



# Sequential auctions with randomly arriving buyers<sup>☆</sup>

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## ABSTRACT

We analyze a dynamic market in which buyers compete in a sequence of private-value auctions for differentiated goods. New buyers and new objects may arrive at random times. Since objects are imperfect substitutes, buyers' values are not persistent. Instead, each buyer's private value for a new object is a new independent draw from the same distribution.

We consider the use of second-price auctions for selling these objects, and show that there exists a unique symmetric Markov equilibrium in this market. In equilibrium, buyers shade their bids down by their continuation value, which is the (endogenous) option value of participating in future auctions. We characterize this option value and show that it depends not only on the number of buyers currently present on the market and the distribution of their values, but also on anticipated market dynamics.

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## 1. Introduction

In this paper, we consider a model of dynamic markets in which buyers and sellers arrive randomly to the market and examine the equilibrium behavior of market participants in response to these arrivals. In particular, we investigate the role of current and anticipated future market conditions in determining endogenous outside options in this setting. We characterize the influence of these outside options on equilibrium price determination, and study the manner in which equilibrium payoffs and behavior are affected by both current conditions and future dynamics.

In our model, buyers face an infinite sequence of auctions for heterogeneous but stochastically equivalent objects—each buyer's private value for a new object is a new independent draw from a fixed distribution—that arrive at random times. Moreover, new buyers probabilistically arrive on the market in each period. Thus, in any given auction, buyers are presented with the outside option of participating in a future auction for an imperfectly substituted “equivalent” object, but with a potentially different number of competitors. We describe the unique symmetric Markov equilibrium of this model by deriving a precise characterization of this option value.

Essentially, losing in an auction today yields the opportunity to participate in another auction in the future; however, the potential for entry by additional buyers and the random arrival times of auctions implies, in contrast to much of the

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literature on sequential auctions, that the competitive environment in the future may differ significantly from the present one. Thus, the expected payoff of a buyer currently present on the market is directly linked to her expected payoffs with a different number of competitors present in future periods. This leads to a difference equation characterizing the (endogenously determined in equilibrium) “outside option” available to each buyer. While this difference equation admits, in general, an infinite number of solutions, we show that there is a unique bounded solution corresponding to the unique symmetric Markov equilibrium of the game. This implies that each buyer’s equilibrium outside option is, in fact, an appropriately discounted sum of expected payoffs from participating in each of the infinite sequence of auctions with differing numbers of participants, where the weight on each auction is a combination of pure time discounting and the likelihood of market dynamics leading to the corresponding state.

Our model of sequential auctions captures several salient features of many real-world markets. Consider, for instance, a buyer in the housing market. When deciding on a bid to place on a particular home, the buyer must consider the market dynamics—offers will be affected if a large number of homes are expected to come onto the market in the near future, or, alternately, if future demand is anticipated to be high due to (for instance) rapid job creation in the region. Moreover, houses are heterogeneous: although houses within a particular neighborhood are similar, they are typically sufficiently differentiated that different buyers will rank them (and value them) differently. Therefore, homebuyers’ offers will depend not only on the currently available housing stock, but also expectations about current and future market tightness.

Similarly, the model applies to settings in which firms participate repeatedly in pure common-values auctions; for instance, firms that bid in oil rights auctions. Such firms have private estimates of the value of a particular tract, upon which bids will be based. Such bids will not only take into account the winner’s curse, but also the frequency of future auctions for similar tracts and expectations about the competitiveness of such auctions. In the presence of capacity constraints on the number of new projects that can be undertaken simultaneously, accounting for the option value of future auctions is a crucial part of the bid determination process.

The present work follows in a long literature of models examining dynamic markets with randomly arriving buyers and sellers. Wolinsky (1988) considers the steady state of a dynamic market where buyers are randomly matched to sellers. As in our model, buyers’ values are redrawn from the same distribution whenever they face a new seller; sellers, however, are long-lived and strategic. De Fraja and Sákovic (2001) examine a similar dynamic market, with the crucial difference that objects are homogeneous, and hence buyers’ values do not vary over time. More recent contributions to the literature include Satterthwaite and Shneyerov (2007, 2008), who show that, in the presence of persistent frictions, the steady-state equilibrium arising from forward-looking bidding in a dynamic market converges to the static Walrasian equilibrium.

