eta_t \hat{c}_t^{1-\rho} - \eta_t n_t^{1+\phi} + \hat{\mathcal{B}}_t.$$

Here \tilde{q}_t^k is the marginal utility weighted price of the geometrically decaying coupon bond with decay rate δ^k , which must satisfy

$$\tilde{q}_{t-1}^k = \beta \mathbb{E}_{t-1} \left[m_t^{\frac{\rho-\gamma}{1-\gamma}} e^{-\rho g_{\theta,t}} (\hat{c}_t^{-\rho} + (1 - \delta^k) \tilde{q}_t^k) \right].$$

and effective returns \mathcal{R}_t^k are given by

$$\mathcal{R}_t^k \equiv \frac{e^{-g_{\theta,t}} \eta_t (\hat{c}_t^{-\rho} + (1 - \delta^k) \tilde{q}_t^k)}{\beta \mathbb{E}_{t-1} \left[m_t^{\frac{\rho-\gamma}{1-\gamma}} e^{-\rho g_{\theta,t}} (\hat{c}_t^{-\rho} + (1 - \delta^k) \tilde{q}_t^k) \right]}$$

Two Bellman equations characterize a Ramsey problem. One value function pertains to a continuation planning problem for $t \geq 1$ that has state variables $\{\hat{\mathcal{B}}_{t-1}, \tilde{\mathbf{q}}_{t-1}, \eta_t\}$; the other pertains to a $t = 0$ problem with given initial condi-

tions \hat{B}_-, η_{-1} . To express a scaled version of the continuation planning problem, let $\hat{V}_{t-1}(\hat{\mathcal{B}}_{t-1}, \tilde{\mathbf{q}}_{t-1}, \eta_t)$ be the planner's optimal value function. It satisfies the Bellman equation⁵

$$\hat{V}_{t-1}(\hat{\mathcal{B}}_{t-1}, \tilde{\mathbf{q}}_{t-1}, \eta_t) = \max_{\hat{W}_t, m_t, \hat{c}_t, n_t, \hat{\mathbf{q}}_t, \hat{\mathcal{B}}_{t-1}} \mathbb{E}_{t-1} \left[\left(e^{g\theta_t} \hat{W}_t \right)^{1-\gamma} \right]^{\frac{1}{1-\gamma}} \quad (13)$$

where the maximization is subject to

$$\hat{\mathcal{B}}_t = \sum_k \mathcal{R}_t^k \hat{\mathcal{B}}_{t-1}^k - \eta_t \hat{c}_t^{1-\rho} + \eta_t n_t^{1+\phi} \quad (14a)$$

$$\tilde{q}_{t-1}^k = \beta \mathbb{E}_{t-1} \left[m_t^{\frac{\rho-\gamma}{1-\gamma}} e^{-\rho g\theta, t} (\hat{c}_t^{-\rho} + (1-\delta^k) \tilde{q}_t^k) \right] \quad (14b)$$

$$m_t = \frac{\left(e^{g\theta, t} \hat{W}_t \right)^{1-\gamma}}{\mathbb{E}_{t-1} \left[\left(e^{g\theta, t} \hat{W}_t \right)^{1-\gamma} \right]} \quad (14c)$$

$$n_t = \hat{c}_t + \hat{g}_t \quad (14d)$$

$$\hat{W}_t = \left((1-\beta)\eta_t \left[\hat{c}_t^{1-\rho} - (1-\rho) \frac{n_t^{1+\phi}}{1+\phi} \right] + \beta \hat{V}_t(\hat{\mathcal{B}}_t, \tilde{\mathbf{q}}_t, \eta_{t+1})^{1-\rho} \right)^{\frac{1}{1-\rho}} \quad (14e)$$

$$\hat{\mathcal{B}}_{t-1} = \sum_k \hat{\mathcal{B}}_{t-1}^k \quad (14f)$$

$$\log(\eta_{t+1}) = \rho_\eta \log(\eta_t) + \epsilon_{\eta, t} \quad (14g)$$

4.1 Computational Method

This subsection describes our computational approach to the Ramsey problem. Solving numerically the Ramsey problem on page 17 is difficult with the conventional numerical techniques because the state space consists of K endogenous variables. To overcome this curse of dimensionality to adopt the numerical methods developed in Evans (2014) and Bhandari et al. (2017b). Here we briefly outline main ideas behind this approach. Let μ_t , $\boldsymbol{\lambda}_t$, ν_t , and ξ_t be the Lagrange multipliers associated with constraints (14a)-(14d) respectively. Problem (13) is recursive in the effective debt

⁵Following Albuquerque et al. (2016) we assume that the discount factor shock, η_t , is known at time $t-1$.

$\hat{\mathcal{B}}_{t-1}$, the discount factor shock η_t , and the marginal utility weighted prices of the long maturity bonds $\tilde{\mathbf{q}}_{t-1}$. Following Marcet and Marimon (1994), we replace the vector $\tilde{\mathbf{q}}_{t-1}$ with its co-state $\boldsymbol{\lambda}_{t-1}$ and search for solutions that are recursive in $\hat{\mathcal{B}}_{t-1}, \boldsymbol{\lambda}_t, \eta_{t-1}$. We begin by classifying variables as:

- $\mathbf{z}_{t-1} \equiv \{\boldsymbol{\lambda}_{t-1}, \eta_{t-1}\}$,
- $\mathbf{y}_t \equiv \{\hat{\mathcal{B}}_t, \eta_t, \boldsymbol{\lambda}_t, \hat{c}_t, n_t, \mathcal{R}_t, \hat{W}_t, m_t, \nu_t, \xi_t, \mu_t, \hat{V}_{t-1}, \hat{V}_{x,t-1}, \tilde{\mathbf{q}}_{t-1}\}$
- $\mathbf{e}_t \equiv \{\mathbb{E}_{t-1}[\mathcal{R}_t], \mathbb{E}_{t-1}[\mathcal{R}_t \mu_t], \mathbb{E}_{t-1}[W_t^{1-\gamma}], \mathbb{E}_{t-1}[m_t \nu_t]\}$
- $\boldsymbol{\epsilon}_t \equiv \{\epsilon_{\theta,t}, \epsilon_{g,t} \epsilon_{\eta,t}\}$.

