Inequality, Business Cycles and Monetary-Fiscal Policy*

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Abstract

We study monetary and fiscal policy in a heterogeneous agents model with incomplete markets and nominal rigidities. We develop numerical techniques that allow us to approximate Ramsey plans in economies with substantial heterogeneity. In a calibrated model that captures features of income inequality in the US, we study optimal responses of nominal interest rates and labor tax rates to productivity and cost-push shocks. Optimal policy responses are an order of magnitude larger than in a representative agent economy, and for cost-push shocks are of opposite signs. Taylor rules poorly approximate optimal nominal interest rates.

Key words: Sticky prices, heterogeneity, business cycles, monetary policy, fiscal policy

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1 Introduction

An empirical labor literature has documented that dispersions of labor earnings, assets, and other measures of inequality co-move with aggregate business cycle fluctuations. Meanwhile, a quantitative macroeconomics literature that studies optimal monetary and/or fiscal policy over business cycles relies almost exclusively either on a representative agent assumption or oversimplified models of heterogeneity. We want to know how those simplified treatments of heterogeneity affect quantitative prescriptions. Therefore, this paper studies optimal monetary and fiscal policies in a workhorse New Keynesian model augmented to capture rich heterogeneities across agents and empirical facts about co-movements of aggregate variables and measures of inequality.

We study a New Keynesian economy populated by a continuum of heterogeneous agents who are subject to idiosyncratic wage risks. Agents differ in skills and their exposure to aggregate shocks that are calibrated to emulate dynamics of the distribution of U.S. labor earnings documented by Guvenen et al. (2014). Financial markets are incomplete with agents differing in their holdings of stocks, bonds and their access to financial markets. We study how a Ramsey planner adjusts nominal interest rates, transfers, and proportional labor taxes and transfers in response to aggregate shocks.

In studying a Ramsey planner's best policies, we confront substantial computational challenges. Existing studies of economies with heterogeneities, incomplete markets, and aggregate shocks typically approximate the ergodic distributions of competitive equilibrium prices and quantities under a given set of policies and cannot easily be extended to answer normative questions. Ramsey problems bring additional challenges. First, because they assign important roles to Lagrange multipliers on individual and aggregate forward looking constraints, they have additional state variables that summarize history dependencies making the state vectors larger than what are often used to characterize recursive competitive equilibria in applied studies. Second, due to market incompleteness, some state variables exhibit very slow rates of mean-reversion, implying that approximations around a mean of an invariant distribution poorly approximate an optimal policy during a transition from a given initial distribution.

This paper contributes a new computational technique that allows us to obtain good approximations to optimal government policies for economies with such large state spaces. Our numerical methods build on perturbation theory that uses small noise expansions with respect to a one-dimensional parameterization of uncertainty as in Fleming (1971) and Fleming and Souganidis (1986) that has been applied earlier in economics by Anderson et al. (2012). These are related to but differ from expansions in Judd and Guu (1993, 1997) and Judd (1996, 1998) that employ small noise expansions with respect to shocks and state variables about a deterministic steady state. A key step is that at each date, we take a Taylor expansion of policy functions around the current value of state vector with respect to a parameter that scales both idiosyncratic and aggregate shocks.

The current state vector can include a distribution of idiosyncratic states. We thus update the point around which local approximations are taken each period, which allows our approximations to remain accurate even in settings where transition dynamics are slow.

To manage heterogeneity, we approximate the distribution of individual state variables using a discrete grid with a sufficiently large number of points. Our contribution here is to develop a general framework and derive explicit formulas for coefficients occurring in the Taylor expansions of individual agents' and aggregate policy functions. A key step that makes our analysis tractable is a "factorization theorem" using which we show that these formulas require matrix inversions only of manageable dimensions, often equal to the number of *aggregate* variables, and that they can be efficiently computed. In this way, our procedure allows fast approximations even for a large number of agents. That allows us to construct nonlinear impulse responses that describe how distributions across agents respond to an aggregate shock. In Section 3.3, we describe the steps comprising our algorithm and how our method compares to other approaches.

Applying our approach to a calibrated New Keynesian economy with heterogeneous agents, we find that attitudes about inequality induce the planner to use fiscal and monetary tools to redistribute resources toward agents who are especially adversely affected by recessions. We study two types of shocks: shocks to the growth rate of productivity that also change the distribution of labor earnings in ways documented by Guvenen et al. (2014) and cost-push shocks that we model as a shock to the elasticity of substitution between goods that leads to an increase in the the desired mark-ups for the firms. We compare our results to optimal outcomes from a representative agent economy and to outcomes under a competitive equilibrium with simple fiscal and monetary policies rules (such as a Taylor rule) used in existing studies that investigate the transmission of monetary shocks.

In response to a negative productivity shock, we find that optimal monetary policy lowers nominal rates while keeping expected inflation near zero. But the planner also engineers high unanticipated inflation in recessions because that is a good way to transfer resources from agents with high bond holdings toward agents with low holdings. This transfer makes up for the inability of agents fully to insure against aggregate shocks. An optimal plan induces that surprise inflation by increasing the tax rate, which raises real wages and marginal costs for firms. Furthermore, as in data, recessions in our calibrated economy are accompanied by persistent increases inequality. This generates a motive to redistribute labor income from productive agents by increasing transfers. The planner achieves this by keeping marginal labor tax rates high long after output has recovered. We find that in response to a productivity shock that lowers output growth by 3%, there is a nearly permanent increase in the labor tax rate of about 0.5 percentage points and a 0.25 percentage points jump in inflation for one period. As a point of comparison, the optimal tax rate and inflation rate in an economy without heterogeneity are constant for permanent productivity shock and an order of magnitude lower if we impose zero lump-sum transfers.

In response to a "cost-push" shock an optimal policy calls for a significant decrease in nominal interest rates that generates an increase in inflation and output. This policy response is opposite from that found in a representative agent economy. The explanation for this difference is that in response to a cost-push shock, firms want to increase their prices, but the presence of nominal rigidities makes that costly. So in a representative agent economy, a Ramsey planner increases nominal interests rates to reduce output and marginal costs enough to offset that force for inflation, a response that Galí (2015) dubs "leading against the wind". The markup shock also decreases the labor share and increases the profit share, which in heterogeneous agent economies redistributes resources from agents who mainly obtain income from wages to agents to with large stock holdings. Leaning against the wind exacerbates this effect. When we calibrate the distribution of equity ownership to U.S. data, we find that a 1 percentage point positive shock calls for a -0.5 percentage point decrease in the nominal interest rate compared to 0.05 percentage point increase with a representative agent calibration.

We also investigate to what extent Taylor rules approximate an optimal policy. We find that in heterogeneous agent economies Taylor rules do a substantially worse job than in a representative agent counterpart. In absence of heterogeneity, the main trade-off for optimal policy is to stabilize inflation and a Taylor rule with sufficiently large loading on inflation can meet this objective well. In our economy, the optimal policy requires inflation that is large and but very short lived. The allocation with Taylor rules implies that interest rates and inflation share the same persistence and co-move positively. This means that high inflation necessarily comes at an cost of expected high inflation in the future and such behavior is sub-optimal.

As a part of robustness we do several alternatives that modify one feature at a time relative to our baseline, with the key ones being - adding borrowing frictions, alternative ways of setting Pareto weights, different specification for stochastic processes of idiosyncratic shocks. We find that with borrowing frictions that are calibrated to observed distribution of marginal propensities to consume, there is a larger role for fiscal policy - especially in the optimal timing of lump-sum transfers, alternative Pareto weights mainly affect the steady state level of distortionary taxes but not much how monetary and fiscal policies respond to aggregate shocks and the response of nominal rates, tax rates, and inflation are lower when aggregate productivity shocks are not accompanied with shifts in the skill distribution.

We begin by describing our model and some properties of the Ramsey allocation in Section 2. The numerical method and its comparison to alternatives are discussed in Section 3. We use our method to obtain quantitative results in the calibrated economy in Section 4 and 5. Section 6 concludes.

2 Environment

A continuum of infinitely lived households face idiosyncratic shocks to their productivities. Individual i's preferences over stochastic processes for a final consumption good $\{c_{i,t}\}$ and labor supply $\{n_{i,t}\}$ are ordered by

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t u\left(c_{i,t}, n_{i,t}\right)$$

where

$$u\left(c,n\right) = \frac{c^{1-\nu}}{1-\nu} - \frac{n^{1+\gamma}}{1+\gamma}, \tag{1}$$

$\mathbb{E}_t$ is a mathematical expectations operator conditioned on time t information and $\beta \in (0,1)$ is a time discount factor. We use u_c, u_n to denote partial derivaties with the respect to c and n, and higher order derivatives are denoted analogously.

The economy is subject to aggregate and idiosyncratic shocks. In our baseline specification we focus on only one source of aggregate uncertainty – the aggregate productivity shock Θ_t – that follows a stochastic process described by

$$\ln \Theta_t = \rho_\Theta \ln \Theta_{t-1} + \mathcal{E}_{\Theta,t},$$

where $\mathcal{E}_{\Theta,t}$ is a mean-zero, i.i.d. random variables and $\rho_\Theta \in [0,1)$. Aggregate and idiosyncratic shocks relate to individual i's labor productivity $\theta_{i,t}$ by

$$\ln \theta_{i,t} = \ln \Theta_t + e_{i,t} + \varsigma_{i,t}, \tag{2}$$

$$e_{i,t} = \rho_e e_{i,t-1} + \left(1-\rho_e\right)\bar{e} + f\left(e_{i,t-1}\right)\mathcal{E}_{\Theta,t} + \eta_{i,t}, \tag{3}$$

where $\varsigma_{i,t}$, $\eta_{i,t}$ are also mean-zero, i.i.d. random variables. This specification of idiosyncratic shocks builds closely on formulations used in labor literature, e.g. Storesletten et al. (2001), Low et al. (2010) where $\varsigma_{i,t}$ and $\eta_{i,t}$ correspond to transitory and persistent shocks to individual productivity. The function $f\left(e_{i,t-1}\right)$ individuals' skills vary with aggregate shocks. It allows us to match the business cycles in cross sections that are documented by Guvenen et al. (2014). We assume that all shocks take values in a compact set.

Agent i supplies $\theta_{i,t} n_{i,t}$ units of effective labor to a competitive labor market at nominal wage $P_t W_t$, where P_t is the nominal price of the final consumption good at time t. There is a common proportional labor tax rate τ_t and a common lump transfer $T_t P_t$. Agents trade a one-period risk-free nominal bond with price Q_t with each other and with the government. We use $P_t b_{i,t}$, $P_t B_t$ to denote bond holdings of agent i and the debt position of the government respectively, and ι_t,π_t to denote the nominal interest rate and inflation. Finally, $d_{i,t}$ are dividends from intermediate goods producers measured in units of the final good. We take as given an initial price level $P_{-1} < \infty$ and set $\iota_{-1} = \beta^{-1} - 1$.

Agent i's budget constraint can be written as

$$c_{i,t} + Q_t b_{i,t} = (1 - \tau_t) W_t \theta_{i,t} n_{i,t} + T_t + d_{i,t} + \frac{b_{i,t-1}}{1 + \pi_t}. \tag{4}$$

The government's budget constraint at time t is

$$\bar{G} + T_t + \left(\frac{1 + \iota_{t-1}}{1 + \pi_t} \right) B_{t-1} = \tau_t W_t \int_i \theta_{i,t} n_{i,t} di + B_t,$$

where $\bar{G}$ is the level of government non-transfer expenditures.

A final good Y_t is produced by competitive firms that use a continuum of intermediate goods $\{y_t(j)\}_{j \in [0,1]}$ in a production function

$$Y_t = \left[\int_0^1 y_t(j)^{\frac{\epsilon - 1}{\epsilon}} dj \right]^{\frac{\epsilon}{\epsilon - 1}}.$$

The final good producer takes the final good prices P_t and intermediate goods prices $\{p_t(j)\}_j$ as given and solves

$$\max_{\{y_t(j)\}_{j \in [0,1]}} P_t \left[\int_0^1 y_t(j) dj \right]^{\frac{\epsilon}{\epsilon - 1}} - \int_0^1 p_t(j) y_t(j) dj. \tag{5}$$

Outcomes of optimization problem (5) are a demand function for intermediate goods

$$y_t(j) = \left(\frac{p_t(j)}{P_t} \right)^{-\epsilon} Y_t,$$

and a nominal price satisfying

$$P_t = \left(\int_0^1 p_t(j)^{1-\epsilon} \right)^{\frac{1}{1-\epsilon}}.$$

Intermediate goods $y_t(j)$ are produced by monopolists having decreasing returns to scale technology

$$y_t(j) = \left[n_t^D(j) \right]^{\alpha},$$

where $n_t^D(j)$ is the amount of effective labor hired by firm j and $\alpha \in (0, 1]$. These monopolists face downward sloping demand curves $\left(\frac{p_t(j)}{P_t} \right)^{-\epsilon} Y_t$ and choose prices $p_t(j)$ while bearing quadratic Rotemberg (1982) price adjustment costs $\frac{\psi}{2} \left(\frac{p_t(j)}{p_{t-1}(j)} - 1 \right)^2$ measured in units of the final consumption good. Firm j chooses prices $\{p_t(j)\}_t$ that solve

$$\max_{\{p_t(j)\}_t} \mathbb{E}_0 \sum_t \beta^t \left(\frac{C_t}{C_0}\right)^{-\nu} \left\{ \left[\frac{p_t(j)}{P_t} - W_t \left(\left(\frac{p_t(j)}{P_t}\right)^{-\epsilon} Y_t \right)^{\frac{1-\alpha}{\alpha}} \right] \left(\frac{p_t(j)}{P_t}\right)^{-\epsilon} Y_t - \frac{\psi}{2} \left(\frac{p_t(j)}{p_{t-1}(j)} - 1 \right)^2 \right\}, \tag{6}$$

where for convenience we have imposed that each firm values profit streams with a stochastic discount factor that is driven by aggregate consumption $C_t = \int c_{i,t} di$.[1]

In the symmetric equilibrium $p_t(j) = P_t$, $y_t(j) = Y_t$ for all j and market clearing conditions in labor, goods, and bond markets are:

$$N_t = \int \theta_{i,t} n_{i,t} di, \qquad D_t = Y_t - W_t N_t - \frac{\psi}{2}\pi_t^2, \tag{7}$$

$$y_t(j) = Y_t = N_t^{\alpha}, \tag{8}$$

$$C_t + \bar{G} = Y_t - \frac{\psi}{2}\pi_t^2, \tag{9}$$

$$\int_i b_{i,t} di = B_t. \tag{10}$$

Each agent in period 0 is characterized by a triple $(e_{i,-1}, b_{i,-1}, s_i)$ where e_i is agent i persistent productivity component, $b_{i,-1}$ is her initial holdings of debt, and s_i is her stock ownership. Agent i dividends in period t are given by $d_{i,t} = s_i D_t$.

Definition 1. An *allocation* is a sequence $\{c_{i,t}, n_{i,t}\}_{i,t}$. A *bond profile* is a sequence $\left\{\{b_{i,t}\}_i, B_t\right\}_t$. A *price* system is a sequence $\{W_t, P_t\}_t$. A monetary policy is a sequence $\{Q_t, T_t\}_t$. A *monetary-fiscal policy* is a sequence $\{Q_t, T_t, \tau_t\}_t$. *An initial condition* is the distribution $\{\bar{e}_i, b_{i,-1}, s_i\}_i$ and the initial price levels $p_{-1}(j) = P_{-1}$ for all j,

Definition 2. Given an initial condition, a *competitive equilibrium* is a monetary-fiscal policy $\{Q_t, T_t, \tau_t\}_t$ and a sequence $\left\{\{c_{i,t}, n_{i,t}, b_{i,t}\}_i, B_t, W_t, P_t\right\}_t$ such that: (i) $\{c_{i,t}, n_{i,t}, b_{i,t}\}_{i,t}$ maximize (1) subject to (4) and natural debt limits; (ii) final goods firms choose $\{y_t(j)\}_j$ to maximize (5); (iii) intermediate goods producers' prices solve (6) and satisfy $p_t(j) = P_t$; and (iv) market clearing conditions (8), (9) and (10) are satisfied.

A utilitarian Ramsey planner orders allocations by

$$\mathbb{E}_0 \int \sum_{t=0}^{\infty} \beta^t \left[\frac{c_{i,t}^{1-\nu}}{1-\nu} - \frac{n_{i,t}^{1+\gamma}}{1+\gamma} \right] di. \tag{11}$$

[1] In economies with heterogeneous agents and incomplete markets one has to take a stand on how firms are valued. Using aggregate consumption to drive a stochastic discount factor process allows us to get a representative agent economy as a special case of our heterogeneous agent economy by appropriately setting some of our parameters. This choice aligns with Kaplan et al. (2016).

Definition 3. Given an initial condition and a tax sequence $\tau_t = \bar{\tau}$ for some $\bar{\tau}$, *an optimal monetary policy* is a sequence $\{Q_t, T_t\}_t$ that supports a competitive equilibrium allocation that maximizes (11). *An optimal monetary-fiscal policy* given an initial condition is a sequence $\{Q_t, T_t, \tau_t\}_t$ that support a competitive equilibrium allocation that maximizes (11). A maximizing monetary or monetary-fiscal policy is called a *Ramsey plan*; an associated allocation is called a *Ramsey allocation.*

The distinction between optimal monetary and monetary-fiscal policies is that the former takes tax rates as given while the latter also optimizes with respect to tax rates. A common argument is that institutional constraints make it difficult to adjust tax rates in response to typical business cycle shocks, leaving nominal interest rates as the government's only tool for responding to such shocks. We will capture that argument by studying optimal monetary policy when tax rates $\{\tau_t\}_t$ are fixed at some level $\bar{\tau}$. The monetary-fiscal Ramsey plan evaluates the optimal policies when this restriction is dropped.

Standard arguments (e.g. Lucas and Stokey (1983), Galí (2015)) establish that a sequence $\left\{\{c_{i,t}, n_{i,t}, b_{i,t}\}_i, B_t, W_t, P_t, Q_t, \pi_t, \tau_t\right\}_t$ is a competitive equilibrium if and only if it satisfies (4), (7)-(10) and

$$Q_{t-1}u_{c,i,t-1} = \mathbb{E}_{t-1}\frac{u_{c,i,t}}{1+\pi_t},\ (1-\tau_t)W_t\theta_{i,t}u_{c,i,t} = -u_{n,i,t} \text{ for all } i, \tag{12}$$

$$\left(\frac{Y_t\left[1-\epsilon\left(1-\frac{W_t}{\alpha N_t^{\alpha-1}}\right)\right]}{\psi}\right) - \pi_t(1+\pi_t) + \beta\mathbb{E}_t\left(\frac{C_{t+1}}{C_t}\right)^{-\nu}\pi_{t+1}(1+\pi_{t+1}) = 0. \tag{13}$$

Finding an optimal policy amounts to finding a competitive equilibrium that maximizes welfare (11). We describe how we numerically finding such equilibria next.

3 Computational strategy

The Ramsey problem is difficult to analyze using existing numerical techniques as our model features extensive heterogeneity which makes it hard to find even the competitive equilibrium associated with an exogenously given government policy. The key challenge is that the underlying state of the economy is the distribution of individual characteristics, which is a large dimensional object. Many existing approaches[2] to analyzing such models solve for the invariant cross-sectional distribution in a competitive equilibrium without aggregate shocks, and then approximate aggregate responses around that distribution. Our problem is more complex since we also need to find the optimal policies themselves and so the invariant distribution is unknown. Moreover, since solutions to Ramsey problem often feature martingale-like components (see, e.g. Aiyagari et al. (2002)) the rate of convergence to the invariant distribution may be extremely slow, and therefore the behavior of the economy around that distribution may not be informative about its behavior around a given

[2] We discuss the comparison to other methods in the literature in Section 3.3

state. These problems make the existing numerical methods inapplicable to our settings.

To overcome these challenges we build on the ideas in Fleming (1971), Fleming and Souganidis (1986), and Anderson et al. (2012) and develop a new approach that involves constructing a power series approximation to policy rules at each period around the aggregate state that prevails in the economy in that period. This approach does not require us to known anything about an invariant distribution, captures transition dynamics, and is fast and highly parallelizable. As an additional benefit, it readily extends to second- and higher-order approximations. The second order approximations allow us to capture the effects of aggregate risk on the competitive equilibrium dynamics in economies with significant heterogeneity that has been a challenge for the existing literature.

Our plan for this section is as follows. We first explain in Section 3.1 how our approach works in simpler settings where we fix government policy at an exogenously given level and approximate the competitive equilibrium. This application is both more transparent than a more involved Ramsey problem, and is also useful in its own right as it show our approach can overcome challenges faced by the existing methods. Then in Section 3.2 we show that the same techniques extend to Ramsey settings with minimal changes. Finally, in Section 3.3 we discuss the relationship of our method to existing alternatives. The discussion in these subsections is self-contained. A reader not interested in computational techniques may skip directly to Section 4.

3.1 Basic ideas: competitive equilibrium for given a policy rule

We first consider the problem of finding the competitive equilibrium for a given government policy. To make our exposition most transparent, we assume that government has no expenditures, sets taxes $\tau_t = T_t = 0$ and implements inflation $\pi_t = 0$ in all periods. We assume all agents have equal ownership of firms and that there is no permanent component of labor productivity, $e_{i,t} = 0$. These assumptions represent a simple non-trivial case that allows us to explain our approach in a transparent way. It also has a natural and interesting interpretation: this is a version of Huggett (1993) economy with natural borrowing limits extended to allow for endogenous labor supply, decreasing returns to scale in production, and aggregate shocks. In addition, the computational algorithm and techniques described are general enough to make it operational to extend to the Ramsey problem.

Individual decisions in our economy can be characterized recursively. The aggregate variables in a given period depend on the realized aggregate shock Θ and the beginning of the period distribution of assets Ω. Since we assumed that government has no revenues in this example, it also cannot issue debt, and so the distribution Ω satisfies $\int b d\Omega = 0$. We denote the space of such distributions by $\mathcal{W}$. We use tildes to denote policy functions, and let $\tilde{X} = \begin{bmatrix} \tilde{Q} & \tilde{W} & \tilde{D} \end{bmatrix}^T$ be a vector of aggregate policy functions capturing interest rates, wages and dividends. Individual policy functions depend both on aggregate state (Θ, Ω) and on idiosyncratic state (ς, b) that capture the realization ς of the idiosyncratic shock that affects individual with asset b. Let $\tilde{x} = \left[\begin{array}{ccc} \tilde{b} & \tilde{c} & \tilde{n} \end{array} \right]^T$ be the triplet of the individual policy functions. Finally, $\widetilde{\Omega}(\Theta, \Omega) : \mathbb{R} \times \mathcal{W} \to \mathcal{W}$ be the law of motion describing how the aggregate distribution of debt next period is affected by the aggregate shock in the current period.

Individual optimality conditions consist of the budget constraint (4) and the optimality conditions (12). In our recursive notation these conditions read

$$\tilde{c}(\varsigma, \Theta, b, \Omega) + \tilde{Q}(\Theta, \Omega)\tilde{b}(\varsigma, \Theta, b, \Omega) = \tilde{W}(\Theta, \Omega)\exp(\Theta + \varsigma)\tilde{n}(\varsigma, \Theta, b, \Omega) + b + \tilde{D}(\Theta, \Omega), \tag{14a}$$

$$\beta\mathbb{E}\left\{ u_c\left[\tilde{c}\left(\cdot, \cdot, \tilde{b}(\Theta, b, \Omega), \widetilde{\Omega}(\Theta, \Omega)\right)\right] \Big| \Theta, \Omega\right\} = \tilde{Q}(\Theta, \Omega)u_c[\tilde{c}(\varsigma, \Theta, b, \Omega)], \tag{14b}$$

$$\tilde{W}(\Theta, \Omega)\exp(\Theta + \varsigma)u_c[\tilde{c}(\varsigma, \Theta, b, \Omega)] = -u_n[\tilde{c}(\varsigma, \Theta, b, \Omega)], \tag{14c}$$

for all $\varsigma, \Theta, b, \Omega$. The aggregate constraints and the firm's optimality condition (13) can be written as

$$\left[\int \exp(\Theta + \varsigma)\tilde{n}(\varsigma, \Theta, b, \Omega)\, d\Pr(\varsigma)\, d\Omega\right]^{\alpha} = \int \tilde{c}(\varsigma, \Theta, b, \Omega)\, d\Pr(\varsigma)\, d\Omega, \tag{15a}$$

$$\frac{\epsilon - 1}{\epsilon}\alpha\left[\int \exp(\Theta + \varsigma)\tilde{n}(\varsigma, \Theta, b, \Omega)\, d\Pr(\varsigma)\, d\Omega\right]^{\alpha - 1} = \tilde{W}(\Theta, \Omega), \tag{15b}$$

$$\left(1 - \frac{\epsilon - 1}{\epsilon}\alpha\right)\left[\int \exp(\Theta + \varsigma)\tilde{n}(\varsigma, \Theta, b, \Omega)\, d\Pr(\varsigma)\, d\Omega\right]^{\alpha} = \tilde{D}(\Theta, \Omega), \tag{15c}$$

for all Θ, Ω. Finally, the law of motion for the distribution of debts induced by the savings behavior of the agents is given by

$$\widetilde{\Omega}(\Theta, \Omega)(y) = \int \iota\left(\tilde{b}(\varsigma, \Theta, b, \Omega) \le y\right) d\Pr(\varsigma)\, d\Omega \quad \forall y, \tag{16}$$

where ι is the indicator variable. Equations (14), (15), and (16) fully describe the equilibrium behavior of aggregate and individual variables $\tilde{X}$ and $\tilde{x}$.

Our starting point is the perturbation theory of Fleming (1971), Fleming and Souganidis (1986), and Anderson et al. (2012) that uses small noise expansions around the current state Ω. Consider a family of stochastic processes parameterized by a positive scalar σ that scales all shocks (ς, Θ). Let $\tilde{x}(\sigma\varsigma, \sigma\Theta, b, \Omega; \sigma)$ and $\tilde{X}(\sigma\Theta, \Omega; \sigma)$ denote policy functions with scaling parameter σ. The perturbational approach takes first-, second-, and higher order approximations with respect to σ, then evaluates these derivative at $\sigma = 0$. For example, first order approximations take the form

$$\tilde{X}(\sigma\Theta, \Omega; \sigma) = \bar{X}(\Omega) + \sigma\left[\bar{X}_{\Theta}(\Omega)\Theta + \bar{X}_{\sigma}(\Omega)\right] + O\left(\sigma^2\right), \tag{17}$$

$$\tilde{x}(\sigma\varsigma, \sigma\Theta, b, \Omega; \sigma) = \bar{x}(b, \Omega) + \sigma\left[\bar{x}_{\Theta}(b, \Omega)\Theta + \bar{x}_{\varsigma}(b, \Omega)\varsigma + \bar{x}_{\sigma}(b, \Omega)\right] + O\left(\sigma^2\right), \tag{18}$$

where we used $\bar{X}, \bar{x}$ to denote the value of the policy function evaluated at $\sigma = 0$, $\bar{X}_\Theta, \bar{x}_\Theta, \bar{x}_\varsigma$ to denote derivatives with respect to aggregate and idiosyncratic shocks, and $\bar{X}_\sigma$,$\bar{x}_\sigma$ are the derivative of the last argument of the policy function. Bars indicate that all the derivatives have been evaluated at $\sigma = 0$. Observe that all the evaluations are performed around a current state Ω that changes as the aggregate state of the economy changes. The values for the derivatives $\bar{X}_\Theta, \bar{x}_\Theta, \bar{x}_\varsigma$,$\bar{X}_\sigma$,$\bar{x}_\sigma$ are obtained via the implicit function theorem that dictates repeated differentiation (14), (15) and (16) with respect to σ and then applying a method of undetermined coefficients. In simulation, the next period aggregate state $\widetilde{\Omega}$ is then obtained and next period policy functions are then approximated around this new state $\widetilde{\Omega}$.

An immediate challenge for us and one that is not shared by Fleming (1971), Fleming and Souganidis (1986) and Anderson et al. (2012) is a curse of dimensionality that arises because our state-space Ω is infinite dimensional. Suppose that we approximate this distribution on a grid with K points. As we explain below, a direct application of the implicit function theorem, as used by those authors, would require us to invert a $3K \times 3K$ matrix. In our applications K is large,[3] which makes such inversions computationally costly and the direct application of this approach infeasible for heterogeneous agents economies. Our principal contribution is to show that this problem can be split into K independent linear problems, each of which require inversion of a 3×3 matrix. This makes our approach computationally feasible, highly parallelizable, and fast. The crucial intermediate step that enable us to achieve this is a "factorization theorem" that we derive in the next section. This theorem shows that in competitive equilibrium settings there exists a particularly simple relationship between derivatives of individual and aggregate policy functions. Importantly, this theorem and simplifications that it provides extend also to second- and higher-order approximations and to Ramsey settings.

3.1.1 Points of expansion and zeroth order terms

Consider the expansion around deterministic economy with a given distribution of assets Ω. Observe that we have

$$\bar{b}(b, \Omega) = b \text{ for all b,}\Omega. \tag{19}$$

This implies that equations for deterministic economy are

$$\begin{aligned}
\bar{c}(b, \Omega) + \bar{Q}(\Omega) b &= \bar{W}(\Omega) \bar{n}(b, \Omega) + b + \bar{D}(\Omega), && \text{(20a)}\\
\beta &= \bar{Q}(\Omega), && \text{(20b)}\\
\bar{W}(\Omega) u_c[\bar{c}(b, \Omega)] &= -u_n[\bar{n}(b, \Omega)], && \text{(20c)}
\end{aligned}$$

[3] In the full Ramsey problem that we study in Section 4 the distribution of individual characteristics is 3-dimensional, so Ω is a distribution over $\mathbb{R}^3$. We approximate it with $K = 10,000$ grid points.

for all (b, Ω) and aggregate constraints

$$\begin{aligned}\left[\int \bar{n}\left(b,\Omega\right)d\Omega\right]^{\alpha} &= \int \bar{c}\left(b,\Omega\right)d\Omega,\\ \frac{\epsilon-1}{\epsilon}\alpha\left[\int \bar{n}\left(b,\Omega\right)d\Omega\right]^{\alpha-1} &= \bar{W}\left(\Omega\right),\\ \left(1-\frac{\epsilon-1}{\epsilon}\alpha\right)\left[\int \bar{n}\left(b,\Omega\right)d\Omega\right]^{\alpha} &= \bar{D}\left(\Omega\right),\end{aligned} \tag{21}$$

and the law of motion

$$\bar{\Omega}\left(\Omega\right) = \Omega \tag{22}$$

that hold for all Ω.