These models are all focused on the analysis of the steady state of dynamic markets, when the rate of entry to the market is perfectly balanced with the rate of exit. Our paper, on the other hand, is concerned with the behavior of agents in response to the changing market conditions of a dynamic environment. This is in line with work such as Taylor (1995), which examines bargaining power and price formation as it relates to the number of traders on each side of a dynamic market. He assumes that agents are homogeneous—all buyers have the same commonly known value for purchasing an object—and that trade is conducted via posted prices. As in Coles and Muthoo (1998), our model enriches his setting by allowing for heterogeneous buyers and objects. Instead of a combination of ultimatum bargaining and Bertrand competition, however, we model the surplus-division process by explicitly employing an auction mechanism for price determination.<sup>1</sup> In addition, we decouple the arrival of buyers and sellers.

The present work is also closely linked to elements of the theoretical literature on sequential auctions.<sup>2</sup> Engelbrecht-Wiggans (1994) studies sequential auctions in which a fixed number of perfectly patient buyers with single-unit demand compete in a finite sequence of second-price auctions for stochastically equivalent objects. His model, however, does not allow for several features of the present work; in particular, it does not allow for the entry of new buyers or consider the role of market dynamics in price determination. Jofre-Bonet and Pesendorfer (2003) assume stochastically equivalent objects and allow for the entry of new potential buyers, but these agents are short-lived bidders who do not take the future into account. Both Budish (2008) and Zeithammer (2010) consider a similar environment, restricted to two periods (and two objects), and study the efficiency properties of sequential auctions under different information revelation regimes. Said (2011) looks at a setting with entry dynamics similar to the present work, but makes the complementary (and opposite) assumption of independent private values that are *fixed* across time, focusing on the design of optimal mechanisms. As in the present work, outside options are endogenously determined; however, the presence of persistent private values introduces an element of learning that is not present herein.

## 2. The model

We consider the continuous-time limit of an infinite-horizon discrete-time market model; periods of length  $\Delta$  are indexed by  $t \in \mathbb{N}$ . There is a finite number  $n_t \geq 2$  of risk-neutral buyers with single-unit demand on the market in any given

<sup>1</sup> Inderst (2008) and Fuchs and Skrzypacz (2010) also examine the non-steady-state impact of new buyer arrivals on the payoffs of buyers and sellers. However, they do this in a framework of bilateral (one-to-one) bargaining which does not capture the dynamic nature of direct competition between several current and future potential market participants.

<sup>2</sup> There is also a growing empirical literature examining bidding in sequential auctions on eBay. See, among others, Sailer (2008), Zeithammer (2006), Ingster (2009), and Backus and Lewis (2010).

period  $t$ . Since the objects under consideration are differentiated and hence imperfectly substitutable, buyers have different values for different goods. Moreover, since buyers are heterogeneous, their evaluation of these distinct goods differ from one another—in the terminology of Engelbrecht-Wiggans (1994), objects are stochastically equivalent. This also arises if objects are common-value goods and each buyer receive a new unbiased signal about each object.

Thus, each buyer  $i \in \{1, \dots, n_i\}$  has a private valuation  $v_i^t$  for the object available at time  $t$ , where  $v_i^t$  is independently (across buyers  $i$  and periods  $t$ ) drawn from the distribution  $F$  on  $\mathbb{R}_+$ . We assume that  $F$  has finite variance and a continuous density function  $f$ . Finally, buyers discount the future exponentially with discount factor  $\delta = e^{-r\Delta}$ , where  $r > 0$  is the discount rate.

In each period, there is at most one seller present. The arrival of sellers is stochastic; in particular, there is some exogenously fixed probability  $p = \lambda\Delta$ , where  $\lambda > 0$ , that a new seller arrives on the market in each period. Similarly, additional buyers may arrive on the market in each period. For simplicity, we will assume that at most one buyer arrives at a time, and that this arrival occurs with some exogenously given probability  $q = \rho\Delta$ , where  $\rho > 0$ . Note, however, that we assume that this arrival occurs immediately and with probability 1 if there is only a single buyer remaining on the market from previous periods—there are always at least two buyers present. In the event that a seller has arrived on the market, each buyer  $i$  observes both  $v_i^t$  and the number of competing buyers present. The seller then conducts a second-price auction to allocate their object.

