Capital Misallocation and Risk Sharing

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ABSTRACT

We develop an optimal contracting model in which limited enforcement of financial contracts generates dispersion in marginal products of capital across firms. We show that the optimal contract can be implemented using state-contingent transfers and a simple collateral constraint that limits the capital input of firms by a fraction of the financial wealth of the firm owner. Compared to models with exogenous collateral constraint and incomplete markets (for example Moll (2014)), we find that the degree of measured misallocation is *increasing* in the persistence of the idiosyncratic productivity shocks. Under the optimal contract, the possibility to transfer wealth from high productivity states to low productivity states allows firm owners to trade off efficient allocation of consumption against the efficient allocation of capital. We show that for reasonable values of risk aversion, insurance needs more than offset production efficiency concerns.

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1 Introduction

A vast empirical literature (see Hsieh and Klenow (2009) and several others) documents dispersion in marginal products of capital across firms for several countries. These patterns are commonly interpreted as evidence of capital misallocation and responsible for cross-country total factor productivity gaps. To account for the empirically observed capital misallocation, several authors have proposed models of financial frictions where entrepreneurs' borrowing is limited by their wealth. However, existing models typically do not provide an explicit micro-foundation for the financial constraints. In addition, in applied work, these models have had limited success in accounting for the large observed dispersion in marginal products. In this paper, we develop an equilibrium model of investment where financial constraints are derived from agency frictions. We demonstrate that the optimal contract in our setting can be implemented using state-contingent transfers and a collateral constraint similar to the one used in the previous literature. Compared to models with exogenous constraints, we show that our optimal contracting framework amplifies the degree of capital misallocation.

Our environment consists of well-diversified intermediaries who offer long-term contracts to entrepreneurs who own productive technologies but not (enough) wealth. Entrepreneurs are risk averse and subject to idiosyncratic productivity shocks. The optimal contract balances insurance against low productivity states as well as funding investment for high productivity states. The agency friction we consider is limited enforcement of financial contracts – entrepreneurs have an option to renege the current contract, abscond with a fraction of the capital stock, and anonymously enter into a new contract with a financial intermediary.

We obtain two main results in this environment. First, we characterize the optimal lending contract and provide an implementation result. We show that the equilibrium allocation with optimal contracts subject to limited enforcement can be implemented using Arrow securities and a collateral constraint. The constraint is linear in the financial wealth of the entrepreneur, and its tightness (or the slope parameter) is independent of idiosyncratic histories. Our implementation thus mimics the exogenous collateral constraints widely used in applied work but allows entrepreneurs to transfer wealth across states.

Second, we demonstrate that under the optimal contract, measured capital misallocation is increasing in the persistence of idiosyncratic productivity shocks under moderate levels of risk aversion. This result is in sharp contrast with that obtained in models with exogenous collateral constraints, where self-financing undoes capital misallocation in the presence of persistent productivity shocks.

In a seminal work, Moll (2014) shows that in an economy where firm owners' borrowing capacity is determined by their financial wealth and they can borrow and save exclusively using a risk-free bond, the degree of misallocation is decreasing in the persistence of idiosyncratic shocks. In the presence of collateral constraints, capital misallocation occurs if owners of high productivity firms do not accumulate enough wealth to finance the efficient level of capital. When productivity shocks are persistent, owners of high productivity firms typically have experienced a long sequence high productivity, and therefore would have saved out of their financial constraint.

Our result implies that the above intuition depends crucially on the assumption of exogenously incomplete market, that is, the risk-free bond is the only financial asset and entrepreneurs cannot allocate wealth across different productivity states. In our setup, the only friction is limited enforcement; markets are otherwise complete. The optimal contract allows firm owners to trade off the allocation of wealth to insure against adverse income states versus the allocation of wealth to maximize productive efficiency. On the one hand, insurance, i.e., consumption efficiency, implies that entrepreneurs need to borrow from states with high productivity and transfer wealth to states with low productivity. On the other hand, production efficiency requires more wealth in high-productivity states to back the financing of a larger amount of capital and less wealth in low productivity states. These two distinct motives pull in opposite directions. For a given level of risk aversion, as shocks become more persistent, the insurance motives are stronger. Entrepreneurs choose to enter productive states with low levels of wealth; sacrificing productive efficiency in order to attain better consumption insurance. This makes misallocation higher.

Our paper contributes to the literature on optimal contracting and capital misallocation. The optimal contract setup with limited enforcement builds on the classical contributions of Kehoe and Levine (1993), Alvarez and Jermann (2000) and Albuquerque and Hopenhayn (2004). Kehoe and Levine (1993) and Alvarez and Jermann (2000) consider risk sharing problems in endowment economies without production decisions. Albuquerque and Hopenhayn (2004) study optimal lending contracts where entrepreneurs are risk-neutral and focus on production decisions in the presence of limited enforcement. Recently, Rampini and Viswanathan (2010, 2013) study the implications of limited contract enforcement for risk management and capital structure also in the context of risk-neutral agents. There is a parallel literature that studies the impact of limited commitment on consumption risk sharing, for example, Krueger and Perry (2006), Krueger and Perry (2004), and Krueger and Uhlig (2006). In contrast, our setup features risk averse agents in a production economy and we emphasize the trade-off between risk sharing and production efficiency. In addition, our implementation result is novel and provides micro-foundation for the widely used wealth-based collateral constraints used in the literature.

Our paper is related to the literature on capital misallocation. For instance, Banerjee and Moll (2010); Buera et al. (2011); Buera and Shin (2011, 2013); Buera et al. (2015); Midrigan and Xu (2014); Moll (2014). All these papers use a risk-free bond and exogenously-specified collateral constraints. Our paper uses an optimal contracting framework to show that the steady-state level of capital misallocation in the aforementioned papers is to a large extent driven by the assumption of risk-free bond.

On the methodological side, our paper is related to the literature of continuous-time dynamic contracting, especially those focus on limited commitment. Using the continuous-time methodology, Grochulski and Zhang (2011) solve an optimal risk sharing problem with limited commitment in an endowment economy. Ai and Li (2015) study the impact of limited commitment on CEO compensation and investment. Bolton et al. (2019) analyze the implications on limited commitment on corporate liquidity and risk management.

The paper is organized as follows. Section 2 describes the environment and the setup of the model. In Section 3, we present our decentralization result and show that any equilibrium with long-term contracts can be implmented using Arrow securities and a linear collateral constraint. In Section 4, we prove our main result on the relationship between persistence of idiosyncratic risk and capital misallocation. Section 6 concludes. The proofs and other details omitted from the main text are relegated to the Appendix.

2 Model Setup

Preferences and Technology Time is continuous. The economy consists of a unit mass of entrepreneurs and a unit mass of workers. Entrepreneurs are endowed with an idiosyncratic productivity process θ and a technology that combines capital and labor to produce output:

$$y = f(\theta, K, L) = (\theta K)^{\alpha} L^{1-\alpha}, \tag{1}$$

where productivity θ follows a Levy process with

$$d\theta_t = \mu(\theta_t)dt + \sigma(\theta_t)dM_t,$$

where M_t is a (vector) martingale. In this section and the next, we assume M_t is a standard Brownian motion to simplify notation. We will allow M_t to contain jumps in general in Section 4. We assume that $\mu(\theta)$ and $\sigma(\theta)$ satisfy appropriate conditions so that θ_t has a unique stationary distribution with a compact support.

Entrepreneurs have expected utility with constant relative risk averion. That, is, given a consumption sequence $\{C_t\}_{t=0}^{\infty}$, the continuation utlity at time t, denoted U_t is calculated as

$$U_t = \left\{ E_t \left[\int_0^{\infty} e^{-\beta s} \beta \frac{C_{t+s}^{1-\gamma}}{1-\gamma} ds \right] \right\}^{\frac{1}{1-\gamma}}. \tag{2}$$

Workers are endowed with one unit of labor. They are hand-to-mouth, in the sense that worker consume current-period labor income without maximizing any intertemporal utility. This assumption is made for simplicity and for the convenience of comparison to the existing literature which often makes the same simplifying assumption.

