Sophisticated Policies in Models with Endogenous Debt Constraints*

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Abstract

This paper studies the role for policy in uniquely implementing the best competitive equilibrium in financial frictions models with multiple equilibria. I consider an environment in which agents with idiosyncratic productivity shocks trade Arrow securities subject to state contingent debt constraints that are determined endogenously in equilibrium. Agents can choose to default on their debt at any time after which they are barred from financial markets in all future periods but can continue to hold fiat money. Debt constraints are determined in equilibrium to have the property that an agent who has borrowed up to this limit is indifferent between defaulting and having access to financial markets in future periods. I characterize the Ramsey equilibrium when the government can control the money supply and show that the policy departs from the Friedman rule. However, I show that the Ramsey policy is also consistent with another equilibrium in which no Arrow securities are traded and debt constraints are zero in all periods. I consider an expanded policy space where in addition to money supply rules, the government can offer some agents access to an overnight risk-free deposit facility with specified interest rates. I show that the set of competitive equilibria is identical to the set when policy only consists of money supply rules. I prove that there exist *Sophisticated* policies as in Atkeson, Chari, and Kehoe (2010) that uniquely implement any particular competitive equilibrium. These policies are contingent on histories of private actions and require that all continuation outcomes constitute continuation competitive equilibria. The policy stipulates that the government allow access to this deposit scheme only after particular histories with an interest rate that is greater than the return on money.

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1 Introduction

Is there a role for government intervention in financial markets? If so, what kinds of policies lead to desirable outcomes? To answer these questions, I study a complete markets model with financial frictions in which agents are subject to state contingent debt constraints that are determined endogenously in equilibrium. The key friction underlying this model is that at any time agents can choose to default on their debt and lose access to financial markets in all future periods. Debt constraints are chosen in equilibrium so that an agent who has borrowed up to the limit is indifferent between defaulting and paying back his debt. This allows for the severity of the financial friction to be an equilibrium object which is in sharp contrast to much of the literature that exogenously assumes the form and tightness of debt constraints. A crucial feature of this model is that there are multiple equilibria. For example, there is an equilibrium in which financial markets work well and another in which these frictions are very tight. This generates a potentially important role for policy. The main result of this paper is that one can construct feasible and simple policies that uniquely implement the desired equilibrium.

In the model, agents have stochastic access to a production technology that transforms one unit of labor into one unit of the consumption good. Agents can either be productive or unproductive in which case they do not have access to this technology. All agents are subject to a cash in advance constraint in which the single consumption good must be purchased with money. In addition, they can trade Arrow securities subject to state contingent debt constraints. At the beginning of each period after the state is realized, agents can default on their existing obligations and subsequently be barred from trading Arrow securities in all future periods. They can however continue to hold fiat money after default. As mentioned in the previous paragraph, debt constraints are determined so that the value of paying an amount equal to this limit equals the value of defaulting. These debt constraints are termed "not-too-tight" and were first studied by Alvarez and Jermann (2000). The first main result of the paper concerns the Ramsey outcome when the government can control the money supply and impose uniform lump sum taxes on all agents. I show in this case that the Ramsey policy departs from the Friedman Rule. In general, the government will want to choose the money supply so that the real return on money is lower than the real interest rate in the economy. The reason for this is that agents can hold fiat money even after defaulting. By lowering the real return on money, the government worsens the outcomes for the agent after default. This intuition is similar to much of the work in models with limited commitment where increasing the severity of default punishments allows greater risk-sharing to be sustained in equilibrium. The optimal policy trades off the benefit of harsher punishments after default with worsening the friction due to cash-in-advance constraints on path. However, I show in this environment that the Ramsey policy is also consistent with another competitive equilibrium in which no Arrow securities are traded. Debt constraints are zero for all agents and the prices of the securities adjust so that no agent wishes to save. Agents' only use fiat money to smooth consumption. This equilibrium is similar to the one characterized by Scheinkman and Weiss (1986).