We assume that sellers are nonstrategic—they are unable to set a reserve price and cannot remain on the market for more than one period. Conversely, buyers must participate in each auction that takes place when they are present on the market, but may submit bids less than their true values.<sup>3</sup> There are two main advantages to assuming that reserve prices are not used. First, we are able to avoid the issues of pooling (partial or otherwise) that can occur at the reserve price when there are allocational externalities—see, for instance, Haile (2000) and Jehiel and Moldovanu (2000). Second, equilibrium in the presence of reserve prices is characterized by a *nonlinear* second-order difference equation. Solving such a difference equation numerically, let alone analytically, is typically an exercise in futility. And, computational issues aside, the lack of a closed-form solution precludes many avenues of additional investigation, including (for instance) standard comparative statics exercises. This is in sharp contrast to, for instance, Coles and Muthoo (1998), who arrive at a *linear* second-order (homogeneous) difference equation in their model, in part due to their assumptions of binary values and price-setting via Bertrand competition.

As is standard,  $Y_k^{(n)}$  denotes the  $k$ -th highest of  $n$  independent draws from  $F$ , with  $G_k^{(n)}$  and  $g_k^{(n)}$  denoting the corresponding distribution and density functions, respectively. In addition, we will define, for all  $n \in \mathbb{N}$ ,

$$\hat{Y}(n) := \mathbb{E}[Y_1^{(n)}] - \mathbb{E}[Y_1^{(n-1)}].$$

This is the expected difference between the highest of  $n$  and  $n - 1$  independent draws from  $F$ , where, by convention, we let  $\mathbb{E}[Y_1^{(0)}] = 0$ . It is useful to note that  $\hat{Y}(n)$  is decreasing in  $n$ .

### 3. Symmetric Markov equilibrium

Let  $V(v_i^t, n)$  denote the expected payoff to a bidder when her valuation is  $v_i^t$  and she is one of  $n$  bidders present on the market. Recall that a seller must be currently present on the market for buyers to be aware of their valuations. Furthermore, let  $W(n)$  denote the expected value to a buyer when she is one of  $n$  buyers present on the market at the beginning of a period, *before* the realization of the buyer and seller arrival processes. At the beginning of a period when there are  $n \geq 2$  buyers present, there are four possible outcomes: with probability  $pq$ , both a buyer and a seller arrive, leading to an auction with  $n + 1$  participants; with probability  $p(1 - q)$  only a seller arrives, yielding an auction with  $n$  participants; with probability  $(1 - p)q$  only a buyer arrives, leading to the next period starting with  $n + 1$  participants; or, with the remaining probability  $(1 - p)(1 - q)$ , neither a buyer nor a seller arrive, leading to the next period being identical to the current one. Thus, for all  $n \geq 2$ ,

$$W(n) := pq\mathbb{E}[V(v_i^t, n + 1)] + p(1 - q)\mathbb{E}[V(v_i^t, n)] + (1 - p)q\delta W(n + 1) + (1 - p)(1 - q)\delta W(n). \quad (1)$$

Let us now consider the problem facing buyer  $i$  when there are  $n \geq 2$  buyers on the market and an object is currently available (and, hence, an auction is “about” to occur). This buyer, with valuation  $v_i^t$ , must choose her bid  $b_i^t$ . If she wins the auction, she receives a payoff of  $v_i^t$  less the second-highest bid. On the other hand, if she loses, she remains on the market as one of  $n - 1$  buyers tomorrow, yielding her a payoff of  $\delta W(n - 1)$ . Therefore,

$$V(v_i^t, n) = \max_{b_i^t} \left\{ \Pr(b_i^t > \max_{j \neq i} \{b_j^t\}) \mathbb{E}[v_i^t - \max_{j \neq i} \{b_j^t\} \mid b_i^t > \max_{j \neq i} \{b_j^t\}] + \Pr(b_i^t < \max_{j \neq i} \{b_j^t\}) \delta W(n - 1) \right\}.$$

We may use this expression in order to determine equilibrium bid functions, as demonstrated in the following result. Note that we focus on the unique symmetric Markov equilibrium of this dynamic game. Other equilibria certainly exist; however,

<sup>3</sup> It is easy to see that, in equilibrium (given a particular discount factor), bids will always be positive if the support of the value distribution is sufficiently bounded away from zero.

as all other equilibria are ruled out by interim dominance-type arguments, it is natural to focus on the symmetric Markov equilibrium.