Contracts For a prevailing wage w, an entrepreneur with productivity θ and capital stock K generates an income

$$\Pi(\theta, K; w) = \max_L \left\{ (\theta K)^{\alpha} L^{1-\alpha} - wL \right\}. \tag{3}$$

There is a perfectly competitive financial intermediary sector that offers long-term lending contracts to entrepreneurs. A contract specifies the sequence of consumption for the entrepreneur, C_t and the process of cumulative investment I_t such that given I_t, the process of capital is determined by

$$dK_t = dI_t - \delta K_t dt,$$

where dI_t is the amount of investment made at time t, and δ is the rate of depreciation. Given a contract $\{C_t, I_t, K_t\}_{t=0}^{\infty}$, the entrepreneur's uitlity is computed using (2). For an interest rate r, the financial intermediary's profit is given by

$$E\left[\int_0^{\infty} e^{-rt} \left\{ (\Pi(\theta_t, K_t; w) - C_t)\, dt - dI_t \right\} \right].$$

At any time t, an entrepreneur can default on the contract, abscond with a fraction λ of the capital stock under operation, and anonymously enter into another contract with a financial intermediary. Let $\bar{U}(\theta_t, K_t)$ be the maximum utility that an entrepreneur with productivity θ and capital stock K can achieve on the anonymous market. Incentive compatibility implies that in order for the entrepreneur not to have an incentive to default, the continuation utility promised to entrepreneurs satisfies

$$U_t \geq \bar{U}(\theta_t, \lambda K_t) \tag{4}$$

for all t.

We define $V(\theta, K, U)$ to be the maximum value a financial intermediary can achieve by offering an incentive compatible contract with an initial promised utility U to an entrepreneur with productivity θ and capital stock K. Because financial intermediaries are perfectly competitive, the maximum utility an entrepreneur can achieve on the competitive financing market is

$$\bar{U}(\theta, K) = \max\{U : V(\theta, K, U) \geq 0\}. \tag{5}$$

As is standard in the dynamic contracting literature, we use promised utility as a state variable in the design of the optimal contract. Without loss of generality, the optimal contract can be specified in two steps. First, we specify consumption and investment as functions of the state variables: $C(\theta, K, U)$, $I(\theta, K, U)$. Second, we specify the law of motion of continuation utilities as

$$dU = \left[\frac{\beta}{1-\gamma}\left(U - C^{1-\gamma}U^{\gamma}\right) + \frac{1}{2}\frac{G(\theta, U, K)^2}{U}\right]dt + G(\theta, U, K)\,dM_t,$$

where $G(\theta, U, K)$ is the policy function that specifies the sensitivity of continuation utility with respect to the Brownian motion dM_t.

Equilibrium with limited enforcement Formally, an stationary competitive equilibrium with limited enforcement consists of i) an interest rate r and a wage rate w; ii) individual policy functions $\{C(\theta, K, U), I(\theta, K, U), L(\theta, K), G(\theta, K, U)\}$; iii) outside option $\bar{U}(\theta, K)$; and iv) the stationary distribution $\Phi(\theta, K, U)$, such that:

1. Given the equilibrium wage w, the outside option $\bar{U}(\theta, K)$, and the equilibrium interest rate r, the policy functions $\{C(\theta, K, U), I(\theta, K, U), L(\theta, K), G(\theta, K, U)\}$ solves the optimal contracting problem subject to the limited enforcement constraint (4).

2. The equilibrium outside options are determined by the perfect competition condition (5).

3. The goods market, the labor market, and the capital market clear:

$$\begin{aligned}\int\left[C\left(\theta,K,U\right)+\delta K+wL\left(\theta,K,U\right)\right]d\Phi\left(\theta,K,U\right) &= \int\left(\theta K\right)^{\alpha}L\left(\theta,K\right)^{1-\alpha}d\Phi\left(\theta,K,U\right);\\ \int L\left(\theta,K,U\right) &= 1;\\ \int\left[dI\left(\theta,K,U\right)-\delta K\right]d\Phi\left(\theta,K,U\right) &= 0.\end{aligned}$$

   In the first equation, $C\left(\theta,K,U\right)$ is the consumption of the entrepreneur, δK is investment, which equals depreciation in steady state, and $wL\left(\theta,K,U\right)$ is the consumption of the hand-to-mouth workers. We normalize the total supply of labor of the economy to 1 and the third equation requires investment equals depeciation for the steady state.

4. The stationary distribution $\Phi\left(\theta,K,U\right)$ is consistent the law of motion of state variables implied by the optimal contract.

3 Decentralization

In this section, we decentralize the equilibrium with optimal contracts using Arrow securities and collateral constraints.

Let A_t denote the financial asset of an entrepreneur at time t, the utility maximization problem of an entrepreneur can be written as:

$$\begin{gathered}\max_{C_t,\,g_t,K_t} E\left[\int_0^{\infty}e^{-\beta t}\beta\frac{C_t^{1-\gamma}}{1-\gamma}dt\right]\\ \text{s.t. } dA_t=\left[rA_t+\Pi\left(\theta_t,K_t;w_t\right)-C_t\right]dt+g_tA_tdM_t,\\ K_t\leq\lambda A_t.\end{gathered}\tag{6}$$

In the above formulation, entrepreneurs make consumption (C_t) and production (K_t) decisions subject to the collateral constraint, $K_t\leq\lambda A_t$. We allow the entrepreneur to save at the risk-free interest r and purchase a vector of Arrow securities denoted by process g_t. Note that the above problem is almost identical to the one typically modeled in the literature of capital misallocation with collateral constraint, for example, Moll (2014), with only one difference. Here, we allow the entrepreneur to purchase a vector of Arrow securities whereas the model of Moll (2014) assumes exogenous incomplete markets, which corresponds to the restriction of $g_t = 0$ in our setup.

We now define a stationary competitive equilibrium with Arrow securities and collateral constraints.

Equilibrium with Arrow securities and collateral constraints For a given constant risk-free interest r, the above utility maximization problem can be formulated recursively using (θ, A) as the state variables. We denote $\{\mathtt{C}(\theta, A), \mathtt{K}(\theta, A), \mathtt{g}(\theta, A)\}$ as the policy functions of the entrepreneur's maximization problem. A stationary equilibrium with Arrow securities and collateral constraints consists of i) prices, which include interest rate r and wage rate w; ii) policy functions $\{\mathtt{C}(\theta, A), \mathtt{L}(\theta, A), \mathtt{K}(\theta, A), \mathtt{g}(\theta, A)\}$; and iii) the stationary distribution of types $\Psi(\theta, A)$, such that:

1. Given the equilibrium wage w and the equilibrium interest rate r, the policy functions $\{\mathtt{C}(\theta, A,), \mathtt{L}(\theta, A), \mathtt{K}(\theta, A), \mathtt{g}(\theta, A)\}$ solve the optimal consumption and production problem in (6).

2. Consumption, labor and capital markets clear.

3. The stationary distribution $\Psi(\theta, A)$ is consistent the law of motion of wealth implied by the optimal policies.

The main result in this section is that any equilibrium with limited enforcement and long-term contracts can be implemented by an equilibrium with Arrow securities and collateral constraints. To state our main result, we start with a stationary equilibrium with limited enforcement, which we denote as $\mathcal{E}$. We show that for any $\mathcal{E}$, there exists a competitive equilibrium with Arrow securities and collateral constraints, denoted $\mathtt{E}$, such that prices and allocations in $\mathcal{E}$ can be constructed from $\mathtt{E}$.