We stack the first order conditions of the Ramsey problem listed in Appendix A into the functions f , g and F that satisfy

$$\mathbf{e}_{t-1} = \mathbb{E}_{t-1}[f(\mathbf{y}_t)]. \quad (15a)$$

$$g(\mathbf{e}_{t-1}) = 0. \quad (15b)$$

$$F(\hat{\mathcal{B}}_{t-1}, \mathbf{z}_{t-1}, \mathbf{y}_t, \mathbf{e}_{t-1}, \mathbf{y}_{t+1}, \hat{\mathcal{B}}_{t-1}, \sigma \boldsymbol{\epsilon}_t) = 0 \quad (15c)$$

In our problem f is given by the definition of \mathbf{e}_{t-1} above, g is given by the first order conditions with respect to $\{\hat{\mathcal{B}}_{t-1}^k\}_{k \geq 2}$ and F is given by the remaining equations. We list all the equations in Appendix A. The optimal allocation can be represented by functions (y, B)

$$\begin{aligned} \mathbf{y}_t &= y(\hat{\mathcal{B}}_{t-1}, \mathbf{z}_{t-1}, \sigma \boldsymbol{\epsilon}_t | \sigma) \\ \hat{\mathcal{B}}_{t-1} &= B(\hat{\mathcal{B}}_{t-1}, \mathbf{z}_{t-1} | \sigma) \end{aligned}$$

that solve the system of equations (15).

We approximate (y, B) around $\hat{\mathcal{B}}_{t-1} = \hat{\mathcal{B}}_-$ and $\sigma = 0$ using a second order Taylor expansion of the form:

$$\begin{aligned} y(\hat{\mathcal{B}}_-, z_-, \sigma \boldsymbol{\epsilon} | \sigma) &= \bar{y}(\hat{\mathcal{B}}_-) + y_z(\hat{\mathcal{B}}_-)(\mathbf{z}_- - \bar{\mathbf{z}}) + y_\epsilon(\hat{\mathcal{B}}_-)\sigma \boldsymbol{\epsilon} + \frac{1}{2} \left(y_{zz}(\hat{\mathcal{B}}_-)(\mathbf{z}_- - \bar{\mathbf{z}})(\mathbf{z}_- - \bar{\mathbf{z}}) \right. \\ &\quad \left. + 2y_{z\epsilon}(\hat{\mathcal{B}}_-)(\mathbf{z}_- - \bar{\mathbf{z}})\sigma \boldsymbol{\epsilon} + y_{\epsilon\epsilon}(\hat{\mathcal{B}}_-)\sigma \boldsymbol{\epsilon} \sigma \boldsymbol{\epsilon} + y_{\sigma\sigma}(\hat{\mathcal{B}}_-)\sigma^2 \right) + \mathcal{O}(\sigma^3) \\ &\equiv \check{y}(\mathcal{B}_-, z_-, \sigma \boldsymbol{\epsilon}_t | \sigma) + \mathcal{O}(\sigma^3) \end{aligned} \quad (16)$$

and

$$\begin{aligned} B(\hat{\mathcal{B}}_-, z_- | \sigma) &= \bar{\mathcal{B}}(\hat{\mathcal{B}}_-) + B_z(\hat{\mathcal{B}}_-)(\mathbf{z}_- - \bar{\mathbf{z}}) + B_\sigma(\hat{\mathcal{B}}_-)\sigma + \mathcal{O}(\sigma^2) \\ &\equiv \check{B}(\hat{\mathcal{B}}_-, z_- | \sigma) + \mathcal{O}(\sigma^2). \end{aligned} \quad (17)$$

Here $\bar{\mathbf{y}}(\hat{\mathcal{B}}_-)$ satisfies $\bar{y} = y(\hat{\mathcal{B}}_-, \bar{z}(\hat{\mathcal{B}}_-), 0|0)$, $\bar{z}(\hat{\mathcal{B}}_-) = I_y^z \bar{\mathbf{y}}(\hat{\mathcal{B}}_-)$ where I_y^z is a matrix that selects variables \mathbf{z} from the vector \mathbf{y} , and $\bar{\mathcal{B}}(\hat{\mathcal{B}}_-) = \lim_{\sigma \rightarrow 0} B(\hat{\mathcal{B}}_-, \bar{z}(\hat{\mathcal{B}}_-) | \sigma)$. The remaining coefficients are derivatives of y and B evaluated at the non-stochastic steady state associated with $\hat{\mathcal{B}}_{t-1} = \hat{\mathcal{B}}_-$ and $\sigma = 0$.

The immediate hurdle in applying a perturbation method for problem with portfolio choice is that at $\sigma = 0$, all assets earn the same returns and so $\bar{\mathcal{B}}(\hat{\mathcal{B}}_-)$ is not pinned down when solving for non stochastic steady state. Furthermore all the derivative terms in expansion (16) and (17) are evaluated at $\hat{\mathcal{B}}_{t-1} = \hat{\mathcal{B}}_-$ and $\sigma = 0$ and implicitly depend on the choice of the steady state portfolio $\bar{\mathcal{B}}(\hat{\mathcal{B}}_-)$. To see this notice that non-stochastic steady state Ramsey plan, $\bar{\mathbf{y}}(\hat{\mathcal{B}}_-)$, can be found by solving the system of equations

$$F(\hat{\mathcal{B}}_-, I_y^z \bar{\mathbf{y}}, f(\bar{\mathbf{y}}), \bar{\mathbf{y}}, \hat{\mathcal{B}}, 0) = 0 \quad (18)$$

for any choice of $\hat{\mathcal{B}}$, and the function g given by (15b) will be identically 0 when $\sigma = 0$. In Appendix A, we show with sufficient smoothness of y and B that $\bar{\mathcal{B}}(\hat{\mathcal{B}}_-)$ solve the non-linear equation

$$g_e \mathbb{E} [f_{yy}(y_\epsilon \epsilon, y_\epsilon \epsilon)] = 0, \quad (19)$$

where y_ϵ implicitly depends on the choice of $\bar{\mathcal{B}}$. A special feature of (19) is that only the first order derivatives of y are required to determine $\bar{\mathcal{B}}$. In the same spirit we can show a corresponding result for B_z and B_σ that says that they satisfy a linear equation which depends on only second order derivatives of y .