**Lemma (Equilibrium bids).** *In equilibrium, a buyer with value  $v_i^t$  who is one of  $n \geq 2$  buyers on the market bids  $b_i^t = b^*(v_i^t, n)$ , where*

$$b^*(v_i^t, n) := v_i^t - \delta W(n - 1). \tag{2}$$

**Proof.** Note that, since

$$\Pr(b_i^t < \max_{j \neq i} \{b_j^t\}) = 1 - \Pr(b_i^t > \max_{j \neq i} \{b_j^t\}),$$

we may rewrite  $V(v_i^t, n)$  as

$$\max_{b_i^t} \left\{ \Pr(b_i^t > \max_{j \neq i} \{b_j^t\}) \mathbb{E} \left[ v_i^t - \delta W(n - 1) - \max_{j \neq i} \{b_j^t\} \mid b_i^t > \max_{j \neq i} \{b_j^t\} \right] + \delta W(n - 1) \right\}.$$

As the trailing  $\delta W(n - 1)$  in the above expression is merely an additive constant, the maximization problem above corresponds to that of a second-price auction with  $n$  bidders in which each bidder  $i$ 's valuation is given by  $v_i^t - \delta W(n - 1)$ . The standard dominance argument for private value second-price auctions then implies that  $b^*(v_i^t, n) = v_i^t - \delta W(n - 1)$ .  $\square$

Given this bidding strategy and the symmetry of continuation payoffs across buyers in a symmetric equilibrium, the probability that  $i$  wins the period- $t$  auction is simply  $\Pr(v_i^t > \max_{j \neq i} \{v_j^t\})$ , and her surplus in this case becomes  $v_i^t - \max_{j \neq i} \{v_j^t\}$ . Therefore, we may rewrite  $V$  as

$$\begin{aligned} V(v_i^t, n) &= \Pr(v_i^t > \max_{j \neq i} \{v_j^t\}) \mathbb{E} \left[ v_i^t - \max_{j \neq i} \{v_j^t\} \mid v_i^t > \max_{j \neq i} \{v_j^t\} \right] + \delta W(n - 1) \\ &= \Pr(Y_1^{(n-1)} < v_i^t) \mathbb{E} [v_i^t - Y_1^{(n-1)} \mid Y_1^{(n-1)} < v_i^t] + \delta W(n - 1) \\ &= G_1^{(n-1)}(v_i^t) (v_i^t - \mathbb{E}[Y_2^{(n)} \mid Y_1^{(n)} = v_i^t]) + \delta W(n - 1), \end{aligned} \tag{3}$$

where the final equality follows from well-known properties of order statistics—see, for example, David and Nagaraja (2003, Chapter 3).

Note that, ex ante, any one of the  $n \geq 2$  buyers present on the market in any period is equally likely to have the highest value amongst her competitors (and hence win the object). We may use this fact, along with the result above, to show that the ex ante expected utility of a buyer when there is an object available for sale is simply the sum of her probability of winning the object multiplied by her expected payoff (conditional on winning) and the discounted option value of losing the object and remaining on the market in the next period. Formally, we are able to prove the following result.

**Lemma (Expected auction payoffs).** *The expected payoff to a bidder from an auction with  $n \geq 2$  participants is*

$$\mathbb{E}[V(v_i^t, n)] = \hat{Y}(n) + \delta W(n - 1). \tag{4}$$

**Proof.** Recall that Eq. (3) provides an expression for  $V(v_i^t, n)$ . Taking the expectation of this expression with respect to  $v_i^t$  yields

$$\begin{aligned} \mathbb{E}[V(v_i^t, n)] &= \int_{-\infty}^{\infty} ((x - \mathbb{E}[Y_2^{(n)} \mid Y_1^{(n)} = v_i^t]) G_1^{(n-1)}(x) + \delta W(n - 1)) f(x) dx \\ &= \frac{1}{n} \left( \int_{-\infty}^{\infty} x g_1^{(n)}(x) dx - \int_{-\infty}^{\infty} \mathbb{E}[Y_2^{(n)} \mid Y_1^{(n)} = v_i^t] g_1^{(n)}(x) dx \right) + \delta W(n - 1). \end{aligned}$$