Our construction relies on a mapping between promised utility and wealth. We denote this mapping as $U = U(A|\,\theta)$ and the inverse of the mapping as $A = A(U|\,\theta) \equiv U^{-1}(U|\,\theta)$. That is, for every θ, $U(\cdot|\,\theta)$ is a mapping from wealth to promised utility, and $A(\cdot|\,\theta)$ is the inverse of the mapping. A equilibrium with long-term contracts $\mathcal{E}$ is a collection of prices $\{r, w\}$, policy functions $\{C(\theta, K, U), I(\theta, K, U), L(\theta, K), G(\theta, K, U)\}$, entrepreneurs' outside option, $\bar{U}(\theta, K)$, and distribution $\Phi(\theta, K, U)$. Given E, we can construct $\mathtt{E}$ by letting

$$\begin{aligned}
\mathtt{K}(\theta, A) &= \lambda A \text{ if } \theta = \theta_H, \, 0 \text{ otherwise}, & (7)\\
\mathtt{L}(\theta, A) &= L(\theta, \mathtt{K}(\theta, A)), & (8)\\
\mathtt{C}(\theta, A) &= C(\theta, \mathtt{K}(\theta, A), U(\theta, A)), & (9)\\
\mathtt{g}(\theta, A) &= G(\theta, \mathtt{K}(\theta, A), U(\theta, A)), & (10)
\end{aligned}$$

and we show that $\{r, w\}$; $\{\mathtt{C}(\theta, A,), \mathtt{L}(\theta, A), \mathtt{K}(\theta, A), \mathtt{g}(\theta, A)\}$; $\Psi(\theta, K, U)$ is a stationary equilibrium for some . We complete our contruction by demontrating that $\mathcal{E}$ can be recovered from $\mathtt{E}$ using the following procedure:

$$\begin{aligned}
C(\theta, K, U) &= \mathtt{C}(\theta, A(U|\theta)), & (11)\\
dI(\theta, K, U) &= \mathtt{K}(\theta, A(U|\theta)) - K, & (12)\\
L(\theta, K, U) &= \mathtt{L}(\theta, A(U|\theta)), & (13)\\
G(\theta, K, U) &= \mathtt{g}(\theta, A(U|\theta)). & (14)
\end{aligned}$$

We summarize our result by the following proposition.

Theorem 1. *Given a stationary equilibrium with long-term contracts,*

$$\mathcal{E} = \left\{\{r, w\}, \{C(\theta, K, U), I(\theta, K, U), L(\theta, K), G(\theta, K, U)\}, \bar{U}(\theta, K), \Phi(\theta, K, U)\right\},$$

there exists a one-to-one mapping $U = U(A|\theta)$ such that

$$\mathtt{E} = \{\{r, w\}, \{\mathtt{C}(\theta, A,), \mathtt{L}(\theta, A), \mathtt{K}(\theta, A), \mathtt{g}(\theta, A)\}, \Psi(\theta, A)\}$$

is a competitive equilibrium with Arrow securities and collateral constraints, where the policy fucntions $\{\mathtt{C}(\theta, A,), \mathtt{L}(\theta, A), \mathtt{K}(\theta, A), \mathtt{g}(\theta, A)\}$ are given by (7)-(10). Furthermore, $\mathcal{E}$ can be recovered from $\mathtt{E}$ using (11)-(14).

Given the above proposition, we will focus on the competitive equilibrium with Arrow securities and collateral constraints for the rest of the paper. For a given equilibrium, we can define the efficient level of output as

$$\begin{aligned}
Y^* &= \max \int (\theta_i K_i)^\alpha L_i^{1-\alpha} di\\
\textit{subject to} &: \int K_i di \leq \int K d\Phi(\theta, K, U); \quad \int L_i di \leq 1.
\end{aligned}$$

The efficiency ratio is defined as

$$EF = \frac{\int (\theta K(\theta, A))^{\alpha} L(\theta, A)^{1-\alpha} d\Psi(\theta, A)}{Y^*}.$$

4 Misallocation and persistence

Our main interest is to characterize equililbriums with collateral constraints and investigate the relationship between EF and the persistence of the productivity shock θ_t. To derivation closed form solutions, in this section, we assume that θ_t follows a two state Markov chain with state space $\{\theta_H, \theta_L\}$ and with an instantaneous switching rate of κ. Formally, the law of motion of θ can be described as

$$d\theta_t = (\theta_H - \theta_L)\left[-I_H(\theta_t) dN_{H,t} + I_L(\theta_t) dN_{L,t}\right]. \tag{15}$$

where $I_\theta\left(\tilde{\theta}_t\right)$ is the indicator function that takes value of 1 when $\tilde{\theta}_t = \theta$, and the processes $\{dN_{\theta,t}\}_\theta$ are independent Poisson processes with a common intensity κ. Given these assumptions, $1-\kappa$ is a measure of one-dimensional measure of the persistence in θ. Furthermore, changing κ keeps the unconditional distribution of θ_t unchanged. This will be useful when we do comparative statics with persistence.

Given sequences of prices $\{r_t, w_t\}$, the entrepreneur whether or not to operate the technology, factor choices capital and labor, a (non-negative) consumption process , and a portfolio of Arrow securities $\{g_{HL,t}W_t,\ g_{LH,t}W_t\}$. The process $g_{HL,t}$ represents the fraction of wealth W_t that is carried from state $\theta_t = \theta_H$ to state $\theta_t = \theta_L$, and $g_{LH,t}$ is the fraction of wealth that is carried from state $\theta_t = \theta_L$ to state $\theta_t = \theta_H$. The entrepreneurs value

$$\begin{aligned} V_t(\theta, W) &= \max_{C_t, g_{H,t}, g_{L,t}} E_t\left[\int_t^\infty e^{-\beta(s-t)} u(C_s)\, ds\right] \\ \text{s.t. } dW_t &= \left[r_t W_t + \Pi(\theta, W; w_t, r_t) - C_t\right] dt + I_H(\theta_t) W_t g_{HL}\left[dN_{H,t} - \kappa dt\right] \\ &\quad + I_L(\theta_t) W_t g_{LH}\left[dN_{L,t} - \kappa dt\right]. \end{aligned}$$

The drift ("dt") term budget constraint is the change in wealth coming from risk-free financial return, business profits net of entrepreneurial consumption and the state-contingent returns arise from the terms multiplying the martingale $[dN_{\theta,t} - \kappa]$. The next lemma summarizes the optimal consumption savings plan in a stationary equilibrium.

Lemma 1. *The value function* $V(\theta, W)$ *and the profit function* $\Pi(\theta, W; w, r)$ *satisfy*

$$V(\theta, W) = \frac{1}{1-\gamma} H(\theta) W^{1-\gamma}$$

$$\Pi(\theta, W; w, r) = W \pi(\theta; r, w)$$

with

$$H(\theta) = \left\{ \frac{1}{\gamma} \left[\beta + (\gamma - 1)(r + \pi_\theta(r, w)) + \gamma \kappa (1 - \omega) \right] \right\}^{-\gamma}$$

$$\pi_\theta(r, w) = \left[\alpha \left(\frac{1-\alpha}{w} \right)^{\frac{1-\alpha}{\alpha}} \theta - r - \delta \right]^{+} \lambda$$

and the optimal consumption and portfolio rules are given by

$$\frac{C(\theta, W)}{W} = x(\theta) = H(\theta)^{-\frac{1}{\gamma}}, \quad g_{HL} = \omega - 1, \quad g_{LH} = \omega^{-1} - 1$$

with $\omega = \left[\frac{H(\theta_H)}{H(\theta_L)} \right]^{-\frac{1}{\gamma}}$.

Proof. See Appendix $\square$

The derivation of the optimal policies comes from a standard Merton-like problem. The explicit solutions arise thanks to the homotheticity properties of preferences and technology. Since $\pi_H > \pi_L$, consumption smoothing across states is achieved by committing more wealth to states with low business income, i.e., set $g_{HL} > g_{LH}$.

Let us suppose for now that both types produce in the stationary equilibrium. The next lemma shows that a sufficient (distributional) statistic for the degree of misallocation is the ratio of aggregate wealth of entrepreneurs with type θ_H to entrepreneurs with type θ_L.