Using these insights we proceed along the following steps

1. Given current scaled effective debt, $\hat{\mathcal{B}}_{t-1}$ and states \mathbf{z}_{t-1} , obtain $\bar{\mathbf{y}}$ and $\bar{\mathcal{B}}$ by solving (18) and (19).
2. Use the implicit function theorem to totally differentiate equations (15) around $(\bar{\mathbf{y}}, \bar{\mathcal{B}})$ and solve for the remaining derivatives in (16) and (17).
3. Use expansions in (16) and (17) to simulate the policy rules for one period and

obtain \mathbf{y}_t .

4. Restart from 1 with a new level of effective debt $\hat{\mathcal{B}}_t$ and states \mathbf{z}_t .

A sample path for the optimal Ramsey allocation is then simulated by repeating steps 1-4 until a desired simulation length has been reached.

5 Quantitative analysis

5.1 Calibration

We now turn to the quantitative analysis of the optimal portfolio in a calibrated economy. Our strategy is to choose parameters that describe preferences and shocks so that our model economy with a tax policy $\tau_t = \bar{\tau}$ produces moments that match the patterns in returns that we outlined in the previous section along with other business cycle facts for the US. After estimating these parameters we compute the Ramsey allocation.⁶

The stochastic process for the shocks is parameterized as

$$\log \frac{\theta_t}{\theta_{t-1}} = g_{\theta,t}, \quad \log \frac{g_t}{\theta_t} = \hat{g}_t$$

and

$$\begin{bmatrix} g_{\theta,t} \\ \hat{g}_t \\ \log \eta_t \end{bmatrix} = \mu + A \begin{bmatrix} g_{\theta,t-1} \\ \hat{g}_{t-1} \\ \log \eta_{t-1} \end{bmatrix} + \Sigma \begin{bmatrix} \epsilon_{\theta,t} \\ \epsilon_{g,t} \\ \epsilon_{\eta,t} \end{bmatrix}$$

We restrict A so that $g_{\theta,t}$ and \hat{g}_t are serially uncorrelated after conditioning on $\{\eta_j\}_{j=0}^t$.⁷ We use $K = 2$ and set $p^1 = p^2 = 1$ and $\delta^2 = 0.2$ capturing a maturity of the 5 year bond.⁸

⁶We have experimented with several tax rules, for example specifications where tax rates depend on output growth, expenditures and debt levels. We estimated these specifications using observed average marginal tax rates as in Barro and Redlick (2011). Our parameter estimates for preferences and shocks are not too sensitive to what tax rules we use.

⁷This implies that $\mu = \begin{bmatrix} \mu_\theta \\ \bar{g} \\ 0 \end{bmatrix}$, $A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \rho_\eta \end{bmatrix}$ and $\Sigma = \begin{bmatrix} \sigma_\theta & 0 & \sigma_{\eta\theta} \\ 0 & \sigma_g & \sigma_{\eta g} \\ \sigma_{\eta\theta} & \sigma_{\eta g} & \sigma_\eta \end{bmatrix}$

⁸We chose a two bond specification as this is the most transparent model of maturity management. 5 year bonds are chosen as these are the longest bonds for which Fama-Bliss data is available. Our numerical methods extend straightforwardly to the portfolio problems with more bonds, or with bonds of finite maturity.

Parameters	Values
Preferences:	
β	0.98
ϕ	2
ρ, γ	0.29, 23.22
Shocks:	
\bar{g}, σ_g	0.4, 0.023
$\mu_\theta, \sigma_\theta$	0.015, 0.02
ρ_η, σ_η	0.98, 0.03
$\sigma_{\eta g}, \sigma_{\eta \theta}$	0.21, 0.03

Table III: Estimated parameters

Holding period returns	Data		Macro quantities	Data	
		Model			Model
mean 1 yr bond	1.63%	1.39%	mean output growth	1.50%	1.50%
mean 5 yr bond	2.85%	2.40%	std. output growth	2.31%	2.32%
std 1 year bond	2.61%	3.14%	mean exp. output ratio	17.00%	16.00%
std. 5 year bond	6.42%	4.00%	std govt. exp growth	4.17%	4.11%
corr 1yr, 5yr	0.66	0.80	corr output, govt. exp	-0.06	0.00
corr 1/5 yr, output growth	0.04, 0.02	0.02, 0.03	Frisch elasticity	-	0.5
corr 1/5 yr, govt. exp growth	0.11, 0.15	0.09, 0.13			

Table IV: Targeted moments and model fit

We estimate $\beta, \gamma, \rho, \phi$ along with μ, A, Σ using a simulated method of moments procedure. Our target moments are listed in table IV. For a given choice of parameters we simulate 10,000 samples of length 50 and compute a distance metric that weights all the moments equally, with the caveat that deviations of means and standard deviations from the targets are measured as percent deviation. Our model fit is described in “Model” column of table IV. The underlying estimates are in table III.

Our estimation favors the Epstein-Zin specification with fairly persistent and large discount rate shocks. We find a large risk aversion around 23 and IES of 3. The model replicates excess returns on a long bond and correlations of returns and output growth and government expenditures.