For a given Ω, we solve the system of equations above for $\bar{x}(b,\Omega), \bar{X}(\Omega)$. Although this is a nonlinear system of equations, it can be solved efficiently using existing numerical techniques, say by discretizing the state space Ω. In what follows, we take these solutions as known, and focus on explaining how one can find the derivatives that appear in (17) and (18). We call these $\bar{x}(b,\Omega), \bar{X}(\Omega)$ the zeroth order expansion. Also, whenever it does not cause confusion, we drop Ω from the argument and simply use $\bar{x}(b), \bar{X}$.

To obtain the coefficients of the first order expansion of the policy functions, we will need to know how policy functions are affected by perturbations to the aggregate state Ω. These effects can be derived using equations (20)-(22). Let $\partial\bar{Q}$ denote the Frechet derivative of $\tilde{Q}$ with respect to perturbations of state Ω evaluated at $\sigma = 0$, and let $\partial\bar{Q}\cdot\Delta$ denote the value of that derivative in the direction $\Delta \in \mathcal{W}$.[4] We extend similarly the notation of Frechet derivatives for other policy functions and triplets $\tilde{X}$ and $\tilde{x}$. Our key result is the following theorem.

Theorem 1. *(Factorization theorem) For each b there exists a loading matrix $\mathsf{C}(b)$ with coefficients known from the zeroth order expansion such that*

$$\partial\bar{x}(b) = \mathsf{C}(b)\partial\bar{X}. \tag{23}$$

For any perturbation $\Delta \in \mathcal{W}$ we have

$$\partial\bar{X}\cdot\Delta = \mathsf{D}^{-1}\int \mathsf{E}(b)d\Delta \tag{24}$$

for a 3×3 matrix D and 3×1 vectors $\mathsf{E}(b)$, all with coefficients known from the zeroth order expansion.

[4] In particular, $\partial\bar{Q}(\Omega)$ is a linear operator such that $\lim_{\|\Delta\|\to 0}\frac{|\bar{Q}(\Omega+\Delta)-\bar{Q}(\Omega)-\partial\bar{Q}(\Omega)\cdot\Delta|}{\|\Delta\|} = 0$. If we restrict attention to changes Δ that only assigns positive masses to K levels of debts, then $\partial\bar{Q}$ is a $K\times 1$ vector that captures the effect of changing masses in each of the K points of the distribution Ω; Δ is also a $K\times 1$ vector, and $\partial\bar{Q}\cdot\Delta$ is a dot product. When Δ is a function, then $\partial\bar{Q}\cdot\Delta$ is an integral of that function with density $\partial\bar{Q}$.

This theorem fundamental to our approach and it will play a critical role in allowing us to solve for first- (and, ultimately, second- and higher-) order approximations computationally efficiently, allowing us to handle rich heterogeneity. The economic content of this theorem is as follows. Function $\partial\bar{x}(\cdot)$ captures how any perturbation of the distribution of asset holdings affects the savings, consumption and labor decisions of each agent. In competitive equilibrium agents do not care about the asset distribution per se; what affects their decision are prices ($\bar{Q}, \bar{W}$ in our example) and lumpsum income ($\bar{D}$). Thus, matrix $\partial\bar{x}(\cdot)$ can be factorized in two terms: how changes in distribution affect prices and lump-sum income, $\partial\bar{Q}$, $\partial\bar{W}$, $\partial\bar{D}$, and how individual decisions load on these aggregate variables, matrix $\mathsf{C}(b)$. This loading matrix itself is known, in a sense that it can be constructed directly from zeroth order calculations of $\bar{x}$ and $\bar{X}$, and gives equation (23). This equation then allows on to compute the effect of any perturbation Δ on aggregate variables using a linear equation (24). Explicit construction of matrices $\mathsf{C}(b)$, D and $\mathsf{E}(b)$ is given in the proof.

Proof. Observe from (19) that $\bar{b}\,(b)$ does not depend on Ω and therefore $\partial\bar{b}(b) = \mathbf{0}$. Differentiate (20a) and (20c) to get a linear system

$$\begin{bmatrix} 1 & -\bar{W} \\ \bar{W}u_{cc}\left[\bar{c}(b)\right] & u_{nn}\left[\bar{n}(b)\right] \end{bmatrix} \begin{bmatrix} \partial\bar{c}(b) \\ \partial\bar{n}(b) \end{bmatrix} = \begin{bmatrix} -b & \bar{n}(b) & 1 \\ 0 & -u_c\left[\bar{c}\,(b)\right] & 0 \end{bmatrix} \begin{bmatrix} \partial\bar{Q} \\ \partial\bar{W} \\ \partial\bar{D} \end{bmatrix}.$$

Observe that all terms in the 2×2 matrices that appear in this expression can be directly constructed from the zeroth order variables $\bar{X}$,$\bar{x}(b)$. Inverting the matrix on the left hand side, we obtain an expression for $\begin{bmatrix} \partial\bar{c}(b) & \partial\bar{n}(b) \end{bmatrix}^T$ as a linear function of $\partial\bar{X}$. Adding the first raw of zeros to this matrix, we obtain $\mathsf{C}(b)$ that appear in equation (23). Also observe that the same steps allow us to find the derivative of policy function with respect to individual debt level, $\bar{x}_b(b)$, as

$$\bar{x}_b(b) \equiv \begin{bmatrix} \bar{b}_b(b) \\ \bar{c}_b(b) \\ \bar{n}_b(b) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \bar{Q}-1 & 1 & \bar{W} \\ 0 & \bar{W}u_{cc}\left[\bar{c}\,(b)\right] & u_{nn}\left[\bar{n}\,(b)\right] \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ \bar{D} \\ 0 \end{bmatrix}, \tag{25}$$

where again all the terms on the right hand side are known from the zeroth order expansion.

We obtain derivative $\partial\bar{X}$ by differentiating feasibility constraints (21). For simplicity, suppose that $\Delta \in \mathcal{W}$ has density δ and $\omega(b)$ is the density of Ω. Consider the Frechet derivative of the aggregate consumption:

$$\begin{aligned} \partial\left(\int \bar{c}\,(b,\Omega)\,d\Omega\right)\cdot\Delta &= \lim_{\alpha\to0}\frac{1}{\alpha}\left[\int \bar{c}\,(b,\Omega+\alpha\Delta)\,(\omega\,(b)+\alpha\delta(b))\,db - \int \bar{c}\,(b,\Omega)\,\omega\,(b)\,db\right] \\ &= \int\left(\partial\bar{c}\,(b,\Omega)\cdot\Delta\right)d\Omega - \int \bar{c}_b\,(b)\,d\Delta, \end{aligned}$$

where in the last line we use integration by parts. Substitute equation (23) to get

$$\int \left(\partial \bar{c}\left(b, \Omega\right) \cdot \Delta\right) d\Omega = \left(\partial \bar{Q} \cdot \Delta\right) \int \mathsf{C}_{21}(b) d\Omega + \left(\partial \bar{W} \cdot \Delta\right) \int \mathsf{C}_{22}(b) d\Omega + \left(\partial \bar{D} \cdot \Delta\right) \int \mathsf{C}_{23}(b) d\Omega,$$

where $\mathsf{C}_{ij}(b)$ is the ij^{th} element of the loading matrix $\mathsf{C}_{ij}(b)$. Since matrices $\mathsf{C}(b)$ are known from the zero order expansion, so are all the integrals on the right hand side of this expression and their value does not depend on individual b. Thus, a derivative $\partial \left(\int \bar{c}\left(b, \Omega\right) d\Omega\right) \cdot \Delta$ is equal to a weighted sum of derivatives $\partial \bar{Q} \cdot \Delta$, $\partial \bar{W} \cdot \Delta$, and $\partial \bar{D} \cdot \Delta$ minus the integral $\int \bar{c}_b\left(b\right) d\Delta$, where function $\bar{c}_b\left(b\right)$ was already found in (25). Similarly, the Frechet derivative of the aggregate labor supply can also be written as a linear relationship between $\partial \bar{X}$ and $\int \bar{n}_b\left(b\right) d\Delta$. Using these observations, we can differentiate (21) and construct matrices D,$\mathsf{E}(b)$ that appear in (24). □

3.1.2 First order expansion

We now consider first order expansions of equations (14) and (15). These equations depends on policy functions in both current and the next period, with the latter appearing in the expectation in (14b). The expansion of current period policy functions is straightforward and is given in equations (17), (18). The expansion of the expectation of the next period policy function $\tilde{c}$ requires some care because the aggregate distribution of asset is affected by aggregate shocks.

The aggregate distribution in the next period is given by $\widetilde{\Omega}\left(\sigma\Theta, \Omega; \sigma\right)$ and satisfies, in light of equation (22),

$$\widetilde{\Omega}\left(\sigma\Theta, \Omega; \sigma\right) = \Omega + \sigma\left[\bar{\Omega}_{\Theta}\Theta\right] + O\left(\sigma^{2}\right),$$

where $\bar{\Omega}_{\Theta}$ is a distribution in $\mathcal{W}$.[5] A typical element of $\bar{\Omega}_{\Theta}$, denoted by $\bar{\Omega}_{\Theta}(b)$, can be found from (16):

$$\bar{\Omega}_{\Theta}(b) = -\omega(b)\bar{b}_{\Theta}\left(b\right) \text{ for all } b. \tag{26}$$

Using this insight, we have

$$\begin{aligned}\mathbb{E}\left\{\tilde{c}\left(\cdot, \cdot, \tilde{b}\left(\sigma\varsigma, \sigma\Theta, b, \Omega; \sigma\right), \widetilde{\Omega}\left(\sigma\Theta, \Omega; \sigma\right); \sigma\right) \middle| \Theta, \Omega\right\} &= \bar{c}(b) + \sigma\left\{\bar{c}_b(b)\bar{b}_{\varsigma}(b)\varsigma\right. \\ &+ \left(\rho_{\Theta}\bar{c}_{\Theta}(b) + \bar{c}_b(b)\bar{b}_{\Theta}(b) - \partial\bar{c}(b)\cdot\left(\omega\bar{b}_{\Theta}\right)\right)\Theta + \left.\left(\bar{c}_{\sigma}(b) + \bar{c}_b(b)\bar{b}_{\sigma}(b)\right)\right\} + O\left(\sigma^{2}\right).\end{aligned} \tag{27}$$

Note that $\bar{c}_b(b)$ and $\partial\bar{c}(b)$ are the same objects we computed in the previous section. This equation show how idiosyncratic and aggregate shocks in the current period affect consumption (and hence the marginal utility of consumption) in the next period. These effects are grouped into three groups. The first group, that is multiplied by ς shows the effect of the idiosyncratic shocks. Current period idiosyncratic shocks also affects savings in the current period, $\bar{b}_{\varsigma}(b)\varsigma$ and therefore consumption

[5] Asset market clearing implies that $\bar{\Omega}_{\sigma} = 0$.

tomorrow is also affected by ς through the effect on savings, $\bar{c}_b(b)\bar{b}_\varsigma(b)\varsigma$.[6] The second group of terms capture the effect of the aggregate shocks. The aggregate shocks are persistent and their expected value next period is $\rho_\Theta\Theta$, thus the term $\bar{c}_b(b)\bar{b}_\Theta(b)$ captures the direct effect of the shock. The term $\rho_\Theta\bar{c}_\Theta(b)$ analogously to its idiosyncratic counterpart, captures how changes in savings today affects consumption tomorrow. The term, which is equal to $\partial\bar{c}(b)\cdot\bar{\Omega}_\Theta$ from (26), captures how changes in the aggregate distribution of debt caused by the aggregate shock affects individual agent's consumption. Finally, the last set of terms captures how the presence of risk affects the *level* of consumption, savings and the law of motions for aggregate debt even in the absence of any shock.

We are now ready to find the derivatives in the expansion that appear in (17) and (18). Consider the first order expansion with respect to σ of equations (15) and (14) and use method of undetermined coefficients to find the derivatives that multiply $\sigma\varsigma$ and $\sigma\Theta$. The derivatives $\bar{x}_\varsigma(b)$ are easy to find since they cancel out from the expansions of the feasibility constraints and appear only in the individual optimality conditions

$$\bar{c}_\varsigma(b) + \bar{Q}\bar{b}_\varsigma(b) = \bar{W}\left(\bar{n}(b) + \bar{n}_\varsigma(b)\right), \tag{28a}$$

$$\bar{Q}u_{cc}[\bar{c}(b)]\bar{c}_\varsigma(b) = \beta u_{cc}[\bar{c}(b)]\bar{c}_b(b)\bar{b}_\varsigma(b), \tag{28b}$$

$$\bar{W}\left(u_c[\bar{c}(b)] + u_{cc}[\bar{c}(b)]\bar{c}_\varsigma(b)\right) = -u_{nn}[\bar{n}(b)]\bar{n}_\varsigma(b). \tag{28c}$$

for all b. All variables apart from $\bar{b}_\varsigma(b), \bar{c}_\varsigma(b), \bar{n}_\varsigma(b)$, are known from the zeroth order expansion. Thus we can find $\bar{b}_\varsigma(b), \bar{c}_\varsigma(b), \bar{n}_\varsigma(b)$ separately for each b by solving this 3×3 system of equations. In the direct analogy with (25) we can write this solution compactly as

$$\bar{x}_\varsigma(b) = \mathsf{K}(b)^{-1}\mathsf{L}(b) \tag{29}$$

for matrices $\mathsf{K}(b)$,$\mathsf{L}(b)$ known from the zeroth order expansion. Similarly we show that $\bar{X}_\sigma, \bar{x}_\sigma$ are all zero vectors.

Solving for the effect of the aggregate shocks is more complicated because they affect the distribution of debts and the aggregate constraints. Using (27), the individual constraints imply that

$$\bar{c}_\Theta(b) + \bar{Q}\bar{b}_\Theta(b) + \bar{Q}_\Theta\bar{b}(b) = \bar{W}\left(\bar{n}(b) + \bar{n}_\Theta(b)\right) + \bar{W}_\Theta\bar{n}(b) + \bar{D}_\Theta, \tag{30a}$$

$$\beta u_{cc}[\bar{c}(b)]\left(\rho_\Theta\bar{c}_\Theta(b) + \bar{c}_b(b)\bar{b}_\Theta(b) - \partial\bar{c}(b)\cdot\left(\omega\bar{b}_\Theta\right)\right) = \bar{Q}u_{cc}[\bar{c}(b)]\bar{c}_\Theta(b) + u_c[\bar{c}(b)]\bar{Q}_\Theta, \tag{30b}$$

$$\bar{W}\left(u_c[\bar{c}(b)] + u_{cc}[\bar{c}(b)]\bar{c}_\Theta(b)\right) + \bar{W}_\Theta u_c[\bar{c}(b)] = -u_{nn}[\bar{n}(b)]\bar{n}_\Theta(b). \tag{30c}$$

[6]Remember that we assumed that ς are i.i.d. in our simple example. With persistent idiosyncratic shocks there would be an additional term capturing persistence, analogous to $\rho_\Theta\bar{c}_\Theta(b)$ for the aggregate shock.

The aggregate constraints are

$$\begin{aligned} \alpha\bar{N}^{\alpha-1}\int\left(\bar{n}\left(b\right)+\bar{n}_{\Theta}\left(b\right)\right)d\Omega &= \int\bar{c}_{\Theta}\left(b\right)d\Omega, \\ \frac{\epsilon-1}{\epsilon}\alpha\left(\alpha-1\right)\bar{N}^{\alpha-2}\int\left(\bar{n}\left(b\right)+\bar{n}_{\Theta}\left(b\right)\right)d\Omega &= \bar{W}_{\Theta}, \\ \left(1-\frac{\epsilon-1}{\epsilon}\alpha\right)\alpha\bar{N}^{\alpha-1}\int\left(\bar{n}\left(b\right)+\bar{n}_{\Theta}\left(b\right)\right)d\Omega &= \bar{D}_{\Theta}, \end{aligned} \tag{31}$$

where $\bar{N}$ is the aggregate labor supply in the zeroth order expansion.

The direct way to proceed (e.g. along the lines of Anderson et al. (2012)) would be to use equations (30) to express individual derivatives $\bar{x}_{\Theta}$ as a linear function of $\bar{X}_{\Theta}$ and then substitute those expressions in (31) to solve for $\bar{X}_{\Theta}$. However, in heterogeneous agents economies like ours this approach is infeasible but for the simplest models of heterogeneity. The problem lies in the fact that the individual condition (30) for a given b contains unknown variables $\bar{b}_{\Theta}\left(\cdot\right)$ for all debts levels due to the integral $\partial\bar{c}(b)\cdot\left(\omega\bar{b}_{\Theta}\right)$. So if the distribution Ω contains K elements, then expression x_{Θ} as a function of $\bar{X}_{\Theta}$ from (30) would require inversion of a $3K\times 3K$ matrix, which is computationally costly for K large.

Theorem 1 significantly simplifies the analysis, by allowing us to break one system of $3K\times 3K$ equations into K system of 3×3 equations. Using Theorem 1 we can write

$$\partial\bar{x}(b)\cdot\left(\omega\bar{b}_{\Theta}\right)=\mathsf{C}(b)\partial\bar{X}\cdot\left(\omega\bar{b}_{\Theta}\right)\equiv\mathsf{C}(b)\bar{X}'_{\Theta}.$$

Variable $\bar{X}'_{\Theta}$ captures how aggregate variable tomorrow are affected by the changes in the aggregate distribution tomorrow. Using (24) we can write it explicitly as

$$\bar{X}'_{\Theta} = \mathsf{D}^{-1}\int\mathsf{E}(b)\omega(b)\bar{b}_{\Theta}(b)db. \tag{32}$$

Now equation (30) defines a linear relationship between policy functions $\bar{x}_{\Theta}(b)$ and aggregate variables $\begin{bmatrix}\bar{X}_{\Theta} & \bar{X}'_{\Theta}\end{bmatrix}$ which is independent of any other $\bar{x}_{\Theta}(\hat{b})$. Thus, we can write (30) for each b as

$$\mathsf{M}(b)\bar{x}_{\Theta}(b)=\mathsf{N}(b)\cdot\begin{bmatrix}\bar{X}_{\Theta} & \bar{X}'_{\Theta}\end{bmatrix}^{\mathsf{T}}, \tag{33}$$

where $\mathsf{M}(b)$ is a 3×3 matrix, and all coefficients of $\mathsf{M}(b)$ and $\mathsf{N}(b)$ are known from the zeroth order expansion. Thus we can solve for each $\bar{x}_{\Theta}(b)$ as a function of aggregate variables independently for each b, which is a system of 3×3 equations that can be solved independently for each of K levels of debt.

Now we can substitute the obtained expressions for $\bar{x}_{\Theta}(\cdot)$ into equations (31) and (32) to solve for $\bar{X}_\Theta, \bar{X}'_\Theta$. This is an 6×6 system of equations of the form

$$\mathsf{O} \cdot \begin{bmatrix} \bar{X}_\Theta & \bar{X}'_\Theta \end{bmatrix}^{\mathsf{T}} = \mathsf{P} \tag{34}$$

that can be easily handled numerically.

This completes the solution for the first order responses to the aggregate shock Θ. It is insightful to review the economics behind these equations. We know from Theorem 1 that any changes in the individual variables are proportional to changes in aggregate variables with loading given by matrix $\mathsf{C}(b)$. Thus, instead of searching for how the aggregate distribution changes tomorrow, it is sufficient to search for how how aggregate variables change tomorrow, $\bar{X}'_\Theta$. For each level b, that implies that we can find a linear mapping from $\begin{bmatrix} \bar{X}_\Theta & \bar{X}'_\Theta \end{bmatrix}$ into $\bar{b}_\Theta(b)$, which can be solved independently for each b by inverting a 3×3 matrix. $\begin{bmatrix} \bar{X}_\Theta & \bar{X}'_\Theta \end{bmatrix}$ themselves must satisfy resource constraints which constitute an 6×6 matrix.

3.1.3 Second and higher order terms

Our approach extends easily to higher-order approximations. The key insight is that factorization property shown in Theorem 1 for the first order expansions preserves for higher order expansions as well. In the appendix we prove the second order corollary of Theorem 1 and equation (24) and provide their explicit coefficients.

Corollary 1. *The second order Frechet derivative $\partial^2 \bar{x}(b)$ is a linear map of $\partial^2 \bar{X}$ and $\partial \bar{X}$ with all coefficients of that map known from the lower-order expansions. For any $\Delta_1, \Delta_2 \in \mathcal{W}$, the second order Frechet derivative $\partial^2 \bar{X} \cdot (\Delta_1, \Delta_2)$ is a bilinear map of Δ_1, Δ_2 with all coefficients known from the lower-order expansions.*

The second order expansion of policy functions have additional terms $\bar{x}_{\varsigma\varsigma}(b), \bar{x}_{\varsigma\sigma}(b), \bar{x}_{\Theta\Theta}(b), \bar{x}_{\Theta\sigma}(b)$, $\bar{x}_{\sigma\sigma}(b), \bar{X}_{\Theta\Theta}, \bar{X}_{\Theta\sigma}, \bar{X}_{\sigma\sigma}$. Calculations of $\bar{x}_{\varsigma\varsigma}$,$\bar{x}_{\Theta\Theta}$,$\bar{X}_{\Theta\Theta}$ then proceeds analogously to their first order counterparts, while the cross-partials $\bar{x}_{\varsigma\sigma}, \bar{x}_{\Theta\sigma}, \bar{X}_{\Theta\sigma}$ are all zeros. Unlike the first order expansion, the intercept terms $\bar{x}_{\sigma\sigma}, \bar{X}_{\sigma\sigma}$ are no longer zero. They depend on $\text{var}(\varsigma), \text{var}(\Theta)$ and capture such effects as precautionary savings. Solution for $\bar{x}_{\sigma\sigma}(b), \bar{X}_{\sigma\sigma}$ involves steps similar to those used to solve for $\bar{x}_\Theta(b), \bar{X}_\Theta$ in the previous section. We provide those details in the appendix.

3.1.4 A remark on the choice of the state space

Before concluding this section, we want to add a remark about the choice of the states pace. There exist many equivalent state space representations that allow one recursively to construct a competitive equilibrium. The state space representation that works best for our approach is the one that satisfies some version of the stationarity condition (19). The distribution of debts Ω that we used in our example as a state space sometimes is not ideal. Although the law of motion $\tilde{\Omega}$ satisfied the stationarity condition in our example, in other applications it often won't, for example if the $\sigma = 0$ economy has a deterministic dynamics. A more universal choice of the state space would be the distribution of ratios of agents' marginal utilities or stochastic Pareto-Nigishi weights, defined as $m_{i,t} \propto u_{c,i,t}$, $\int m_{i,t} di = 1$. In many economies a distribution of such weights M would satisfy property (19) even if the distribution of debts Ω does not.

3.1.5 Numerical approximation

To implement our algorithm numerically, we approximate Ω with discrete distributions with K points $\{b_k\}_k$ with masses $\{\omega_k\}_k$. All the integrals in our expressions collapse then to sums. For example, our expression (32) for X'_Θ becomes

$$\bar{X}'_\Theta = \mathsf{D}^{-1} \sum_k \mathsf{E}(b_k) \omega_k \bar{b}_\Theta(b_k).$$

All intermediate terms, such as $\mathsf{E}(b_k)$, can be computed independently for each k, making the algorithm highly parallelizable. Once we compute approximations of the policy functions in the current period, we use Monte-Carlo methods to obtain the next period distribution of assets $\tilde{\Omega}$, for which we repeat this procedure.

We now discuss the numerical accuracy of our approximations. To compare our approximated policy rules with those solved via global methods we shut down the aggregate shocks and compute the steady state distribution of assets the model presented in 3.1. Optimal policies of each agent are computed using the endogenous grid method of Carol (2005). The steady-state distribution is approximated using a histogram and computed using the transition matrix constructed from the policy rules following Young (2010). We then compare the approximated policy rules using our method, around the same stationary distribution, to those of the global solution. As noted in section 3.1.2, even in absence of aggregate shocks our techniques are still necessary, for a second order expansion, to determine the effect of the presence idiosyncratic risk on policies through $\bar{x}_{\sigma\sigma}$ and $\bar{X}_{\sigma\sigma}$.

We calibrate the 6 parameters of this model, ν,γ,β, α, ϵ and σ_ς, as follows. We set $\nu = 1$ and $\gamma = 2$ to match of calibration in Section 4. β is set to target an interest rate of 2%. $\epsilon = 6$ targets a markup of 20% and the decreasing returns to scale parameter, α, is chosen to target a labor share of 0.66. The standard deviation σ_ς is set at .25 to match our calibration in section 4. We choose very loose ad-hoc borrowing constraints, $\underline{b} = -100$ to approximate a natural debt limit. To ensure an accurate approximation to the global policy rules, we approximated the consumption and labor policy rules using cubic splines with 200 grid points and the steady-state distribution with a histogram with 1000 bins.

The policies of the agents and aggregate variables are then approximated using a second order approximation around the stationary distribution $\bar{\Omega}$ with the expressions derived in 3.1.1-3.1.3. The

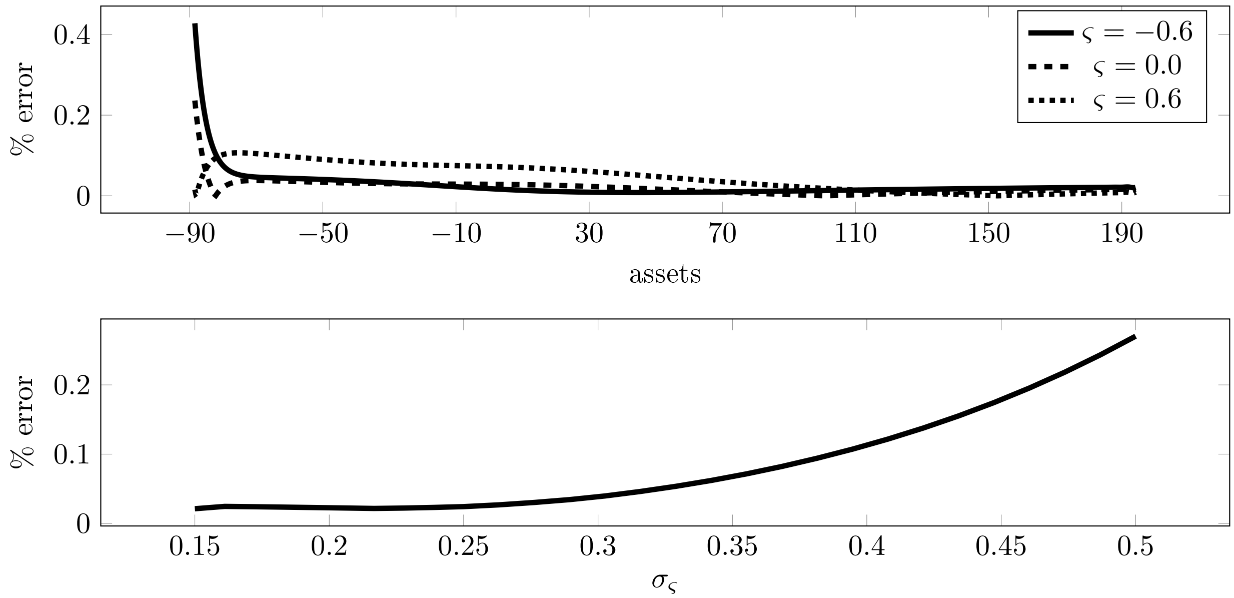

Figure I: Percentage error of consumption policy functions relative to global solution in top panel. Bottom panel plots average (with respect to the stationary distribution) absolute consumption error relative to global solution as σ_ς varies.

equilibrium policy rules and aggregates, around this distribution of 1000 points, can be approximated using single processor in 0.8 seconds. The equilibrium interest rate found by perturbation code was 2.0013%. The percent error in the aggregate labor supply was 0.036%. Finally, we can evaluate the policy errors for the individual agents. In the first panel of Figure I we plot the percentage errors for consumption relative to the global solution for a median shock to ς along with a ± 2.5 standard deviation shock. For almost all of the agents, the perturbation methods perform well with errors less that 0.1% which would correspond making an error in consumption of less than a dollar on every $1000 dollars spent. As is expected, since the approximation in Section 3.1 assumes natural borrowing limits, the accuracy deteriorates near the borrowing limit assumed by the global solution. The range plotted in Figure I contains 99.2% of the agents in the stationary distribution. As a robustness check we also observe how accuracy behaves as we change σ_ς. For values of $\sigma_\varsigma \in [0.1, 0.5]$, we compute the average absolute percentage error as

$$\int\int \frac{\left|c^{global}(\sigma_\varsigma\varsigma, b; \sigma_\varsigma) - c^{pertub}(\sigma_\varsigma\varsigma, b; \sigma_\varsigma)\right|}{c^{global}(\sigma_\varsigma\varsigma, b; \sigma_\varsigma)} d\Pr(\varsigma) d\Omega$$

and plot it in the bottom panel of Figure I. We see that the average error remains very low around the level of our calibration and increases moderately for higher σ_ς.

3.2 Ramsey problems

We now extend our approach to Ramsey problems. The problem of finding the optimal monetary-fiscal policy consists of finding $\left\{\{c_{i,t}, n_{i,t}, b_{i,t}\}_i, B_t, W_t, P_t, Q_t, \pi_t, \tau_t\right\}_t$ that maximize welfare (11) subject to (4), (7) - (10) and (12) - (13). The optimal monetary problem imposes, in addition, a constraint $\tau_t = \bar{\tau}$ for all t. In either case, this problem can be written recursively (see Appendix). Relative to competitive equilibrium problem, Ramsey problems add additional state variables. Using our remark in Section 3.1.4, we show in the appendix that a convenient choice of state variables will be the joint distribution, Z, of agents' marginal utilities in the previous period, the Lagrange multipliers on the implementability constraints (4) and the persistent component of productivity, $e_{i,t}$,[7] as well as the last period realization of the aggregate shock Θ and the Lagrange multiplier on the Phillips curve equation (13), Λ.[8] Similarly to Section 3.1, we use $\tilde{x}(\sigma\varsigma, \sigma\Theta, \Lambda, z, Z; \sigma)$ to denote the vector of idiosyncratic variables and $\tilde{X}(\sigma\Theta, Z, \Lambda; \sigma)$ to denote the aggregate variables. In the Ramsey problem $\tilde{x}$ consists of $\tilde{z}$, $\tilde{c}$, $\tilde{n}$, $\tilde{b}$ as well as the Lagrangians on the individual optimality conditions, and $\tilde{X}$ consists of $\tilde{Q}$, $(1-\tilde{\tau})\tilde{W}$, $\tilde{T}+\tilde{D}$, $\tilde{\pi}$, $\tilde{N}$, $\tilde{\Lambda}$. We expand with respect to σ around any given aggregate state (Z, Λ) with all policy functions around $\sigma = 0$ denoted by $\bar{x}(z), \bar{X}$.