This motivates the study of *Sophisticated Policies* as defined by Atkeson, Chari, and Kehoe

(2010). These policies can differ on and off the equilibrium path and have the property that all continuation outcomes constitute continuation competitive equilibria. The first step in the construction of such policies is to consider a related environment with an equilibrium set that is identical to the one with debt constraints. The reason for this is that it is unclear in the context of this equilibrium concept as to what these not-too-tight debt constraints correspond to. They are neither optimality nor market clearing conditions. In the related setup, competitive intermediaries offer insurance contracts to agents who can default on them at any time. While the state of the world is observable to the intermediaries, agents' actions are not. I show that the set of equilibrium allocations is the same in both environments. This equivalence is useful as it allows us to interpret the "not-too-tight" debt constraints as a consequence of the profit maximizing contracts offered by intermediaries to agents. The policies I consider internalize how these contracts change in anticipation of government intervention after certain histories.

The next step is to consider the set of competitive equilibria when the government has an expanded set of instruments. In addition to money supply rules, I consider a policy instrument which allows the government to offer some agents access to an overnight risk-free deposit facility. Eligible agents can deposit cash with the government overnight at a specified interest rate. Uniform lump sum taxes can be levied to balance the government's budget. The government specifies the contracts eligible for the scheme and allows all agents signed to these contracts who haven't defaulted to use this deposit facility. I prove that the set of competitive equilibria given these policies is identical to the set when the government can only control the money supply. As a result, as long as these policies can only depend on the exogenous states of the world, there are multiple equilibria and the government cannot use the deposit facility to implement equilibria that Pareto-dominate the best competitive equilibrium.

The main result in the paper states that when we allow these policies to also depend on histories of private but publicly observable actions, unique implementation is possible. Given a competitive equilibrium, we can construct Sophisticated policies so that there exists a unique Sophisticated equilibrium corresponding to it. More specifically, under this policy, the government commits to offering a subset of agents access to an overnight risk free savings technology (a risk free bond) after certain histories. As is standard in cash-in-advance environments agents who are productive sell their consumption goods for cash which they hold overnight. Under this savings scheme, agents can deposit this cash with the government and receive a return that is greater than that of money. With the usual restriction on policies, multiple equilibria exist due the strategic complementarities in the actions of intermediaries. In particular, if prices are such that all other intermediaries offer a particular contract $\tilde{C}$ different from the desired one, an individual intermediary finds it optimal to also offer this contract. The idea behind this implementation technique is that policy is chosen so that an individual intermediary no longer has an incentive to go along with contract $\tilde{C}$. As an example suppose the government commits to intervening after the following history; all intermediaries besides a small subset offer $\tilde{C}$ while this small subset offers contract $\hat{C}$ with the property that some agent receives more insurance than that provided by $\tilde{C}$. The government offers all agents receiving $\hat{C}$ who haven't defaulted access to this risk free technology with returns that exceed the return from holding fiat money. Now consider if all intermediaries offering $\tilde{C}$ is an equilibrium. One can show that an individual intermediary has an incentive to offer a contract like $\hat{C}$ which will make both him and the agent strictly better off. Moreover, given this policy agents will not want to default if the return on the risk free savings technology is sufficiently attractive. A similar argument rules out other kinds of equilibria including those with sunspots.

An attractive feature of this policy is its institutional simplicity. The implementation only relies on a risk free instrument . In particular, it does not involve the government having to lend to distressed agents which in general might subject to various moral hazard problems. Moreover, it does not assume that government has a better monitoring technology and so is not subject to the same frictions as private intermediaries. This is contrast to much of the literature that assumes in crisis times that government has the ability to relax financial frictions and enter financial markets and increase the amount of credit in the economy.

Literature: There is a large and growing literature on macroeconomic models with financial frictions. Often, these frictions take the form of collateral constraints as first introduced by Kiyotaki and Moore (1997). Recent examples include Gertler and Kiyotaki (2010), Shourideh and Zetlin-Jones (2012) and Buera and Moll (2012). While these models are useful for understanding how exogenous shocks to the tightness of the collateral constraints affect the aggregate economy, they are less useful for understanding the role of policy to mitigate financial crises. This is because the friction is exogenously imposed and as a result the only beneficial policies are those that involve the government having a special ability to relax these frictions in bad times. If we assume that such frictions are derived from a contracting problem with limited commitment or moral hazard, it is unclear why the government should have these abilities when private agents do not.