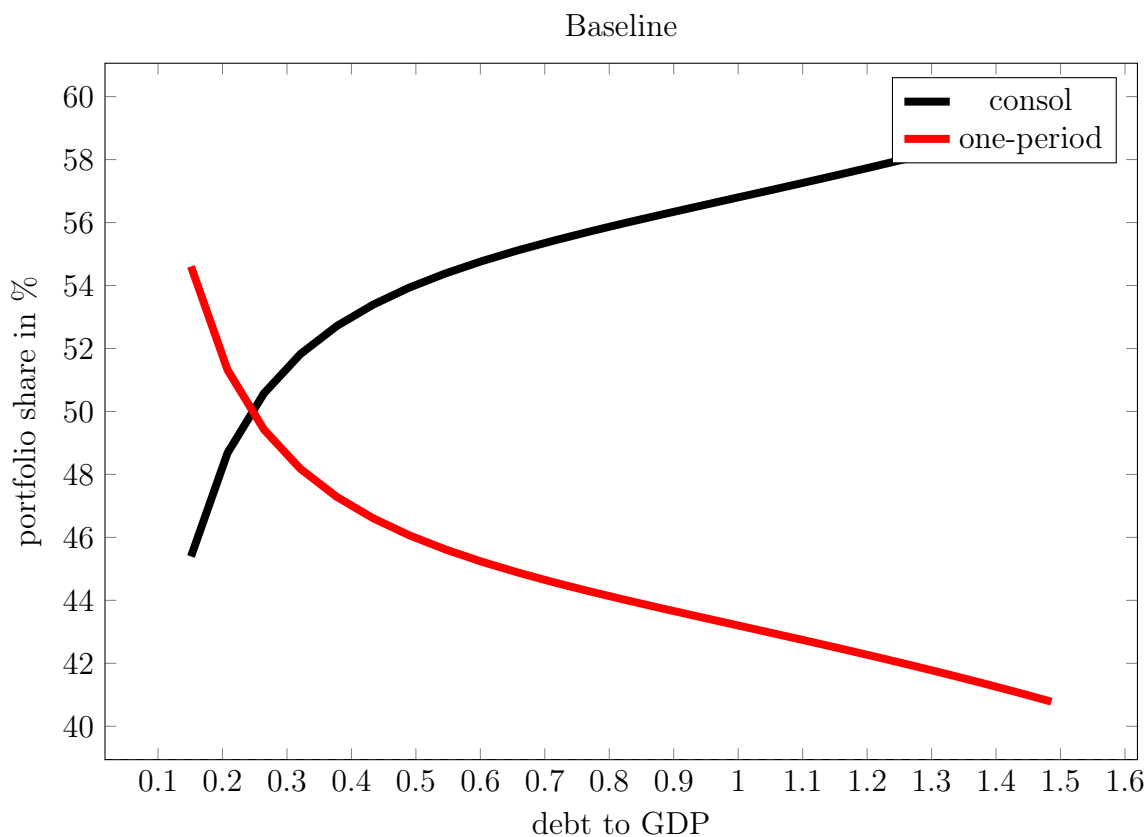


Figure II: Percentage of total debt held in the short maturity bond and the long maturity bond.

5.2 Results

In this section we describe the government’s optimal portfolio. There are two central findings: (a) the optimal portfolio is long in both the short maturity asset and the long maturity asset with about equal shares, and (b) there is little re-balancing of the portfolio in response to shocks.

Figure IV plots the portfolio holdings in the short and the long maturity bond respectively as a function of the debt to GDP ratio. The share of the total debt in the long maturity bond ranges between 46% and 58% as we vary debt between 30% and 120% of output.

The key intuition for this optimal portfolio comes from forces that working through formula (11). Returns induced by the optimal allocations retain the patterns of returns in the data, i.e., they are volatile and uncorrelated with shocks that drive the primary deficits. A portfolio with large holdings especially in the long-maturity

bond has the disadvantage of exposing the governments budget constraint to large fluctuations in returns induced by discount factor shocks. Hence it is optimal to hold moderate positions. Furthermore, the returns on the long maturity bonds are negatively correlated with the marginal utility of consumption. This implies that the volatility of effective returns is lower than the volatility of returns for the long bond and larger for the short bond. In terms of their effective returns, the two securities are more alike in their variability and this feature is reflected in the approximately 50 – 50 share that the optimal portfolio displays.

Next we turn to how the portfolios are re-balanced in response to shocks. In figure III we plot the impulse response of total debt, the share of the total debt in the long maturity, and the tax rate to a one standard deviation shock to ϵ_θ , ϵ_g and ϵ_η . We see that the portfolio is fairly stable in response to all these shocks. For example, after a 2% increase productivity growth, the government reduces the debt to GDP ratio by 1.5%, with almost no change in the portfolio structure. Secondly, for all shocks tax rates are fairly smooth but not constant. This is because with two assets, the planner cannot implement the complete markets allocation.

5.2.1 Comparison to conventional RBC models

The stochastic process for productivity and expenditure shocks in our model as well as parameters of the Frisch elasticity of labor supply and the IES are very similar to those typically used in the Ramsey models, such as Chari et al. (1994), Buera and Nicolini (2004), Schmitt-Grohe and Uribe (2004), Farhi (2010), Faraglia et al. (2012). The crucial difference is that in our model risk aversion is not equal to the IES, and substantial fluctuations in asset returns are driven by shocks orthogonal to short run productivity and expenditure shocks. The conventional representative agent models can be captured in our model by setting $\gamma = \rho$ and $\sigma_\eta = 0$. In this section we contrast our findings with those obtained under such calibrations. We refer to our model specification as the “baseline calibration” and a version in which we set $\gamma = \rho = 2$ and $\sigma_\eta = 0$ as the “RBC calibration”. This will also help to explain why our conclusions about portfolio management differ from those of Buera and Nicolini (2004), Faraglia et al. (2012) and Debortoli et al. (2016 forthcoming).

In figure IV, we overlay the optimal portfolio as a function of the debt to GDP ratio for the RBC calibration on top of the graph for the baseline calibration. The optimal portfolio now is about 10 to 20 times larger with large long positions in the