The following Lemma shows that our algorithm from Section 3.1 applies essentially as is to the much more complicated Ramsey problem. The two key results that achieve that is the stationary in the appropriately defined state variables and factorization of individual policy functions into aggregate response and individual loadings, which are the analogues of equation (19), Theorem 1 and its corollary.

Lemma 1. *In the Ramsey problem $\bar{z}(z) = z$ for all Z, Λ. The first and second order derivatives $\partial\bar{x}(z), \partial^2\bar{x}(z), \partial\bar{X}, \partial^2\bar{X}$ satisfy Theorem 1 and its corollary, with all the matrices are known from the lower-order expansions. $\bar{x}_\varsigma(z)$, $\bar{x}_\Theta(z)$, $\bar{x}_\sigma(z)$, $\bar{X}_\Theta$, $\bar{X}_\sigma$ and their second order analogues satisfy equivalent equations from Section 3.1.*

The key take away of this lemma is that the mathematical structure of the Ramsey problem remains essentially the same for our problem as that of Section 3.1. The analytical expressions used for matrices are more cumbersome but their computations as well as the computations of the expansion of policy functions with respect to the aggregate and idiosyncratic shocks continue to be as straightforward as they were in Section 3.1.

3.3 Comparison to Other Methods

Our method is related to perturbation techniques of Judd and Guu (1993, 1997) and Judd (1996, 1998) that were subsequently extended to heterogeneous agent economies by Campbell (1998),

[7] For our approximations we assume that $\rho_e \to 1$ as $\sigma \to 0$.

[8] There also time-invariant parts of the state space $s_i, \bar{e}_i$ but since they are constants and we omit them from discussion for simplicity.

Reiter (2009), Mertens and Judd (2013), Ahn et al. (2017), Winberry (2016), and Legrand and Ragot (2017).

Those approaches approximate responses to aggregate shocks by using first-order expansions of policy rules around a steady state Z^{SS} obtained by shutting down aggregate shocks. Ahn et al. (2017), which represents the current frontier,[9] extends Reiter (2009) into continuous time and performs a sophisticated form of model reduction by projecting a distribution of individual states onto a lower-dimensional subspace that is designed to do a good job of approximating impulse response functions of key variables like prices. In doing so, they can incorporate a larger number of individual state variables than was previously possible. Lastly, except for Legrand and Ragot (2017), the studies cited earlier in this paragraph focus on competitive equilibria under fixed policies, rather than finding optimal policies.

We differ from these contributions in two ways. First, our points of approximation, Z_{t-1}, are dynamic and history dependent. By building on Fleming (1971), Fleming and Souganidis (1986), and Anderson et al. (2012), we take Taylor expansions with respect to uncertainty at each date as aggregate shocks push the economy through time.

There are several reasons that approximating around Z^{SS} is not a good way to approximate economies like ours. First, computing Z^{SS} in a Ramsey setting is difficult. Z^{SS} is an endogenous object that depends on a key object to be computed, an optimal policy. That requires jointly solving for agents' optimal behaviors, which depend on the government's policies, and optimal optimal policies. Even for a deterministic setting, that would require using computationally challenging non-linear solution methods. We are not aware of methods to do that quickly. But even if Z^{SS} could be found, it would be unlikely to be a good point of expansion. That is because in Ramsey settings with incomplete markets, speeds of mean-reversion to Z^{SS} are typically extremely slow because state variables are driven by martingale-like dynamics that drift slowly.[10] From a computational point of view, using perturbation around the fixed point provides a poor approximation for the optimal policy in many states that are away from Z^{SS}.

Secondly, paralleling Evans (2015), in Theorem 1 and Corollary 1 we are able to characterize the derivatives used in a small noise expansion in terms of matrices of small (typically $N_x \times N_x$) dimension. This allows us to perform not only first-order but also higher-order expansions quickly. As mentioned earlier, higher order terms are required to compute transition paths and accurate responses of aggregate variables to shocks because agents' responses to the idiosyncratic risks that

[9] Mertens and Judd (2013) approximate around a point of no heterogeneity. Winberry (2016) uses a variant in which parametric forms capture the steady state distributions rather than the histograms used by Reiter (2009). Legrand and Ragot (2017) study an optimal fiscal policy problem with idiosyncratic risk and aggregate shocks after truncating individual histories, which limits the amount of heterogeneity that they can consider.

[10] To give an extreme example, debt follows a random walk in a canonical incomplete market model of Barro (1979), so that the speed of the mean reversion is 0. Aiyagari et al. (2002) showed that a slow martingale-like component is generally typically present in a Ramsey plan in an incomplete market economy. In Bhandari et al. (2017) we compute analytically the speed of mean reversion for several incomplete markets economies.

they face puts slow drifts into equilibrium distributions of their state variables.

Alternatives to perturbation methods the literature have also used *projection methods* like Krusell and Smith (1998), Den Haan (1997), Algan et al. (2010). Projection methods summarize the infinite dimensional state variable using a subset of moments and approximate value functions and policy functions by using functional approximations and simulations for aggregate laws of motion that describe the ergodic behavior of moments.

Like the perturbation methods cited above, projection methods that approximate around the long run ergodic distribution Z^{SS} are problematic in Ramsey settings. Projections methods work well only when an economy exhibits what Krusell and Smith termed an "approximate aggregation" property in which a function of the first moment of Z predicts next period's prices accurately. In our setting, Z is distributed over $\mathbb{R}^3$, which makes it much more difficult to summarize in terms of one or two dimensional statistics. Moreover, there is little reason to believe that our economy with heterogeneous loading on aggregate shocks (and, in later extensions, heterogeneous participation in asset markets) exhibits approximation aggregation.

4 Quantitative Application

We apply our equilibrium approximation algorithm to an economy whose initial conditions are calibrated to recent U.S. data, assess quantitatively the properties of Ramsey policies, and contrast them with those of benchmark representative agent settings, and in particular, how interest rates and tax rates respond to aggregate fluctuations in productivity and markups.

4.1 Calibration

We set $\rho_\Theta = 1$, implying that the aggregate productivity follows an i.i.d growth process.[11] The mean and standard deviation of the growth rate in productivity Θ_t are set at 2% and 3%, respectively, to match the mean and standard deviation of growth rate in output per hour in the U.S. We set $\nu = 1$ so that the preferences are consistent with balanced growth and $\gamma = 2$ to attain an Frisch elasticity of labor supply equal to 0.5.

In addition to productivity shocks, here we also introduce fluctuations in desired aggregate markups. As in Galí (2015), time varying desired markups are implemented by replacing the

[11] An i.i.d growth rate process approximates output per worker well for the US and has several convenient properties. As we explain later the first best allocation has constant real rates and in absence of markup shocks and heterogeneity can be implemented using standard Taylor rules. Since the economy is growing, we scale menu costs and exogenous government spending by Θ_t.

constant elasticity of substitution ϵ with a stochastic process[12]

$$\ln(\epsilon_t) = (1 - \rho_\epsilon) \ln(\bar{\epsilon}) + \rho_\epsilon \ln(\epsilon_{t-1}) + \mathcal{E}_{\epsilon,t}.$$

We set the steady state elasticity of substitution at $\bar{\epsilon} = 6$ to target a value-added markup of 20%, normalize $\mathcal{E}_\epsilon$ so that one standard deviation innovation to $\ln(\epsilon)$ changes markups by 1% and following Smets and Wouters (2007) set the persistence of $\ln(\epsilon_t)$ to be 0.65.

Remaining parameters are calibrated by insisting that competitive equilibrium outcomes given policies $\{\iota_t, \tau_t\}_t$ match stylized facts about U.S. policies. In particular, we set

$$\tau_t = \bar{\tau} \tag{35}$$

with $\bar{\tau} = 24\%$ to match the federal average marginal income tax estimated by Barro and Redlick (2011). We set ι_t to follow a Taylor rule

$$\iota_t = \left(\frac{1}{\beta \mathbb{E} exp\left\{-\bar{\Theta}\right\}} - 1 \right) + 1.5\pi_t \tag{36}$$

Our choice of constant tax rates and Taylor rule that responds more than one to one to inflation and has a constant intercept features a desirable property that it can implement the optimal allocation in the absence of heterogeneity and markup shocks.[13] Our baseline calibration is not sensitive to alternative specifications for fiscal and monetary policy that allow the nominal interest rate and tax rate to feedback on output or the output gap.[14] We set the discount factor β to match an interest rate of 4% per year. Following Schmitt-Grohé and Uribe (2004), we use estimates of Sbordone (2002) to calibrate the menu cost $\psi\Theta_t$.[15] We set $\alpha = 1$ to model firms as having constant returns to scale production. The mean of government expenditures divided by labor productivity $\bar{G}$ is

[12]The New Keynesian literature has emphasized cost-push shocks in accounting for business cycle fluctuations (see, e.g., Smets and Wouters (2007)) and studied their implications for optimal monetary policy in a representative agent framework (see, e.g., Clarida et al. (2001), Galí (2015), Woodford (2003)). As in Galí (2015) we interpret cost-push shocks as changes in desired markups arising from fluctuations in the elasticity of substitution parameter ϵ and use the phrase "markup-shock" and "cost-push shock" interchangeably. Other alternatives such as fluctuations in wage markups are left for future work.

[13]The reason is that the real interest rate is constant in a flexible price version of such an economy given our choice of preferences and the lack of predictability in aggregate productivity growth. See also related discussion of the "divine coincidence" property in these class of models as in Blanchard and Gali (2007).

[14]An empirical literature about Taylor rules typically estimates loading on output that are small and statistically close to zero. See for example Bhandari et al. (2017) for a discussion of fiscal policy rules and Clarida et al. (2000) for a discussion of monetary policy rules.

[15]Using quarterly data inflation and measures for marginal cost mc_t, Sbordone estimates a relationship $\pi_t = \alpha_1 \mathbb{E}_t \pi_{t+1} + \alpha_0 mc_t$ with $\alpha_1 \approx 1$ and α_0^{-1} in the range of 10 - 20 depending on particular measure of marginal cost. In a linearized version of Phillps curve equation (13) and converting to annual calibration, $\frac{1}{4}\alpha_0 = \frac{\psi}{\epsilon - 1}$. Using $\epsilon = 6$ and $\alpha_0^{-1} = 15$ implies $\psi = 18.75$. As explained in Sbordone (2002), in a Calvo type price setting friction, this estimate corresponds to firms changing prices every 9 months. In appendix D we show how our results change when we vary ψ.

calibrated to match the average government current expenditures net of transfer payments (federal) to annual GDP.[16]

The stochastic process for idiosyncratic shocks and their loading on aggregate shocks are chosen to match several stylized facts about labor earnings. We set auto correlation and standard deviation of of e_{it} and standard deviation of $\varsigma_{i,t}$ to match the authorization of earnings, standard deviation of one year change in one year log earnings along with the standard deviation of 5 year change in log earnings. As in Storesletten et al. (2004) and Low et al. (2010) we find a fairly persistent skill process with the $\rho_e = 0.99$ and the standard deviations of the persistent components and transitory component to be 12% and 24%. The loading function $f(e)$ is constructed to match the evidence in Guvenen et al. (2014) on how recessions affects households in different parts of labor earning distribution. Following the empirical procedure in Guvenen et al. (2014), we rank workers by their average log labor earnings 5 years prior to simulating a 3% fall in aggregate output. We then compute the percent income loss for each worker following the recession and calibrate the parameters of a quadratic function for $f(e) = f_0 + f_1 e + f_2 e^2$ to match income losses of the 5^{th}, 50^{th} and 95^{th} percentiles. The right panel of Figure II compares earnings loss patterns simulated from our model with corresponding data summarized by moments in Guvenen et al. (2014).All the parameters of our baseline specification are summarized in Table I.

For reporting the optimal policies we calibrate the initial distribution of debt, stock holdings and the persistent component of productivities using on wages, debt and stock holdings from Survey of Consumer Finances 2007 (SCF). The SCF measures wage income, hours worked, debt and stock holdings.[17] We restrict sample to married households who report at least 100 hours for the year. We draw $e_{i,-1}$ from the ergodic distribution implied by our calibrated process (3) which also produces initial wages that are similar to those in our SCF sample. We next use the observed distribution of stock holdings in the SCF to draw $s_{i,-1}$ and for initial debt $b_{i,-1}$ we use a fitted values from the regression

$$debt = a_0 + a_w \times wages + a_s \times share \quad of \quad stocks$$

Parameters a_0, a_w, a_s are estimated using debt, wages, and stock holdings from the SCF. We find that 30% fraction our sample holds zero stocks and remaining distribution of stock holdings are right skewed with the top 10% households holding about 60% of the total stocks. The point estimates of a_w and a_s (standard errors in the parenthesis) are 0.82(0.03) and 1.01(0.01) indicating a positive relationship between debt holdings, wages, and stock holdings.

[16]The data is obtained from NIPA and averaged for the period 1960-2016.

[17]We sum direct holdings plus indirect holdings through liquid assets (net of unsecured credit), government bond mutual funds (taxable and nontaxable), saving bonds, money market accounts, and components of retirement accounts that are invested in government bonds. We add liquid assets to debt holdings because b in our model also includes private IOUs. It also makes sense when we calibrate the hand to mouth types where we set$b_{i,g,-1} = 0$. We interpret stock holdings to be the sum of direct stock and mutual funds and indirect holdings in the through retirement accounts.

Parameters	Values	Targeted moment	Values
Preferences, technology			
ν^{-1}	1	inter temporal elasticity	1
γ^{-1}	2	Frisch elasticity	2
β	.98	risk free rate	4%
ϵ	6	average markups	20%
ψ	18.75	estimates from Sbordone (2002)	see footnote 15
Aggregate shocks			
$\bar{\Theta}$	2%	mean labor productivity growth	2%
$\bar{G}$	7%	mean govt. expenditure to labor productivity	7%
$\bar{\epsilon}$	6	mean markup	20%
s.d of $\mathcal{E}_\Theta$	3%	s.d of log change in labor productivity	3%
s.d of $\mathcal{E}_\epsilon$,ρ_ϵ	4%,0.65	normalize to target the size and persistence of changes in markups, estimates from Smets and Wouters (2007)	1%,0.65
Idiosyncratic shocks			
s.d of ς	0.24	s.d. of one year and 5 year log earnings change, autocorr. of annual earnings	0.55, 0.72,0.99
s.d. of η	0.12		
ρ_e	0.99		
f_2, f_1, f_0	0.28,−0.52,0.00	earnings losses $5^{th}, 50^{th}$ & 95^{th} percentiles	

Table I: Baseline calibration

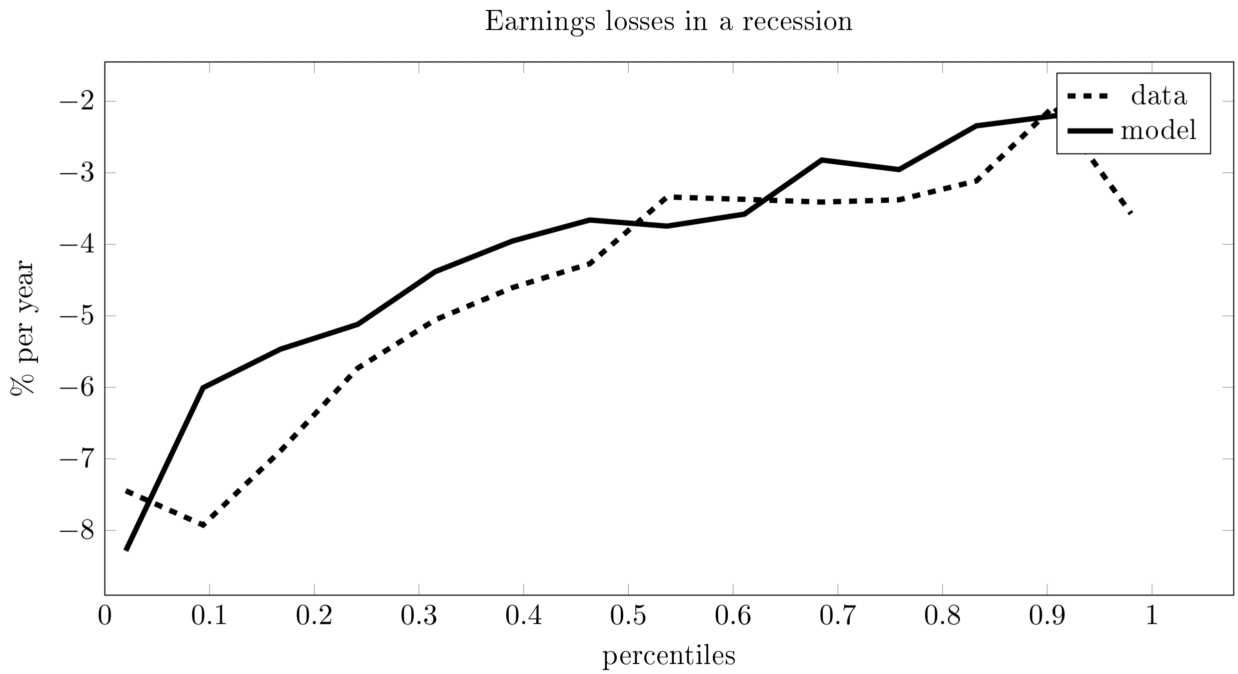

Figure II: Annual earnings losses using simulated earnings from the model and data in Guvenen et al. (2014).

5 Results

We focus on the optimal responses to productivity and desired markup shocks. When we consider purely a monetary policy response, we fix the tax rate at $\tau_t = \tau^*$, where τ^* is the optimal tax rate in the non-stochastic environment. Since Ramsey policies at time 0 typically differ from continuation Ramsey policies at $t \geq 1$, we report impulse responses for a shock that occurs at a later, $t = 5$.[18] We begin with the optimal response to a one standard deviation negative labor productivity shock.[19]

[18] The effects of $t = 0$ policies lasts longer than one period in our economy is shared with several Ramsey models - in real models like Lucas and Stokey (1983), Aiyagari et al. (2002) it shows up in the dynamics of the Lagrange multiplier on government's present value budget constraint and in monetary models such as Schmitt-Grohé and Uribe (2004) additionally as in the dynamics of the Lagrange multipliers on the Phillips curve. Our choice of $t = 5$ ensures that the results are not driven by these time 0 considerations. Other choices with the shock occurring at $t = 10$ and $t = 15$ give similar results and we report then in the online Appendix to keep the main text short.

[19] To compute the impulse responses we draw a path of length 25 for $\{\mathcal{E}_{\Theta,t}\}$ simulate the economy twice changing only the shock at period 10: for the first path $\mathcal{E}_{\Theta,10} = -0.03$ and for the second $\mathcal{E}_{\Theta,10} = 0$. The impulse response is the difference in the path of the endogenous variables. To account for non-linearity in the decisions rules we repeat this exercise 20 times with different draws of $\{\mathcal{E}_{\Theta,t}\}$ and report the mean path. The distribution of the IRF is quite tight for our case and so we do not report the standard error bands around the mean path. All variables show deviations in percentage points.

5.1 Optimal Response to Productivity Shock

We first consider an optimal monetary policy. Figure IV depicts responses to a one-time, one standard deviation negative impulse to aggregate productivity $\mathcal{E}_\Theta$ occurring at $t = 10$. On impact, this shock induces a drop in growth rate of output of about 3 percentage points. The solid lines represent responses in our calibrated heterogeneous agent New Keynesian (HANK) economy, the dashed line show responses in its representative agent counterpart (RANK). Here we have set $b_{1,-1} = B_{-1}$, $\theta_{i,t} = \Theta_t$ for all i and $f = 0$.

In the representative agent version, the economy's response to a productivity shock is efficient without policy adjustments. As a result, the Ramsey planner keeps nominal interest rates unchanged to keep inflation stable. Tax rates, which are unchanged by assumption. Such a hands-off monetary policy is not optimal when agents are heterogeneous. A productivity shock affects different agents differently and because markets are incomplete, agents cannot insure those risks. A monetary policy response provides missing insurance, partly compensating for market incompleteness.

Productivity shocks differentially affect agents for two reasons. One arises from wealth heterogeneity. Because an adverse productivity shock permanently lowers all wages, the consumption of agents having few financial assets falls by more than consumption of agents with more financial assets. To provide insurance against that adverse aggregate shock, the planner desires to lower returns on assets. She can achieve this in two ways. First, the planner can reduce the *ex post* realized real return on debt by engineering a surprise inflation at the time of the shock. Second, the planner can tilt the path of future nominal interest rates to reduce *ex ante* returns on savings going forward. Both of these effects appear in Figure IV. The planner cuts interests rates on impact of the shock, thereby generating a spike in inflation and a drop in *ex post* real asset returns, and then promises higher nominal interest rates that lowers aggregate demand and consumption in the future and lower equilibrium real rates in the current period.

The second motive for government intervention comes from productivity shocks having different effects on high-wage and low-wage agents. As we saw in panel B of Figure II, low-wage agents are more strongly affected by an adverse productivity shock. To provide insurance, the government would like to design a policy that redistributes resources from high-wage earners to low-wage earners. Monetary policy can redistribute by affecting returns on financial assets. Recall from Section 4.1 that wages and asset holdings are positively correlated, both in the data and in our model. Thus, a policy that reduces asset returns effectively redistributes resources from high-wage to low-wage individuals. A desire to use this channel reinforces the direction of the optimal response that discussed above.

When a government also has access to fiscal policy, increasing progressiveness of the labor taxes also induces redistributions by offsetting differential affects of an adverse productivity shock on the distribution of wages. Figure IV shows the optimal response of monetary-fiscal policy to one standard deviation negative aggregate labor productivity shock. Because an adverse TFP shock

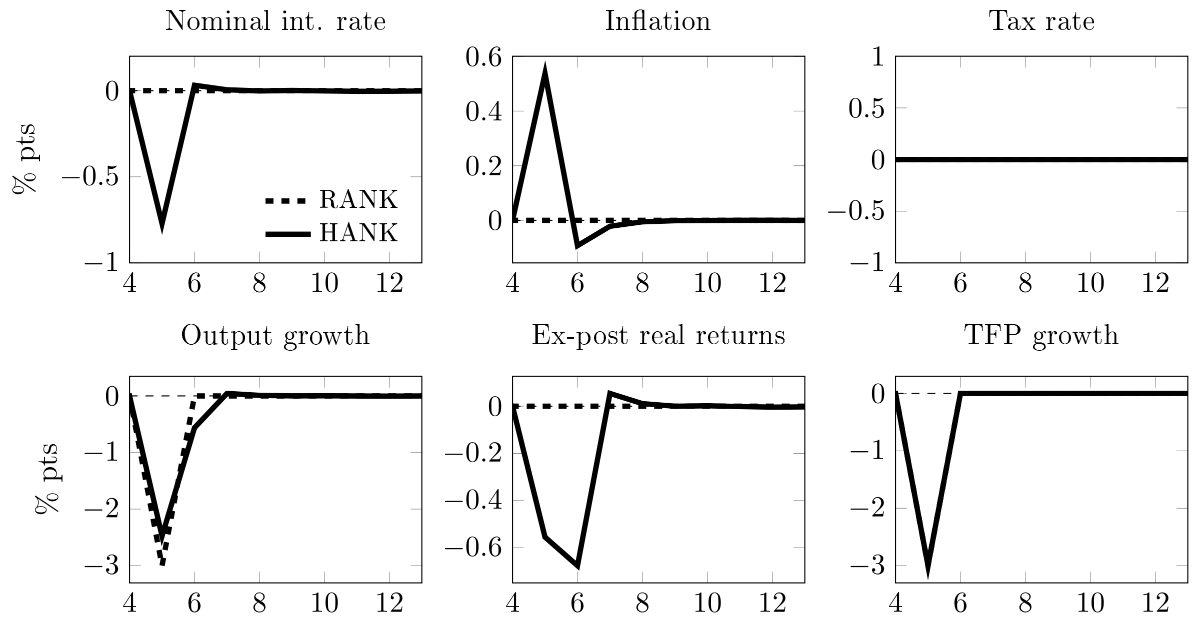

Figure III: Optimal monetary response to a productivity shock

is associated with an persistent increase in the dispersion of log wages, a government optimally responds by permanently increasing tax rates one period after the shock. The delayed increased in tax rates helps to lower real rate at the time of the shock (for the same reason tax rates are temporarily decreased on the impact of the shock). As the result, the path of nominal rates and inflation is smoother when fiscal policy is active, which helps the planner to reduce to costs of price adjustments.

5.2 Optimal Responses to Markup Shocks

Now we turn to the optimal response to an increase in desired markups. Experiments were constructed in similar ways to those conducted in Section 5. We start with differences in monetary policy when labor taxes are fixed at τ^* and then study the case with optimal monetary and fiscal policy. A key finding is that the trade-offs faced by the policy maker in a heterogeneous agents setting differ substantially from those in a representative agent economy, leading to policy prescriptions that an order of magnitude larger in the HANK economy and can also have opposite signs from those in the RANK economy.

Optimal monetary responses to a one standard deviation positive shock to $\mathcal{E}_\epsilon$ are shown in Figure V. Although the representative agent model calls for a moderate tightening of monetary policy following a cost-push shock, the heterogeneous agents economy requires a substantial decrease in nominal interest rates and a positive spike in inflation.

To understand this result, it is useful first to analyze the optimal response in the RANK model.

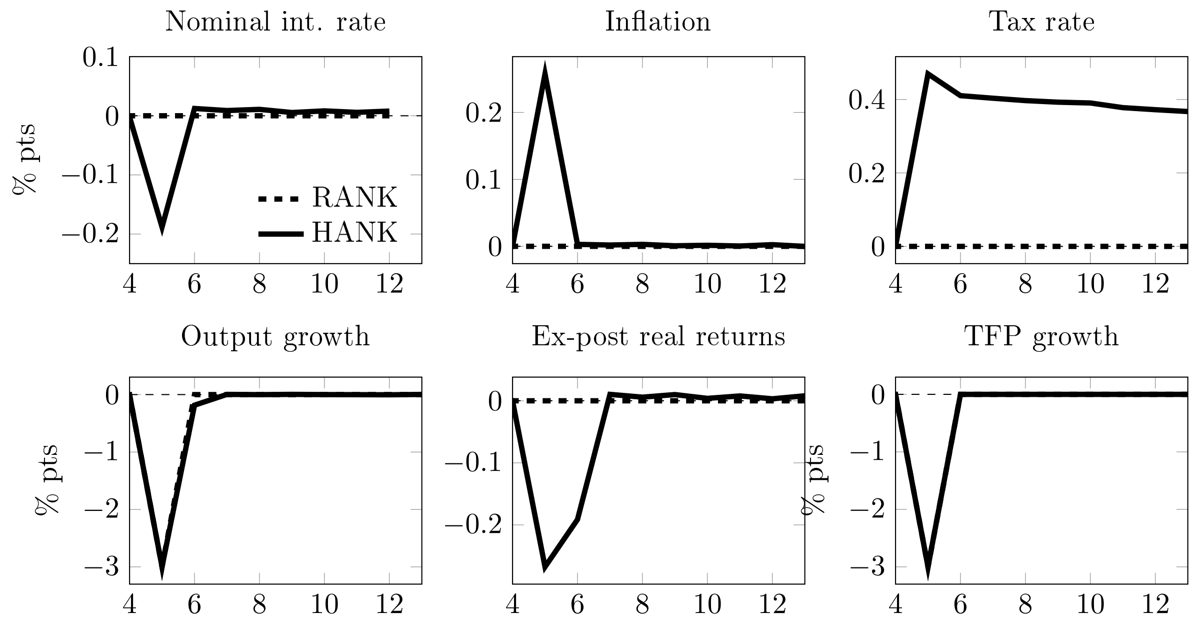

Figure IV: Optimal monetary-fiscal response to a productivity shock

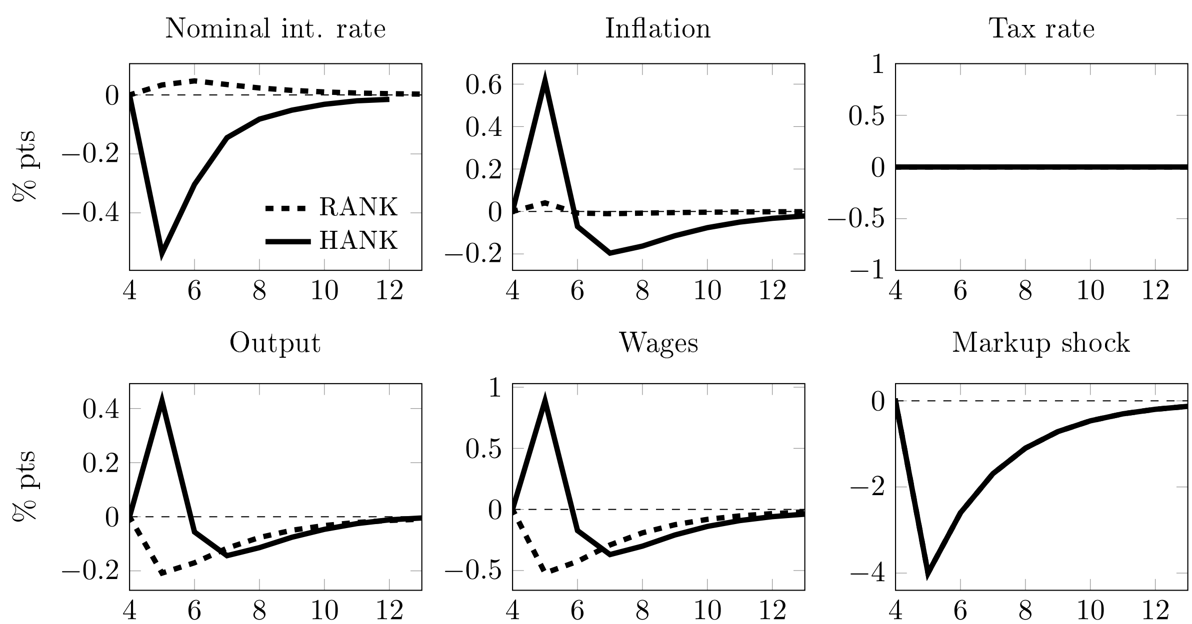

Figure V: Optimal monetary response to a markup shock

A positive cost-push shock increases firms' desired mark ups over marginal costs. Since price changes are costly in New Keynesian models, the planner offsets this effect by lowering marginal costs and thereby pushes output below its natural level.[20] Galí (2015) dubs this policy "leaning against the wind". The reduction in output is achieved by committing to a tight monetary policy in the future that lowers aggregate demand.