Notice, however, that

$$\frac{1}{n} \left( \int_{-\infty}^{\infty} x g_1^{(n)}(x) dx - \int_{-\infty}^{\infty} \mathbb{E}[Y_2^{(n)} \mid Y_1^{(n)} = v_i^t] g_1^{(n)}(x) dx \right) = \frac{1}{n} (\mathbb{E}[Y_1^{(n)}] - \mathbb{E}[Y_2^{(n)}]) = \mathbb{E}[Y_1^{(n)}] - \mathbb{E}[Y_1^{(n-1)}].$$

This is exactly the quantity previously defined as  $\hat{Y}(n)$ , implying (as desired) that

$$\mathbb{E}[V(v_i^t, n)] = \hat{Y}(n) + \delta W(n - 1). \quad \square$$

With this result in hand, we may rewrite Eq. (1) for  $n \geq 2$  in terms of  $W$  and  $\hat{Y}$  alone:

$$W(n+1) = \frac{1 - \delta pq - \delta(1-p)(1-q)}{\delta(1-p)q} W(n) - \frac{p(1-q)}{(1-p)q} W(n-1) - \frac{p}{\delta(1-p)} \hat{Y}(n+1) - \frac{p(1-q)}{\delta(1-p)q} \hat{Y}(n).$$

Recall, however, that a single buyer remaining from period  $t$  will *always* be joined by another buyer at time  $t + 1$ . Thus, when  $n = 1$ , we have

$$W(1) = p(\hat{Y}(2) + \delta W(1)) + (1-p)\delta W(2) = \frac{p\hat{Y}(2) + \delta(1-p)W(2)}{1 - \delta p}.$$

Thus, a buyer's expected payoff is given by a solution to the second-order nonhomogeneous linear difference equation and boundary condition in the two equations above. While one could solve this system directly, the continuous-time limit is significantly more tractable. Recalling that  $\delta = e^{-r\Delta}$ ,  $p = \lambda\Delta$ , and  $q = \rho\Delta$  and taking the limit as the period length  $\Delta$  goes to zero yields

$$W(n+1) = \frac{r + \lambda + \rho}{\rho} W(n) - \frac{\lambda}{\rho} (\hat{Y}(n) + W(n-1)) \quad \text{for all } n \geq 2, \tag{5}$$

and

$$W(1) = W(2). \tag{6}$$

We may then rewrite this second-order difference equation as a first-order system of difference equations. In particular, we have, for all  $k > 0$ ,

$$\begin{pmatrix} W(k+2) \\ W(k+1) \end{pmatrix} = \begin{bmatrix} \frac{r+\lambda+\rho}{\rho} & -\frac{\lambda}{\rho} \\ 1 & 0 \end{bmatrix} \begin{pmatrix} W(k+1) \\ W(k) \end{pmatrix} + \begin{pmatrix} -\frac{\lambda}{\rho} \hat{Y}(k+1) \\ 0 \end{pmatrix}. \tag{7}$$

Note that, in general, there are an infinite number of solutions to this difference equation, each of which is parametrized by the values of  $W(1)$  and  $W(2)$ ; imposing the boundary condition in Eq. (6) reduces this to a continuum of possible solutions. However, we are able to rule out solutions in which expected utility diverges to infinity as the number of buyers on the market grows—there exists a unique bounded (and hence “sensible”) solution to the difference equation.

To characterize this solution, we define

$$\zeta_1 := \frac{r + \lambda + \rho - \sqrt{(r + \lambda + \rho)^2 - 4\lambda\rho}}{2\rho} \quad \text{and} \quad \zeta_2 := \frac{r + \lambda + \rho + \sqrt{(r + \lambda + \rho)^2 - 4\lambda\rho}}{2\rho}.$$

These two constants are the eigenvalues of the “transition” matrix in Eq. (7). It is straightforward to show that  $\zeta_1 \zeta_2 = \frac{\lambda}{\rho}$  and  $0 < \zeta_1 < 1 < \zeta_2$  for all  $r, \lambda, \rho > 0$ .