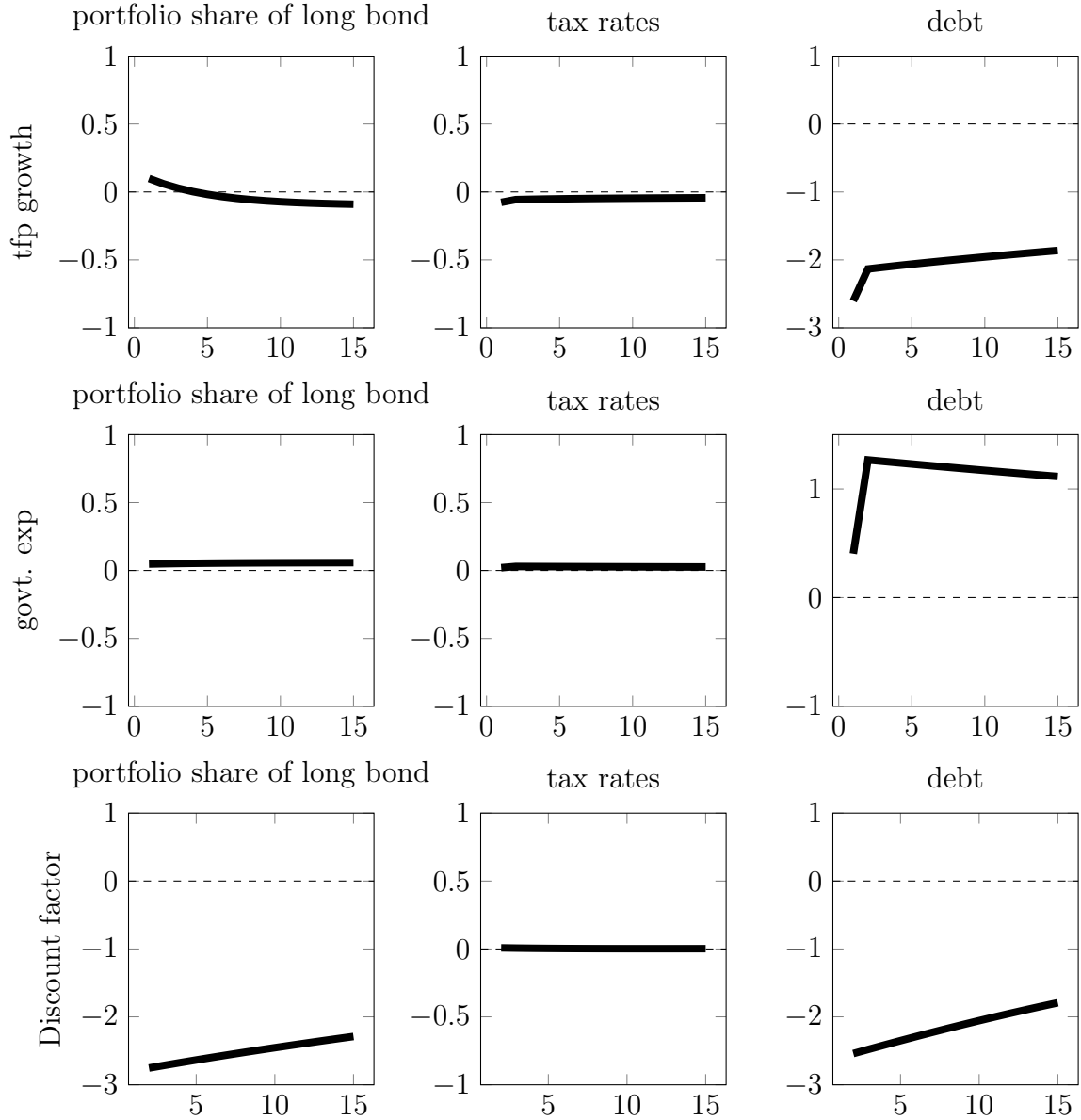


Figure III: Impulse response functions for baseline calibration for a one standard deviation shock to $\epsilon_\theta, \epsilon_g, \epsilon_\eta$ respectively. The units on the y-axis are in percentage points.

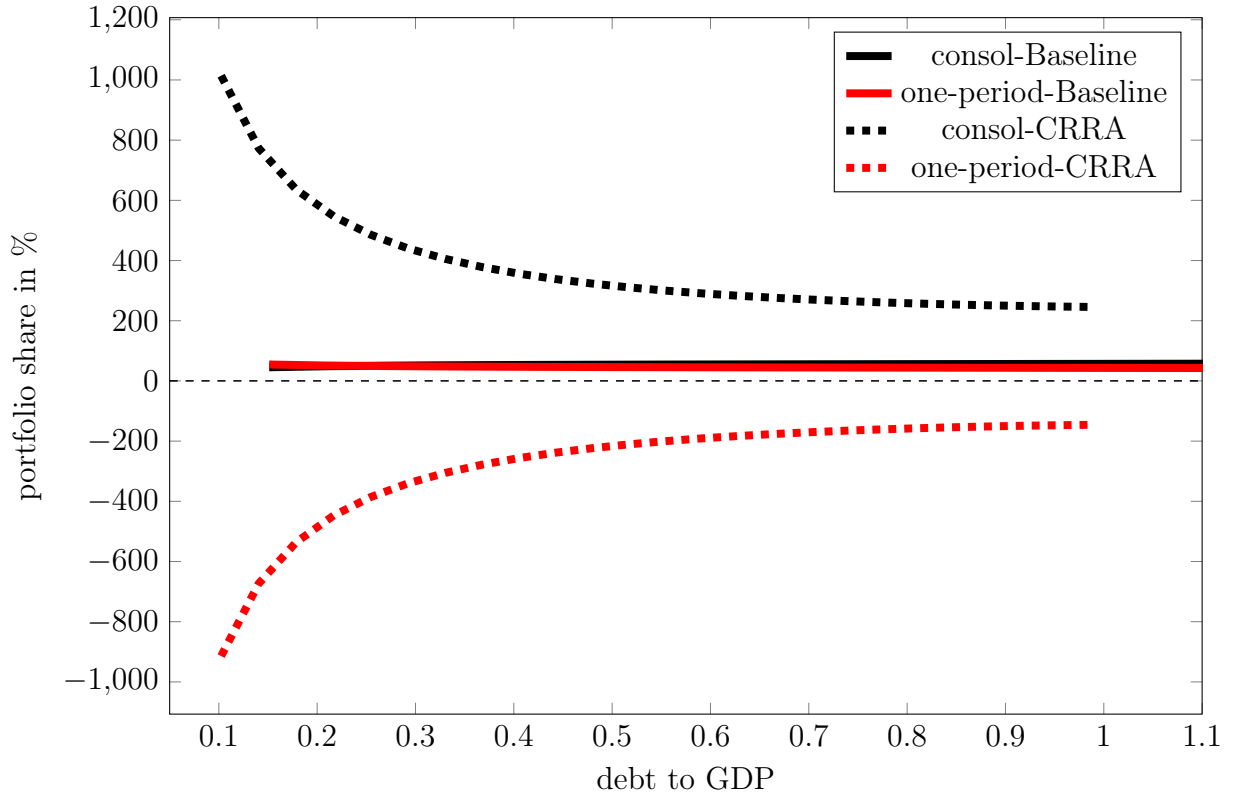


Figure IV: Percentage of total debt held in the short maturity bond and the long maturity bond for the baseline calibration (solid lines) and the CRRA-no discount factor shocks specification (dashed lines)

5 yr bond and offsetting short positions in the one year bond. In figure V we plot the impulse responses for the RBC calibration and compare them to the baseline calibration. The top panel of figure V shows that in response to a 2% increase in productivity, the response of total debt and tax rates is similar across both settings but the response of the portfolio holdings are about 10 times larger in the RBC calibration.