However positive mark-up shocks not only induce inflationary pressure, they also decrease labor's share and increase profit's share (see the last row in Figure V). In the representative agent economy, these effects on factor shares are of second-order since workers and firm owners are the same person. In the HANK economy stock ownership is heterogeneous and, thus, a mark-up shock naturally redistributes resources from agents with low stock ownership to agents with high stock ownership. Since stock ownership and labor earnings are correlated, this effectively redistributes resources from low wage workers to high wage workers. Leaning against the wind policies exacerbate this effect.

When markets are incomplete, agents cannot insure against the cost push shock and the Ramsey planner sets policies indirectly to provide insurance by offsetting the distributional effects of a cost-push shock. Quantitatively, this consideration dominates the planner's desire to reduce costs of price changes. The planner induces a desired redistribution by significantly lowering interest rates immediately and committing to low interest rates in the future. That boosts aggregate demand and thereby raises wages and lowers dividends. A notable feature of the optimal policy is the increases in wages that occur in the period that the shock hits. Postponing wage increases would be detrimental because firms would respond to anticipated wage increases by raising current prices thereby generating extra inflation, which is costly, while not generating welfare gains.

With fixed tax rates, a cost-push shock sets up a tension between movements in the labor wedge, i.e., deviations in $(1-\tau)W$ from one, and costs of inflation. In a representative agent economy, when the planner has access to fiscal policy, a first-best allocation characterized by $\pi_t = 0$ and $(1-\tau)W = 1$ is feasible and can be implemented using a labor tax subsidy that offsets the time varying markup and nominal rates that do not respond to to the cost-push shock. This summarizes the dashed lines in Figure VI.

Optimal fiscal policy in a HANK economy also stands starkly in contrast to what it is in a RANK economy. In the HANK economy, the planner *raises* taxes in response to an adverse cost-push shock. The aim of planner is still to transfer resources from high-wage owners of the firms to lower-wage agents but, with labor taxes available, the planner has a more direct way of influencing wages and firm profits. Higher tax rates contracts labor supply, raise wages, and lowers dividends. That arrests some of the adverse distributional effects of markup shocks. As before, tax changes are concentrated on impact of the shock in order to make the wage increase and the resulting inflation both be unanticipated. Following the shock, the planner implements a tax subsidy like the one used

[20] It is feasible to implement an allocation that sets $\pi_t = 0$ but that requires that wages adjust to offset markups and associated deviations of $(1-\tau)W$ from one, which is costly. We return to this consideration when we discuss an optimal monetary-fiscal response to cost-push shocks.

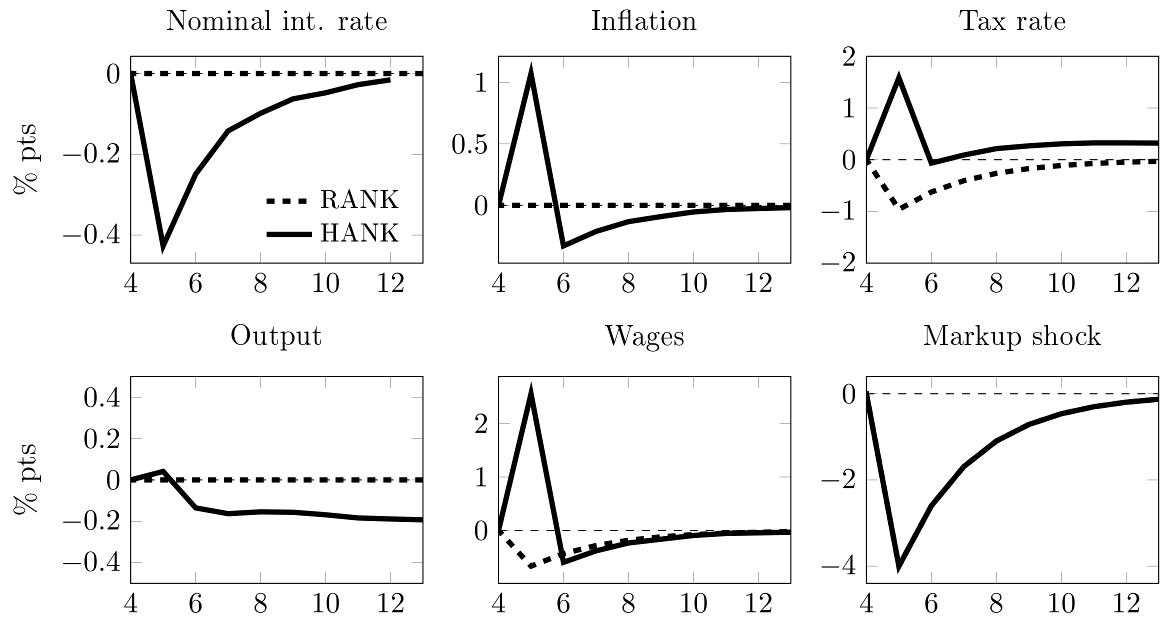

Figure VI: Optimal fiscal monetary response to a markup shock

in the representative agent economy. The nominal rate closely tracks the real rate, which is also low on impact. These responses are summarized in the solid lines of Figure VI.

A novel aspect of the optimal plan that is captured by our solution method is visible in dynamics of tax rates and output which have a almost a permanent change that lasts after the transitory shock wears off. This occurs because the shock shifts the distributional state variable which has almost martingale like dynamics. The new level of tax rates and consequently output reflect this permanent shift in planners' state variables. Although this effect is theoretically present in all our exercises, its quantitatively largest for markup shocks with fiscal and monetary policy.

5.3 Taylor Rules

In this section we study how well standard Taylor rules approximate the optimal policy in heterogeneous agents settings. We impose a Taylor rule of the form (36) and compare responses to TFP shocks and cost-push shocks with responses under an optimal policy.[21]

We begin with the RANK economy. As explained before, the Taylor rule economy implements an optimal allocation in an economy with only productivity shocks. It also leads to outcomes similar to those for an optimal policy in response to a cost-push shock. A key feature of the optimal response is low and stable prices that can be implemented by a Taylor rule that features a sufficiently large response of nominal rates to inflation. These findings confirm conclusions of Woodford (2003) and

[21] As in sections 5.1 and 5.2, we we keep tax rates constant at optimal τ^* value and focus on monetary responses for all the experiments.

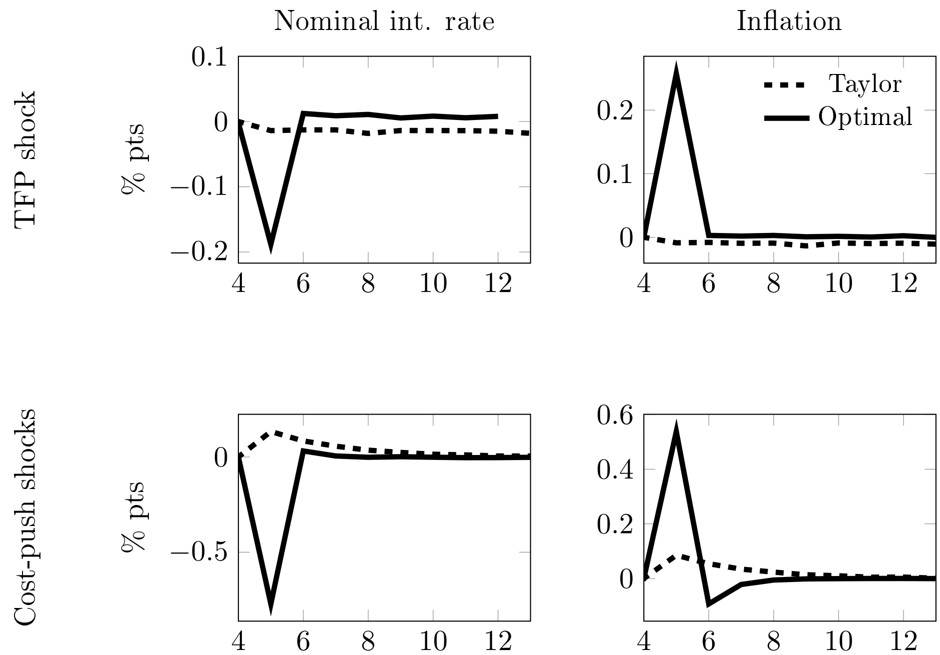

Figure VII: Comparing optimal monetary responses to Taylor rule in HANK model. The solid line is the optimal response and the dashed line is the response in a competitive equilibrium with $i_t = \bar{i} + 1.5\pi_t$.

Galí (2015) who also find that Taylor rules are to being optimal rules.

Things differ in the HANK model. In response to both types of adverse shocks – a negative aggregate productivity shock or a positive cost-push shock that raises markups – the optimal plan seeks to transfer resources from high wage earners who receive asset income to low wage earners who rely primarily on wage income. Transfers are implemented by lowering nominal rates, thereby raising aggregate demand, wages, and the price level. Under the Taylor rule, the responses of aggregates in the HANK economy are very close to those of RANK economy. The Taylor rule continues to recommend no response to productivity shocks, and contrary to an optimal response, an increase in interest rates after a cost-push shock. Thus, Taylor rules do a poor job of approximating an optimal policy. Figure VII summarizes these findings.

5.4 Robustness

We now discuss the robustness of our baseline findings and the roles of several assumptions. We begin with our assumption about borrowing limits. In the baseline we imposed natural borrowing limits which meant that the agents consumption behavior is well approximated by permanent income hypothesis and they feature a fairly a uniform distribution of the marginal propensity to consume of out transitory or persistent income shocks. As pointed out by Auclert (2017), Kaplan et al. (2016), matching the distribution of marginal propensities to consume is important to study the effects of monetary shocks. We relax the natural debt limit assumption and follow Campbell and Mankiw (1989), Jappelli and Pistaferri (2014) to add credit frictions in the form of segmented markets. In particular, we introduce agents who are "hand-to-mouth" types - they have zero initial debt holding and for all periods thereafter consume their after-tax wage income plus dividends from the firm. We use then use evidence from Jappelli and Pistaferri (2014) to calibrate the distribution of these agents. Using the 2010 Survey of Household Income and Wealth, Jappelli and Pistaferri find that marginal propensity to consume out of transitory income are higher (lower) for agents with small (large) cash on hand[22]and a parsimonious description of their data in figure 4 is a relationship

$$mpc_i = 0.6 - 0.4p(i) \tag{37}$$

where $p(i)$ is percentile of cash on hand to which agent i belongs. We populate our economy with hand-to-mouth types targeting the slope and the intercept of the relationship (37) in our competitive equilibrium at $t = 0$. The optimal monetary responses with hand-to-mouth agents are then reported in Figure VIII.

In comparison to the baseline without hand-to-mouth types, we find that the optimal interest rate response is lower for markup shocks but about the same for productivity shocks. A key difference is the path of transfers which is no longer indeterminate and is chosen optimally in conjugation with the path for the nominal rates. Timing of transfers in our setting with borrowing constrained agents can be used for two objectives - to influence the path of aggregate demand and to smooth consumption of constrained agents over time. Which of these two objectives is salient depends on the type of shocks.

. As emphasized before, with cost push shocks and higher desired markups, the planner aims to increase wages and lower dividends. In the baseline this was achieved by exclusively lowering interest rates but with with hand-to-mouth agents, the planner uses lower nominal rates and higher transfers to achieve this goal. Lower nominal rate increases desired consumption for agents who are on their Euler equations and higher transfers increases the consumption for constrained agents who have a higher marginal propensity to consume. We see in bottom panel of Figure VIII that the drop in nominal rates is much lower than the baseline but the resulting inflation similar to the benchmark without constrained agents.

With lower aggregate productivity that disproportionately affects low skill workers, the planner provides insurance against that shock by lowering asset returns and transferring resources from savers to borrowers. However when a sufficiently large fraction of the economy is constrained, they do not directly benefit from lower returns on debt and this reduces the incentive to lower interest rates. At the same time, a smaller fraction agents who participate in the asset markets implies

[22]They define cash on hand as as the sum of household disposable income and financial wealth, net of consumer debt.

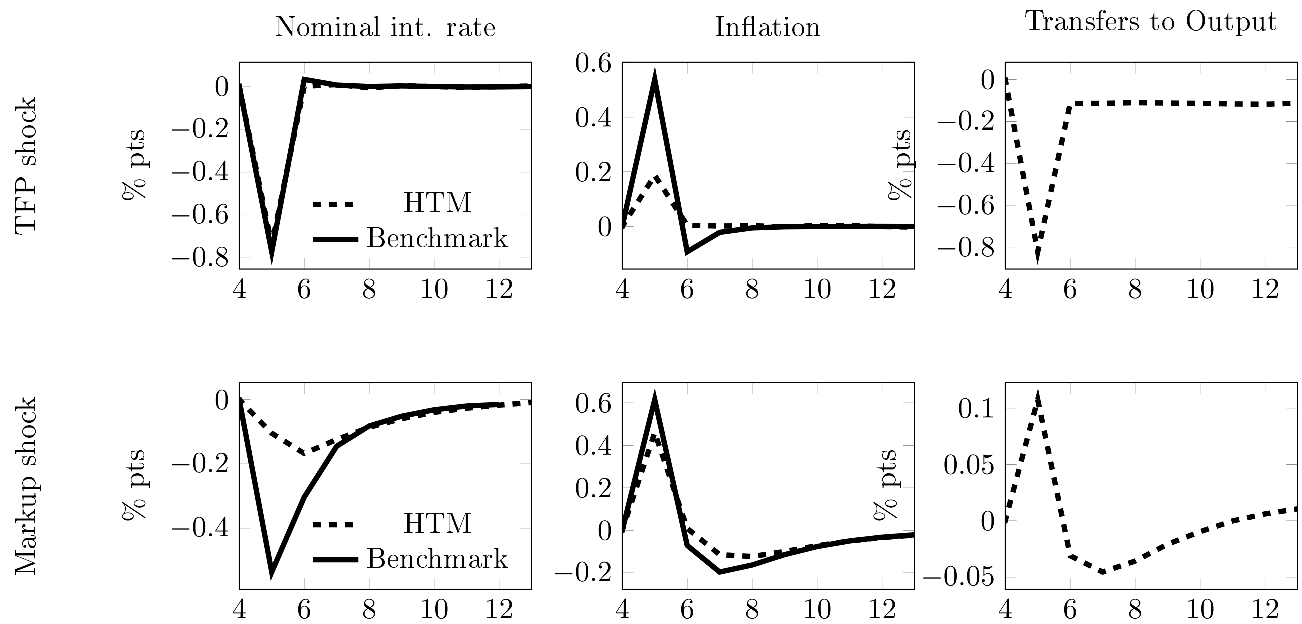

Figure VIII: Optimal responses with hand to mouth agents.

that nominal interest rates are less potent to affect aggregate demand and inflation. Together these forces approximately net out and the planner keeps maintains the same contractionary path for nominal rates as in the baseline. Now given taxes are fixed, the present value of transfers needs to adjust to satisfy the budget constraint and in the optimal allocation. In the optimal allocation, the planner lowers transfers more on impact and less in future to smooth the transitory increase in wages that accompanies the contractionary monetary policy.

Other robustness exercises include alternative Pareto weights, isolating the role of inequality shocks, lowering menu costs. Since agents are heterogeneous, a Ramsey planner wants to redistribute resources; how and how much depends on the Pareto weights. However, Pareto weights affect the *average* levels of the interest rate, tax rate, and transfers and policy responses to shocks are mainly driven by the planner's desire to provide insurance which are distinct from its desire to redistribute. As a result, policy responses to aggregate shocks remain quite similar when we assign Pareto weights other than those assigned by the utilitarian in government objective function (11), or if we were to have made alternative assumptions about $\bar{\tau}$ in our analysis of the optimal monetary policy.

The planner's preference for supplying insurance is driven by the assumption that aggregate shocks affect marginal utilities of consumption of agents differentially. This arises from two features of our baseline calibration: agents differ in their holdings of the nominal bonds as well as in their exposures to the aggregate shocks through the loading function that we have f calibrated to match the evidence in Guvenen et al. (2014). In Figure IX of Appendix D, we indicate optimal responses in a economy in which $f = 0$ and find that asset heterogeneity alone contributes to 50% of the policy responses.

We calibrated menu costs to match the the slope of the Phillips curve. Lower costs of changing prices imply that lowering ex-post returns through inflation is a cheaper tool for the planner to insure agents. We use Figure X of Appendix D to study the role of price stickiness by computing the optimal monetary-fiscal policy when menu cost parameter $\psi = 0$. We find that an inflation response of about 5 percentage points, which is about 10 times larger than in the benchmark economy.

In our experiments the planner chooses transfers T_t at each t. Authors of RANK models including Schmitt-Grohé and Uribe (2004) and Siu (2004) typically restrict $T_t = 0$ for all t. When we re-compute optimal policies in the RANK version or our economy with $T_t = 0$ for all t, we recover outcomes like those discovered by Schmitt-Grohé and Uribe and Siu, namely, that paths of inflation and interest rates in that economy, while not longer constant, are extremely smooth. This reaffirms those authors' insight that the cost of price changes in calibrated RANK economies is sufficiently high that a Ramsey planner chooses to abstain from using inflation fluctuations to smooth distortions coming from aggregate shocks. The peak change in nominal interest rates and inflation is 0.05% and 0.03% in the RANK economy, which is an order of magnitude smaller than our baseline HANK outcomes.

6 Concluding Remarks

James Tobin described macroeconomics as a field that explains aggregate quantities and prices while ignoring distribution effects. Tobin's characterization also describes much work subsequent to his in the real business cycle, asset pricing, Ramsey tax and debt, and New Keynesian research traditions. In each of these lines of research, an assumptions of complete markets and/or of a representative consumer allows the analyst to compute aggregate quantities and prices without also determining distributions across agents.

This paper departs from Tobin's "aggregative economics" in two ways. First, we assume incomplete markets – which means that aggregate quantities, prices, and allocations across agents must be determined jointly, not recursively as in complete markets models. And second, we specify technology and relative skills shocks in a way that makes contact with findings of Guvenen et al. (2014) that US cross section distributions of labor earnings have moved systematically over business cycles. A common shock affects *both* an aggregate technology shock and the cross-section distribution of skills. Cross-section dispersions in labor earnings and asset holdings shape both aggregate outcomes and choices confronting a Ramsey planner.

Finally, an incomplete markets model goes a long way toward framing an optimal policy problem when it sets the menu of assets. By specifying that the only asset traded in our model is a risk-free *nominal* bond, we activate a beneficial role for fiscal and monetary policy to make nominal interest rates fluctuate in ways that hedge inequality-increasing shocks to distributions of labor earnings.[23]

[23] It is fruitful to compare our assumptions with those of Musto and Yilmaz (2003), who focus on how markets that allow citizens to insure outcomes of voting affect the efficacy of redistribution.

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A System of Equations

The planner's problem is to choose an allocation that maximizes

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \int \left(\frac{c_{i,t}^{1-\nu}}{1-\nu} - \frac{n_{i,t}^{1+\gamma}}{1+\gamma} \right) di$$

subject to the necessary conditions for a competitive equilibrium (under constant returns to scale)

Some Simplifications and change of variables

Will work with the assumption that agents have equal shares of the firm so that $s_i = 1$. Under this assumption we can show that the Philips' curve is slack. At the end of the appendix we show how our method extends to when the Phillip's curve is not slack. Define $\mathcal{T}_t = D_t + T_t$ and $\mathcal{W}_t = (1-\tau_t)W_t$. Then the maximization problem can be rewritten as max

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \int \left(\frac{c_{i,t}^{1-\nu}}{1-\nu} - \frac{n_{i,t}^{1+\gamma}}{1+\gamma} \right) di$$

subject to the necessary conditions for a competitive equilibrium

$$\begin{aligned} c_{i,t} + b_{i,t} Q_t &= \mathcal{W}_t \theta_{i,t} n_{i,t} + \mathcal{T}_t + \frac{b_{i,t-1}}{1+\pi_t} \\ c_{i,t}^{-\nu} \mathcal{W}_t \theta_{i,t} &= n_{i,t}^{\gamma} \\ c_{i,t}^{-\nu} Q_t &= \beta \mathbb{E}_t \left[c_{i,t+1}^{-\nu} (1+\pi_{t+1})^{-1} \right] \\ \int c_{i,t} di &= C_t \\ C_t + \bar{G} &= \int \theta_{i,t} n_{i,t} - \frac{\psi}{2} \pi_t^2 di. \end{aligned}$$

Finally define $a_{i,t} = b_{i,t} Q_t c_{i,t}^{-\nu}$ then the budget constraint can be rewritten as (after substituting for Q_t and $\mathcal{W}_t$)

$$c_{i,t}^{1-\nu} + a_{i,t} = n_{i,t}^{1+\gamma} + c_{i,t}^{-\nu} \mathcal{T}_t + \frac{a_{i,t-1} c_{i,t}^{-\nu} (1+\pi_t)^{-1}}{\beta \mathbb{E}_{t-1} \left[c_{i,t}^{-\nu} (1+\pi_t)^{-1} \right]}. \tag{38}$$

Applying $\mathbb{E}_{t-1}$ to both sides we have

$$a_{i,t-1} = \beta \mathbb{E}_{t-1} \left[c_{i,t}^{1-\nu} - c_{i,t}^{-\nu} \mathcal{T}_t - n_{i,t}^{1+\gamma} + a_{i,t} \right].$$

Iterating forward we then have

$$a_{i,t} = \mathbb{E}_t \left[\sum_{s=1}^{\infty} \beta^s \left(c_{i,t+s}^{1-\nu} - c_{i,t+s}^{-\nu} \mathcal{T}_{t+s} - n_{i,t+s}^{1+\gamma} \right) \right].$$

This implies that the incomplete markets restriction can instead be interpreted as a sequence of measurability constraints on the allocation:

$$\frac{a_{i,t-1} c_{i,t}^{-\nu} (1+\pi_t)^{-1}}{\beta \mathbb{E}_{t-1} \left[c_{i,t}^{-\nu} (1+\pi_t)^{-1} \right]} = \mathbb{E}_t \left[\sum_{s=0}^{\infty} \beta^s \left(c_{i,t+s}^{1-\nu} - c_{i,t+s}^{-\nu} \mathcal{T}_{t+s} - n_{i,t+s}^{1+\gamma} \right) \right].$$

Finally we define $m_{i,t}$ as the time-t Pareto-Negishi weights such that $c_{i,t} = m_{i,t}^{1/\nu} C_t$. Defining $\mathcal{Q}_t = C_t^{-\nu} Q_t$, we have that the Euler equation can be written as

$$\mathcal{Q}_{t-1} = \beta m_{i,t-1} \mathbb{E}_{t-1} \left[c_{i,t}^{-\nu} (1+\pi_{t+1})^{-1} \right].$$

Note that implicit in this definition is the restriction that $\int m_{i,t}^{1/\nu} di = 1$.

Planner's Problem and Choice of State

The planner's problem is then

$$\sup_{\substack{\{c_{i,t}, n_{i,t}, m_{i,t}, a_{i,t-1}\} \\ \{C_t, \pi_t \mathcal{W}_t, \mathcal{T}_t, \mathcal{Q}_{t-1}\}}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \int \left(\frac{c_{i,t}^{1-\nu}}{1-\nu} - \frac{n_{i,t}^{1+\gamma}}{1+\gamma} \right) di$$

subject to

$$\begin{aligned} c_{i,0}^{-\nu} b_{i,0} &= \mathbb{E}_0 \left[\sum_{s=0}^{\infty} \beta^s \left(c_{i,s}^{1-\nu} - c_{i,s}^{-\nu} \mathcal{T}_s - n_{i,s}^{1+\gamma} \right) \right] \qquad (39) \\ c_{i,0}^{-\nu} \mathcal{W}_0 \theta_{i,0} &= n_{i,0}^{\gamma} \\ m_{i,0}^{-1/\nu} c_{i,0} &= C_0 \\ \int c_{i,0} di &= C_0 \\ C_0 + \bar{G} &= \int \theta_{i,0} n_{i,0} - \frac{\psi}{2} \pi_0^2 di \end{aligned}$$

and

$$\frac{a_{i,t-1}c_{i,t}^{-\nu}(1+\pi_t)^{-1}}{\beta\mathbb{E}_{t-1}\left[c_{i,t}^{-\nu}(1+\pi_t)^{-1}\right]} = \mathbb{E}_t\left[\sum_{s=0}^{\infty}\beta^s\left(c_{i,t+s}^{1-\nu} - c_{i,t+s}^{-\nu}\mathcal{T}_{t+s} - n_{i,t+s}^{1+\gamma}\right)\right] \tag{40}$$

$$c_{i,t}^{-\nu}\mathcal{W}_t\theta_{i,t} = n_{i,t}^{\gamma}$$

$$\mathcal{Q}_{t-1} = \beta m_{i,t-1}\mathbb{E}_{t-1}\left[c_{i,t}^{-\nu}(1+\pi_{t+1})^{-1}\right]$$

$$m_{i,t}^{-1/\nu}c_{i,t} = C_t$$

$$\int c_{i,t}di = C_t$$

$$C_t + \bar{G} = \int \theta_{i,t}n_{i,t} - \frac{\psi}{2}\pi_t^2 di$$

for $t \geq 1$. We proceed by letting $\nu_{i,0}$ be the Lagrange multiplier on (39) and $\beta^t\nu_{i,t}$ be the Lagrange Multiplier on (40). Adding these to the objective function and defining $\mu_{i,0} = \nu_{i,0}$ and $\mu_{i,t} = \mu_{i,t-1} + \nu_{i,t}$ we have

$$\inf_{\mu_{i,0},\{\nu_{i,t}\}} \sup_{\substack{\{c_{i,t},n_{i,t},m_{i,t},a_{i,t-1}\}\\ \{C_t,\pi_t,\mathcal{W}_t,\mathcal{T}_t,\mathcal{Q}_{t-1}\}}} \int\left(\frac{c_{i,t}^{1-\nu}}{1-\nu} - \frac{n_{i,t}^{1+\gamma}}{1+\gamma}\right) + \left(c_{i,0}^{1-\nu} - c_{i,0}^{-\nu}(\mathcal{T}_0 + b_{i,0}) - n_{i,0}^{1+\gamma}\right)\mu_{i,0}di$$

$$+\mathbb{E}_0\sum_{t=1}^{\infty}\beta^t\int\left(\frac{c_{i,t}^{1-\nu}}{1-\nu} - \frac{n_{i,t}^{1+\gamma}}{1+\gamma}\right) + \left(c_{i,t}^{1-\nu} - c_{i,t}^{-\nu}(\mathcal{T}_t) - n_{i,0}^{1+\gamma}\right)\mu_{i,t} - \frac{a_{i,t-1}c_{i,t}^{-\nu}(1+\pi_t)^{-1}\nu_{i,t}}{\beta\mathbb{E}_{t-1}\left[c_{i,t}^{-\nu}(1+\pi_t)^{-1}\right]}di$$

subject to

$$c_{i,0}^{-\nu}\mathcal{W}_0\theta_{i,0} = n_{i,0}^{\gamma}$$

$$m_{i,0}^{-1/\nu}c_{i,0} = C_0$$

$$\int c_{i,0}di = C_0$$

$$C_0 + \bar{G} = \int \theta_{i,0}n_{i,0} - \frac{\psi}{2}\pi_0^2 di.$$

and

$$
\begin{aligned}
c_{i,t}^{-\nu}\mathcal{W}_t\theta_{i,t} &= n_{i,t}^{\gamma} \\
\mathcal{Q}_{t-1} &= \beta m_{i,t-1}\mathbb{E}_{t-1}\left[c_{i,t}^{-\nu}(1+\pi_{t+1})^{-1}\right] \\
m_{i,t}^{-1/\nu}c_{i,t} &= C_t \\
\int c_{i,t}di &= C_t \\
C_t + \bar{G} &= \int \theta_{i,t}n_{i,t} - \frac{\psi}{2}\pi_t^2 di \\
\mu_{i,t} &= \mu_{i,t-1} + \nu_{i,t}
\end{aligned}
$$

for $t \geq 1$. We then split the inf sup into two parts

$$
\begin{aligned}
&\inf_{\mu_{i,0}} \sup_{\substack{c_{i,0}, n_{i,0}, m_{i,0} \\ C_0, \pi_0, \mathcal{W}_0, \mathcal{T}_0}} \int \left(\frac{c_{i,t}^{1-\nu}}{1-\nu} - \frac{n_{i,t}^{1+\gamma}}{1+\gamma}\right) + \left(c_{i,0}^{1-\nu} - c_{i,0}^{-\nu}(\mathcal{T}_0 + b_{i,0}) - n_{i,0}^{1+\gamma}\right)\mu_{i,0}di \\
&\quad + \inf_{\{\nu_{i,t}\}} \sup_{\substack{\{c_{i,t}, n_{i,t}, m_{i,t}, a_{i,t-1}\} \\ \{C_t, \pi_t, \mathcal{Q}_{t-1}, \mathcal{W}_t, \mathcal{T}_t\}}} \beta\mathbb{E}_0 \sum_{t=1}^{\infty} \beta^{t-1} \int \left(\frac{c_{i,t}^{1-\nu}}{1-\nu} - \frac{n_{i,t}^{1+\gamma}}{1+\gamma}\right) + \left(c_{i,t}^{1-\nu} - c_{i,t}^{-\nu}(\mathcal{T}_t) - n_{i,0}^{1+\gamma}\right)\mu_{i,t} \\
&\qquad - \frac{a_{i,t-1}c_{i,t}^{-\nu}(1+\pi_t)^{-1}\nu_{i,t}}{\beta\mathbb{E}_{t-1}\left[c_{i,t}^{-\nu}(1+\pi_t)^{-1}\right]}di
\end{aligned}
$$

where the first inf sup is subject to

$$
\begin{aligned}
c_{i,0}^{-\nu}\mathcal{W}_0\theta_{i,0} &= n_{i,0}^{\gamma} \\
m_{i,0}^{-1/\nu}c_{i,0} &= C_0 \\
C_0 &= \int c_{i,0}di \\
C_0 + \bar{G} &= \int \theta_{i,0}n_{i,0} - \frac{\psi}{2}\pi_0^2 di.
\end{aligned}
$$

and the second subject to

$$c_{i,t}^{-\nu}\mathcal{W}_t\theta_{i,t} = n_{i,t}^{\gamma} \tag{41}$$

$$\mathcal{Q}_{t-1} = \beta m_{i,t-1}\mathbb{E}_{t-1}\left[c_{i,t}^{-\nu}(1+\pi_{t+1})^{-1}\right] \tag{42}$$

$$m_{i,t}^{-1/\nu}c_{i,t} = C_t \tag{43}$$

$$\int c_{i,t}di = C_t \tag{44}$$

$$C_t + \bar{G} = \int \theta_{i,t}n_{i,t} - \frac{\psi}{2}\pi_t^2 di \tag{45}$$

$$\mu_{i,t} = \mu_{i,t-1} + \nu_{i,t}.$$

The state for the continuation problem is the joint distribution of $(m_{i,0}, \mu_{i,0}, e_{i,0})$, where $e_{i,0}$ is the persistent component of productivity. For the remainder of this note we'll focus on the first order conditions for the continuation problem and the solutions to those recursive in the distribution over (m_i, μ_i, e_i). Let $\beta^t\phi_{i,t}, \beta^t\rho_{i,t-1}, \beta^t\varphi_{i,t}, \beta^t\chi_t, \beta\Xi_t$ be the Lagrange multipliers on (41)-(45) respectively.