Endogenous debt constraints of the form studied here were first introduced by Alvarez and Jermann (2000). As in this paper debt constraints are set to be not-too-tight and chosen so that an agent who has borrowed up to the limit the previous period is indifferent between paying back and defaulting. Alvarez and Jermann (2000) show that the efficient allocation from Kehoe and Levine (1993) can be decentralized using complete markets and not-too-tight debt constraints. However, the decentralization is weak as autarky is always an equilibrium in their environment. This is not true in the model I consider. While there is equilibrium multiplicity, the best equilibrium is in general inefficient. This because of the presence of pecuniary externalities in the agent's problem. Since agents can hold money after default, the return on money affects the value of default which is not internalized by agents. This is not true in Alvarez and Jermann (2000) since they assume that after default, agents can only consume their endowments. This is similar to the literature on inefficiencies arising due to prices in the consumption set, a seminal example of which is Golosov and Tsyvinski (2007). However, I show that the set of equilibria is identical to that of a decentralized contracting environment similar to that of Prescott and Townsend (1984) and Golosov and Tsyvinski (2007) in which agents have hidden actions. This provides a similar interpretation for these constraints in the setup I consider. In another related paper, Hellwig and

Lorenzoni (2009) study a decentralized environment similar to Alvarez and Jermann (2000) but in which after default, agents can save in Arrow securities. In contrast, I assume that after default agents can save in an uncontingent asset (money). As a result, the punishment after default is more severe than Hellwig and Lorenzoni (2009) but less than Alvarez and Jermann (2000).

This paper uses techniques and language developed by Chari and Kehoe (1990) and expounded upon by Atkeson, Chari, and Kehoe (2010) (henceforth ACK) which allows us to think about how policy can uniquely implement a desired competitive equilibrium. ACK define the concepts of *Sophisticated policies* and *Sophisticated equilibrium* in which policies depend on the history of private actions and are explicit about how they differ on and off the equilibrium path. This approach requires that all continuation outcomes (including those following a deviation) constitute a continuation competitive equilibrium. It is in sharp contrast to much of the literature that considers policies with the property that no equilibrium exists following a private deviation. This is sometimes referred to as implementation via nonexistence. In a related paper, Bassetto (2002) demonstrates another way in which policy can ensure uniqueness in models with price level indeterminacy. As in ACK, he is explicit about government strategies on and off the equilibrium path and shows that there exist strategies that lead to an equilibrium price level that is pinned down by fiscal variables. However, unlike the standard fiscal theory of the price level, government budget constraints hold both on and off the equilibrium path.

Finally, this paper is related to the literature on multiple equilibria in general equilibrium models. Woodford (1986a) has a simple example of an economy with financing constraints and shows the existence of self-fulfilling fluctuations similar to overlapping generations economies. Moreover he shows that the persistence of these fluctuations is similar to those of business cycle fluctuations. In another important paper, Woodford (1986b) provides conditions in order for a steady state of non-linear model to be indeterminate and proves that indeterminacy is a necessary and sufficiency condition for the existence of sunspot equilibria. In a recent paper Benhabib and Wang (2013) show that one can generate endogenous fluctuations in models with collateral constraints that can match U.S. time series data. Gu, Mattesini, Monnet, and Wright (2013) also prove the existence of multiple equilibria in a model with endogenous debt limits and show how one can generate endogenous fluctuations with sunspot dynamics.

The rest of the paper proceeds as follows: Section 2 lays out the model with cash-in-advance and not-too-tight debt constraints and characterizes the multiplicity of equilibria that occurs. In Section 3, I set up the contracting problem and prove an equivalence result, after which I proceed to the construction of Sophisticated Policies. Section 4 contains a discussion of some of the modelling assumptions and Section 5 concludes.

2 Model

The economy is populated by a two types of agents $I = \{i, j\}$ with each type of equal measure 1. Time is discrete $t = 0, 1, 2, ..$The aggregate state space is $S = \{i, j\}$ with $\Pr[s_0 = i] = \frac{1}{2}$ and subsequently $\Pr\left[s^{t+1}=j \mid s^t=i\right] = \Pr\left[s^{t+1}=i \mid s^t=j\right] = \lambda$. I denote the unconditional probabilities of histories denoted by $\pi\left(s^t\right)$ and the conditional probabilities by $\pi\left(s^{t+1} \mid s^t\right)$. The symbol $\succeq$ is used the denote the partial order on histories. For example, $s^{t'} \succeq s^t$ for $t' \geq t$ denotes a possible continuation of history s^t. Given a random variable x, I use the notation $\{x\}_{t'}$ to denote the stochastic process $\left\{x_t\left(s^t\right); \forall\, t' \leq t \leq \infty,\ s^t \in S^t\right\}$.There is a single divisible, nonstorable consumption good. In state i agent i is productive; he can transform 1 unit of labor into 1 unit of the consumption good while j cannot. All agents have preferences over consumptions streams and labor. Agent $i's$ utility in period t and state s^t is denoted by $u\left(c_t^i\left(s^t\right)\right) - l_t^i\left(s^t\right)$ where $u: R_+ \rightarrow R$ is strictly increasing, strictly concave and continuously differentiable.