The source of the big differences in portfolio positions can be inferred from table V where we compare the covariance matrix of effective returns and the present value of effective deficits across both calibrations. Compared to the baseline calibration, the RBC calibration features both a lower volatility of effective returns and a much greater correlation of effective returns with innovations to the present value of effective deficits. This higher correlation provides the planner a much greater ability to hedge innovations to the present value of deficits. For instance, in figure V, the volatility

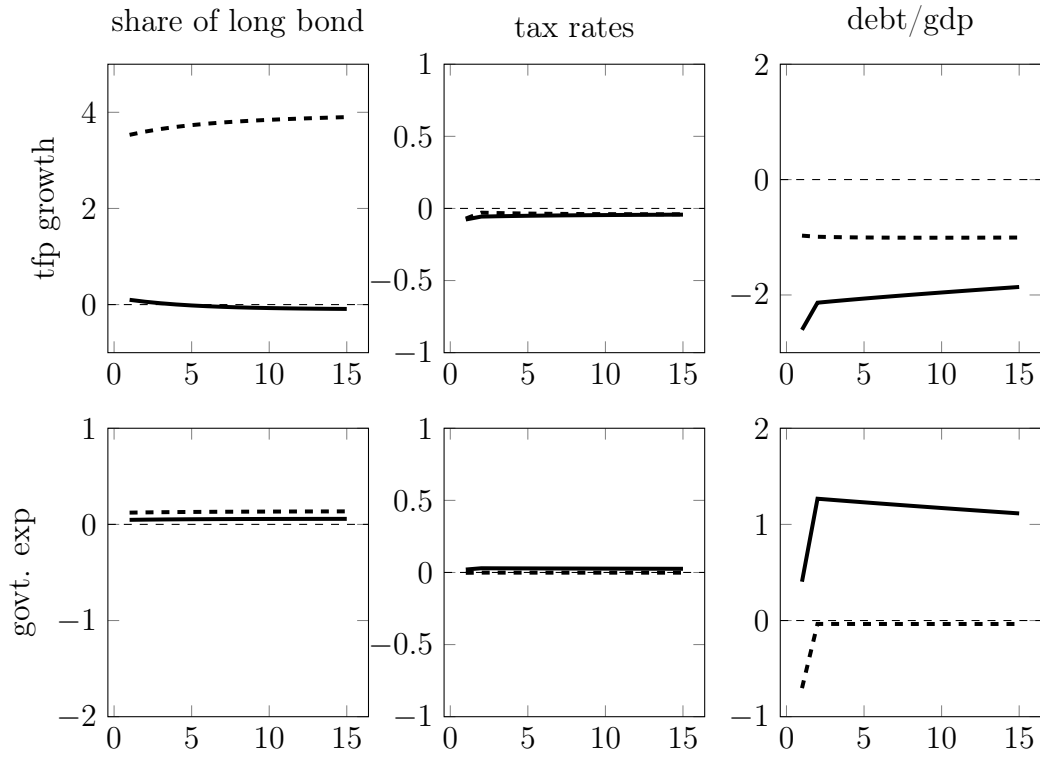
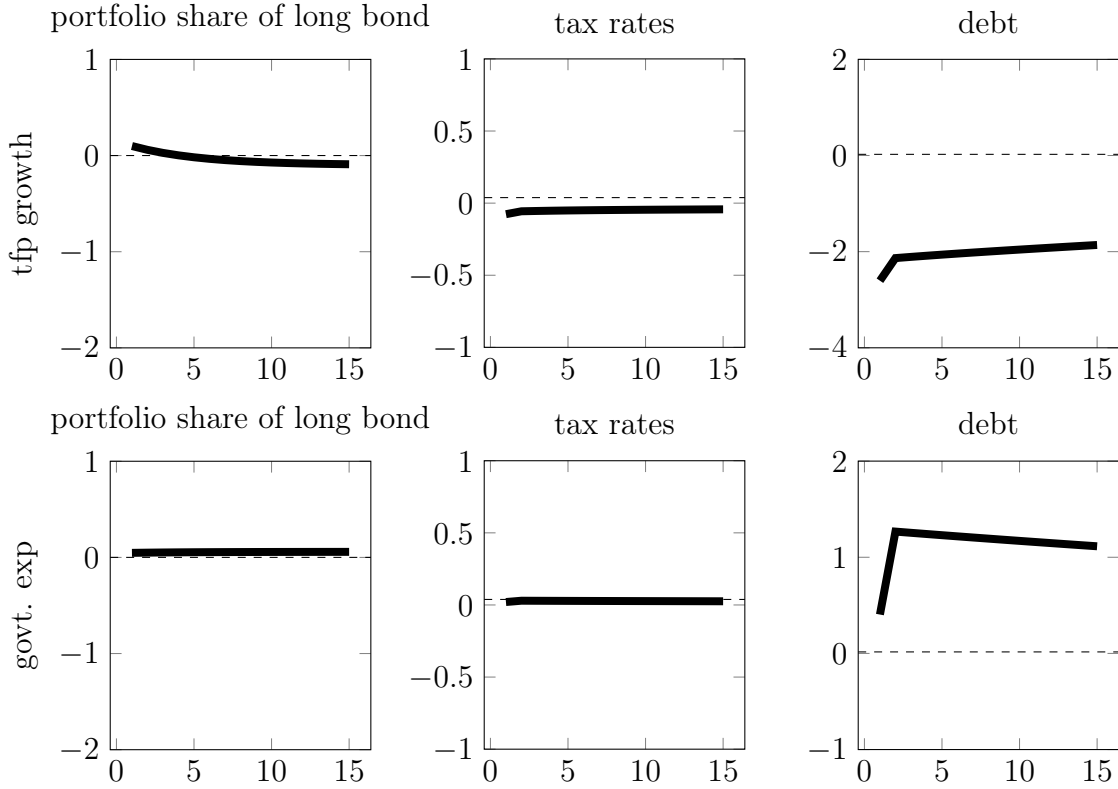


Figure V: Impulse response functions for CRRA-no discount factor shock specification (dashed) and baseline calibration (solid) for a one standard deviation shock to $\epsilon_\theta, \epsilon_g$ respectively. The units on the y-axis are in percentage points.



of tax rates and debt is much lower in the RBC calibration relative to the baseline calibration. Combined with low volatility of returns, the RBC planner necessarily has to take much larger portfolio positions in order to achieve this hedging.