Lemma 2. *Suppose a stationary equilibrium exists with* $\pi_L(r, w) = 0$. *The degree of misallocation is given by*

$$1 - EF = 1 - \left[\frac{\theta_L (\eta(r, w) + 1 - \lambda \eta(r, w)) + \theta_H \lambda \eta(r, w)}{\theta_H (\eta(r, w) + 1)} \right]^{\alpha} \tag{16}$$

where $\eta(r, w) \equiv \frac{\int W d\Phi(W|\theta_H; r, w)}{\int W \varphi d\Phi(W|\theta_L; r, w)}$.

Proof. For an entrepreneur who is operating with non-zero capital, factor demands are linear in wealth and given by

$$K(W,\theta) = \lambda W$$

$$L(W,\theta) = \left(\frac{1-\alpha}{w}\right)^{\frac{1}{\alpha}} \theta\lambda W$$

Let φ be the fraction of θ_L entrepreneurs who are active. Substituting the factor demands, we get

$$\begin{aligned}
EF &= \frac{\int (\theta_i K_i)^\alpha L_i^{1-\alpha} di}{\theta_H^\alpha \left(\int K_i di\right)^\alpha \left(\int L_i di\right)^{1-\alpha}} \\
&= \frac{\int \left(\frac{1-\alpha}{w}\right)^{\frac{1-\alpha}{\alpha}} \lambda\theta_L W_i I_{\{\theta_{i,t}=\theta_L\}} \varphi di + \int \left(\frac{1-\alpha}{w}\right)^{\frac{1-\alpha}{\alpha}} \lambda\theta_H W_i I_{\{\theta_{i,t}=\theta_H\}} di}{\theta_H^\alpha \left[\int \lambda W_i I_{\{\theta_{i,t}=\theta_L\}} \varphi di + \int \lambda W_i I_{\{\theta_{i,t}=\theta_H\}} di\right]^\alpha \cdot 1} \\
&= \frac{\left(\frac{1-\alpha}{w}\right)^{\frac{1-\alpha}{\alpha}} \lambda \left(\theta_L W_L \varphi + \theta_H W_H\right)}{\theta_H^\alpha \lambda^\alpha \left(W_L\varphi + W_H\right)^\alpha} = \left(\frac{1-\alpha}{w}\right)^{\frac{1-\alpha}{\alpha}} \lambda^{1-\alpha} \frac{\theta_L W_L \varphi + \theta_H W_H}{\theta_H^\alpha \left(W_L\varphi + W_H\right)^\alpha} \\
&= \left(\frac{1-\alpha}{w}\right)^{\frac{1-\alpha}{\alpha}} \lambda^{1-\alpha} W_L^{1-\alpha} \frac{\theta_L \varphi + \theta_H \eta}{\theta_H^\alpha \left(\varphi + \eta\right)^\alpha} \\
&= \left[\frac{\theta_L \left(\eta + 1 - \lambda\eta\right) + \theta_H \lambda\eta}{\theta_H \left(\eta + 1\right)}\right]^\alpha .
\end{aligned}$$

□

In view of Lemma 2, we only need to characterize how the aggregate wealth shares evolve in the stationary equilibrium. Let $W_{H,t} = \int W_i I_{\{\theta_{i,t}=\theta_H\}} di$ and $W_{L,t} = \int W_i I_{\{\theta_{i,t}=\theta_L\}} di$. A heuristic derivation of the dynamics of $dW_{\theta,t}$ is as follows. At time $t+\Delta$, $e^{-\kappa\Delta}$ fraction of entrepreneurs will remain at $\theta_{t+\Delta} = \theta_H$. Conditional on being in state θ_H, that is, conditioning on $dN_{Ht} = 0$ over the interval $(t, t+\Delta)$,

$$dW_{H,t} = W_{H,t}\left[r + \pi(\theta; r, w) - x(\theta_H) - \kappa(\omega - 1)\right] dt$$

from the flow budget constraint. At the same time, $\left(1 - e^{-\kappa_L\Delta}\right)$ fraction experience a regime switch and transition from $\theta_L \to \theta_H$ and when they become θ_H, their wealth $W_{L,t} \to \omega^{-1} W_{L,t}$. Putting both of these together, and taking limits as $\Delta \to 0$, we get

$$dW_{H,t} = \left[r + \pi(\theta_H; r, w) - x(\theta_H) - \kappa\omega\right] W_{H,t} dt + \kappa\omega^{-1} W_{L,t} dt \tag{17}$$

The law of motion of the wealth share of type θ_L is obtained analogously and we have

$$dW_{L,t} = \left[r + \pi_L\left(\theta_L; r, w\right) - x\left(\theta_L\right) - \kappa\omega^{-1}\right] W_{L,t}dt + \kappa\omega W_{H,t}dt \tag{18}$$

Since we study the steady-state equilibrium, stationarity requires

$$\frac{dW_{H,t}}{W_{H,t}} = \frac{dW_{L,t}}{W_{L,t}} = 0. \tag{19}$$

From the capital market clearing condition, $d\mathbf{K}_t = dW_{H,t} + dW_{L,t}$ and therefore we get

$$0 = \left[r + \pi\left(\theta_H; r, w\right) - x\left(\theta_H\right)\right] W_{H,t}dt + \left[r + \pi\left(\theta_L; r, w\right) - x\left(\theta_L\right)\right] W_{L,t}dt$$

which can be simplified to obtain

$$\eta(r, w) = -\frac{r + \pi\left(\theta_L; r, w\right) - x\left(\theta_L\right) - \kappa\omega^{-1}}{\kappa\omega}.$$

In the final part of this section, we show that for sufficiently high risk aversion, misllocation is increasing in persistence. Given our discussion above, this is true if and only if $\frac{\partial\eta}{\partial\kappa} > 0$. To formally prove our claim, we need to make the following technical assumption that restrict the parameter space for the economy:

Assumption 1. $\lambda \in \left(1, \min\left\{2, \frac{2\beta}{\delta\left(\frac{\theta_H}{\theta_L} - 1\right)}\right\}\right)$ *and* $\frac{2\beta}{\delta\left(\frac{\theta_H}{\theta_L} - 1\right)} > 1.$

The next theorem states our main result.

Theorem 2. *Under assumption 1, $\exists\,\bar{\gamma}$ such that for $\forall\gamma > \bar{\gamma}$, $\frac{\partial\eta}{\partial\kappa} > 0$ for all $\kappa > 0$.*

Proof. See Appendix □

The intuition for the result is as follows. Our economy features a trade off between production efficiency and insurance. To maximize production in presence of a collateral constraint, the entrepreneur would want to bring more wealth in the high productivity states, i.e., states with $\theta_t = \theta_H$. This would make the constraint less likely to bind. On the other hand, to smooth consumption across states, the entrepreneur would like to borrow against states of the world with $\theta_t = \theta_H$ and transfer wealth to states with $\theta_t = \theta_L$. Thus insurance concerns force an allocation of wealth across states in a way opposite of what is desired for production efficiency.

For a given level of risk aversion, as shocks become more persistent, they amplify insurance requirements. The entrepreneur would want to bring a larger fraction of wealth to $\theta_t = \theta_L$ to insure against the adverse shock in a present value sense. Since this means a lower wealth in states which $\theta_t = \theta_H$, more entrepreneurs will be constrained and aggregate misallocation will be larger. We term this the "stock effect" of increasing persistence. There is another "flow effect" which has to be acknowledged. More persistent shocks, mean that productive entrepreneurs earn high profits for longer duration and on this account will accumulate more wealth. Theorem 2 asserts that the flow effect is not sufficiently strong for large values of risk aversion. More generally, the degree of misallocation will be an inverse U-shaped function of persistence. For a given value of risk aversion and low levels of persistence, misallocation increases with persistence due to the stock effect but eventually when shocks are very persistent, the flow effect dominates. The persistence that generates the highest degree of misallocation is a function of the level of risk aversion and it moves towards $\kappa = 0$ (or when θ_t approaches a unit root) for a finite but "large" value of γ.