First Order Conditions

The first order conditions with respect to $m_{i,t}, c_{i,t}, n_{i,t}, a_{i,t-1}$ are

$$\beta\mathbb{E}_t\left[c_{i,t+1}^{-\nu}(1+\pi_{t+1})^{-1}\right]\rho_{i,t} + \frac{1}{\nu}m_{i,t}^{-1/\nu-1}c_{i,t}\varphi_{i,t} = 0$$

$$c_{i,t}^{-\nu} + \left((1-\nu)c_{i,t}^{-\nu} - \nu c_{i,t}^{-\nu-1}\mathcal{T}_t\right)\mu_{i,t} + \frac{\nu a_{i,t-1}c_{i,t}^{-\nu-1}(1+\pi_t)-1}{\beta\mathbb{E}_{t-1}\left[c_{i,t}^{-\nu}(1+\pi_t)^{-1}\right]}(\nu_{i,t} - \frac{\mathbb{E}_{t-1}\left[c_{i,t}^{-\nu}(1+\pi_t)^{-1}\nu_{i,t}\right]}{\mathbb{E}_{t-1}\left[c_{i,t}^{-\nu}(1+\pi_t)^{-1}\right]})$$

$$-\beta\nu m_{i,t-1}c_{i,t}^{-\nu}(1+\pi_{t+1})^{-1}\rho_{i,t-1} + \nu c_{i,t}^{-\nu-1}\mathcal{W}_t\theta_{i,t}\phi_{i,t} + m_{i,t}^{-1/\nu}\varphi_{i,t} - \chi_t = 0$$

$$-n_{i,t}^{\gamma} - (1+\gamma)n_{i,t}^{\gamma}\mu_{i,t} + \gamma n_{i,t}^{\gamma-1}\phi_{i,t} + \theta_{i,t}\Xi_t = 0$$

$$\frac{\mathbb{E}_{t-1}\left[c_{i,t}^{-\nu}(1+\pi_t)^{-1}\nu_{i,t}\right]}{\mathbb{E}_{t-1}\left[c_{i,t}^{-\nu}(1+\pi_t)^{-1}\right]} = 0$$

The first order conditions with respect to C_t,π_t,$\mathcal{T}_t$, $\mathcal{W}_t$ and $\mathcal{Q}_{t-1}$ give

$$\int \varphi_{i,t} di + \chi_t - \Xi_t = 0$$

$$-\psi \pi_t \Xi_t + \int \frac{a_{i,t-1} c_{i,t}^{-\nu} (1+\pi_t)^{-2}}{\beta \mathbb{E}_{t-1}\left[c_{i,t}^{-\nu}(1+\pi_t)^{-1}\right]} (\nu_{i,t} - \frac{\mathbb{E}_{t-1}\left[c_{i,t}^{-\nu}(1+\pi_t)^{-1}\nu_{i,t}\right]}{\mathbb{E}_{t-1}\left[c_{i,t}^{-\nu}(1+\pi_t)^{-1}\right]}) di$$

$$-\beta \int m_{i,t-1} c_{i,t}^{-\nu} (1+\pi_t)^{-2} \rho_{i,t-1} di = 0$$

$$\int c_{i,t}^{-\nu} \mu_{i,t} di = 0$$

$$\int c_{i,t}^{-\nu} \theta_{i,t} \phi_{i,t} di = 0$$

$$\int \rho_{i,t-1} di = 0$$

There are two things we can do, to simplify these expressions. First

$$\frac{\mathbb{E}_{t-1}\left[c_{i,t}^{-\nu}(1+\pi_t)^{-1}\nu_{i,t}\right]}{\mathbb{E}_{t-1}\left[c_{i,t}^{-\nu}(1+\pi_t)^{-1}\right]} = 0$$

can be used with $\mu_{i,t} = \mu_{i,t-1} + \nu_{i,t}$ to get

$$\mu_{i,t-1} = \frac{\mathbb{E}_{t-1}\left[c_{i,t}^{-\nu}(1+\pi_t)^{-1}\mu_{i,t}\right]}{\mathbb{E}_{t-1}\left[c_{i,t}^{-\nu}(1+\pi_t)^{-1}\right]}.$$

Second, multiplying the FOC w.r.t $m_{i,t}$ by $m_{i,t}$gives

$$\mathcal{Q}_t \rho_{i,t} + \frac{1}{\nu} C_t \varphi_{i,t} = 0$$

which immediately implies

$$\int \varphi_{i,t} di = 0$$

and thus $\Xi_t = \chi_t$. Finally we will find it convenient to define $\varrho_{i,t} = c_{i,t}^{-\nu}(1+\pi_t)^{-1}\rho_{i,t-1}$ and $r_{i,t} = \frac{c_{i,t}^{-\nu}(1+\pi_t)}{\beta\mathbb{E}_{t-1}\left[c_{i,t}^{-\nu}(1+\pi_t)^{-1}\right]}$ then our first order conditions become

$$\beta\mathbb{E}_t\left[\varrho_{i,t+1}\right] + \frac{1}{\nu}m_{i,t}^{-1/\nu-1}c_{i,t}\varphi_{i,t} = 0$$

$$c_{i,t}^{-\nu} + \left((1-\nu)c_{i,t}^{-\nu} - \nu c_{i,t}^{-\nu-1}\mathcal{T}_t\right)\mu_{i,t} + \frac{\nu a_{i,t-1}r_{i,t}}{c_{i,t}}(\mu_{i,t}-\mu_{i,t-1})$$

$$-\beta\nu m_{i,t-1}\frac{\varrho_{i,t}}{c_{i,t}} + \nu c_{i,t}^{-\nu-1}\mathcal{W}_t\theta_{i,t}\phi_{i,t} + m_{i,t}^{-1/\nu}\varphi_{i,t} - \chi_t = 0$$

$$-n_{i,t}^{\gamma} - (1+\gamma)n_{i,t}^{\gamma}\mu_{i,t} + \gamma n_{i,t}^{\gamma-1}\phi_{i,t} + \theta_{i,t}\chi_t = 0$$

$$\varrho_{i,t} - c_{i,t}^{-\nu}(1+\pi_t)^{-1}\rho_{i,t-1} = 0$$

$$\mu_{i,t-1} - \beta\mathbb{E}_{t-1}\left[r_{i,t}\mu_{i,t}\right] = 0$$

$$r_{i,t} - \frac{c_{i,t}^{-\nu}(1+\pi_t)}{\beta\mathbb{E}_{t-1}\left[c_{i,t}^{-\nu}(1+\pi_t)^{-1}\right]} = 0$$

and

$$-\psi\pi_t\chi_t + (1+\pi_t)^{-1}\int a_{i,t-1}r_{i,t}(\mu_{i,t}-\mu_{i,t-1}) - \beta\varrho_{i,t}di = 0$$

$$\int \rho_{i,t-1}di = 0$$

$$\int c_{i,t}^{-\nu}\mu_{i,t}di = 0$$

$$\int c_{i,t}^{-\nu}\theta_{i,t}\phi_{i,t}di = 0$$

System of Equations

Combining the FOC with equilibrium conditions we have

$$
\begin{aligned}
n_{i,t}^{1+\gamma} + c_{i,t}^{-\nu}\mathcal{T}_t + a_{i,t-1}r_{i,t} - c_{i,t}^{1-\nu} - a_{i,t} &= 0 \\
c_{i,t}^{-\nu}\mathcal{W}_t\theta_{i,t} - n_{i,t}^{\gamma} &= 0 \\
m_{i,t}^{-1/\nu}c_{i,t} - C_t &= 0 \\
\beta\mathbb{E}_t\left[\varrho_{i,t+1}\right] + \frac{1}{\nu}m_{i,t}^{-1/\nu-1}c_{i,t}\varphi_{i,t} &= 0 \\
c_{i,t}^{-\nu} + \left((1-\nu)c_{i,t}^{-\nu} - \nu c_{i,t}^{-\nu-1}\mathcal{T}_t\right)\mu_{i,t} + \frac{\nu a_{i,t-1}r_{i,t}}{c_{i,t}}(\mu_{i,t} - \mu_{i,t-1}) & \\
-\beta\nu m_{i,t-1}\frac{\varrho_{i,t}}{c_{i,t}} + \nu c_{i,t}^{-\nu-1}\mathcal{W}_t\theta_{i,t}\phi_{i,t} + m_{i,t}^{-1/\nu}\varphi_{i,t} - \chi_t &= 0 \\
-n_{i,t}^{\gamma} - (1+\gamma)n_{i,t}^{\gamma}\mu_{i,t} + \gamma n_{i,t}^{\gamma-1}\phi_{i,t} + \theta_{i,t}\chi_t &= 0 \\
\varrho_{i,t} - c_{i,t}^{-\nu}(1+\pi_t)^{-1}\rho_{i,t-1} &= 0 \\
\mu_{i,t-1} - \beta\mathbb{E}_{t-1}\left[r_{i,t}\mu_{i,t}\right] &= 0 \\
r_{i,t} - \frac{c_{i,t}^{-\nu}(1+\pi_t)^{-1}}{\beta\mathbb{E}_{t-1}\left[c_{i,t}^{-\nu}(1+\pi_t)^{-1}\right]} &= 0 \\
\mathcal{Q}_{t-1} - \beta m_{i,t-1}\mathbb{E}_{t-1}\left[c_{i,t}^{-\nu}(1+\pi_{t+1})^{-1}\right] &= 0 \\
e_{i,t} - (\rho_e e_{i,t-1} + \varsigma_{i,t}) &= 0
\end{aligned}
$$

and

$$
\begin{aligned}
\int c_{i,t}^{-\nu}\mu_{i,t}di &= 0 \\
C_t - \int c_{i,t}di &= 0 \\
\int \theta_{i,t}n_{i,t} - \frac{\psi}{2}\pi_t^2 di - C_t - \bar{G} &= 0 \\
-\psi\pi_t\chi_t + (1+\pi_t)^{-1}\int a_{i,t-1}r_{i,t}(\mu_{i,t} - \mu_{i,t-1}) - \beta\varrho_{i,t}di &= 0 \\
\int c_{i,t}^{-\nu}\theta_{i,t}\phi_{i,t}di &= 0 \\
\int \rho_{i,t-1}di &= 0 \\
\mathbb{E}_{t-1}\mathcal{T}_t &= 0
\end{aligned}
$$

Along with the implicit restrictions that $\mathcal{Q}_{t-1}, a_{i,t-1}$ and $\rho_{i,t-1}$ do not depend on period t shocks. The additional constraint $\mathbb{E}_{t-1}\mathcal{T}_{t-1}$ was due to Ricardian equivalences. Without it there would be multiple solutions to the FOC corresponding to different paths of government debt.

As one final note, we can make one additional normalization by defining $\hat{m}_{i,t} = m_{i,t}^{1/\nu} - 1$ and $\hat{\mu}_{i,t} = \frac{\mu_{i,t}}{m_{i,t}}$ which satisfy $\int \hat{m}_{i,t} di = \int \hat{\mu}_{i,t} di = 0$ and search for a solution that is recursive in these variables. Not one can always recover $(m_{i,t}, \mu_{i,t})$ directly from $(\hat{m}_{i,t}, \hat{\mu}_{i,t})$ as $m_t^i = (\hat{m}_{i,t} + 1)^\nu$ and $\mu_{i,t} = \frac{\hat{\mu}_{i,t}}{(\hat{m}_{i,t}+1)^\nu}$. Thus, let $z = (\hat{m}, \hat{\mu}, e)$ and Ω be the joint distribution over these variables. We can then write the individual constraints recursively. We will assume that Θ follows an AR (1) process, due to the timing of the planner's problem (at time t but before shocks have been realized) policy functions will have to be a function of both the previous level of TFP (Θ) and the innovation to TFP $\mathcal{E}$.

$$\begin{aligned} &\tilde{n}(\varsigma,\mathcal{E},\Theta,z,\Omega)^{1+\gamma} + \tilde{c}(\varsigma,\mathcal{E},\Theta,z,\Omega)^{-\nu}\tilde{\mathcal{T}}(\mathcal{E},\Theta,\Omega) + \tilde{a}(\Theta,z,\Omega)\tilde{r}(\varsigma,\mathcal{E},\Theta,z,\Omega) \\ &\quad - c(\varsigma,\mathcal{E},\Theta,z,\Omega)^{1-\nu} - \tilde{a}(\rho_\Theta\Theta + \mathcal{E}, \tilde{z}(\varsigma,\mathcal{E},\Theta,z,\Omega), \tilde{\Omega}(\mathcal{E},\Theta,\Omega)) = 0 \end{aligned} \tag{46}$$

$$\tilde{c}(\varsigma,\mathcal{E},\Theta,z,\Omega)^{-\nu}\tilde{\mathcal{W}}(\mathcal{E},\Theta,\Omega)\exp(\mathcal{E} + \rho_\Theta\Theta + \rho_e e(z) + \varsigma) - \tilde{n}(\varsigma,\mathcal{E},\Theta,z,\Omega)^\gamma = 0 \tag{47}$$

$$\tilde{m}(\varsigma,\mathcal{E},\Theta,z,\Omega)^{-1/\nu}\tilde{c}(\varsigma,\mathcal{E},\Theta,z,\Omega) - \tilde{C}(\mathcal{E},\Theta,\Omega) = 0 \tag{48}$$

$$\beta\mathbb{E}\left[\tilde{\varrho}(\cdot,\cdot,\rho_\Theta\Theta + \mathcal{E}, \tilde{z}(\varsigma,\mathcal{E},\Theta,z,\Omega), \tilde{\Omega}(\mathcal{E},\Theta,\Omega)\right] + \frac{1}{\nu}\tilde{m}(\varsigma,\mathcal{E},\Theta,z,\Omega)^{-1/\nu-1}\tilde{c}(\varsigma,\mathcal{E},\Theta,z,\Omega)\tilde{\varphi}(\varsigma,\mathcal{E},\Theta,z,\Omega) = 0 \tag{49}$$

$$\begin{aligned} &\tilde{c}(\varsigma,\mathcal{E},\Theta,z,\Omega)^{-\nu} + \Big((1-\nu)\tilde{c}(\varsigma,\mathcal{E},\Theta,z,\Omega)^{-\nu} - \nu\tilde{c}(\varsigma,\mathcal{E},\Theta,z,\Omega)^{-\nu-1}\tilde{\mathcal{T}}(\mathcal{E},\Theta,\Omega)\Big)\tilde{\mu}(\varsigma,\mathcal{E},\Theta,z,\Omega) \\ &- \beta\nu\tilde{m}(\varsigma,\mathcal{E},\Theta,z,\Omega)\frac{\tilde{\varrho}(\varsigma,\mathcal{E},\Theta,z,\Omega)}{\tilde{c}(\varsigma,\mathcal{E},\Theta,z,\Omega)} + \frac{\nu\tilde{a}(\Theta,z,\Omega)\tilde{r}(\varsigma,\mathcal{E},\Theta,z,\Omega)}{\tilde{c}(\varsigma,\mathcal{E},\Theta,z,\Omega)}(\tilde{\mu}(\varsigma,\mathcal{E},\Theta,z,\Omega) - \frac{\hat{\mu}(z)}{(\hat{m}(z)+1)^\nu}) \\ &+ \nu\tilde{c}(\varsigma,\mathcal{E},\Theta,z,\Omega)^{-\nu-1}\tilde{\mathcal{W}}(\mathcal{E},\Theta,\Omega)\exp(\mathcal{E} + \rho_\Theta\Theta + \rho_e e(z) + \varsigma)\tilde{\phi}(\varsigma,\mathcal{E},\Theta,z,\Omega) \\ &+ \tilde{m}(\varsigma,\mathcal{E},\Theta,z,\Omega)^{-1/\nu}\tilde{\varphi}(\varsigma,\mathcal{E},\Theta,z,\Omega) - \tilde{\chi}(\mathcal{E},\Theta,\Omega) = 0 \end{aligned} \tag{50}$$

$$\begin{aligned} &-\tilde{n}(\varsigma,\mathcal{E},\Theta,z,\Omega)^\gamma - (1+\gamma)\tilde{n}(\varsigma,\mathcal{E},\Theta,z,\Omega)^\gamma\tilde{\mu}(\varsigma,\mathcal{E},\Theta,z,\Omega) \\ &+ \gamma\tilde{n}(\varsigma,\mathcal{E},\Theta,z,\Omega)^{\gamma-1}\tilde{\phi}(\varsigma,\mathcal{E},\Theta,z,\Omega) + \exp(\mathcal{E} + \Theta + \rho_e e(z) + \varsigma)\tilde{\chi}(\mathcal{E},\Theta,\Omega) = 0 \end{aligned} \tag{51}$$

$$\tilde{\varrho}(\varsigma,\mathcal{E},\Theta,z,\Omega) - \tilde{c}(\varsigma,\mathcal{E},\Theta,z,\Omega)^{-\nu}(1+\tilde{\pi}(\mathcal{E},\Theta,\Omega))^{-1}\tilde{\rho}(z,\Omega) = 0 \tag{52}$$

$$\tilde{r}(\varsigma,\mathcal{E},\Theta,z,\Omega) - \frac{\tilde{c}(\varsigma,\mathcal{E},\Theta,z,\Omega)^{-\nu}(1+\pi(\mathcal{E},\Theta,\Omega))^{-1}}{\beta\mathbb{E}\left[\tilde{c}(\cdot,\cdot,\Theta,z,\Omega)^{-\nu}(1+\pi(\cdot,\Theta,\Omega))^{-1}\right]} = 0 \tag{53}$$

$$\frac{\hat{\mu}(z)}{(\hat{m}(z)+1)^\nu} - \beta\mathbb{E}\left[\tilde{r}(\cdot,\cdot,\Theta,z,\Omega)\tilde{\mu}(\cdot,\cdot,\Theta,z,\Omega)\right] = 0 \tag{54}$$

$$\tilde{\mathcal{Q}}(\Omega) - \beta\left(\hat{m}(z)+1\right)^\nu\mathbb{E}\left[\tilde{c}(\cdot,\cdot,\Theta,z,\Omega)^{-\nu}(1+\pi(\cdot,\Theta,\Omega))^{-1}\right] = 0 \tag{55}$$

$$\tilde{\hat{\mu}}(\varsigma,\mathcal{E},\Theta,z,\Omega) - \frac{\tilde{\mu}(\varsigma,\mathcal{E},\Theta,z,\Omega)}{\tilde{m}(\varsigma,\mathcal{E},\Theta,z,\Omega)} = 0$$

$$\tilde{\hat{m}}(\varsigma,\mathcal{E},\Theta,z,\Omega) - \tilde{m}(\varsigma,\mathcal{E},\Theta,z,\Omega)^{1/\nu} + 1 = 0$$

while the aggregate constraints are

$$\int\int \tilde{c}(\varsigma,\mathcal{E},\Theta,z,\Omega)^{-\nu}\tilde{\mu}(\varsigma,\mathcal{E},\Theta,z,\Omega)d\Pr(\varsigma)d\Omega=0 \tag{56}$$

$$\tilde{C}(\mathcal{E},\Theta,\Omega)-\int\int \tilde{c}(\varsigma,\mathcal{E},\Theta,z,\Omega)d\Pr(\varsigma)d\Omega=0 \tag{57}$$

$$\int\int \exp(\mathcal{E}+\rho_\Theta\Theta+\varsigma)\tilde{n}(\varsigma,\mathcal{E},\Theta,z,\Omega)di-\frac{\psi}{2}\tilde{\pi}(\mathcal{E},\Theta,\Omega)^2-\tilde{C}(\mathcal{E},\Theta,\Omega)-\bar{G}d\Pr(\varsigma)d\Omega=0 \tag{58}$$

$$(1+\tilde{\pi}(\mathcal{E},\Theta,\Omega))^{-1}\int\int \tilde{a}(z,\Omega)\tilde{r}(\varsigma,\mathcal{E},\Theta,z,\Omega)(\tilde{\mu}(\varsigma,\mathcal{E},\Theta,z,\Omega)-\mu(\cdot))-\beta\tilde{\varrho}(\varsigma,\mathcal{E},\Theta,z,\Omega)$$
$$-\psi\tilde{\pi}(\mathcal{E},\Theta,\Omega)\tilde{\chi}(\mathcal{E},\Theta,\Omega)d\Pr(\varsigma)d\Omega=0 \tag{59}$$

$$\int\int \tilde{c}(\varsigma,\mathcal{E},\Theta,z,\Omega)^{-\nu}\exp(\mathcal{E}+\rho_\Theta\Theta+\rho_e e(z)+\varsigma)\tilde{\phi}(\varsigma,\mathcal{E},\Theta,z,\Omega)d\Pr(\varsigma)d\Omega=0 \tag{60}$$

$$\int \tilde{\rho}(\Theta,z,\Omega)d\Omega=0 \tag{61}$$

$$\mathbb{E}\left[\tilde{\mathcal{T}}(\cdot,\Theta,\Omega)\right]=0 \tag{62}$$

B Expansion

Definition of variables and F and R notation

Let $\tilde{x}$ denote the individual variables and $\tilde{X}$ denote the aggregate variables. Equations (46) - (55) can be summarized by the following function

$$F\left(z,\mathbb{E}\tilde{x}(\cdot,\cdot,\Theta,z,\Omega),\tilde{x}(\varsigma,\mathcal{E},\Theta,z,\Omega),\mathbb{E}\left[\tilde{x}(\cdot,\cdot,\rho_\Theta\Theta+\mathcal{E},\tilde{z}(\varsigma,\mathcal{E},\Theta,z,\Omega),\tilde{\Omega}(\mathcal{E},\Theta,\Omega)\right],\tilde{X}(\mathcal{E},\Theta,\Omega),\varsigma,\mathcal{E},\Theta\right)=0 \text{ for all}$$

and equations (56)-(62) can be defined by

$$\int\int R\left(z,\tilde{x}(\varsigma,\mathcal{E},\Theta,z,\Omega),\tilde{X}(\mathcal{E},\Theta,\Omega),\varsigma,\mathcal{E},\Theta\right)d\Pr(\varsigma)d\Omega=0.$$

We define the law of motion for $\tilde{\Omega}$ by

$$\widetilde{\Omega}\left(\mathcal{E},\Theta,\Omega\right)(y)=\int \iota\left(\tilde{z}\left(\varsigma,\mathcal{E},\Theta,z,\Omega\right)\leq y\right)d\Pr\left(\varsigma\right)d\Omega \quad \forall y,$$

where $\iota\left(\tilde{z}\left(\varsigma,\mathcal{E},\Theta,z,\Omega\right)\leq y\right)$ is 1 iff all elements of $\tilde{z}\left(\varsigma,\mathcal{E},\Theta,z,\Omega\right)$ are less than all elements of y and zero otherwise.

As in the main text, we will consider a positive scalar σ that scales all shocks and $1-\rho_e$.[24] We let $\tilde{x}(\sigma\varsigma,\sigma\mathcal{E},\sigma\Theta,z,\Omega;\sigma)$ and $\tilde{X}(\sigma\mathcal{E},\sigma\Theta,z,\Omega;\sigma)$ denote the policy rules with scaling parameters σ.

[24] For simplicity of exposition we assume that $\rho_e\to 1$ as $\sigma^2\to 0$.

Zeroth Order Expansion

We'll begin by showing that when $\sigma = 0$, $\tilde{z}(0,0,0,z,\Omega;0) = z$ and hence that $\tilde{\Omega}(0,0,\Omega;0) = \Omega$. This is true by assumption for e. From (53) it is clear that $\tilde{r}(0,0,0,z,\Omega;0) = \beta$ and thus equation (54) gives us

$$\tilde{\mu}(0,0,0,z,\Omega;0) = \mu(z)$$

directly. In absence of shocks equations (48) and (55) imply

$$\tilde{c}(0,0,0,z,\Omega;0) = \tilde{m}(0,0,0,z,\Omega;0)^{1/\nu}\tilde{C}(0,0,\Omega;0)$$

and

$$\tilde{c}(0,0,0,z,\Omega;0) = m(z)^{\frac{1}{\nu}}\left(\frac{\beta}{\tilde{\mathcal{Q}}(0,\Omega;0)(1+\pi(0,0,\Omega;0))}\right)^{1/\nu}$$

As $\mathbb{E}[m(\cdot)^{\frac{1}{\nu}}] = 1$, this immediately implies $\left(\frac{\beta}{\tilde{\mathcal{Q}}(0,\Omega;0)(1+\pi(0,0,\Omega;0))}\right)^{1/\nu} = \tilde{C}(0,0,\Omega;0)$ and hence

$$\tilde{c}(0,0,0,z,\Omega;0) = m(z)^{\frac{1}{\nu}}\tilde{C}(0,0,\Omega;0).$$

Combined we see that $\tilde{m}(0,0,0,z,\Omega;0) = m(z)$. As this holds for m and μ it will also hold for $\hat{m}$ and $\hat{\mu}$ and hence $\tilde{\Omega}(0,0,\Omega;0) = \Omega$ and $\tilde{z}(0,0,0,z,\Omega;0) = z$. Note that we obtained this by exploiting that expectations (48) and (55) are just equalities, this implies that $\tilde{z}(0,0,\Theta,z,\Omega;0) = z$ and $\tilde{\Omega}(0,\Theta,\Omega;0) = \Omega$ as well.

Frechet Derivatives

We being by differentiating F and R with respect to z to obtain

$$\bar{F}_{\mathbf{x}}\bar{x}_z + \bar{F}_z = 0$$

where $\bar{F}_{\mathbf{x}} = \bar{F}_{x-} + \bar{F}_x + \bar{F}_{x+}$ after exploiting $\bar{z}_z = I$. This yields $\bar{x}_z(z) = \bar{F}_x(z)^{-1}\bar{F}_z$.

Next we need the Frechet derivatives with respect to Ω. Differentiating, F with respect to Ω yields

$$\bar{F}_{\mathbf{x}}\partial\bar{x} + \bar{F}_X\partial\bar{X} = 0$$

or

$$\partial\bar{x}(z) = -\bar{F}_{\mathbf{x}}(z)^{-1}\bar{F}_X\partial\bar{X} \equiv \mathsf{C}(z)\partial\bar{X}.$$

To solve for $\partial \bar{X}$ we differentiate R in the direction Δ, with density δ to get

$$\begin{aligned}0 &= \lim_{\alpha \to 0} \frac{1}{\alpha} \left[\int \bar{R}(z, \bar{x}(z, \Omega + \alpha\Delta), \bar{X}(\Omega + \alpha\Delta))(\omega(z) + \alpha\delta(z))dz - \int \bar{R}(z)\omega(z)dz \right] \\ &= \int \bar{R}(z)\delta(z)dz + \int \left(\bar{R}_x(z)\partial\bar{x}(z) \cdot \Delta + \bar{R}_X(z)\partial\bar{X} \cdot \Delta \right) \omega(z)dz.\end{aligned}$$

Substituting for $\partial\bar{x}(z) = \mathsf{C}(z)\partial\bar{X}$, we get

$$\begin{aligned}\partial\bar{X} \cdot \Delta &= -\left(\int \left(\bar{R}_x(z)\mathsf{C}(z) + \bar{R}_X(z) \right) d\Omega \right)^{-1} \int \bar{R}(z)d\Delta \\ &\equiv \mathsf{D}^{-1} \int \mathsf{E}(z)d\Delta.\end{aligned}$$

First Order

Next we differentiate both F and R with respect to σ and use the method of undetermined coefficients to find the derivatives that multiply ς, $\mathcal{E}$ and Θ. For ς, this yields

$$\bar{F}_x \bar{x}_\varsigma + \bar{F}_{x+} \bar{x}_z \mathsf{p} \bar{x}_\varsigma + \bar{F}_\varsigma = 0$$

or

$$\bar{x}_\varsigma(z) = -\left(\bar{F}_x(z) + \bar{F}_{x+}(z)\bar{x}_z(z)\mathsf{p} \right)^{-1} \bar{F}_\varsigma(z) \equiv \mathsf{K}(z)^{-1}\mathsf{L}(z).$$

For Θ, we find that (after noting that the same analysis in the Zeroth Order Expansion can yield $\bar{\Omega}_\Theta = 0$ and $\bar{z}_\Omega = 0$)

$$\bar{F}_{x-}\bar{x}_\Theta + \bar{F}_x \bar{x}_\Theta + \bar{F}_{x+} \left(\bar{x}_\Theta \rho_\Theta \right) + \bar{F}_X \bar{X}_\Theta + \bar{F}_\Theta = 0$$

or

$$\bar{x}_\Theta = -\left(\bar{F}_{x-} + \bar{F}_x + \rho_\Theta \bar{F}_{x+} \right)^{-1} (\bar{F}_X \bar{X}_\Theta + \bar{F}_\Theta).$$

and

$$\int \bar{R}_x(z)\bar{x}_\Theta(z) + \bar{R}_X(z)\bar{X}_\Theta + \bar{R}_\Theta(z)d\Omega = 0.$$

Plugging in for $\bar{x}_\Theta$ yields a direct expression for $\bar{X}_\Theta$.