6 Conclusion

In this paper we revisit the predictions of Ramsey models for optimal management of government portfolios. For a range of settings, we show that the covariance of

Baseline Calibration	short bond	long bond	deficits	RBC Calibration	short bond	long bond	deficits
short bond	0.48	-0.54	1.55	short bond	1.01	0.24	4.22
long bond	-0.54	5.55	-0.16	long bond	0.24	0.05	0.97
deficits	1.55	-0.16	19.69	deficits	4.22	0.97	42.83

Table V: Covariance matrix of effective returns and innovations to present value of effective primary deficits

marginal utility adjusted returns with primary government deficits is a key determinant of the portfolio structure. Using US data on bond prices, we document that returns on government bonds are volatile and have low correlation with macro aggregates such as output growth and primary government deficits. Findings from the asset pricing literature induce us to use recursive preference and richer shocks processes to serve the goal of studying optimal government policy in a setting that is potentially consistent with observed returns. We find that the optimal portfolio loads evenly on short and long maturities and is fairly stable with respect to business cycle shocks. A byproduct of our analysis is a computational method allows for fast and reliable solutions for Ramsey models with arbitrarily rich asset markets.

Our point of departure is a body of work that dealt with this topic in canonical Real Business Cycle type settings which feature expected utility preferences and exogenous fluctuations in productivity or government purchases. A common prescription from these models is for the government to issue large amounts of debt in the long maturities with offsetting positions in the shorter maturities. Furthermore, these models also suggest that the government should actively re-balance its portfolios. We show that the failure of RBC models to match bond returns and more generally dynamics of risk premia is exactly what drives their implications about portfolio management.

A natural extension to our exercise is to allow for inflation and study joint portfolio and monetary policy in the direction of Lustig et al. (2008). We leave this for future work.

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A First Order Conditions of The Ramsey Problem

Here we describe first-order conditions for the $t \geq 1$ continuation Ramsey problem. Define $U(\hat{c}_t, n_t) = \eta_t \left[\frac{\hat{c}_t^{1-\rho}}{1-\rho} - \frac{n_t^{1+\phi}}{1+\phi} \right]$ and its respective derivatives by $\hat{U}_{c,t}$, $\hat{U}_{n,t}$, $\hat{U}_{cc,t}$, and $\hat{U}_{nn,t}$. We begin by documenting the following derivatives of \hat{W}_t with respect to \hat{c}_t , n_t , $\hat{\mathcal{B}}_t$, and $\tilde{\mathbf{q}}_t$:

$$\begin{aligned}\hat{W}_{c,t} &= (1 - \beta)\eta_t \hat{W}_t^{-\rho} \hat{U}_{c,t} \\ \hat{W}_{n,t} &= -(1 - \beta)\eta_t \hat{W}_t^{-\rho} \hat{U}_{n,t} \\ \hat{W}_{\mathcal{B},t} &= \beta \hat{W}_t^{-\rho} \hat{V}_t^{-\rho} \hat{V}_{\mathcal{B},t} \\ \hat{W}_{\mathbf{q},t} &= \beta \hat{W}_t^{-\rho} \hat{V}_t^{-\rho} \hat{V}_{\mathbf{q},t}.\end{aligned}$$

After multiplying the implementability constraint of the Ramsey problem by $e^{(1-\rho)g_{\theta,t}} m_t^{\frac{\rho-\gamma}{1-\gamma}}$ and defining $\mathcal{R}_t^1 = m_t^{\frac{\rho-\gamma}{1-\gamma}} e^{-\rho g_{\theta,t}} \hat{U}_{c,t}$ and $\mathcal{R}_t^k = m_t^{\frac{\rho-\gamma}{1-\gamma}} e^{-\rho g_{\theta,t}} (\hat{U}_{c,t} + (1 - \delta^k) \tilde{q}_t^k)$ for $k \geq 2$ the recursive version of the Ramsey problem becomes

$$\hat{V}_{t-1}(\hat{\mathcal{B}}_{t-1}, \tilde{\mathbf{q}}_{t-1}, \eta_t) = \max_{\hat{W}_t, m_t, \hat{c}_t, n_t, \tilde{\mathbf{q}}_t, \hat{\mathcal{B}}_t, \hat{\mathcal{B}}_{t-1}} \mathbb{E}_{t-1} \left[\left(e^{g_{\theta,t}} \hat{W}_t \right)^{1-\gamma} \right]^{\frac{1}{1-\gamma}}$$

subject to

$$\hat{\mathcal{B}}_t e^{(1-\rho)g_{\theta,t}} m_t^{\frac{\rho-\gamma}{1-\gamma}} = \frac{(\hat{\mathcal{B}}_{t-1} - \sum_{k \geq 2} \hat{\mathcal{B}}_{t-1}^k) \mathcal{R}_t^1}{\beta \mathbb{E}_{t-1} [\mathcal{R}_t^1]} + \sum_{k \geq 2} \frac{\hat{\mathcal{B}}_{t-1}^k \mathcal{R}_t^k}{\beta \mathbb{E}_{t-1} [\mathcal{R}_t^k]} - (\hat{c}_t \hat{U}_{c,t} + n_t \hat{U}_{n,t}) e^{(1-\rho)g_{\theta,t}} m_t^{\frac{\rho-\gamma}{1-\gamma}} \quad (20a)$$

$$\tilde{q}_{t-1}^k = \beta \mathbb{E}_{t-1} [\mathcal{R}_t^k] \quad (20b)$$

$$m_t = \frac{\left(e^{g_{\theta,t}} \hat{W}_t \right)^{1-\gamma}}{\mathbb{E}_{t-1} \left[\left(e^{g_{\theta,t}} \hat{W}_t \right)^{1-\gamma} \right]} \quad (20c)$$