A natural question is how large is "large". We approach using numerical analysis in two ways. Here, we study how the two shock calibrated economy behaves for a range of (κ, γ) and in section 5, we use a Ornstein-Uhlenbeck process with a reflecting barrier as in Moll (2014) and compare allocations in both complete markets and incomplete markets (bond) economy. We set $\beta = \delta = 0.05$. We calibrate $\theta_H = 1.2$, $\theta_L = 0.8$ to hit match the unconditional volatility of productivity to 4%. Then for $\lambda = 1.2$, we plot $1 - EF$ for several values of κ and γ.

In the left panel of Figure 1, we see that for risk aversion of as low as 5, misllocation is increasing for almost all feasible values of $1 - \kappa$. The magnitude of misallocation ranges between 2% – 6%. These results and insights can be contrasted with an economy where only the only asset is a risk-free bond. In our setup, this would mean adding an additional constraint

$$g_{HL,t} = g_{LH,t} = 0.$$

Keeping all the parameters the same, we solve for the stationary equilibrium and measure misllocation using the same way, i.e., $1 - EF$ in the bond economy. In right panel of Figure 1, we plot the misallocation as a function of persistence for several values of γ. The plots show that patterns are quite different. In the bond-economy steady-state misallocation is always decreasing in persistence. This should be expected given the analysis in Moll (2014). In absence of state-contingent returns,

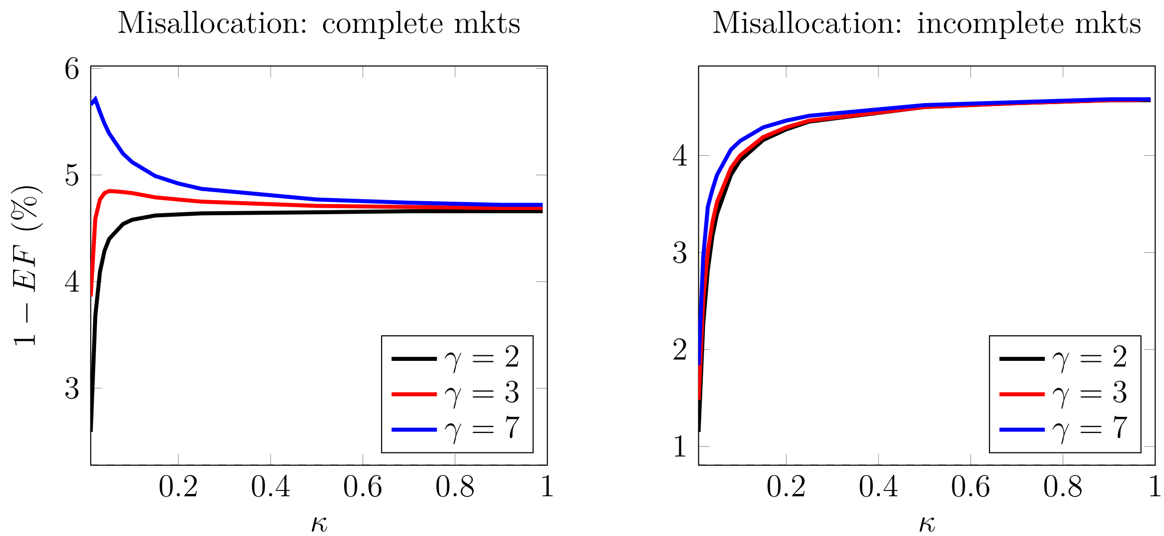

Figure 1: Misallocation for several values of risk aversion, $\gamma \in \{2, 3, 7\}$ in the two-shock CRS economy. The left panel has a complete market structure and the right panel has a single risk-free bond market structure.

the only response to more persistent shocks is to self-insure and save more. But more wealth also relaxes the collateral constraints. Thus the insurance requirements go in the same direction as production efficiency.

5 Calibrated Economy

In the previous section, we used a simplified shock process where θ_t took only two values. This helped us with closed-form characterization of the wealth distribution and all equilibrium quantities. However, the insights are more general. In this section, we study them numerically using a more "standard" process for θ_t. We demonstrate that our main take-away that for reasonable levels of risk aversion, the patterns of misallocation in the complete markets version are diametrically opposite to those in the incomplete markets version.

The productivity θ follows

$$d\theta_t = \mu(\theta_t)\, dt + \sigma(\theta_t)\, dB_t,$$

where

$$\mu(\theta) = 2\kappa\left(\bar{\theta} - \theta\right); \quad \sigma(\theta) = \sqrt{\kappa}\sigma\left(\theta_H - \theta\right)\left(\theta - \theta_L\right).$$

The process has several desirable properties. The support of θ_t is bounded in

Discount factor	β	0.05
Capital share	α	1/3
Depreciation	δ	0.05
Collateral constrain	λ	1.2
std. of idio risk	σ	0.6
Persistence	κ	(0.01,0.95)
Risk aversion	γ	{2,3,5,7,9}

Table 1: Calibration

$[\theta_L, \theta_H]$. The parameter κ fully determines persistence and stationary distribution is independent of κ.

We follow Moll (2014) to calibrate our economy. See Table 1 for the parameter values . In figure 2, we plot the resulting misallocation as a function κ for several values of γ for our economy and the economy with only a risk-free bond (or the market structure assumed in Moll (2014)). Qualitatively we see the same patterns as in Figure 1, confirming our main insights.

6 Conclusion

This paper shows that factor misallocation is closely tied to the risk-sharing avenues available to firm owners. In contrast to the commonly studied bond-only economy with collateral constraints (for example Moll (2014)), we find that keeping fixed the nature of financial frictions, the degree of misallocation is *increasing* in persistence of the idiosyncratic risk when firms have access to state-contingent contracts. Allowing the possibility to transfer wealth from states where productivity is high to states where productivity is low generates a force that works against efficient allocation of capital. We show that for reasonable values of risk aversion, insurance needs more than offset efficiency concerns. A rigorous empirical examination of the extent of explicit and implicit insurance available to entrepreneurs will require us to study consumption patterns of firm-owners. We leave this for future work.