Finally for $\mathcal{E}$, we find

$$\bar{F}_x \bar{x}_\mathcal{E} + \bar{F}_{x+} \left(\bar{x}_\Theta + \bar{x}_z \mathsf{p} \bar{x}_\mathcal{E} + \partial\bar{x} \cdot \bar{\Omega}_\mathcal{E} \right) + \bar{F}_X \bar{X}_\mathcal{E} + \bar{F}_\mathcal{E} = 0$$

and

$$\int \bar{R}_x(z)\bar{x}_\mathcal{E}(z) + \bar{R}_X(z)\bar{X}_\mathcal{E} + \bar{R}_\mathcal{E}(z)d\Omega = 0.$$

Substituting for $\partial\bar{x}$, we obtain and defining $\bar{X}'_{\mathcal{E}} = \partial\bar{X}\cdot\bar{\Omega}_{\mathcal{E}}$

$$\bar{F}_x(z)\bar{x}_{\mathcal{E}}(z) + \bar{F}_{x+}(z)\left(\bar{x}_\Theta(z) + \bar{x}_z(z)\mathsf{p}\bar{x}_{\mathcal{E}}(z) + \mathsf{C}(z)\bar{X}'_{\mathcal{E}}\right) + \bar{F}_X(z)\bar{X}_{\mathcal{E}} + \bar{F}_{\mathcal{E}}(z) = 0,$$

or

$$\mathsf{M}(z)\bar{x}_{\mathcal{E}}(z) = \mathsf{N}(z)\left[\begin{array}{ccc} I & \bar{X}_{\mathcal{E}} & \bar{X}'_{\mathcal{E}} \end{array}\right]^{\mathsf{T}}$$

Solving for these derivatives first requires determining $\bar{\Omega}_{\mathcal{E}}$. $\tilde{\Omega}$ is defined by

$$\widetilde{\Omega}\left(\mathcal{E},\Theta,\Omega\right)(y) = \int\prod_i \iota\left(\tilde{z}^i\left(\varsigma,\mathcal{E},\Theta,z,\Omega\right)\leq y^i\right)d\Pr\left(\varsigma\right)d\Omega,$$

where z^i represents the i^{th} element of z. Differentiating with respect to $\mathcal{E}$ yields

$$\tilde{\Omega}_{\mathcal{E}}(\mathcal{E},\Theta,\Omega)(y) = -\int\sum_i \delta(\tilde{z}^i\left(\varsigma,\mathcal{E},\Theta,z,\Omega\right) - y^i)\prod_{j\neq i}\iota\left(\tilde{z}^j\left(\varsigma,\mathcal{E},\Theta,z,\Omega\right)\leq y^j\right)\tilde{z}^i_{\mathcal{E}}\left(\varsigma,\mathcal{E},\Theta,z,\Omega\right)d\Pr(\varsigma)d\Omega$$

which evaluated at $\sigma = 0$ gives

$$\bar{\Omega}_{\mathcal{E}}(y) = -\int\sum_i\delta(z^i - y^i)\prod_{j\neq i}\iota(z^j\leq y^j)\bar{z}^i_{\mathcal{E}}(z)d\Omega = -\sum_i\int\delta(z^i - y^i)\prod_{j\neq i}\iota(z^j\leq y^j)\bar{z}^i_{\mathcal{E}}(z)d\Omega.$$

The density of $\bar{\Omega}_{\mathcal{E}}$ is given by $\bar{\omega}_{\mathcal{E}}(y) = \frac{\partial^{n_z}}{\partial y^1\partial y^2\cdots\partial y^{n_z}}\bar{\Omega}_{\mathcal{E}}(y)$ so

$$\bar{\omega}_{\mathcal{E}}(y) = -\sum_i\frac{\partial}{\partial y^i}\int\prod_j\delta(z^j - y^j)\bar{z}^i_{\mathcal{E}}(z)d\Omega = -\sum_i\frac{\partial}{\partial y^i}\left(\bar{z}^i_{\mathcal{E}}(y)\omega(y)\right).$$

Plugging in for the definition of $\bar{X}'_{\mathcal{E}}$ we find

$$\begin{aligned}
\bar{X}'_{\mathcal{E}} &= -\mathsf{D}^{-1}\int\bar{R}(z)\sum_i\frac{\partial}{\partial z^i}\left(\bar{z}^i_{\mathcal{E}}(z)\omega(z)\right)dz\\
&= \mathsf{D}^{-1}\int\sum_i\left(\frac{\partial}{\partial z^i}\bar{R}(z)\right)\bar{z}^i_{\mathcal{E}}(z)\omega(z)dz\\
&= \mathsf{D}^{-1}\int\sum_i\left(\bar{R}_{z^i}(z) + \bar{R}_x(z)\bar{x}_{z^i}(z)\right)\bar{z}^i_{\mathcal{E}}d\Omega\\
&= \mathsf{D}^{-1}\int\left(\bar{R}_z(z) + \bar{R}_x(z)\bar{x}_z(z)\right)\mathsf{q}\bar{x}_{\mathcal{E}}(z)d\Omega
\end{aligned}$$

Substituting for $\bar{x}_{\mathcal{E}}(z)$ in this plus

$$\int\bar{R}_x(z)\bar{x}_{\mathcal{E}}(z) + \bar{R}_X(z)\bar{X}_{\mathcal{E}} + \bar{R}_{\mathcal{E}}(z)d\Omega = 0$$

yields a linear system

$$\mathsf{O}\cdot\begin{bmatrix}\bar{X}_{\mathcal{E}} & \bar{X}'_{\mathcal{E}}\end{bmatrix}^{\mathsf{T}} = \mathsf{P}$$

which is easy to solve.

Second Order

Derivatives of the states

We continue as before, differentiating with respect to the states z and Ω. As before $\bar{x}_{zz}(z)$ is easy to solve. $\partial\bar{x}_z(z)$ can be computed by differentiating F w.r.t. z and then taking the Frechet derivative to find [25]

$$\bar{F}_{\mathbf{x}}(z)\partial\bar{x}_z(z)+\bar{F}_{\mathbf{xx}}(z)\cdot(\bar{x}_0(z),\partial\bar{x}(z))+\bar{F}_{\mathbf{x}X}(z)\cdot\left(\bar{x}_0(z),\partial\bar{X}\right)+\bar{F}_{z\mathbf{x}}(z)\cdot(I,\partial\bar{x}(z))+\bar{F}_{zX}\cdot\left(I,\partial\bar{X}\right)=0,$$

where I represents the identity matrix. This linear system is easily solved to give

$$\partial\bar{x}_z(z)=\mathsf{Q}(z)\cdot\left(I,\partial\bar{X}\right)$$

where $\mathsf{Q}(z)$ is $n_x\times n_z\times n_X$.

Finally the second order Frechet derivative of F yields

$$\bar{F}_{\mathbf{x}}(z)\partial^2\bar{x}(z)+\bar{F}_X(z)\partial^2\bar{X}+\bar{F}_{\mathbf{xx}}\cdot(\partial\bar{x}(z),\partial\bar{x}(z))+\bar{F}_{\mathbf{x}X}(z)\cdot\left(\partial\bar{x}(z),\partial\bar{X}\right)+\bar{F}_{X\mathbf{x}}(z)\cdot\left(\partial\bar{X},\partial\bar{x}(z)\right)+\bar{F}_{XX}(z)\cdot\left(\partial\bar{X},\partial\bar{X}\right)=0$$

Substituting for $\partial\bar{x}(z)=\mathsf{C}(z)\partial\bar{X}$ and solving for $\partial^2\bar{x}^k$ then gives

$$\partial^2\bar{x}(z)=\mathsf{C}(z)\partial^2\bar{X}+\mathsf{R}(z)\cdot\left(\partial\bar{X},\partial\bar{X}\right).$$

[25] We extend the definition of $\bar{F}_{\mathbf{x}}$ to $\bar{F}_{\mathbf{xx}},\bar{F}_{z\mathbf{x}}$ and $\bar{F}_{\mathbf{x}X}$ as follows

$$\bar{F}_{z\mathbf{x}}=\bar{F}_{zx-}+\bar{F}_{zx}+\bar{F}_{zx+}$$

$$\bar{F}_{\mathbf{x}X}=\bar{F}_{x-X}+\bar{F}_{xX}+\bar{F}_{x+X}$$

and

$$\bar{F}_{\mathbf{xx}}=\bar{F}_{x-x-}+\bar{F}_{x-x}+\bar{F}_{x-x+}+\bar{F}_{xx-}+\bar{F}_{xx}+\bar{F}_{xx+}+\bar{F}_{x+x-1}+\bar{F}_{x+x}+\bar{F}_{x+x+}$$

To find $\partial^2 \bar{X}$ we differentiate R w.r.t Ω in the direction Δ_1 and then Δ_2 to find

$$\int \left(\bar{R}_x(z)\partial\bar{x}(z)\cdot\Delta_1 + \bar{R}_X(z)\partial\bar{X}\cdot\Delta_1\right)d\Delta_2 + \int \left(\bar{R}_x(z)\partial\bar{x}(z)\cdot\Delta_2 + \bar{R}_X(z)\partial\bar{X}\cdot\Delta_2\right)d\Delta_1$$
$$+\int \Bigg(\bar{R}_{xx}(z)\cdot(\partial\bar{x}(z)\cdot\Delta_1, \partial\bar{x}(z)\cdot\Delta_2) + \bar{R}_{xX}(z)\cdot\left(\partial\bar{x}(z)\cdot\Delta_1, \partial\bar{X}\cdot\Delta_2\right)$$
$$\bar{R}_{Xx}(z)\cdot\left(\partial\bar{X}\cdot\Delta_1, \partial\bar{x}(z)\cdot\Delta_2\right) + \bar{R}_{XX}(z)\cdot\left(\partial\bar{X}\cdot\Delta_1, \partial\bar{X}\cdot\Delta_2\right)\Bigg)d\Omega$$
$$+\int \left(\bar{R}_x(z)\partial^2\bar{x}(z)\cdot(\Delta_1,\Delta_2) + \bar{R}_X(z)\partial^2\bar{X}\cdot(\Delta_1,\Delta_2)\right)d\Omega = 0$$

A bit of rearranging and substituting for $\partial^2\bar{x}$ yields

$$\mathsf{D}\partial^2\bar{X}\cdot(\Delta_1,\Delta_2) = \int \mathsf{S}(z)\partial\bar{X}\cdot\Delta_2 d\Delta_1 + \int \mathsf{S}(z)\partial\bar{X}\cdot\Delta_1 d\Delta_2 + \mathsf{T}\cdot\left(\partial\bar{X}\cdot\Delta_1, \partial\bar{X}\cdot\Delta_2\right).$$

Derivatives of Shocks

Next we proceed by taking derivatives w.r.t. σ and then use the method of undetermined coefficients as in the first order approach. We note that solving for $\bar{x}_{\varsigma\varsigma}, \bar{x}_{\varsigma\Theta}$ and $\bar{x}_{\varsigma\mathcal{E}}$ are trivial as these interactions do not effect aggregates. We'll begin with $\Theta\Theta$ (again with the knowledge that $\bar{z}_{\Theta\Theta} = 0$ and $\bar{\Omega}_{\Theta\Theta} = 0$) which implies

$$\bar{F}_{x-}\bar{x}_{\Theta\Theta} + \bar{F}_x\bar{x}_{\Theta\Theta} + \bar{F}_X\bar{X}_{\Theta\Theta} + \bar{F}_{x+}\left(\bar{x}_{\Theta\Theta}\rho_\Theta^2\right)$$
$$+\bar{F}_{x-x-}\cdot(\bar{x}_\Theta,\bar{x}_\Theta) + 2\bar{F}_{x-x}\cdot(\bar{x}_\Theta,\bar{x}_\Theta) + 2\bar{F}_{x-X}(\bar{x}_\Theta,\bar{X}_\Theta) + 2\bar{F}_{x-x+}\cdot(\bar{x}_\Theta,\bar{x}_\Theta\rho_\Theta) + 2\bar{F}_{x-\Theta}\bar{x}_\Theta$$
$$+F_{xx}\cdot(\bar{x}_\Theta,\bar{x}_\Theta) + 2\bar{F}_{xX}\cdot(\bar{x}_\Theta,\bar{X}_\Theta) + 2\bar{F}_{xx+}\cdot(\bar{x}_\Theta,\bar{x}_\Theta\rho_\Theta) + 2\bar{F}_{x\Theta}\bar{x}_\Theta + \bar{F}_{XX}\cdot(\bar{X}_\Theta,\bar{X}_\Theta) + 2\bar{F}_{Xx+}\cdot(\bar{X}_\Theta,\bar{x}_\Theta\rho_\Theta)$$
$$+2\bar{F}_{X\Theta}\bar{X}_\Theta + \bar{F}_{x+x+}\cdot(\bar{x}_\Theta\rho_\Theta,\bar{x}_\Theta\rho_\Theta) + 2F_{x+\Theta}\bar{x}_\Theta\rho_\Theta + F_{\Theta\Theta} = 0$$

and

$$\int \Bigg(\bar{R}_x(z)\bar{x}_{\Theta\Theta}(z) + \bar{R}_X(z)\bar{X}_{\Theta\Theta} + \bar{R}_{xx}(z)\cdot(\bar{x}_\Theta(z),\bar{x}_\Theta(z))$$
$$+ 2\bar{R}_{xX}(z)\cdot\left(\bar{x}_\Theta(z),\bar{X}_\Theta\right) + 2\bar{R}_{x\Theta}(z)\bar{x}_\Theta(z)$$
$$+ \bar{R}_{XX}(z)\cdot\left(\bar{X}_\Theta,\bar{X}_\Theta\right) + 2\bar{R}_{X\Theta}(z)\bar{X}_\Theta + \bar{R}_{\Theta\Theta}(z)\Bigg)d\Omega = 0.$$

From the first set of equations implies

$$\mathsf{U}(z)\bar{x}_{\Theta\Theta}(z) = \mathsf{V}_1(z) + \mathsf{V}_2(z)\bar{X}_{\Theta\Theta}$$

which we can then plug into the R equation to yield

$$\mathsf{W}(z)\bar{X}_{\Theta\Theta} = \mathsf{X}(z).$$

Before computing $\bar{x}_{\mathcal{E}\mathcal{E}}$ and $\bar{x}_{\mathcal{E}\Theta}$ we will also need $\bar{x}_{z\Theta}(z)$ and $\partial\bar{x}_\Theta$. The first is straightforward as changes in an individual state z do not effect aggregates. For the latter we differentiate F and R (as well as exploiting that $\partial\bar{z}_\Theta = 0$) to obtain

$$\begin{aligned}
&\bar{F}_{x-}\partial\bar{x}_\Theta + \bar{F}_x\partial\bar{x}_\Theta + \bar{F}_X\partial\bar{X}_\Theta + \bar{F}_{x+}\left(\partial\bar{x}_\Theta\rho_\Theta\right)\\
&+\bar{F}_{x-x-}\cdot(\bar{x}_\Theta,\partial\bar{x}) + \bar{F}_{x-x}\cdot(\bar{x}_\Theta,\partial\bar{x}) + \bar{F}_{x-X}(\bar{x}_\Theta,\partial\bar{X}) + \bar{F}_{x-x+}\cdot(\bar{x}_\Theta,\partial\bar{x}) + \bar{F}_{xx-}\cdot(\bar{x}_\Theta,\partial\bar{x})\\
&+F_{xx}\cdot(\bar{x}_\Theta,\partial\bar{x}) + \bar{F}_{xX}\cdot(\bar{x}_\Theta,\partial\bar{X}) + \bar{F}_{xx+}\cdot(\bar{x}_\Theta,\partial\bar{x}) + \bar{F}_{Xx-}\cdot(\bar{X}_\Theta,\partial\bar{x}) + \bar{F}_{Xx}\cdot(\bar{X}_\Theta,\partial\bar{x})\\
&+\bar{F}_{Xx+}\cdot(\bar{X}_\Theta,\partial\bar{x}) + \bar{F}_{x+x-}\cdot(\bar{x}_\Theta\rho_\Theta,\partial\bar{x}) + \bar{F}_{x+x}\cdot(\bar{x}_\Theta\rho_\Theta,\partial\bar{x}) + \bar{F}_{x+X}\cdot(\bar{x}_\Theta\rho_\Theta,\partial\bar{X})\\
&+\bar{F}_{x+x+}\cdot(\bar{x}_\Theta\rho_\Theta,\partial\bar{x}) + \bar{F}_{\Theta x-}\partial\bar{x} + \bar{F}_{\Theta x}\partial\bar{x} + \bar{F}_{\Theta X}\partial\bar{X} + \bar{F}_{\Theta x+}\partial\bar{x} = 0
\end{aligned}$$

and

$$\begin{aligned}
0 = &\int\left(\bar{R}_\Theta(z) + \bar{R}_x(z)\bar{x}_\Theta(z) + \bar{R}_X(z)\bar{X}_\Theta\right)\delta(z)dz\\
&+\int\Bigg(\bar{R}_x(z)\partial\bar{x}_\Theta(z)\cdot\Delta + \bar{R}_X(z)\partial\bar{X}_\Theta\cdot\Delta + \bar{R}_{xx}(z)\cdot(\bar{x}_\Theta(z),\partial\bar{x}(z)\cdot\Delta)\\
&+\bar{R}_{xX}(z)\cdot(\bar{x}_\Theta\partial\bar{X}\cdot\Delta) + \bar{R}_{Xx}(z)\cdot(\bar{X}_\Theta,\partial\bar{x}(z)\cdot\Delta) + \bar{R}_{XX}(z)\cdot(\bar{X}_\Theta,\partial\bar{X}\cdot\Delta)\\
&+\bar{R}_{\Theta x}(z)\partial\bar{x}(z)\cdot\Delta + \bar{R}_{\Theta X}(z)\partial\bar{X}\cdot\Delta\Bigg)d\Omega
\end{aligned}$$

From F equations we immediately obtain

$$\mathsf{Y}(z)\partial\bar{x}_\Theta(z) = \mathsf{Z}_1(z)\partial\bar{X}_\Theta + \mathsf{Z}_2(z)\partial\bar{X}$$

which can be plugged into the equations from R to obtain

$$\mathsf{AA}\partial\bar{X}_\Theta\cdot\Delta = \mathsf{BB}\partial\bar{X}\cdot\Delta + \int\left(\bar{R}_\Theta(z) + \bar{R}_x(z)\bar{x}_\Theta(z) + \bar{R}_X(z)\bar{X}_\Theta\right)\delta(z)dz$$

For $\bar{x}_{\mathcal{E}\Theta}$ we see (after defining $\bar{x}^+_{\mathcal{E}} = \bar{x}_\Theta + \partial\bar{x}\cdot\bar{\Omega}_{\mathcal{E}} + \bar{x}_z\bar{z}_{\mathcal{E}}$)

$$\begin{aligned}
\bar{F}_x\bar{x}_{\mathcal{E}\Theta} + \bar{F}_X\bar{X}_{\mathcal{E}\Theta} + \bar{F}_{x+}\left(\bar{x}_{\Theta\Theta}\rho_\Theta + \bar{x}_{z\Theta}\bar{x}_{\mathcal{E}}\rho_\Theta + \partial\bar{x}_\Theta\cdot\bar{\Omega}_{\mathcal{E}} + \bar{x}_z\mathsf{p}\bar{x}_{\mathcal{E}\Theta} + \partial\bar{x}\cdot\bar{\Omega}_{\mathcal{E}\Theta}\right)\\
\bar{F}_{xx-}\cdot(\bar{x}_{\mathcal{E}},\bar{x}_\Theta) + F_{xx}\cdot(\bar{x}_{\mathcal{E}},\bar{x}_\Theta) + \bar{F}_{xX}\cdot(\bar{x}_{\mathcal{E}},\bar{X}_\Theta) + \bar{F}_{xx+}\cdot(\bar{x}_{\mathcal{E}},\bar{x}_\Theta\rho_\Theta) + \bar{F}_{x\Theta}\bar{x}_{\mathcal{E}}\\
+\bar{F}_{Xx-}\cdot(\bar{X}_{\mathcal{E}},,\bar{x}_\Theta) + \bar{F}_{Xx}\cdot(\bar{X}_{\mathcal{E}},\bar{x}_\Theta) + \bar{F}_{Xx+}\cdot(\bar{X}_{\mathcal{E}},\bar{x}_\Theta\rho_\Theta) + \bar{F}_{X\Theta}\bar{X}_\Theta + \bar{F}_{x+x-}\cdot(\bar{x}^+_{\mathcal{E}},\bar{x}_\Theta)\\
+\bar{F}_{x+x}\cdot(\bar{x}^+_{\mathcal{E}},\bar{x}_\Theta) + \bar{F}_{x+X}\cdot(\bar{x}^+_{\mathcal{E}},\bar{X}_\Theta) + \bar{F}_{x+x+}\cdot(\bar{x}^+_{\mathcal{E}},\bar{x}_\Theta\rho_\Theta) + \bar{F}_{x+\Theta}(\bar{x}^+_{\mathcal{E}})\\
+\bar{F}_{\mathcal{E}x-}\bar{x}_\Theta + \bar{F}_{\mathcal{E}x}\bar{x}_\Theta + \bar{F}_{\mathcal{E}X}\bar{X}_\Theta + \bar{F}_{\mathcal{E}x+}\bar{x}_\Theta\rho_\Theta + \bar{F}_{\mathcal{E}\Theta} = 0
\end{aligned}$$

Most of these terms are already known, but there are a few that are new and need to be computed: $\partial\bar{x}_\Theta\cdot\bar{\Omega}_{\mathcal{E}}$ and $\partial\bar{x}\cdot\bar{\Omega}_{\mathcal{E}\Theta}$. For the first we have

$$\partial\bar{x}_\Theta(z)\cdot\bar{\Omega}_{\mathcal{E}} = \mathsf{Y}(z)^{-1}\mathsf{Z}_1(z)\partial\bar{X}_\Theta\cdot\bar{\Omega}_{\mathcal{E}} + \mathsf{Y}(z)^{-1}\mathsf{Z}_2(z)X'_{\mathcal{E}}$$

where

$$\begin{aligned}
\partial\bar{X}_\Theta\cdot\bar{\Omega}_{\mathcal{E}} &= \mathsf{AA}^{-1}\mathsf{BB}X'_{\mathcal{E}} + \mathsf{AA}^{-1}\int\left(\bar{R}_\Theta(z) + \bar{R}_x(z)\bar{x}_\Theta(z) + \bar{R}_X(z)\bar{X}_\Theta\right)\bar{\omega}_{\mathcal{E}}(z)dz\\
&= \mathsf{AA}^{-1}\mathsf{BB}X'_{\mathcal{E}} + \mathsf{AA}^{-1}\int\left(\frac{d}{dz}\bar{R}_\Theta(z) + \frac{d}{dz}\bar{R}_x(z)\bar{x}_\Theta(z) + \bar{R}_x(z)\bar{x}_{\Theta z}(z) + \frac{d}{dz}\bar{R}_X(z)\bar{X}_\Theta\right)\bar{z}_{\mathcal{E}}(z)\omega(z)dz
\end{aligned}$$

where in the last term is easily computable from known terms[26] . Finally $\partial\bar{x}\cdot\bar{\Omega}_{\mathcal{E}\Theta} = \mathsf{C}(z)\partial\bar{X}\cdot\bar{\Omega}_{\mathcal{E}\Theta} = \mathsf{C}(z)\bar{X}'_{\mathcal{E}\Theta}$. Thus we can show that the derivative $\bar{x}_{\mathcal{E}\Theta}$ solves the following system of equations

$$\mathsf{CC}(z)\bar{x}_{\mathcal{E}\Theta} = \mathsf{DD}(z)\begin{bmatrix} I & \bar{X}_{\mathcal{E}\Theta} & \bar{X}'_{\mathcal{E}\Theta}\end{bmatrix}.$$

Recall we had

$$\tilde{\Omega}_{\mathcal{E}}(\mathcal{E},\Theta,\Omega)(y) = -\int\sum_i\delta(\tilde{z}^i(\varsigma,\mathcal{E},\Theta,z,\Omega)-y^i)\prod_{j\neq i}\iota\left(\tilde{z}^j(\varsigma,\mathcal{E},\Theta,z,\Omega)\leq y^j\right)\tilde{z}^i_{\mathcal{E}}(\varsigma,\mathcal{E},\Theta,z,\Omega)\,d\Pr(\varsigma)d\Omega.$$

[26]Note we do need to expand the left hand side. For example $\frac{d}{dz}\bar{R}_\Theta(z) = \bar{R}_{\Theta z}(z) + \bar{R}_{\Theta x}(z)\bar{x}_z(z)$.

Thus

$$\tilde{\Omega}_{\mathcal{E}\Theta}(\mathcal{E},\Theta,\Omega)(y) = -\int \sum_i \delta(\tilde{z}^i(\varsigma,\mathcal{E},\Theta,z,\Omega) - y^i) \prod_{j\neq i} \iota\left(\tilde{z}^j(\varsigma,\mathcal{E},\Theta,z,\Omega) \leq y^j\right) \tilde{z}^i_{\mathcal{E}\Theta}(\varsigma,\mathcal{E},\Theta,z,\Omega)\, d\Pr(\varsigma) d\Omega$$

$$- \int \sum_i \delta'(\tilde{z}^i(\varsigma,\mathcal{E},\Theta,z,\Omega) - y^i) \prod_{j\neq i} \iota\left(\tilde{z}^j(\varsigma,\mathcal{E},\Theta,z,\Omega) \leq y^j\right) \tilde{z}^i_{\mathcal{E}}(\varsigma,\mathcal{E},\Theta,z,\Omega)\, \tilde{z}^i_{\Theta}(\varsigma,\mathcal{E},\Theta,z,\Omega)\, dP$$

$$- \int \Bigg(\sum_i \sum_{j\neq i} \delta(\tilde{z}^i(\varsigma,\mathcal{E},\Theta,z,\Omega) - y^i) \delta(\tilde{z}^j(\varsigma,\mathcal{E},\Theta,z,\Omega) - y^j)$$

$$\prod_{k\neq i,j} \iota\left(\tilde{z}^k(\varsigma,\mathcal{E},\Theta,z,\Omega) \leq y^k\right) \tilde{z}^j_{\mathcal{E}}(\varsigma,\mathcal{E},\Theta,z,\Omega)\, \tilde{z}^i_{\Theta}(\varsigma,\mathcal{E},\Theta,z,\Omega) \Bigg) d\Pr(\varsigma)\ d\Omega$$

Evaluating this term at $\sigma = 0$ and exploiting $\bar{z}_\Theta = 0$ we have

$$\bar{\Omega}_{\mathcal{E}\Theta} = -\sum_i \int \delta(z^i - y^i) \prod_{j\neq i} \iota(z^j \leq y^j) \bar{z}^i_{\mathcal{E}\Theta}(z) d\Omega$$

and thus

$$\bar{\omega}_{\mathcal{E}\Theta}(y) = -\sum_i \frac{\partial}{\partial y^i} \int \prod_j \delta(z^j - y^j) \bar{z}^i_{\mathcal{E}\Theta}(z) d\Omega = -\sum_i \frac{\partial}{\partial y^i} \left(\bar{z}^i_{\mathcal{E}\Theta}(y)\omega(y)\right)$$

and hence

$$\begin{aligned}
\bar{X}'_{\mathcal{E}\Theta} &= -\mathsf{D}^{-1} \int \bar{R}(z) \sum_i \frac{\partial}{\partial z^i} \left(\bar{z}^i_{\mathcal{E}\Theta}(z)\omega(z)\right) dz \\
&= \mathsf{D}^{-1} \int \sum_i \left(\frac{\partial}{\partial z^i} \bar{R}(z)\right) \bar{z}^i_{\mathcal{E}\Theta}(z)\omega(z) dz \\
&= \mathsf{D}^{-1} \int \sum_i \left(\bar{R}_{z^i}(z) + \bar{R}_x(z)\bar{x}_{z^i}(z)\right) \bar{z}^i_{\mathcal{E}\Theta} d\Omega \\
&= \mathsf{D}^{-1} \int \left(\bar{R}_z(z) + \bar{R}_x(z)\bar{x}_z(z)\right) \mathsf{q}\bar{x}_{\mathcal{E}\Theta}(z) d\Omega
\end{aligned}$$

which combined with the second derivative of R w.r.t. $\mathcal{E}\Theta$

$$\begin{aligned}
\int \Bigg(& \bar{R}_x(z)\bar{x}_{\mathcal{E}\Theta}(z) + \bar{R}_X(z)\bar{X}_{\mathcal{E}\Theta} + \bar{R}_{xx}(z) \cdot (\bar{x}_{\mathcal{E}}(z), \bar{x}_\Theta(z)) \\
& + \bar{R}_{xX}(z) \cdot \left(\bar{x}_{\mathcal{E}}(z), \bar{X}_\Theta\right) + \bar{R}_{x\Theta}(z)\bar{x}_{\mathcal{E}}(z) \\
& + \bar{R}_{XX}(z) \cdot \left(\bar{X}_{\mathcal{E}}, \bar{X}_\Theta\right) + \bar{R}_{X\Theta}(z)\bar{X}_{\mathcal{E}} + \bar{R}_{\mathcal{E}\Theta}(z) \Bigg) d\Omega = 0
\end{aligned}$$

gives a linear relationship

$$\mathsf{EE}\left[\begin{array}{cc}\bar{X}_{\mathcal{E}\Theta} & \bar{X}'_{\mathcal{E}\Theta}\end{array}\right]=\mathsf{FF}$$

to solve for $\bar{X}_{\mathcal{E}\Theta}$.