$$n_t = \hat{c}_t + \hat{g}_t \quad (20d)$$

$$\hat{W}_t = \left((1 - \beta)\eta_t \left[\hat{c}_t^{1-\rho} - (1 - \rho) \frac{n_t^{1+\phi}}{1 + \phi} \right] + \beta \hat{V}_t(\hat{\mathcal{B}}_t, \tilde{\mathbf{q}}_t, \eta_{t+1})^{1-\rho} \right)^{\frac{1}{1-\rho}} \quad (20e)$$

$$\log(\eta_{t+1}) = \rho_\eta \log(\eta_t) + \epsilon_{\eta,t} \quad (20f)$$

Letting μ_t , λ_{t-1} , ν_t , and ξ_t be the multipliers of constraints (20a)-(20d) respectively, the envelope conditions for the planner's problem imply

$$\hat{V}_{\mathcal{B},t-1} = \frac{\mathbb{E}_{t-1}[\mathcal{R}_t^1 \mu_t]}{\beta \mathbb{E}_{t-1}[\mathcal{R}_t^1]} \quad (21)$$

$$\hat{V}_{\mathcal{q},t-1} = -\lambda_{t-1}. \quad (22)$$

The first order conditions with respect to $\hat{\mathcal{B}}_t$, \hat{c}_t , n_t , $\hat{\mathcal{q}}_t$, and m_t are then given by

$$e^{(1-\gamma)g_{\theta,t}} \hat{V}_{t-1}^\gamma \hat{W}_t^{-\gamma} \hat{W}_{\mathcal{B},t} - m_t^{\frac{\rho-\gamma}{1-\gamma}} e^{(1-\rho)g_{\theta,t}} \mu_t + \frac{(1-\gamma)e^{(1-\gamma)g_{\theta,t}} \hat{W}_t^{-\gamma} \hat{W}_{\mathcal{B},t}}{\mathbb{E}_{t-1} \left[\left(e^{g_{\theta,t}} \hat{W}_t \right)^{1-\gamma} \right]} (\nu_t - \mathbb{E}_{t-1}[m_t \nu_t]) = 0 \quad (23)$$

$$\begin{aligned} & e^{(1-\gamma)g_{\theta,t}} \hat{V}_{t-1}^\gamma \hat{W}_t^{-\gamma} \hat{W}_{c,t} + \frac{(1-\gamma)e^{(1-\gamma)g_{\theta,t}} \hat{W}_t^{-\gamma} \hat{W}_{c,t}}{\mathbb{E}_{t-1} \left[\left(e^{g_{\theta,t}} \hat{W}_t \right)^{1-\gamma} \right]} (\nu_t - \mathbb{E}_{t-1}[m_t \nu_t]) \\ & + \left(\frac{(\hat{\mathcal{B}}_{t-1} - \sum_{k \geq 2} \hat{\mathcal{B}}_{t-1}^k) \mathcal{R}_{c,t}^1}{\beta \mathbb{E}_{t-1}[\mathcal{R}_t^1]} + \sum_{k \geq 2} \frac{\hat{\mathcal{B}}_{t-1}^k \mathcal{R}_{c,t}^k}{\beta \mathbb{E}_{t-1}[\mathcal{R}_t^k]} \right) (\mu_t - \beta \hat{V}_{\mathcal{B},t-1}) \\ & - (\hat{c}_t \hat{U}_{cc,t} + \hat{U}_{c,t}) e^{(1-\rho)g_{\theta,t}} m_t^{\frac{\rho-\gamma}{1-\gamma}} \mu_t + \beta \sum_{k \geq 2} \lambda_{t-1}^k \mathcal{R}_{c,t}^k - \xi_t = 0 \end{aligned} \quad (24)$$

$$e^{(1-\gamma)g_{\theta,t}} \hat{V}_{t-1}^\gamma \hat{W}_t^{-\gamma} \hat{W}_{n,t} - (n_t \hat{U}_{nn,t} + \hat{U}_{n,t}) e^{(1-\rho)g_{\theta,t}} m_t^{\frac{\rho-\gamma}{1-\gamma}} \mu_t + \xi_t = 0 \quad (25)$$

$$\begin{aligned} & e^{(1-\gamma)g_{\theta,t}} \hat{V}_{t-1}^\gamma \hat{W}_t^{-\gamma} \hat{W}_{q^k,t} + \frac{(1-\gamma)e^{(1-\gamma)g_{\theta,t}} \hat{W}_t^{-\gamma} \hat{W}_{q^k,t}}{\mathbb{E}_{t-1} \left[\left(e^{g_{\theta,t}} \hat{W}_t \right)^{1-\gamma} \right]} (\nu_t - \mathbb{E}_{t-1}[m_t \nu_t]) \\ & + \frac{\hat{\mathcal{B}}_{t-1}^k \mathcal{R}_{q^k,t}^k}{\beta \mathbb{E}_{t-1}[\mathcal{R}_t^k]} (\mu_t - \beta \hat{V}_{\mathcal{B},t-1}) + \beta \lambda_{t-1}^k \mathcal{R}_{q^k,t}^k = 0 \end{aligned} \quad (26)$$

$$- \left(\frac{(\hat{\mathcal{B}}_{t-1} - \sum_{k \geq 2} \hat{\mathcal{B}}_{t-1}^k) \mathcal{R}_{m,t}^1}{\beta \mathbb{E}_{t-1}[\mathcal{R}_t^1]} + \sum_{k \geq 2} \frac{\hat{\mathcal{B}}_{t-1}^k \mathcal{R}_{m,t}^k}{\beta \mathbb{E}_{t-1}[\mathcal{R}_t^k]} \right) \beta \hat{V}_{\mathcal{B},t-1} + \sum_{k \geq 2} \beta \mathcal{R}_{m,t}^k \lambda_{t-1}^k - \nu_t = 0 \quad (27)$$

while the first order conditions with respect to the portfolio choice $\hat{\mathbf{B}}_{t-1}$ are

$$\frac{\mathbb{E}_{t-1} [\mathcal{R}_t^k \mu_t]}{\mathbb{E}_{t-1} [\mathcal{R}_t^k]} - \frac{\mathbb{E}_{t-1} [\mathcal{R}_t^1 \mu_t]}{\mathbb{E}_{t-1} [\mathcal{R}_t^1]} = 0. \quad (28)$$