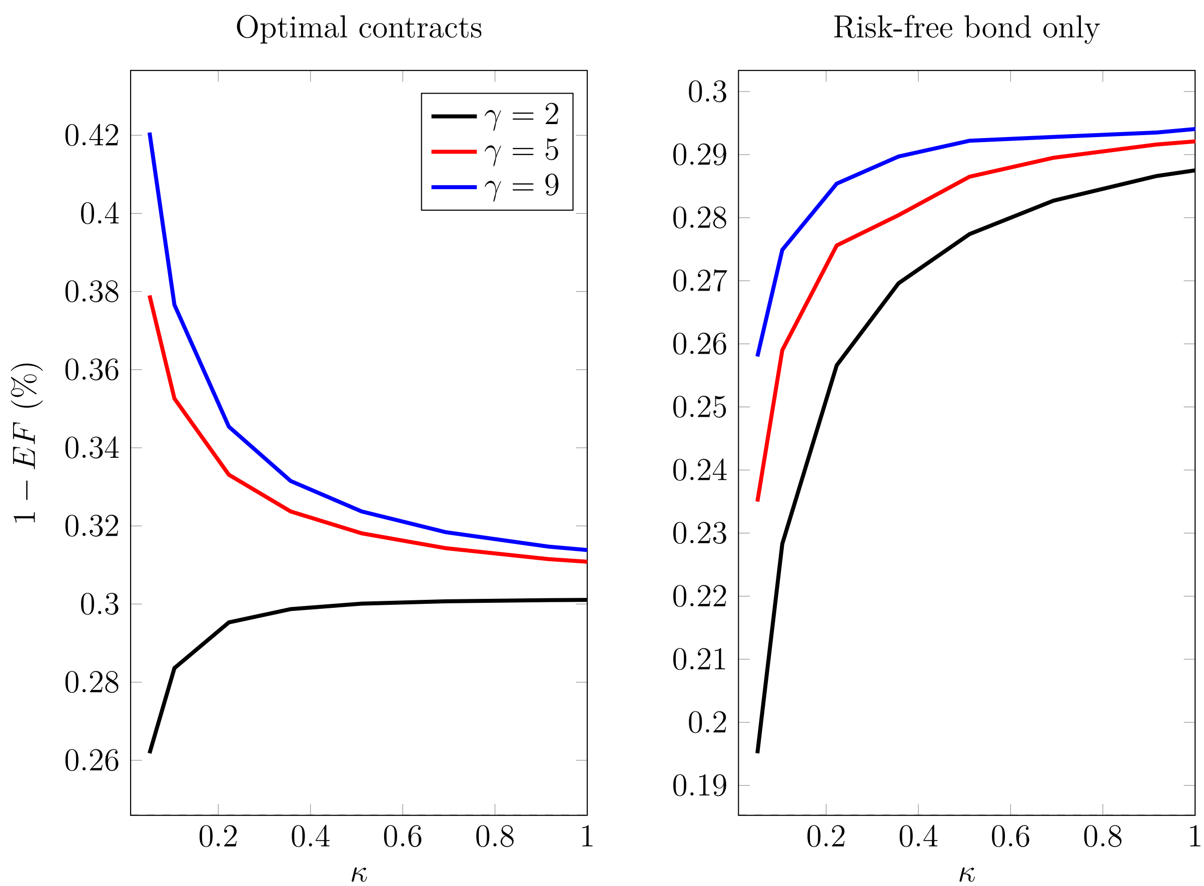

Figure 2: The left panel plots the degree of misallocation in the economy with optimal contracts. The right panel plots the degree of misallocation in the economy with a risk-free bond. The lines blue, red and black are settings with different values of risk aversion.

References

Achdou, Yves, Francisco J Buera, Jean-Michel Lasry, Pierre-Louis Lions, and Benjamin Moll. 2014. "Partial differential equation models in macroeconomics." *Philosophical Transactions of the Royal Society A: Mathematical, Physical and Engineering Sciences*, 372(2028): , p. 20130397.

Achdou, Yves, Jiequn Han, Jean-Michel Lasry, Pierre-Louis Lions, and Benjamin Moll. 2017. "Income and wealth distribution in macroeconomics: A continuous-time approach."Technical report, National Bureau of Economic Research.

Aguiar, Mark, and Mark Bils. 2015. "Has consumption inequality mirrored income inequality?" *American Economic Review*, 105(9): 2725–56.

Ai, Hengjie, Kai Li, and Fang Yang. 2016. "Financial intermediation and capital reallocation."Technical report, Working Paper.

Ai, Hengjie, and Rui Li. 2015. "Investment and CEO Compensation under Limited Commitment." *Journal of Financial Economics.*, 116 452–472.

Albuquerque, Rui, and Hugo Hopenhayn. 2004. "Optimal Lending Contracts and Firm Dynamics." *Review of Economic Studies*, 71 285–315.

Alvarez, Fernando, and Urban J. Jermann. 2000. "Efficiency, Equilibrium, and Asset Pricing with Risk of Default." *Econometrica*, 68(4) 775–797.

Angeletos, George-Marios. 2007. "Uninsured idiosyncratic investment risk and aggregate saving." *Review of Economic dynamics*, 10(1): 1–30.

Asker, John, Allan Collard-Wexler, and Jan De Loecker. 2014. "Dynamic inputs and resource (mis) allocation." *Journal of Political Economy*, 122(5): 1013–1063.

Banerjee, Abhijit V, and Benjamin Moll. 2010. "Why does misallocation persist?" *American Economic Journal: Macroeconomics*, 2(1): 189–206.

Bolton, Patrick, Neng Wang, and Jinqiang Yang. 2019. "Optimal Contracting, Corporate Finance, and Valuation with Inalienable Human Capital." *Journal of Finance*, 74(3): 1363–1429.

Buera, Francisco J, Joseph P Kaboski, and Yongseok Shin. 2011. "Finance and development: A tale of two sectors." *American Economic Review*, 101(5): 1964–2002.

Buera, Francisco J, Joseph P Kaboski, and Yongseok Shin. 2015. "Entrepreneurship and financial frictions: A macrodevelopment perspective." *economics*, 7(1): 409–436.

Buera, Francisco J, and Yongseok Shin. 2011. "Self-insurance vs. self-financing: A welfare analysis of the persistence of shocks." *Journal of Economic Theory*, 146(3): 845–862.

Buera, Francisco J, and Yongseok Shin. 2013. "Financial frictions and the persistence of history: A quantitative exploration." *Journal of Political Economy*, 121(2): 221–272.

David, Joel M, and Venky Venkateswaran. 2017. "The sources of capital misallocation."Technical report, National Bureau of Economic Research.

David, Joel, Lukas Schmid, and David Zeke. 2018. "Risk-adjusted capital allocation and misallocation." *Available at SSRN 3255111*.

Eisfeldt, Andrea L, and Adriano A Rampini. 2006. "Capital reallocation and liquidity." *Journal of monetary Economics*, 53(3): 369–399.

Grochulski, Borys, and Yuzhe Zhang. 2011. "Optimal risk sharing and borrowing constraints in a continuous-time model with limited commitment." *Journal of Economic Theory*, 146 2356–2388.

Guiso, Luigi, Luigi Pistaferri, and Fabiano Schivardi. 2005. "Insurance within the firm." *Journal of Political Economy*, 113(5): 1054–1087.

Haltiwanger, John, Robert Kulick, and Chad Syverson. 2018. "Misallocation measures: The distortion that ate the residual."Technical report, National Bureau of Economic Research.

Hopenhayn, Hugo A. 2014. "On the measure of distortions."Technical report, National Bureau of Economic Research.

Hsieh, Chang-Tai, and Peter J Klenow. 2009. "Misallocation and manufacturing TFP in China and India." *The Quarterly journal of economics*, 124(4): 1403–1448.

Kehoe, Timothy, and David Levine. 1993. “Debt-Constrained Asset Markets.” *Review of Economic Studies*, 60 865–888.

Kiyotaki, Nobuhiro, and John Moore. 1997. “Credit Cycles.” *Journal of Political Economy*, 105 211–284.

Krueger, Dirk, and Fabrizio Perry. 2004. “Understanding Consumption Smoothing: Evidence from the US Consumer Expenditure Data.” *Journal of the European Economic Association*, 3 340–350.

Krueger, Dirk, and Fabrizio Perry. 2006. “Does Income Inequality Lead to Consumption Inequality? Evidence and Theory.” *Review of Economic Studies*, 73) 163–193.

Krueger, Dirk, and Harald Uhlig. 2006. “Competitive Risk Sharing with One-sided Commitment.” *Journal of Monetary Economics*, 53(7) 1661–1691.

Lee, Yoonsoo, and Toshihiko Mukoyama. 2015. “Productivity and employment dynamics of US manufacturing plants.” *Economics Letters*, 136 190–193.

Midrigan, Virgiliu, and Daniel Yi Xu. 2014. “Finance and misallocation: Evidence from plant-level data.” *American economic review*, 104(2): 422–58.

Moll, Benjamin. 2014. “Productivity losses from financial frictions: Can self-financing undo capital misallocation?” *American Economic Review*, 104(10): 3186–3221.

Rampini, Adriano, and S. Viswanathan. 2010. “Collateral, risk management, and the distribution of debt capacity.” *Journal of Finance*, 65 2293–2322.

Rampini, Adriano, and S. Viswanathan. 2013. “Collateral and capital structure.” *Journal of Financial Economics*, 109 466–492.