For $\bar{x}_{\mathcal{E}\Theta}$ we see (after defining $\bar{x}^{+}_{\mathcal{E}}=\bar{x}_{\Theta}+\partial\bar{x}\cdot\bar{\Omega}_{\mathcal{E}}+\bar{x}_z\bar{z}_{\mathcal{E}}$)

$$\begin{aligned}\bar{F}_x\bar{x}_{\mathcal{E}\mathcal{E}}+\bar{F}_X\bar{X}_{\mathcal{E}\mathcal{E}}+\bar{F}_{x+}\Bigg(&\bar{x}_{\Theta\Theta}\rho_{\Theta}^2+2\rho_{\Theta}\bar{x}_{\Theta z}\bar{z}_{\mathcal{E}}+2\rho_{\Theta}\partial\bar{x}_{\Theta}\cdot\bar{\Omega}_{\mathcal{E}}+\bar{x}_{zz}\cdot(\bar{z}_{\mathcal{E}},\bar{z}_{\mathcal{E}})+2\partial\bar{x}_z\cdot(\bar{x}_{\mathcal{E}},\bar{\Omega}_{\mathcal{E}})\\&+\partial^2\bar{x}\cdot(\bar{\Omega}_{\mathcal{E}},\bar{\Omega}_{\mathcal{E}})+\bar{x}_z\mathsf{p}\bar{x}_{\mathcal{E}\mathcal{E}}+\partial\bar{x}\cdot\bar{\Omega}_{\mathcal{E}\mathcal{E}}\Bigg)+F_{xx}\cdot(\bar{x}_{\mathcal{E}},\bar{x}_{\mathcal{E}})+\bar{F}_{xX}\cdot(\bar{x}_{\mathcal{E}},\bar{X}_{\mathcal{E}})\\&+\bar{F}_{xx+}\cdot(\bar{x}_{\mathcal{E}},\bar{x}^{+}_{\mathcal{E}})+\bar{F}_{Xx}\cdot(\bar{X}_{\mathcal{E}},\bar{x}_{\mathcal{E}})+\bar{F}_{Xx+}\cdot(\bar{X}_{\mathcal{E}},\bar{x}^{+}_{\mathcal{E}})+\bar{F}_{x+x}\cdot(\bar{x}^{+}_{\mathcal{E}},\bar{x}_{\mathcal{E}})\\&+\bar{F}_{x+X}\cdot(\bar{x}^{+}_{\mathcal{E}},\bar{X}_{\mathcal{E}})+\bar{F}_{x+x+}\cdot(\bar{x}^{+}_{\mathcal{E}},\bar{x}^{+}_{\mathcal{E}})+\bar{F}_{\mathcal{E}x}\bar{x}_{\mathcal{E}}+\bar{F}_{\mathcal{E}X}\bar{X}_{\mathcal{E}}+\bar{F}_{\mathcal{E}x+}\bar{x}^{+}_{\mathcal{E}}+\bar{F}_{\mathcal{E}\mathcal{E}}=0\end{aligned}$$

Once again almost all of these terms are already known with the exception of $\partial^2\bar{x}\cdot(\bar{\Omega}_{\mathcal{E}},\bar{\Omega}_{\mathcal{E}})$ which is easily computable from the expressions above and $\partial\bar{x}\cdot\bar{\Omega}_{\mathcal{E}\mathcal{E}}$ which we define as $\partial\bar{x}(z)\cdot\bar{\Omega}_{\mathcal{E}\mathcal{E}}=\mathsf{C}(z)\partial\bar{X}\cdot\bar{\Omega}_{\mathcal{E}\mathcal{E}}=\mathsf{C}(z)\bar{X}'_{\mathcal{E}\mathcal{E}}$. Thus, we have that

$$\mathsf{M}(z)\bar{x}_{\mathcal{E}\mathcal{E}}(z)=\mathsf{GG}(z)\left[\begin{array}{ccc}I & \bar{X}_{\mathcal{E}\mathcal{E}} & \bar{X}'_{\mathcal{E}\mathcal{E}}\end{array}\right]^{\mathsf{T}},$$

where $\mathsf{M}(z)$ is the same as the first order expansion above and $\mathsf{GG}(z)$ is computable from the first order terms.

To solve for $\bar{X}_{\mathcal{E}\mathcal{E}}$ and $\bar{X}'_{\mathcal{E}\mathcal{E}}$ we first need $\bar{\Omega}_{\mathcal{E}\mathcal{E}}$ which we find by differentiating the law of motion for Ω to get

$$\begin{aligned}\tilde{\Omega}_{\mathcal{E}\mathcal{E}}(\mathcal{E},\Theta,\Omega)(y)=&-\int\sum_i\delta(\tilde{z}^i(\varsigma,\mathcal{E},\Theta,z,\Omega)-y^i)\prod_{j\neq i}\iota\left(\tilde{z}^j(\varsigma,\mathcal{E},\Theta,z,\Omega)\leq y^j\right)\tilde{z}^i_{\mathcal{E}\mathcal{E}}(\varsigma,\mathcal{E},\Theta,z,\Omega)\,d\Pr(\varsigma)d\Omega\\&-\int\sum_i\delta'(\tilde{z}^i(\varsigma,\mathcal{E},\Theta,z,\Omega)-y^i)\prod_{j\neq i}\iota\left(\tilde{z}^j(\varsigma,\mathcal{E},\Theta,z,\Omega)\leq y^j\right)\tilde{z}^i_{\mathcal{E}}(\varsigma,\mathcal{E},\Theta,z,\Omega)\,\tilde{z}^i_{\mathcal{E}}(\varsigma,\mathcal{E},\Theta,z,\Omega)\,d\mathrm{Pr}\\&-\int\Bigg(\sum_i\sum_{j\neq i}\delta(\tilde{z}^i(\varsigma,\mathcal{E},\Theta,z,\Omega)-y^i)\delta(\tilde{z}^j(\varsigma,\mathcal{E},\Theta,z,\Omega)-y^j)\\&\quad\prod_{k\neq i,j}\iota\left(\tilde{z}^k(\varsigma,\mathcal{E},\Theta,z,\Omega)\leq y^k\right)\tilde{z}^j_{\mathcal{E}}(\varsigma,\mathcal{E},\Theta,z,\Omega)\,\tilde{z}^i_{\mathcal{E}}(\varsigma,\mathcal{E},\Theta,z,\Omega)\,d\mathrm{Pr}\,(\varsigma)\ d\Omega\Bigg)\end{aligned}$$

evaluating at $\sigma = 0$ this becomes

$$\bar{\Omega}_{\mathcal{E}\mathcal{E}}(y) = -\int \sum_i \delta(z^i - y^i) \prod_{j \neq i} \iota(z^j - y^j) \bar{z}^i_{\mathcal{E}\mathcal{E}}(z) d\Omega - \int \sum_i \delta'(z^i - y^i) \prod_{j \neq i} \iota(z^j - y^j) \left[\bar{z}^i_{\mathcal{E}}(z)\right]^2 d\Omega$$
$$+ \int \sum_i \delta(z^i - y^i) \sum_{j \neq i} \delta(z^j - y^j) \prod_{k \neq i,j} \iota(z^k - y^k) \bar{z}^j_{\mathcal{E}}(z) \bar{z}^i_{\mathcal{E}}(z) d\Omega$$

Thus

$$\bar{\omega}_{\mathcal{E}\mathcal{E}}(y) = \frac{\partial^{n_z}}{\partial y^1 \partial y^2 \cdots \partial y^{n_z}} \bar{\Omega}_{\mathcal{E}\mathcal{E}}(y) = -\sum_i \frac{\partial}{\partial y^i} \int \prod_j \delta(z^j - y^j) \bar{z}^i_{\mathcal{E}\mathcal{E}}(z) d\Omega - \sum_i \frac{\partial}{\partial y^i} \int \delta'(z^i - y^i) \prod_{j \neq i} \delta(z^j - y^j) \left(\bar{z}^i_{\mathcal{E}}(z\right.$$
$$+ \sum_i \sum_{j \neq i} \frac{\partial^2}{\partial y^i \partial y^j} \int \prod_j \delta(z^j - y^j) \bar{z}^j_{\mathcal{E}}(z) \bar{z}^i_{\mathcal{E}} d\Omega$$
$$= -\sum_i \frac{\partial}{\partial y^i} \left(\bar{z}^i_{\mathcal{E}\mathcal{E}}(y) \omega(y)\right) + \sum_i \sum_j \frac{\partial^2}{\partial y^i \partial y^j} \left(\bar{z}^i_{\mathcal{E}}(y) \bar{z}^j_{\mathcal{E}}(y) \omega(y)\right).$$

Thus,

$$\bar{X}'_{\mathcal{E}\mathcal{E}} = \int \bar{R}(z) \left(-\sum_i \frac{\partial}{\partial z^i} \left(\bar{z}^i_{\mathcal{E}\mathcal{E}}(z) \omega(z)\right) + \sum_i \sum_j \frac{\partial^2}{\partial z^i \partial z^j} \left(\bar{z}^i_{\mathcal{E}}(z) \bar{z}^j_{\mathcal{E}}(z) \omega(z)\right) \right) dz$$
$$= \int \left(\bar{R}_z(z) + \bar{R}_x(z) \bar{x}_z(z)\right) \mathsf{p}\bar{x}^i_{\mathcal{E}\mathcal{E}}(z) \omega(z) dz$$
$$+ \int \left(\bar{R}_{zz}(z) + \bar{R}_{xz}(z) \cdot (I, \bar{x}_z(z)) + \bar{R}_{zx}(z) \cdot (\bar{x}_z(z), I) + \bar{R}_{xx}(z) \cdot (\bar{x}_z(z), \bar{x}_z(z))\right) \cdot (\mathsf{p}\bar{x}_{\mathcal{E}}(z), \mathsf{p}\bar{x}_{\mathcal{E}}(z))\, \omega(z) dz$$

which combined with the second derivative of R w.r.t $\mathcal{E}\mathcal{E}$

$$\int \Big(\bar{R}_x(z) \bar{x}_{\mathcal{E}\mathcal{E}}(z) + \bar{R}_X(z) \bar{X}_{\mathcal{E}\mathcal{E}} + \bar{R}_{xx}(z) \cdot (\bar{x}_{\mathcal{E}}(z), \bar{x}_{\mathcal{E}}(z))$$
$$+ 2\bar{R}_{xX}(z) \cdot \left(\bar{x}_{\mathcal{E}}(z), \bar{X}_{\mathcal{E}}\right) + 2\bar{R}_{x\mathcal{E}}(z) \bar{x}_{\mathcal{E}}(z)$$
$$+ \bar{R}_{XX}(z) \cdot \left(\bar{X}_{\mathcal{E}}, \bar{X}_{\mathcal{E}}\right) + 2\bar{R}_{X\mathcal{E}}(z) \bar{X}_{\mathcal{E}} + \bar{R}_{\mathcal{E}\mathcal{E}}(z) \Big) d\Omega = 0$$

gives a system of linear equations of the form

$$\mathsf{O} \cdot \left[\begin{array}{cc} \bar{X}_{\mathcal{E}\mathcal{E}} & \bar{X}'_{\mathcal{E}\mathcal{E}} \end{array} \right]^{\mathsf{T}} = \mathsf{HH}.$$

Finally for the second order expansion we need the effect of the presense of risk. Differentiation of F gives

$$\bar{F}_x \bar{x}_{\sigma\sigma} + \bar{F}_X \bar{X}_{\sigma\sigma} + \bar{F}_{x+}\left(\mathbb{E}\left[\bar{x}_{\varsigma\varsigma}\varsigma^2 + \bar{x}_{\mathcal{E}\mathcal{E}}\mathcal{E}^2\right] + \bar{x}_{\sigma\sigma} + \bar{x}_z \mathsf{p}\bar{x}_{\sigma\sigma} + \partial\bar{x}\cdot\bar{\Omega}_{\sigma\sigma}\right) + \bar{F}_{x-}\mathbb{E}\left[\bar{x}_{\varsigma\varsigma}\varsigma^2 + \bar{x}_{\mathcal{E}\mathcal{E}}\mathcal{E}^2\right] + \bar{F}_{\sigma\sigma} = 0.$$

or

$$\bar{F}_x \bar{x}_{\sigma\sigma} + \bar{F}_X \bar{X}_{\sigma\sigma} + \bar{F}_{x+}\left(\bar{x}_{\varsigma\varsigma}\mathrm{var}(\varsigma) + \bar{x}_{\mathcal{E}\mathcal{E}}\mathrm{var}(\mathcal{E}) + \bar{x}_{\sigma\sigma} + \bar{x}_z \mathsf{p}\bar{x}_{\sigma\sigma} + \partial\bar{x}\cdot\bar{\Omega}_{\sigma\sigma}\right) + \bar{F}_{x-}\left(\bar{x}_{\varsigma\varsigma}\mathrm{var}(\varsigma) + \bar{x}_{\mathcal{E}\mathcal{E}}\mathrm{var}(\mathcal{E})\right) + \bar{F}_{\sigma\sigma} = 0.$$

Defining $\partial\bar{x}(z)\cdot\bar{\Omega}_{\sigma\sigma} = \mathsf{C}(z)\partial\bar{X}\cdot\bar{\Omega}_{\sigma\sigma} \equiv \mathsf{C}(z)\bar{X}'_{\sigma\sigma}$, we find that $\bar{x}_{\sigma\sigma}(z)$ solves the linear system

$$\mathsf{II}(z)\bar{x}_{\sigma\sigma}(z) = \mathsf{JJ}(z)\left[\begin{array}{ccc} I & \bar{X}_{\sigma\sigma} & \bar{X}'_{\sigma\sigma}\end{array}\right]^{\mathsf{T}}.$$

To solve for $\bar{X}'_{\sigma\sigma}$ we need to find $\bar{\Omega}_{\sigma\sigma}$ by differentiating the law of motion for Ω

$$\begin{aligned}\tilde{\Omega}_{\sigma\sigma}(\mathcal{E},\Theta,\Omega)(y) = &-\int\sum_i \delta(\tilde{z}^i(\varsigma,\mathcal{E},\Theta,z,\Omega) - y^i)\prod_{j\neq i}\iota\left(\tilde{z}^j(\varsigma,\mathcal{E},\Theta,z,\Omega)\leq y^j\right)\left(\tilde{z}^i_{\varsigma\varsigma}(\varsigma,\mathcal{E},\Theta,z,\Omega)\varsigma^2 + \tilde{z}^i_{\sigma\sigma}(\varsigma,\mathcal{E},\Theta,z,\Omega)\right)d\mathrm{Pr}(\varsigma\\ &-\int\sum_i \delta'(\tilde{z}^i(\varsigma,\mathcal{E},\Theta,z,\Omega) - y^i)\prod_{j\neq i}\iota\left(\tilde{z}^j(\varsigma,\mathcal{E},\Theta,z,\Omega)\leq y^j\right)\tilde{z}^i_{\varsigma}(\varsigma,\mathcal{E},\Theta,z,\Omega)\,\tilde{z}^i_{\varsigma}(\varsigma,\mathcal{E},\Theta,z,\Omega)\,\varsigma^2 d\mathrm{Pr}\,(\varsigma)\;d\Omega\\ &-\int\Bigg(\sum_i\sum_{j\neq i}\delta(\tilde{z}^i(\varsigma,\mathcal{E},\Theta,z,\Omega) - y^i)\delta(\tilde{z}^j(\varsigma,\mathcal{E},\Theta,z,\Omega) - y^j)\\ &\quad\prod_{k\neq i,j}\iota\left(\tilde{z}^k(\varsigma,\mathcal{E},\Theta,z,\Omega)\leq y^k\right)\tilde{z}^j_{\varsigma}(\varsigma,\mathcal{E},\Theta,z,\Omega)\,\tilde{z}^i_{\varsigma}(\varsigma,\mathcal{E},\Theta,z,\Omega)\Bigg)d\mathrm{Pr}\,(\varsigma)\;d\Omega.\end{aligned}$$

Evaluating at $\sigma = 0$ we find

$$\begin{aligned}\bar{\Omega}_{\sigma\sigma}(y) = &-\int\sum_i \delta(z^i - y^i)\prod_{j\neq i}\iota(z^j - y^j)\left(\bar{z}^i_{\sigma\sigma}(z) + \bar{z}^i_{\varsigma\varsigma}\mathrm{var}(\varsigma)\right)d\Omega - \int\sum_i \delta'(z^i - y^i)\prod_{j\neq i}\iota(z^j - y^j)\left[\bar{z}^i_{\varsigma}(z)\right]^2\mathrm{var}(\varsigma)d\Omega\\ &+\int\sum_i\delta(z^i - y^i)\sum_{j\neq i}\delta(z^j - y^j)\prod_{k\neq i,j}\iota(z^k - y^k)\bar{z}^j_{\varsigma}(z)\bar{z}^i_{\varsigma}(z)\mathrm{var}(\varsigma)d\Omega\end{aligned}$$

which gives

$$\bar{\omega}_{\sigma\sigma}(y) = -\sum_i\frac{\partial}{\partial y^i}\left(\left(\bar{z}^i_{\sigma\sigma}(y) + \bar{z}^i_{\varsigma\varsigma}(y)\mathrm{var}(\varsigma)\right)\omega(y)\right) + \sum_i\sum_j\frac{\partial^2}{\partial y^i\partial y^j}\left(\bar{z}^i_{\varsigma}(y)\bar{z}^j_{\varsigma}(y)\mathrm{var}(\varsigma)\omega(y)\right).$$

Thus,

$$\bar{X}'_{\sigma\sigma} = \int \bar{R}(z) \left(-\sum_i \frac{\partial}{\partial z^i} \left((\bar{z}^i_{\sigma\sigma}(z) + \bar{z}^i_{\varsigma\varsigma}(y)\text{var}(\varsigma))\omega(z) \right) + \sum_i \sum_j \frac{\partial^2}{\partial z^i \partial z^j} \left(\bar{z}^i_{\varsigma}(z)\bar{z}^j_{\varsigma}(z)\text{var}(\varsigma)\omega(z) \right) \right) dz$$

$$= \int \left(\bar{R}_z(z) + \bar{R}_x(z)\bar{x}_z(z) \right) \mathsf{p} \left(\bar{z}^i_{\sigma\sigma}(z) + \bar{z}^i_{\varsigma\varsigma}(y)\text{var}(\varsigma) \right) \omega(z) dz$$

$$+ \int \left(\bar{R}_{zz}(z) + \bar{R}_{xz}(z) \cdot (I, \bar{x}_z(z)) + \bar{R}_{zx}(z) \cdot (\bar{x}_z(z), I) + \bar{R}_{xx}(z) \cdot (\bar{x}_z(z), \bar{x}_z(z)) \right) \cdot (\mathsf{p}\bar{x}_\varsigma(z), \mathsf{p}\bar{x}_\varsigma(z)) \, \text{var}(\varsigma)\omega(z$$

which combined with the second derivative of R w.r.t $\sigma\sigma$

$$\int \Bigg(\bar{R}_x(z) \left(\bar{x}_{\sigma\sigma}(z) + \bar{x}_{\varsigma\varsigma}(z)\text{var}(\varsigma) \right) + \bar{R}_X(z)\bar{X}_{\sigma\sigma} + \bar{R}_{xx}(z) \cdot \left(\bar{x}_\varsigma(z), \bar{x}_\varsigma(z) \right) \text{var}(\varsigma)$$

$$2\bar{R}_{x\varsigma}(z)\bar{x}_\varsigma(z)\text{var}(\varsigma) + \bar{R}_{\mathcal{E}\mathcal{E}}(z) \Bigg) d\Omega = 0$$

gives a system of linear equations of the form

$$\mathsf{KK} \cdot \begin{bmatrix} \bar{X}_{\sigma\sigma} & \bar{X}'_{\sigma\sigma} \end{bmatrix}^{\mathsf{T}} = \mathsf{LL}.$$

B.1 Endogeneous Aggregate States

When τ is fixed at $\bar{\tau}$, we add the additional constraint of the Phillip's curve to the planner's problem

$$C_t^{-\nu} \left(Y_t \left[1 - \epsilon(1 - W_t) \right] - \psi\pi_t(1 + \pi_t) \right) + \beta\mathbb{E}_t \left[C_{t+1}^{-\nu} \psi\pi_{t+1}(1 + \pi_{t+1}) \right] = 0$$

where

$$Y_t = \int n_{i,t}\theta_{i,t} di$$

and

$$W_t = \frac{\mathcal{W}_t}{1 - \bar{\tau}}$$

We'll let Λ_t be the Lagrange multiplier on the Phillips curve. We then need to adjust the FOC as follows

$$\begin{aligned}
n_{i,t}^{1+\gamma} + c_{i,t}^{-\nu}\mathcal{T}_t + a_{i,t-1}r_{i,t} - c_{i,t}^{1-\nu} - a_{i,t} &= 0 \\
c_{i,t}^{-\nu}\mathcal{W}_t\theta_{i,t} - n_{i,t}^{\gamma} &= 0 \\
m_{i,t}^{-1/\nu}c_{i,t} - C_t &= 0 \\
\beta\mathbb{E}_t\left[\varrho_{i,t+1}\right] + \frac{1}{\nu}m_{i,t}^{-1/\nu-1}c_{i,t}\varphi_{i,t} &= 0 \\
c_{i,t}^{-\nu} + \left((1-\nu)c_{i,t}^{-\nu} - \nu c_{i,t}^{-\nu-1}\mathcal{T}_t\right)\mu_{i,t} + \frac{\nu a_{i,t-1}r_{i,t}}{c_{i,t}}(\mu_{i,t}-\mu_{i,t-1}) & \\
-\beta\nu m_{i,t-1}\frac{\varrho_{i,t}}{c_{i,t}} + \nu c_{i,t}^{-\nu-1}\mathcal{W}_t\theta_{i,t}\phi_{i,t} + m_{i,t}^{-1/\nu}\varphi_{i,t} - \Xi_t &= 0 \\
-n_{i,t}^{\gamma} - (1+\gamma)n_{i,t}^{\gamma}\mu_{i,t} + \gamma n_{i,t}^{\gamma-1}\phi_{i,t} + \theta_{i,t}\chi_t + C_t^{-\nu}\left[1-\epsilon(1-\frac{\mathcal{W}_t}{1-\bar{\tau}})\right]\theta_{i,t}\Lambda_t &= 0 \\
\varrho_{i,t} - c_{i,t}^{-\nu}(1+\pi_t)^{-1}\rho_{i,t-1} &= 0 \\
\mu_{i,t-1} - \beta\mathbb{E}_{t-1}\left[r_{i,t}\mu_{i,t}\right] &= 0 \\
r_{i,t} - \frac{c_{i,t}^{-\nu}(1+\pi_t)^{-1}}{\beta\mathbb{E}_{t-1}\left[c_{i,t}^{-\nu}(1+\pi_t)^{-1}\right]} &= 0 \\
\mathcal{Q}_{t-1} - \beta m_{i,t-1}\mathbb{E}_{t-1}\left[c_{i,t}^{-\nu}(1+\pi_{t+1})^{-1}\right] &= 0 \\
e_{i,t} - (\rho_e e_{i,t-1} + \varsigma_{i,t}) &= 0
\end{aligned}$$

and, with $\Pi_t = \psi C_t^{-\nu} \pi_t (1+\pi_t)$

$$\int c_{i,t}^{-\nu} \mu_{i,t} di = 0$$

$$C_t^{-\nu} \left(Y_t \left[1 - \epsilon (1 - \frac{\mathcal{W}_t}{1-\bar{\tau}}) \right] \right) - \Pi_t + \beta \mathbb{E}_t \left[\Pi_{t+1} \right] = 0$$

$$C_t - \int c_{i,t} di = 0$$

$$Y_t - \int n_{i,t} \theta_{i,t} di = 0$$

$$\int \theta_{i,t} n_{i,t} - \frac{\psi}{2} \pi_t^2 di - C_t - \bar{G} = 0$$

$$-\psi \pi_t \chi_t + (1+\pi_t)^{-1} \int a_{i,t-1} r_{i,t} (\mu_{i,t} - \mu_{i,t-1}) - \beta \varrho_{i,t} di - \psi C_t^{-\nu} (1 + 2\pi_t)(\Lambda_t - \Lambda_{t-1}) = 0$$

$$\int c_{i,t}^{-\nu} \theta_{i,t} \phi_{i,t} di + \frac{\epsilon}{1-\bar{\tau}} C_t^{-\nu} Y_t \Lambda_t = 0$$

$$\chi_t - \Xi_t - \nu C_t^{-\nu-1} \left(Y_t \left[1 - \epsilon (1 - W_t) \right] \right) \Lambda_t + \nu \frac{\Pi_t}{C_t} (\Lambda_t - \Lambda_{t-1}) = 0$$

$$\int \rho_{i,t-1} di = 0$$

$$\mathbb{E}_{t-1} \mathcal{T}_t = 0$$

With the additional state Λ, the individual constraints can be written as

$$
\begin{aligned}
\tilde{n}(\varsigma,\mathcal{E},\Theta,z,\Omega,\Lambda)+\tilde{c}(\varsigma,\mathcal{E},\Theta,z,\Omega,\Lambda)^{-\nu}\tilde{\mathcal{T}}(\mathcal{E},\Theta,\Omega,\Lambda)+\tilde{a}(\Theta,z,\Omega,\Lambda)\tilde{r}(\varsigma,\mathcal{E},\Theta,z,\Omega,\Lambda)&\\
-c(\varsigma,\mathcal{E},\Theta,z,\Omega,\Lambda)^{1-\nu}-\tilde{a}(\rho_\Theta\Theta+\mathcal{E},\tilde{z}(\varsigma,\mathcal{E},\Theta,z,\Omega,\Lambda),\tilde{\Omega}(\mathcal{E},\Theta,\Omega,\Lambda),\tilde{\Lambda}(\mathcal{E},\Theta,\Omega,\Lambda))&=0\\
\tilde{c}(\varsigma,\mathcal{E},\Theta,z,\Omega,\Lambda)^{-\nu}\tilde{\mathcal{W}}(\mathcal{E},\Theta,\Omega,\Lambda)\exp(\mathcal{E}+\rho_\Theta\Theta+\rho_e e(z)+\varsigma)-\tilde{n}(\varsigma,\mathcal{E},\Theta,z,\Omega,\Lambda)^{\gamma}&=0\\
\tilde{m}(\varsigma,\mathcal{E},\Theta,z,\Omega,\Lambda)^{-1/\nu}\tilde{c}(\varsigma,\mathcal{E},\Theta,z,\Omega,\Lambda)-\tilde{C}(\mathcal{E},\Theta,\Omega,\Lambda)&=0\\
\beta\mathbb{E}\left[\tilde{\varrho}(\cdot,\cdot,\rho_\Theta\Theta+\mathcal{E},\tilde{z}(\varsigma,\mathcal{E},\Theta,z,\Omega,\Lambda),\tilde{\Omega}(\mathcal{E},\Theta,\Omega,\Lambda),\tilde{\Lambda}(\mathcal{E},\Theta,\Omega,\Lambda)\right]&\\
+\frac{1}{\nu}\tilde{m}(\varsigma,\mathcal{E},\Theta,z,\Omega,\Lambda)^{-1/\nu-1}\tilde{c}(\varsigma,\mathcal{E},\Theta,z,\Omega,\Lambda)\tilde{\varphi}(\varsigma,\mathcal{E},\Theta,z,\Omega,\Lambda)&=0\\
\tilde{c}(\varsigma,\mathcal{E},\Theta,z,\Omega,\Lambda)^{-\nu}+\tilde{m}(\varsigma,\mathcal{E},\Theta,z,\Omega,\Lambda)^{-1/\nu}\tilde{\varphi}(\varsigma,\mathcal{E},\Theta,z,\Omega,\Lambda)-\tilde{\chi}(\mathcal{E},\Theta,\Omega,\Lambda)&\\
-\beta\nu\tilde{m}(\varsigma,\mathcal{E},\Theta,z,\Omega,\Lambda)\frac{\tilde{\varrho}(\varsigma,\mathcal{E},\Theta,z,\Omega,\Lambda)}{\tilde{c}(\varsigma,\mathcal{E},\Theta,z,\Omega,\Lambda)}+\frac{\nu\tilde{a}(\Theta,z,\Omega,\Lambda)\tilde{r}(\varsigma,\mathcal{E},\Theta,z,\Omega,\Lambda)}{\tilde{c}(\varsigma,\mathcal{E},\Theta,z,\Omega,\Lambda)}(\tilde{\mu}(\varsigma,\mathcal{E},\Theta,z,\Omega,\Lambda)-\frac{\hat{\mu}(z)}{(\hat{m}(z)+1)^{\nu}})&\\
+\nu\tilde{c}(\varsigma,\mathcal{E},\Theta,z,\Omega,\Lambda)^{-\nu-1}\tilde{\mathcal{W}}(\mathcal{E},\Theta,\Omega,\Lambda)\exp(\mathcal{E}+\rho_\Theta\Theta+\rho_e e(z)+\varsigma)\tilde{\phi}(\varsigma,\mathcal{E},\Theta,z,\Omega,\Lambda)&\\
+\Big((1-\nu)\tilde{c}(\varsigma,\mathcal{E},\Theta,z,\Omega,\Lambda)^{-\nu}-\nu\tilde{c}(\varsigma,\mathcal{E},\Theta,z,\Omega,\Lambda)^{-\nu-1}\tilde{\mathcal{T}}(\mathcal{E},\Theta,\Omega,\Lambda)\Big)\tilde{\mu}(\varsigma,\mathcal{E},\Theta,z,\Omega,\Lambda)&=0\\
-\tilde{n}(\varsigma,\mathcal{E},\Theta,z,\Omega,\Lambda)^{\gamma}-(1+\gamma)\tilde{n}(\varsigma,\mathcal{E},\Theta,z,\Omega,\Lambda)^{\gamma}\tilde{\mu}(\varsigma,\mathcal{E},\Theta,z,\Omega,\Lambda)+\gamma\tilde{n}(\varsigma,\mathcal{E},\Theta,z,\Omega,\Lambda)^{\gamma-1}\tilde{\phi}(\varsigma,\mathcal{E},\Theta,z,\Omega,\Lambda)&\\
+\exp(\mathcal{E}+\rho_\Theta\Theta+\rho_e e(z)+\varsigma)\left(\tilde{\chi}(\mathcal{E},\Theta,\Omega,\Lambda)+\tilde{C}(\mathcal{E},\Theta,\Omega,\Lambda)^{-\nu}\left[1-\epsilon(1-\frac{\tilde{\mathcal{W}}(\mathcal{E},\Theta,\Omega,\Lambda)}{1-\bar{\tau}})\right]\tilde{\Lambda}(\mathcal{E},\Theta,\Omega,\Lambda)\right)&=0\\
\tilde{\varrho}(\varsigma,\mathcal{E},\Theta,z,\Omega,\Lambda)-\tilde{c}(\varsigma,\mathcal{E},\Theta,z,\Omega,\Lambda)^{-\nu}(1+\tilde{\pi}(\mathcal{E},\Theta,\Omega,\Lambda))^{-1}\tilde{\rho}(z,\Omega,\Lambda)&=0\\
\tilde{r}(\varsigma,\mathcal{E},\Theta,z,\Omega,\Lambda)-\frac{\tilde{c}(\varsigma,\mathcal{E},\Theta,z,\Omega,\Lambda)^{-\nu}(1+\pi(\mathcal{E},\Theta,\Omega,\Lambda))^{-1}}{\beta\mathbb{E}\left[\tilde{c}(\cdot,\cdot,\Theta,z,\Omega,\Lambda)^{-\nu}(1+\pi(\cdot,\Theta,\Omega,\Lambda))^{-1}\right]}&=0\\
\frac{\hat{\mu}(z)}{(\hat{m}(z)+1)^{\nu}}-\beta\mathbb{E}\left[\tilde{r}(\cdot,\cdot,\Theta,z,\Omega,\Lambda)\tilde{\mu}(\cdot,\cdot,\Theta,z,\Omega,\Lambda)\right]&=0\\
\tilde{\mathcal{Q}}(\Omega,\Lambda)-\beta\left(\hat{m}(z)+1\right)^{\nu}\mathbb{E}\left[\tilde{c}(\cdot,\cdot,\Theta,z,\Omega,\Lambda)^{-\nu}(1+\pi(\cdot,\Theta,\Omega,\Lambda))^{-1}\right]&=0\\
\tilde{\tilde{\mu}}(\varsigma,\mathcal{E},\Theta,z,\Omega)-\frac{\tilde{\mu}(\varsigma,\mathcal{E},\Theta,z,\Omega)}{\tilde{m}(\varsigma,\mathcal{E},\Theta,z,\Omega)}&=0\\
\tilde{\tilde{m}}(\varsigma,\mathcal{E},\Theta,z,\Omega)-\tilde{m}(\varsigma,\mathcal{E},\Theta,z,\Omega)^{1/\nu}+1&=0
\end{aligned}
$$