**Theorem** (Unique symmetric Markov equilibrium). *The unique symmetric Markov equilibrium of this infinite-horizon sequential auction game is characterized by the ex ante expected payoff function given by*

$$W(1) = \frac{\zeta_1 \zeta_2}{1 - \zeta_1} \sum_{k=1}^{\infty} \zeta_2^{-k} \hat{Y}(k+1) \tag{8}$$

and, for all  $n \geq 2$ ,

$$W(n) = \frac{\zeta_1^{n-1}(\zeta_2 - 1)}{\zeta_2 - \zeta_1} W(1) + \frac{\zeta_1^n \zeta_2}{\zeta_2 - \zeta_1} \sum_{k=1}^{n-1} \zeta_1^{-k} \hat{Y}(k+1) + \frac{\zeta_1 \zeta_2^n}{\zeta_2 - \zeta_1} \sum_{k=n}^{\infty} \zeta_2^{-k} \hat{Y}(k+1). \tag{9}$$

**Proof.** Notice that we may rewrite Eq. (7) as

$$w_{n+1} = A w_n + y_n,$$

where

$$w_m := (W(m+1), W(m))' \quad \text{and} \quad y_m := (-\zeta_1 \zeta_2 \hat{Y}(m+1), 0)' \quad \text{for all } m \in \mathbb{N},$$

and  $A$  is the “transition” matrix from Eq. (7). Elayed (2005, Theorem 3.17) yields the general solution

$$w_n = A^{n-1} w_1 + \sum_{k=1}^{n-1} A^{n-k-1} y_k.$$

Since  $\zeta_1$  and  $\zeta_2$  are the eigenvalues of the matrix  $A$ , it is straightforward to show that

$$A^k = \frac{1}{\zeta_2 - \zeta_1} \begin{bmatrix} \zeta_2^{k+1} - \zeta_1^{k+1} & \zeta_1^{k+1}\zeta_2 - \zeta_1\zeta_2^{k+1} \\ \zeta_2^k - \zeta_1^k & \zeta_1^k\zeta_2 - \zeta_1\zeta_2^k \end{bmatrix}.$$

Combining this with the imposition of the boundary condition of Eq. (6) allows us to (after some rearrangement and simplification) write the solution to this second-order system as

$$W(n) = \frac{\zeta_2^{n-1}}{\zeta_2 - \zeta_1} \left( (1 - \zeta_1)W(1) - \zeta_1\zeta_2 \sum_{k=1}^{n-1} \zeta_2^{-k} \hat{Y}(k+1) \right) + \frac{\zeta_1^{n-1}}{\zeta_2 - \zeta_1} \left( (\zeta_2 - 1)W(1) + \zeta_1\zeta_2 \sum_{k=1}^{n-1} \zeta_1^{-k} \hat{Y}(k+1) \right). \tag{10}$$

Consider the second term in the above expression. Since  $0 < \zeta_1 < 1$ ,  $\frac{\zeta_1^{n-1}}{\zeta_2 - \zeta_1} (\zeta_2 - 1)W(1)$  approaches 0 as  $n$  increases. Moreover,

$$\begin{aligned} \frac{\zeta_1^n \zeta_2}{\zeta_2 - \zeta_1} \sum_{k=1}^{n-1} \zeta_1^{-k} \hat{Y}(k+1) &= \frac{\zeta_2}{\zeta_2 - \zeta_1} \sum_{k=1}^{n-1} \zeta_1^k \hat{Y}(n-k+1) \\ &\leq \frac{\zeta_2}{\zeta_2 - \zeta_1} \sum_{k=1}^{n-1} \zeta_1^k \sigma < \frac{\zeta_2}{\zeta_2 - \zeta_1} \sum_{k=1}^{\infty} \zeta_1^k \sigma < \frac{\zeta_1 \zeta_2 \sigma}{(\zeta_2 - \zeta_1)(1 - \zeta_1)}, \end{aligned}$$

where  $\sigma^2$  is the (assumed finite) variance of the distribution  $F$ . This follows from Arnold and Groeneveld (1979), who show that

$$\mathbb{E}[Y_1^{(m)}] - \mathbb{E}[Y_2^{(m)}] \leq \frac{m\sigma}{\sqrt{m-1}} \text{ for all } m > 1.$$

Recalling the definition of  $\hat{Y}$ , we then have

$$\hat{Y}(m) = \frac{1}{m} (\mathbb{E}[Y_1^{(m)}] - \mathbb{E}[Y_2^{(m)}]) \leq \frac{\sigma}{\sqrt{m-1}} < \sigma$$

for all  $m > 1$ , implying that the second term in Eq. (10) is bounded.