Restuccia, Diego, and Richard Rogerson. 2008. “Policy distortions and aggregate productivity with heterogeneous establishments.” *Review of Economic dynamics*, 11(4): 707–720.

Restuccia, Diego, and Richard Rogerson. 2013. “Misallocation and productivity.”

A Proof for Lemma 1

With the constant returns to scale technology, given the interest rate r and wage w, entreprenurs' profits are

$$\begin{aligned} \Pi(\theta, W; r, w) &= \max_{K,L} \left\{ (\theta K)^{\alpha} L^{1-\alpha} - wL - (r+\delta) K \right\} \\ 0 &\leq K \leq \lambda W. \end{aligned} \tag{20}$$

where the first order conditions imply that

$$L(W, \theta) = \left(\frac{1-\alpha}{w}\right)^{\frac{1}{\alpha}} \theta K(W, \theta), \quad K(W, \theta) = \lambda W \text{ or } 0.$$

Therefore, the profits are linear in wealth:

$$\Pi(\theta, W; r, w) = W \pi(\theta; r, w)$$

where

$$\pi(\theta; r, w) = \left[\alpha \left(\frac{1-\alpha}{w}\right)^{\frac{1-\alpha}{\alpha}} \theta - r - \delta \right]^{+} \lambda.$$

Due to the homogeneity of value function, we assume

$$V(\theta, W) = \frac{1}{1-\gamma} H(\theta) W^{1-\gamma}$$

By applying Ito's Lemma, the HJB equation can be written as

$$\begin{aligned} \beta \frac{1}{1-\gamma} H(\theta_H) W^{1-\gamma} &= \frac{1}{1-\gamma} {C_H}^{1-\gamma} + H(\theta_H) W^{1-\gamma} \left[r + \pi_H - \frac{C_H}{W} - \kappa g_{HL} \right] \\ &\quad + \kappa \left[\frac{1}{1-\gamma} H(\theta_L) \left((1+g_{HL}) W\right)^{1-\gamma} - \frac{1}{1-\gamma} H(\theta_H) W^{1-\gamma} \right]; \end{aligned}$$

$$\begin{aligned} \beta \frac{1}{1-\gamma} H(\theta_L) W^{1-\gamma} &= \frac{1}{1-\gamma} {C_L}^{1-\gamma} + H(\theta_L) W^{1-\gamma} \left[r + \pi_L - \frac{C_L}{W} - \kappa g_{LH} \right] \\ &\quad + \kappa \left[\frac{1}{1-\gamma} H(\theta_H) \left((1+g_{LH}) W\right)^{1-\gamma} - \frac{1}{1-\gamma} H(\theta_L) W^{1-\gamma} \right]; \end{aligned}$$

Denote $x(\theta)$ as the consumption-wealth ratio

$$x(\theta) = \frac{C(\theta, W)}{W},$$

and the first order condition implies

$$x(\theta) = H(\theta)^{-\frac{1}{\gamma}}; \quad 1 + g_{HL} = \left[\frac{H(\theta_H)}{H(\theta_L)}\right]^{-\frac{1}{\gamma}} \quad 1 + g_{LH} = \left[\frac{H(\theta_L)}{H(\theta_H)}\right]^{-\frac{1}{\gamma}}.$$

Combing the policy functions and HJB equations, we can derive

$$\omega = \frac{\beta + (\gamma - 1)[r + \pi_H] + 2\gamma\kappa}{\beta + (\gamma - 1)[r + \pi_L] + 2\gamma\kappa} \tag{21}$$

where we define $\omega = \left[\frac{H(\theta_H)}{H(\theta_L)}\right]^{-\frac{1}{\gamma}}$.

Given r, the value function and policy function can be constructed as:

$$\begin{aligned} H(\theta_H) &= \left\{\frac{1}{\gamma}[\beta + (\gamma - 1)(r + \pi_H(r)) + \gamma\kappa(1 - \omega)]\right\}^{-\gamma}; \\ H(\theta_L) &= \left\{\frac{1}{\gamma}\left[\beta + (\gamma - 1)(r + \pi_L(r)) + \gamma\kappa\left(1 - \omega^{-1}\right)\right]\right\}^{-\gamma}; \end{aligned}$$

$$x(\theta) = H(\theta)^{-\frac{1}{\gamma}}; \quad g_{HL} = \omega - 1; \quad g_{LH} = \omega^{-1} - 1. \tag{22}$$

where ω is defined in equation (21).

B Proof for Theorem 2

Lemma 3. *Under assumption 1, among the firms which receive θ_L, the fraction of producing belongs to $(0, 1)$ for $\forall\gamma > \bar{\gamma}$, where $\bar{\gamma}$ is defined in the theorem 1.*

We will verify the Lemma 3 later during the proof of Theorem 1. Now assume Lemma 3 holds, then the economy is determined by the following five equations:

$$r = \alpha\left(\frac{1 - \alpha}{w}\right)^{\frac{1-\alpha}{\alpha}} \theta_L - \delta$$

$$\omega = \frac{\beta + (\gamma - 1)[r + \pi_H] + 2\gamma\kappa}{\beta + (\gamma - 1)[r + \pi_L] + 2\gamma\kappa}$$

$$\eta = -\frac{r + \pi(\theta_L; r, w) - x(\theta_L) - \kappa\omega^{-1}}{\kappa\omega}$$

$$\varphi = \frac{\eta + 1 - \lambda\eta}{\lambda}$$

where η is fraction of firms produce when they receive θ_L.

$$x(\theta_H)\eta + x(\theta_L) + \delta(\eta+1) = \alpha\left(\frac{1-\alpha}{w}\right)^{\frac{1-\alpha}{\alpha}}\lambda\left[\theta_H\eta + \theta_L\varphi\right]$$

After the algebra, we can find the economy is determined by the following three equations:

$$\omega = 1 + \frac{(\gamma-1)(r+\delta)\left(\frac{\theta_H}{\theta_L}-1\right)\lambda}{\beta+(\gamma-1)r+2\gamma\kappa} \tag{23}$$

$$\eta = \frac{\beta - r + \gamma\kappa}{\gamma\kappa\omega} \tag{24}$$

$$r = \beta - (r+\delta)\left(\frac{\theta_H}{\theta_L}-1\right)\lambda + \frac{\gamma\kappa(\beta-r)}{\beta-r+\gamma\kappa} \tag{25}$$

Assume the limit of ω, r and η exists and finite. Taking the limit of (23), (24), and (25) with respect to γ,

$$\bar{\omega} = 1 + \frac{(\bar{r}+\delta)\left(\frac{\theta_H}{\theta_L}-1\right)\lambda}{\bar{r}+2\kappa}$$

$$\bar{\eta} = \frac{1}{\bar{\omega}}$$

$$\bar{r} = \frac{2\beta - \delta\left(\frac{\theta_H}{\theta_L}-1\right)\lambda}{2+\left(\frac{\theta_H}{\theta_L}-1\right)\lambda}$$

Later on we are going to characterize the *sufficient* condition when this economy is well defined.