while the aggregate constraints are

$$
\begin{aligned}
\int \tilde{c}(\varsigma, \mathcal{E}, \Theta, z, \Omega, \Lambda)^{-\nu} \tilde{\mu}(\varsigma, \mathcal{E}, \Theta, z, \Omega, \Lambda) d\Pr(\varsigma) d\Omega &= 0 \\
\tilde{C}(\mathcal{E}, \Theta, \Omega, \Lambda)^{-\nu} \left(\left[1 - \epsilon (1 - \frac{\tilde{\mathcal{W}}(\mathcal{E}, \Theta, \Omega, \Lambda)}{1 - \bar{\tau}}) \right] \right) - \tilde{\Pi}(\mathcal{E}, \Theta, \Omega, \Lambda) + \beta \mathbb{E} \left[\tilde{\Pi}(\cdot, \rho_\Theta \Theta + \mathcal{E}, \tilde{\Omega}(\mathcal{E}, \Theta, \Omega, \Lambda), \tilde{\Lambda}(\mathcal{E}, \Theta, \Omega, \Lambda)) \right] &= 0 \\
\tilde{C}(\mathcal{E}, \Theta, \Omega, \Lambda) - \int \tilde{c}(\varsigma, \mathcal{E}, \Theta, z, \Omega, \Lambda) d\Pr(\varsigma) d\Omega &= 0 \\
\tilde{Y}(\mathcal{E}, \Theta, \Omega, \Lambda) - \int \exp(\mathcal{E} + \rho_\Theta \Theta + \rho_e e(z) + \varsigma) \tilde{n}(\varsigma, \mathcal{E}, \Theta, z, \Omega, \Lambda) d\Pr(\varsigma) d\Omega &= 0 \\
\tilde{Y}(\mathcal{E}, \Theta, \Omega, \Lambda) - \frac{\psi}{2} \tilde{\pi}(\mathcal{E}, \Theta, \Omega, \Lambda)^2 - \tilde{C}(\mathcal{E}, \Theta, \Omega, \Lambda) - \bar{G} &= 0 \\
(1 + \tilde{\pi}(\mathcal{E}, \Theta, \Omega, \Lambda))^{-1} \int \tilde{a}(z, \Omega, \Lambda) \tilde{r}(\varsigma, \mathcal{E}, \Theta, z, \Omega, \Lambda) (\tilde{\mu}(\varsigma, \mathcal{E}, \Theta, z, \Omega, \Lambda) - \frac{\hat{\mu}(z)}{(\hat{m}(z) + 1)^\nu}) - \beta \tilde{\varrho}(\varsigma, \mathcal{E}, \Theta, z, \Omega, \Lambda) d\Pr(\varsigma) d\Omega & \\
- \psi \tilde{\pi}(\mathcal{E}, \Theta, \Omega, \Lambda) \tilde{\chi}(\mathcal{E}, \Theta, \Omega, \Lambda) - \tilde{C}(\mathcal{E}, \Theta, \Omega, \Lambda)^{-\nu} \psi (1 + 2\tilde{\pi}(\mathcal{E}, \Theta, \Omega, \Lambda)) (\tilde{\Lambda}(\mathcal{E}, \Theta, \Omega, \Lambda) - \Lambda) &= 0 \\
\int \tilde{c}(\varsigma, \mathcal{E}, \Theta, z, \Omega, \Lambda)^{-\nu} \exp(\mathcal{E} + \rho_\Theta \Theta + \varsigma + e(z)) \tilde{\phi}(\varsigma, \mathcal{E}, \Theta, z, \Omega, \Lambda) d\Pr(\varsigma) d\Omega + \frac{\epsilon}{1 - \bar{\tau}} \tilde{C}(\mathcal{E}, \Theta, \Omega, \Lambda)^{-\nu} \tilde{Y}(\mathcal{E}, \Theta, \Omega, \Lambda) \tilde{\Lambda}(\mathcal{E}, \Theta, \Omega, \Lambda) &= 0 \\
\tilde{\chi}(\mathcal{E}, \Theta, \Omega, \Lambda) - \tilde{\Xi}(\mathcal{E}, \Theta, \Omega, \Lambda) - \nu \tilde{C}(\mathcal{E}, \Theta, \Omega, \Lambda)^{-\nu - 1} \left(\tilde{Y}(\mathcal{E}, \Theta, \Omega, \Lambda) \left[1 - \epsilon (1 - \frac{\tilde{\mathcal{W}}(\mathcal{E}, \Theta, \Omega, \Lambda)}{1 - \bar{\tau}}) \right] \right) \tilde{\Lambda}(\mathcal{E}, \Theta, \Omega, \Lambda) & \\
+ \nu \frac{\tilde{\Pi}(\mathcal{E}, \Theta, \Omega, \Lambda)}{\tilde{C}(\mathcal{E}, \Theta, \Omega, \Lambda)} (\tilde{\Lambda}(\mathcal{E}, \Theta, \Omega, \Lambda) - \Lambda) &= 0 \\
\tilde{\Pi}(\mathcal{E}, \Theta, \Omega, \Lambda) - \tilde{C}(\mathcal{E}, \Theta, \Omega, \Lambda)^{-\nu} \psi \tilde{\pi}(\mathcal{E}, \Theta, \Omega, \Lambda) (1 + \tilde{\pi}(\mathcal{E}, \Theta, \Omega, \Lambda)) & \\
\int \tilde{\rho}(\Theta, z, \Omega, \Lambda) d\Pr(\varsigma) d\Omega &= 0 \\
\mathbb{E} \left[\tilde{\mathcal{T}}(\cdot, \Theta, \Omega, \Lambda) \right] &= 0
\end{aligned}
$$

B.1.1 Expansion

Zeroth order terms

The same logic as before gives $\tilde{z}(0, 0, 0, z, \Omega, \Lambda; 0) = z$ and $\tilde{\Omega}(0, 0, \Omega, \Lambda; 0) = \Omega$. There will be transition dynamics for Λ given by $\tilde{\Lambda}(0, 0, \Omega, \Lambda)$. For a given initial state $\Lambda^0 = \Lambda$, we will denote the deterministic path of Λ by Λ^n. We will assume that Λ^n converges to the steady sate $\bar{\Lambda}$ in N periods.

We begin by computing the expansion around $(0, 0, \Omega, \bar{\Lambda}; 0)$ and then extend to the path

First Order Terms - around $(0, 0, \Omega, \bar{\Lambda}; 0)$

Our equations for F and R are adjusted as follows

$$
F \left(z, \mathbb{E} \tilde{x}(\cdot, \cdot, \Theta, z, \Omega, \Lambda), \tilde{x}(\varsigma, \mathcal{E}, \Theta, z, \Omega, \Lambda), \mathbb{E} \left[\tilde{x}(\cdot, \cdot, \rho_\Theta \Theta + \mathcal{E}, \tilde{z}(\varsigma, \mathcal{E}, \Theta, z, \Omega, \Lambda), \tilde{\Omega}(\Theta, z, \Omega, \Lambda), \tilde{\Lambda}(\Theta, z, \Omega, \Lambda) \right], \tilde{X}(\Theta, \Omega
$$

for all z in Ω and

$$
\int R \left(z, \tilde{x}(\varsigma, \Theta, z, \Omega, \Lambda), \tilde{X}(\Theta, \Omega, \Lambda), \mathbb{E}[\tilde{X}(\cdot, \rho_\Theta \Theta + \mathcal{E}, \tilde{\Omega}(\Theta, \Omega, \Lambda), \tilde{\Lambda}(\Theta, \Omega, \Lambda))], \varsigma, \Theta, \Lambda \right) d\Pr(\varsigma) d\Omega = 0.
$$

where $\tilde{\Lambda}$ is contained in $\tilde{X}$ so there is a projection matrix $\mathcal{P}$ such that $\tilde{\Lambda} = \mathcal{P} \tilde{X}$.

There are only a few changes to the first order terms. The first is that we need to solve for the derivative w.r.t Λ. We can exploit our knowledge of $\tilde{z}(0,0,0,z,\Omega,\Lambda;0)=z$ and $\tilde{\Omega}(0,0,\Omega,\Lambda;0)=\Omega$ and hence $\bar{z}_\Lambda=0$ and $\bar{\Omega}_\Lambda=0$. Thus,

$$\bar{F}_{x-}(z)\bar{x}_\Lambda(z)+\bar{F}_x(z)\bar{x}_\Lambda(z)+\bar{F}_{x+}(z)\left(\bar{x}_\Lambda(z)\bar{\Lambda}_\Lambda\right)+\bar{F}_X(z)\bar{X}_\Lambda=0$$

and

$$\int \bar{R}_x(z)\bar{x}_\Lambda(z)+\bar{R}_X(z)\bar{X}_\Lambda+\bar{R}_{X+}(z)\bar{X}_\Lambda\bar{\Lambda}_\Lambda+\bar{R}_\Lambda(z)d\Omega=0.$$

We need to solve for the persistence of the endogenous state Λ: $\rho_\Lambda=\bar{\Lambda}_\Lambda$. This is a nonlinear equation but fortunately still involves only operations involving small matrices. First note that

$$\bar{x}_\Lambda(z)=-(\bar{F}_{x-}(z)+\bar{F}_x(z)+\rho_\Lambda\bar{F}_{x+}(z))^{-1}\bar{F}_X(z)\bar{X}_\Lambda$$

then

$$\bar{X}_\Lambda=-\left(\int\left[-\bar{R}_x(z)\left(\bar{F}_{x-}(z)+\bar{F}_x(z)+\rho_\Lambda\bar{F}_{x+}(z)\right)^{-1}\bar{F}_X(z)+\bar{R}_X(z)+\rho_\Lambda\bar{R}_{X+}(z)\right]d\Omega\right)^{-1}\left(\int\bar{R}_\Lambda(z)d\Omega\right).$$

So ρ_Λ must solve

$$\rho_\Lambda=-\mathcal{P}\left(\int\left[-\bar{R}_x(z)(\bar{F}_{x-}(z)+\bar{F}_x(z)+\rho_\Lambda\bar{F}_{x+}(z))^{-1}\bar{F}_X(z)+\bar{R}_X(z)+\rho_\Lambda\bar{R}_{X+}(z)\right]d\Omega\right)^{-1}\left(\int\bar{R}_\Lambda d\Omega\right).$$

This can be found easily with a 1-dimensional root solver as all the inversions involve small matrices. The Frechet derivative w.r.t Ω is adjusted as follows:

$$\bar{F}_{\mathbf{x}}(z)\partial\bar{x}(z)+\bar{F}_{x+}(z)\bar{x}_\Lambda(z)\mathcal{P}\partial\bar{X}+\bar{F}_X(z)\partial\bar{X}=0$$

which gives

$$\partial\bar{x}(z)=-(\bar{F}_{\mathbf{x}}(z))^{-1}\left(\bar{F}_{x+}(z)\bar{x}_\Lambda(z)\mathcal{P}+\bar{F}_X(z)\right)\partial\bar{X}\equiv\mathsf{C}(z)\partial\bar{X}.$$

To find $\partial\bar{X}$ we then take the Frechet derivative of R to get

$$0=\int\bar{R}(z)\delta(z)dz+\int\left(\bar{R}_x(z)\partial\bar{x}(z)\cdot\Delta+\bar{R}_{X+}(z)\left((\partial\bar{X}\cdot\boldsymbol{\Delta})+\bar{X}_\Lambda\mathcal{P}(\partial\bar{X}\cdot\boldsymbol{\Delta})\right)+\bar{R}_X(z)\partial\bar{X}\cdot\Delta\right)\omega(z)dz.$$

Substituting for $\partial\bar{x}(z) = \mathsf{C}(z)\partial\bar{X}$, we get

$$\begin{aligned}\partial\bar{X}\cdot\Delta &= -\left(\int\left(\bar{R}_x(z)\mathsf{C}(z) + \bar{R}_{X+}(z)(I + \bar{X}_\Lambda\mathcal{P}) + \bar{R}_X(z)\right)d\Omega\right)^{-1}\int\bar{R}(z)d\Delta \\ &\equiv \mathsf{D}^{-1}\int\mathsf{E}(z)d\Delta.\end{aligned}$$

Finally we can compute the response to shock $\mathcal{E}$

$$\bar{F}_x\bar{x}_\mathcal{E} + \bar{F}_{x+}\left(\bar{x}_\Theta + \bar{x}_z\mathsf{p}\bar{x}_\mathcal{E} + \partial\bar{x}\cdot\bar{\Omega}_\mathcal{E} + \bar{x}_\Lambda\mathcal{P}\bar{X}_\mathcal{E}\right) + \bar{F}_X\bar{X}_\mathcal{E} + \bar{F}_\mathcal{E} = 0.$$

andSubstituting for $\partial\bar{x}$, we obtain and defining $\bar{X}'_\mathcal{E} = \partial\bar{X}\cdot\bar{\Omega}_\mathcal{E}$

$$\bar{F}_x(z)\bar{x}_\mathcal{E}(z) + \bar{F}_{x+}(z)\left(\bar{x}_\Theta(z) + \bar{x}_z(z)\mathsf{p}\bar{x}_\mathcal{E}(z) + \mathsf{C}(z)\bar{X}'_\mathcal{E} + \bar{x}_\Lambda\mathcal{P}\bar{X}_\mathcal{E}\right) + \bar{F}_X(z)\bar{X}_\mathcal{E} + \bar{F}_\mathcal{E}(z) = 0,$$

or

$$\mathsf{M}(z)\bar{x}_\mathcal{E}(z) = \mathsf{N}(z)\begin{bmatrix} I & \bar{X}_\mathcal{E} & \bar{X}'_\mathcal{E}\end{bmatrix}^\mathsf{T}.$$

We obtain the same expression for $\bar{X}'_\mathcal{E}$ above

$$\bar{X}'_\mathcal{E} = \mathsf{D}^{-1}\int\left(\bar{R}_z(z) + \bar{R}_x(z)\bar{x}_z(z)\right)\mathsf{q}\bar{x}_\mathcal{E}(z)d\Omega$$

which, when combined with

$$\int\bar{R}_x(z)\bar{x}_\mathcal{E}(z) + \bar{R}_X(z)\bar{X}_\mathcal{E} + \bar{R}_{X+}(z)\left(\bar{X}_\Theta + \partial\bar{X}\cdot\bar{\Omega}_\mathcal{E} + \bar{X}_\Lambda\mathcal{P}\bar{X}_\mathcal{E}\right) + \bar{R}_\mathcal{E}(z)d\Omega = 0,$$

yields the linear system

$$\mathsf{O}\cdot\begin{bmatrix}\bar{X}_\mathcal{E} & \bar{X}'_\mathcal{E}\end{bmatrix}^\mathsf{T} = \mathsf{P}.$$

First Order Terms - Along Path

Here we assume that all the derivatives of $\tilde{x}$ and $\tilde{X}$ are known at the point $(0,0,\Omega,\bar{\Lambda}^{n+1};0)$. We denote derivatives evaluted at those points by $\cdot^{n+1}$. We start with the derivatives w.r.t the states

We can exploit our knowledge of $\tilde{z}(0,0,0,z,\Omega,\Lambda;0) = z$ and $\tilde{\Omega}(0,0,\Omega,\Lambda;0) = \Omega$ and hence $\bar{z}_\Lambda = 0$ and $\bar{\Omega}_\Lambda = 0$. Thus,

$$\bar{F}^n_{x-}(z)\bar{x}^n_\Lambda(z) + \bar{F}^n_x(z)\bar{x}^n_\Lambda(z) + \bar{F}^n_{x+}(z)\left(\bar{x}^{n+1}_\Lambda(z)\mathcal{P}\bar{X}^n_\Lambda\right) + \bar{F}^n_X(z)\bar{X}^n_\Lambda = 0$$

and

$$\int\bar{R}^n_x(z)\bar{x}^n_\Lambda(z) + \bar{R}^n_X(z)\bar{X}^n_\Lambda + \bar{R}^n_{X+}(z)\bar{X}^{n+1}_\Lambda\mathcal{P}\bar{X}^n_\Lambda + \bar{R}^n_\Lambda(z)d\Omega = 0.$$

Next note that

$$\bar{x}^n_\Lambda(z) = -\left(\bar{F}^n_{x-}(z) + \bar{F}^n_x(z)\right)^{-1}\left(\bar{F}^n_{x+}(z)\bar{x}^{n+1}_\Lambda(z)\mathcal{P} + \bar{F}^n_X(z)\right)\bar{X}^n_\Lambda$$

then

$$\bar{X}^n_\Lambda = -\left(\int \left[-\bar{R}^n_x(z)\left(\bar{F}^n_{x-}(z) + \bar{F}^n_x(z)\right)^{-1}\left(\bar{F}^n_{x+}(z)\bar{x}^{n+1}_\Lambda(z)\mathcal{P} + \bar{F}^n_X(z)\right) + \bar{R}^n_X(z) + \bar{R}^n_{X+}(z)\bar{X}^{n+1}_\Lambda\mathcal{P}\right] d\Omega\right)^{-1}\left(\int \bar{R}^n_\Lambda\right.$$

The Frechet derivative w.r.t Ω is adjusted as follows:

$$\bar{F}^n_{x-}(z)\partial\bar{x}^n(z) + \bar{F}^n_x(z)\partial\bar{x}^n(z) + \bar{F}^n_{x+}(z)\partial\bar{x}^{n+1}(z) + \bar{F}^n_{x+}(z)\bar{x}^{n+1}_\Lambda(z)\mathcal{P}\partial\bar{X}^n + \bar{F}^n_X(z)\partial\bar{X}^n = 0$$

which gives

$$\partial\bar{x}^n(z) = -\left(\bar{F}^n_{x-}(z) + \bar{F}^n_x(z)\right)^{-1}\left[\left(\bar{F}^n_{x+}(z)\bar{x}^{n+1}_\Lambda(z)\mathcal{P} + \bar{F}^n_X(z)\right)\partial\bar{X}^n + \bar{F}^n_{x+}(z)\partial\bar{x}^{n+1}(z)\right] \equiv \sum_{j=0}^{N-n}\mathsf{C}^n_j(z)\partial\bar{X}^{n+j}.$$

Where for the last expression we our knowledge that $\partial\bar{X}^N = \partial\bar{X}$ and $\partial\bar{x}^N(z) = \partial\bar{x}(z) = \mathsf{C}(z)\partial\bar{X}$ and then $\partial\bar{x}^n(z) = \sum_{j=0}^{N-n}\mathsf{C}^n_j(z)\partial\bar{X}^{n+j}$ is derived from the recursion.

To find $\partial\bar{X}$ we then take the Frechet derivative of R to get

$$0 = \int \bar{R}^n(z)\delta(z)dz + \int\left(\bar{R}^n_x(z)\partial\bar{x}^n(z)\cdot\Delta + \bar{R}^n_{X+}(z)\left((\partial\bar{X}^{n+1}\cdot\boldsymbol{\Delta}) + \bar{X}^{n+1}_\Lambda\mathcal{P}(\partial\bar{X}^n\cdot\boldsymbol{\Delta})\right) + \bar{R}^n_X(z)\partial\bar{X}^n\cdot\Delta\right)\omega(z)dz$$

Substituting for $\partial\bar{x}^n(z) = \partial\bar{X}^n$, we get

$$\begin{aligned}\partial\bar{X}^n\cdot\Delta &= -\left(\int\left(\bar{R}^n_x(z)\mathsf{C}^n_0(z) + \bar{R}^n_{X+}(z)\bar{X}^{n+1}_\Lambda\mathcal{P} + \bar{R}^n_X(z)\right)d\Omega\right)^{-1}\int\bar{R}^n(z)\delta(z) + \bar{R}^n_x(z)\partial\bar{x}^n(z)\cdot\Delta dz\\ &\equiv(\mathsf{D}^n)^{-1}\left(\int\bar{R}^n(z)d\Delta + \sum_{j=1}^{N-n}\mathsf{E}^n_j\left(\partial\bar{X}^{n+j}\cdot\Delta\right)\right)\end{aligned}$$

Finally we can compute the response to shock $\mathcal{E}$ (note we only need to do this for $n=0$)

$$\bar{F}^0_x\bar{x}^0_\mathcal{E} + \bar{F}^0_{x+}\left(\rho_\Theta\bar{x}^1_\Theta + \bar{x}^1_z\mathsf{p}\bar{x}^0_\mathcal{E} + \partial\bar{x}^1\cdot\bar{\Omega}^0_\mathcal{E} + \bar{x}^1_\Lambda\mathcal{P}\bar{X}^0_\mathcal{E}\right) + \bar{F}^0_X\bar{X}^0_\mathcal{E} + \bar{F}^0_\mathcal{E} = 0.$$

Substituting for $\partial\bar{x}^1$ we obtain, after defining $\bar{X}^j{}_\mathcal{E} = \partial\bar{X}^j\cdot\bar{\Omega}^0_\mathcal{E}$

$$\bar{F}^0_x(z)\bar{x}_\mathcal{E}(z) + \bar{F}^0_{x+}(z)\left(\rho_\Theta\bar{x}^1_\Theta(z) + \bar{x}^1_z(z)\mathsf{p}\bar{x}^n_\mathcal{E}(z) + \sum_{j=0}^{N-1}\mathsf{C}^1_j(z)\bar{X}^{j+1}_\mathcal{E} + \bar{x}^1_\Lambda(z)\mathcal{P}\bar{X}^0_\mathcal{E}\right) + \bar{F}^0_X(z)\bar{X}^0_\mathcal{E} + \bar{F}^0_\mathcal{E}(z) = 0,$$

or

$$\mathsf{M}^0(z)\bar{x}^0_{\mathcal{E}}(z) = \mathsf{N}^0(z)\left[\begin{array}{ccccc} I & \bar{X}^0_{\mathcal{E}} & \bar{X}^1_{\mathcal{E}} & \cdots & \bar{X}^N_{\mathcal{E}} \end{array}\right]^{\mathsf{T}}.$$

We obtain the following expression for $\bar{X}^j_{\mathcal{E}}$ for $j = 1, \ldots, N$

$$\bar{X}^j_{\mathcal{E}} = \left(\mathsf{D}^j\right)^{-1}\left(\int \left(\bar{R}^j_z(z) + \bar{R}^j_x(z)\bar{x}^j_z(z)\right)\mathsf{q}\bar{x}^0_{\mathcal{E}}(z)d\Omega + \sum_{k=1}^{N-j}\mathsf{E}^j_k\bar{X}^{j+k}_{\mathcal{E}}\right)$$

which, when combined with

$$\int \bar{R}^0_x(z)\bar{x}^0_{\mathcal{E}}(z) + \bar{R}^0_X(z)\bar{X}^0_{\mathcal{E}} + \bar{R}^0_{X+}(z)\left(\rho_\Theta\bar{X}^1_\Theta + \bar{X}^1_{\mathcal{E}} + \bar{X}^1_\Lambda\mathcal{P}\bar{X}^0_{\mathcal{E}}\right) + \bar{R}^0_{\mathcal{E}}(z)d\Omega = 0,$$

yields the linear system

$$\mathsf{O}\cdot\left[\begin{array}{ccccc} I & \bar{X}^0_{\mathcal{E}} & \bar{X}^1_{\mathcal{E}} & \cdots & \bar{X}^N_{\mathcal{E}} \end{array}\right]^{\mathsf{T}} = \mathsf{P}.$$

C Simulation and Choice of K

To simulate an optimal policy two tasks, must be accomplished: we must track the evolution of the distribution of individual state variables and discretize it using K points. In order to track the evolution of the distribution we approximate a continuum of agents with a large number $N = 100,000$ of agents, each of whom receives his/her own idiosyncratic shock. In principle, each period it would be possible to approximate the policy rules around this discretized group of agents, but here we can economize on calculations by following a two step procedure. First, we approximate the $N = 100,000$ group of agents with a smaller group of $K = 10,000$ points using a k-means algorithm. Next we approximate policy rules around the Z constructed with the k-means algorithm and simulate 1 period of the economy with the N agents using the procedure presented in section 3. Additionally, the derivatives provided in section B with respect to the state variables allow us partially to correct the errors that the k-means approximation introduces by adding additional terms to the Taylor expansion. The advantage of this approach is that it can reduce the computational time by what turns out to be a factor of 10 in this instance. We choose K so that increasing K does not change the impulse responses reported in section 4.

D Robustness

In this section, we show robustness of our main results.

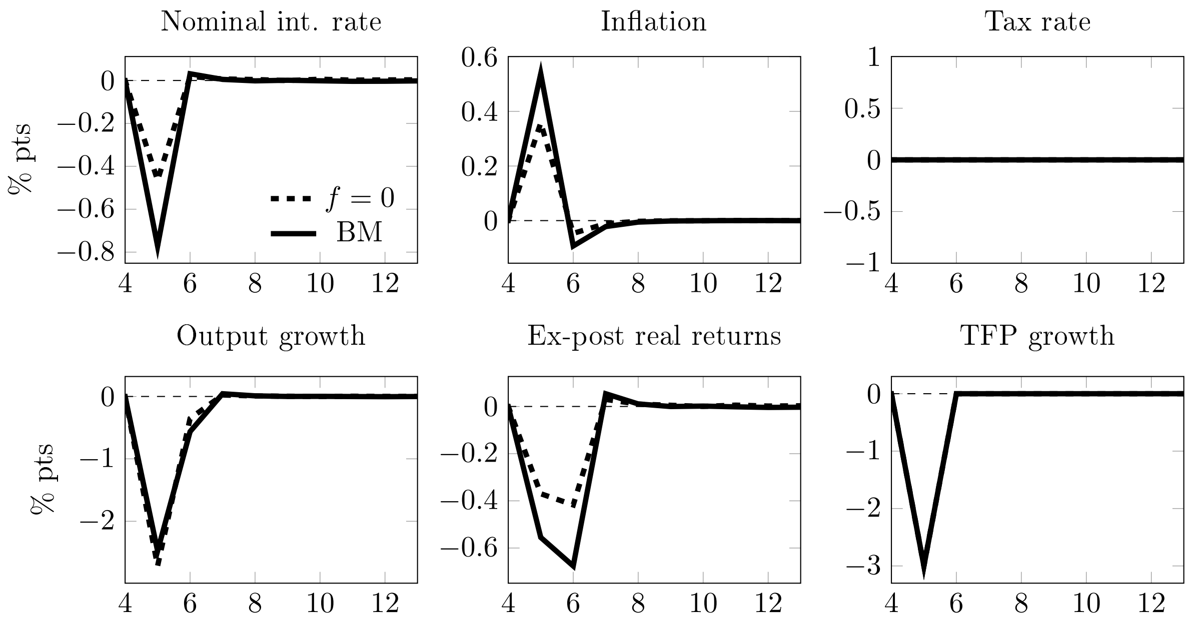

Figure IX: Optimal monetary-fiscal response to a productivity shock with $f = 0$

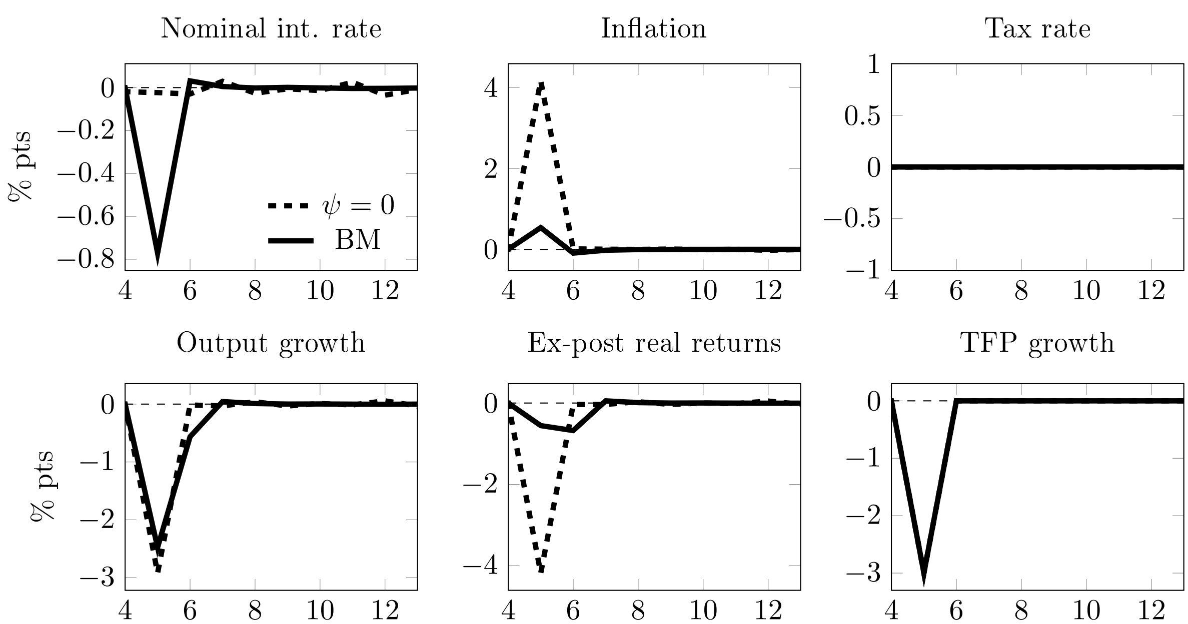

Figure X: Optimal monetary-fiscal response to a productivity shock with $\psi = 0$