The first term in Eq. (10), however, is multiplied by positive powers of  $\zeta_2 > 1$ , implying that an appropriate choice of  $W(1)$  is crucial for ensuring the boundedness of our solution. One such choice is to let

$$W(1) = w^* := \frac{\zeta_1 \zeta_2}{1 - \zeta_1} \sum_{k=1}^{\infty} \zeta_2^{-k} \hat{Y}(k+1).$$

Note that  $w^*$  is well-defined, as  $\zeta_2 > 1$  and  $\hat{Y}(m) < \sigma$  for all  $m > 1$ . Moreover, since  $\zeta_1 < 1$ ,  $w^* > 0$ . The first term in Eq. (10) then becomes

$$\frac{\zeta_1 \zeta_2^n}{\zeta_2 - \zeta_1} \sum_{k=n}^{\infty} \zeta_2^{-k} \hat{Y}(k+1) = \frac{\zeta_1}{\zeta_2 - \zeta_1} \sum_{k=n}^{\infty} \zeta_2^{n-k} \hat{Y}(k+1) = \frac{\zeta_1}{\zeta_2 - \zeta_1} \sum_{k=0}^{\infty} \zeta_2^{-k} \hat{Y}(n+k+1) < \infty,$$

where we again rely on the fact that  $\hat{Y}(m) < \sigma$  for all  $m > 1$  when  $F$  has finite variance. Thus, setting  $W(1) = w^*$  and imposing the boundary condition  $W(2) = W(1)$  leads to a bounded solution (which we will denote by  $W^*(n)$ ) of the difference equation defined in Eq. (7).

To show that this is the *unique* bounded solution, fix any arbitrary  $\alpha \in \mathbb{R}$ , and consider the solution where  $W(1) = W(2) = \alpha w^*$ . The solution in Eq. (10) becomes

$$\begin{aligned} W(n) &= \frac{\zeta_2^{n-1}}{\zeta_2 - \zeta_1} \left( (1 - \zeta_1)\alpha w^* - \zeta_1\zeta_2 \sum_{k=1}^{n-1} \zeta_2^{-k} \hat{Y}(k+1) \right) + \frac{\zeta_1^{n-1}}{\zeta_2 - \zeta_1} \left( (\zeta_2 - 1)\alpha w^* + \zeta_1\zeta_2 \sum_{k=1}^{n-1} \zeta_1^{-k} \hat{Y}(k+1) \right) \\ &= \frac{\zeta_2^{n-1}}{\zeta_2 - \zeta_1} \left( (1 - \zeta_1)w^* - \zeta_1\zeta_2 \sum_{k=1}^{n-1} \zeta_2^{-k} \hat{Y}(k+1) + (1 - \zeta_1)(\alpha - 1)w^* \right) \\ &\quad + \frac{\zeta_1^{n-1}}{\zeta_2 - \zeta_1} \left( (\zeta_2 - 1)w^* + \zeta_1\zeta_2 \sum_{k=1}^{n-1} \zeta_1^{-k} \hat{Y}(k+1) + (\zeta_2 - 1)(\alpha - 1)w^* \right) \\ &= W^*(n) + \frac{\zeta_2^{n-1}(1 - \zeta_1) + \zeta_1^{n-1}(\zeta_2 - 1)}{\zeta_2 - \zeta_1} (\alpha - 1)w^*. \end{aligned}$$

Since  $\zeta_2 > 1 > \zeta_1 > 0$  and  $w^* > 0$ , the above expression remains bounded if, and only if,  $\alpha = 1$ . Thus, the unique bounded solution to the system of difference equations in Eq. (7) satisfying the boundary condition in Eq. (6) is pinned down by the condition  $W(1) = w^*$ . The expression in Eq. (9) follows immediately.  $\square$

Thus, buyers' expected payoffs in the unique symmetric Markov equilibrium of this dynamic auction market are a discounted sum of expected payoffs from participating in each element of the infinite sequence of auctions with differing numbers of bidders. The weight on each auction is a combination of pure time discounting and the likelihood of market dynamics leading to the corresponding state.

Note that this solution is easily generalized to trading institutions other than the sequential second-price auction. By revenue equivalence,  $\hat{Y}(n)$  is the ex ante expected payoff of a buyer in any efficient one-shot auction mechanism with  $n$  buyers. Therefore, Eq. (9) continues to hold, as is, for markets in which objects are sold via, for instance, sequential first-price auctions—while buyers' bids will differ, their expected payoffs will be as in our theorem above.