Taking the partial derivative with respect to κ for equations (23), (24), and (25) :

$$\frac{\partial\omega}{\partial\kappa} = \frac{(\gamma-1)\left(\frac{\theta_H}{\theta_L}-1\right)\lambda\left[(\beta+2\gamma\kappa-(\gamma-1)\delta)\frac{\partial r}{\partial\kappa} - (r+\delta)(\beta+2\gamma)\right]}{(\beta+(\gamma-1)r+2\gamma\kappa)^2} \tag{26}$$

$$\frac{\partial\eta}{\partial\kappa} = \frac{-\gamma\kappa\omega\frac{\partial r}{\partial\kappa} - (\beta-r+\gamma\kappa)\gamma\kappa\frac{\partial\omega}{\partial\kappa} - (\beta-r)\gamma\omega}{(\gamma\kappa\omega)^2} \tag{27}$$

$$\frac{\partial r}{\partial\kappa} = \frac{\gamma(\beta-r)^2}{\left[1+\left(\frac{\theta_H}{\theta_L}-1\right)\lambda + \frac{\gamma^2\kappa^2}{(\beta-r+\gamma\kappa)^2}\right](\beta-r+\gamma\kappa)^2} > 0 \tag{28}$$

Assume the limit of $\frac{\partial \omega}{\partial \kappa}$, $\frac{\partial \eta}{\partial \kappa}$ and $\frac{\partial r}{\partial \kappa}$ exists and finite. Taking the limit, and we can find

$$\lim_{\gamma\to\infty}\frac{\partial \omega}{\partial \kappa}=\frac{1}{\left(\bar{r}+2\kappa\right)^2}\left[\left(\frac{\theta_H}{\theta_L}-1\right)\lambda\left(2\kappa-\delta\right)\lim_{\gamma\to\infty}\frac{\partial r}{\partial \kappa}-2\left(\bar{r}+\delta\right)\left(\frac{\theta_H}{\theta_L}-1\right)\lambda\right]$$

$$\lim_{\gamma\to\infty}\frac{\partial \eta}{\partial \kappa}=\frac{1}{\bar{\omega}^2}\lim_{\gamma\to\infty}\frac{\partial \omega}{\partial \kappa}$$

$$\lim_{\gamma\to\infty}\frac{\partial r}{\partial \kappa}=0$$

Therefore,

$$\lim_{\gamma\to\infty}\frac{\partial \eta}{\partial \kappa}=\frac{2\left(\bar{r}+\delta\right)\left(\frac{\theta_H}{\theta_L}-1\right)\lambda}{\bar{\omega}^2\left(\bar{r}+2\kappa\right)^2}>0$$

since

$$\bar{r}+\delta=\frac{2\beta+2\delta}{2+\left(\frac{\theta_H}{\theta_L}-1\right)\lambda}>0$$

Therefore, $\exists\,\bar{\gamma}_1$ such that for $\forall\gamma>\bar{\gamma}_1$, $\frac{\partial \eta}{\partial \kappa}>0$ for all κ.

Now we are going to show the economy is well defined. First, the wage is positive since

$$\bar{w}=\frac{1-\alpha}{\left(\frac{\bar{r}+\delta}{\alpha\theta_L}\right)^{\frac{\alpha}{1-\alpha}}}>0$$

Next, we show the fraction of firms which receive θ_L and produce in the economy belongs to $(0,1)$.

$$\bar{\varphi}=\frac{\bar{\eta}+1-\lambda\bar{\eta}}{\lambda}$$

It's easy to show

$$\bar{\varphi}=\frac{\bar{\eta}+1-\lambda\bar{\eta}}{\lambda}=\frac{\bar{\eta}+1-\lambda\bar{\eta}-\lambda}{\lambda}+1=\frac{\left(1-\lambda\right)\left(\eta+1\right)}{\lambda}+1<1$$

when $\lambda>1$.

Now we need to show $\bar{\varphi}=\frac{\bar{\eta}+1-\lambda\bar{\eta}}{\lambda}>0\Leftrightarrow\bar{\eta}<\frac{1}{\lambda-1}$. Since $\lambda\in\left(1,\min\left\{2,\ \frac{2\beta}{\delta\left(\frac{\theta_H}{\theta_L}-1\right)}\right\}\right)$ and $\kappa>0$,

$$\bar{r}+2\kappa=\frac{2\beta-\delta\left(\frac{\theta_H}{\theta_L}-1\right)\lambda}{2+\left(\frac{\theta_H}{\theta_L}-1\right)\lambda}+2\kappa>\frac{2\beta-\delta\left(\frac{\theta_H}{\theta_L}-1\right)\lambda}{2+\left(\frac{\theta_H}{\theta_L}-1\right)\lambda}>0$$

, which implies

$$\bar{\omega} = 1 + \frac{(\bar{r}+\delta)\left(\frac{\theta_H}{\theta_L}-1\right)\lambda}{\bar{r}+2\kappa} > 1$$

. Therefore,

$$\bar{\eta} = \frac{1}{\bar{\omega}} < 1 < \frac{1}{\lambda-1},\ \forall\lambda \in \left(1, \min\left\{2,\ \frac{2\beta}{\delta\left(\frac{\theta_H}{\theta_L}-1\right)}\right\}\right)$$

which implies that $\bar{\varphi} \in (0,1)$. Therefore, $\exists\,\bar{\gamma}_2$ such that for $\forall\gamma > \bar{\gamma}_2$, $\varphi \in (0,1)$ and $w > 0$, i.e. the economy is well defined for all $\kappa > 0$ when $\lambda \in \left(1, \min\left\{2,\ \frac{2\beta}{\delta\left(\frac{\theta_H}{\theta_L}-1\right)}\right\}\right)$ and $\frac{2\beta}{\delta\left(\frac{\theta_H}{\theta_L}-1\right)} > 1$. The condition is exactly the assumption 1, which proves Lemma 3.

We can conclude, under assumption 1, $\exists\,\bar{\gamma} = \max\{\bar{\gamma}_1,\ \bar{\gamma}_2\}$ such that for $\forall\gamma > \bar{\gamma}$, $\frac{\partial\eta}{\partial\kappa} > 0$ for all $\kappa > 0$.

C Details for Numerical Algorithm

1. An initial guess r
2. An initial guess wage w
3. Find the cutoff by $\alpha\left(\frac{1-\alpha}{w}\right)^{\frac{1-\alpha}{\alpha}}\underline{\theta} = r+\delta$ and the the policy functions: the labor, capital, marginal profit function
4. Solve the HJB equation and find the optimal consumption and portfolio choice.

$$\begin{aligned}\beta V(\theta, W) = \max_{C_t,\, g_{\theta,t}} \frac{C^{1-\gamma}}{1-\gamma} + V_\theta(\theta, W)\,\mu(\theta) + V_W(\theta, W)\,(r_t W_t + \Pi(\theta, W) - C_t) \\ + \frac{1}{2}V_{\theta\theta}(\theta, W)\,\sigma^2(\theta) + \frac{1}{2}V_{WW}(\theta, W)\,(W g_{\theta,t}\sigma(\theta))^2 + V_{\theta W}(\theta, W)\,W g_{\theta,t}\sigma^2(\theta)\end{aligned} \tag{29}$$

The first order condition implies that

$$C = (V_W(\theta, W))^{-\frac{1}{\gamma}};\ g_{\theta,t} = -\frac{V_{\theta W}(\theta, W)}{V_{WW}(\theta, W)\,W}$$

5. Solve the stationary distribution $\Phi(\theta, W)$ from

$$0 = -\frac{\partial}{\partial \theta}\left[\mu(\theta)\Phi(\theta, W)\right] - \frac{\partial}{\partial W}\left[(rW + \Pi(\theta, W) - C)\Phi(\theta, W)\right]$$
$$+\frac{1}{2}\frac{\partial^2}{\partial \theta^2}\left[\sigma^2(\theta)\Phi(\theta, W)\right] + \frac{1}{2}\frac{\partial^2}{\partial W^2}\left[(W g_\theta \sigma(\theta))^2 \Phi(\theta, W)\right] + \frac{\partial^2}{\partial \theta \partial W}\left[W g_\theta \sigma^2(\theta)\Phi(\theta, W)\right]$$

6. Compute the aggregate consumption/capital/labor/output.

7. Check whether the labor market clears:

$$\int L_t(\theta, W)\, d\Phi_t(\theta, W) = \mathbf{L}$$

   if not, go to step 2; until the labor market clears.

8. Check whether the consumption goods market clears

$$\int C_t(\theta, W)\, d\Phi_t(\theta, W) + \delta \int K_t(\theta, W)\, d\Phi_t(\theta, W) = \alpha \int y_t(\theta, W)\, d\Phi_t(\theta, W)$$

   if not, go to step 1; until consumption goods market clears.