On the other hand, if trade occurs via a different institution, then replacing  $\hat{Y}(n)$  by the appropriate ex ante expected payoff of a buyer in that mechanism would yield the corresponding solution for that institution. For example, suppose that each seller employs some (fixed) multi-lateral bargaining game for allocating her object. Letting  $\tilde{Y}(n)$  denote the ex ante expected payoff to each of  $n$  buyers from participating in the one-shot bargaining game, equilibrium in the induced dynamic market is then characterized by the analogue to Eq. (9) where  $\hat{Y}$  is replaced by  $\tilde{Y}$ .

In addition, it is possible to generalize the model to the case in which market dynamics and the distribution of buyers' values evolve according to a (known) Markov process. This corresponds to, for instance, the case in which there are “buyer's” and “seller's” markets with relatively higher arrival rates of sellers and buyers, respectively. The analysis is analogous to the case we consider above, but with a coupled system of difference equations linking the expected payoffs across the two states of the world. A similar uniqueness result and characterization of expected payoffs follows immediately.

One additional advantage of finding the closed-form solution described in our theorem is that it enables clean comparative statics. In particular, we are able to analytically characterize the response of payoffs to changes in each of the main parameters.

**Corollary (Comparative statics).** For all  $n \geq 2$ ,  $W(n) > W(n+1)$ ,  $W_r(n) < 0$ ,  $W_\lambda(n) > 0$ , and  $W_\rho(n) < 0$ , where  $W_x(\cdot)$  denotes  $\frac{\partial W(\cdot)}{\partial x}$ .

**Proof.** Available on request.  $\square$

As is to be expected, buyers' expected payoffs are decreasing in the competitiveness of the market—the arrival of an additional buyer leads to a decrease in each buyer's surplus. Similarly, expected payoffs are decreasing in the arrival rate of new buyers (as this increases anticipated competitiveness), but are increasing in the discount rate (as buyers become more patient, the potentially infinite stream of auctions becomes more appealing) and the arrival rate of sellers (as auctions become more frequent, competition will be thinned out more quickly). It is easy to see that the reverse is true in each of these cases when considering sellers' expected revenue.<sup>4</sup>

#### 4. Conclusion

This paper characterizes the manner in which current market conditions, as well as anticipated future conditions, create an endogenous option value for bidders in a dynamic market. Since buyers must trade off purchasing in the present against participating in the future, the value of this future option is crucial for current-period bidding; however, the value of the option is itself determined by equilibrium bidding behavior. We prove that there exists a unique symmetric Markov equilibrium, and show that the endogenous option value associated with it is, in fact, the expected discounted sum of the potential payoffs from individual transactions in the infinite sequence of possible states of the world, each differentiated by the potential number of buyers present on the market at that time. When the trading institution is an auction mechanism, buyers are therefore willing to bid their true values less this discounted option value of participating in the future sequence of auctions.

There are several directions for extending our analysis. One possibility is dropping the assumption of stochastic equivalence and endowing each buyer with a fixed private value for obtaining each object. Zeithammer (2010) explores this question when the values may differ across objects. There are several technical difficulties in conducting such an analysis in a model with sealed-bid auctions when values are perfectly correlated across objects, however. These complications are discussed in Said (2011). In particular, the sequential second-price auction is not efficient in the presence of buyer arrivals, so instead we examine a model in which objects are sold using the ascending auction format. Another potentially interesting line of research involves allowing for multiple simultaneous auctions, or allowing sellers to remain on the market for several

<sup>4</sup> Recall that a buyer with value  $v_i^j$  who is one of  $n \geq 2$  buyers present will bid  $b^*(v_i^j, n) = v_i^j - W(n-1)$ . This implies that a seller facing  $n \geq 2$  buyers receives an expected revenue of  $\Pi(n) := \mathbb{E}[Y_2^{(n)}] - W(n-1)$ . The statement in the text follows immediately from this observation.

periods and overlapping with one another. Additional possibilities include endogenizing the entry behavior of buyers and sellers in response to market conditions and dynamics, or allowing for the setting of reserve prices by sellers; in particular, considering the limit behavior of a model with a cap on the number of market participants may be particularly useful in understanding behavior with reserve prices. These extensions are, however, left for future work.

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