In each period, agents trade state contingent Arrow securities, and money in certain ways and subject to various constraints to be laid out. To begin with we study a benchmark environment with complete markets.

2.1 Complete Markets Benchmark

To study the complete markets benchmark I first consider an environment without fiat money in which productive agents can directly consume their produced good. I then show that there exist money supply policies which implement this equilibrium in the environment with fiat money.

Agent i chooses consumption and labor streams $\left\{c_t^i\left(s^t\right), l_t^i\left(s^t\right)\right\}_{s^t \in S^t, t \geq 0}$ to maximize

$$E_0\left[\sum_{t=0}^{\infty} \sum_{s^t \in S^t} \beta^t \pi\left(s^t\right)\left(u\left(c_t^i\left(s^t\right)\right) - l_t^i\left(s^t\right)\right)\right] \tag{1}$$

subject to an Arrow-Debreu budget constraint

$$\sum_{t=0}^{\infty} \sum_{s^t} Q_t\left(s^t\right) c_t^i\left(s^t\right) \leq \sum_{t=0}^{\infty} \sum_{s^t} Q_t\left(s^t\right) l_t^i\left(s^t\right)$$

where $Q_t\left(s^t\right)$ is the Arrow-Debreu price and non-negativity constraints on consumption and labor. Given our assumption on productivities we also have that $l^i\left(s^t\right) = 0$ if $s_t \neq i$ for all $i \in I$.

A competitive equilibrium is defined in a standard fashion. Let $g \equiv u'^{-1}$.

Lemma 1 *If markets are complete, $c_t^i\left(s^t\right) = g(1)$ for all i, t, s^t. The risk free rate $R = \frac{1}{\beta}$.*

Proof. See Appendix. ■

Next, I consider a very similar environment to the one above except that agents have to purchase consumption goods with fiat money. In particular, if agent i is productive he sells his labor services for cash which is used in the following period to purchase consumption goods. One interpretation of this assumption is that there is an aggregate technology which transforms 1 unit of labor services into one unit of the consumption good. A productive agent can then sell his labor services to this technology and is compensated in cash. The timing of actions in each period is as follows: First the state s_t is realized following which the agents trade a complete set of Arrow securities and fiat money. Next, each agent splits into an worker and a shopper; the worker if productive can exchange labor services for money and the shopper must use money to purchase consumption goods. Finally, the entrepreneur and shopper reunite and consumption takes place at the end of the period.

Agent i chooses sequences of consumption, labor, money holdings and Arrow securities $\left\{c_t^i\left(s^t\right), l_t^i\left(s^t\right), m_t^i\left(s^t\right), \left\{a_{s_{t+1}}^i\left(s^t\right)\right\}_{s_{t+1}\in S}\right\}_{s^t\in S^t, t\geq 0}$, where $a_{s_{t+1}}^i\left(s^t\right)$ the total amount of a security which pays 1 unit of the consumption good in state s^{t+1} held by agent i, to maximize (1) subject to period budget constraints (in real terms),

$$p_t\left(s^t\right) m_t^i\left(s^t\right) + \sum_{s^{t+1}} q_{s_{t+1}}\left(s^t\right) a_{s_{t+1}}^i\left(s^t\right)$$
$$\leq \frac{p_t\left(s^t\right)}{p_{t-1}\left(s^{t-1}\right)} l_{t-1}^i\left(s^{t-1}\right) + a_{s_t}^i\left(s^{t-1}\right) + p_t\left(s^t\right)\left[m_{t-1}^i\left(s^{t-1}\right) - \frac{c_{t-1}^i\left(s^{t-1}\right)}{p_{t-1}\left(s^{t-1}\right)}\right] + T_t\left(s^t\right), \text{ if } s_{t-1} = i, \tag{2}$$

$$p_t\left(s^t\right) m_t^i\left(s^t\right) + \sum_{s^{t+1}} q_{s_{t+1}}\left(s^t\right) a_{s_{t+1}}^i\left(s^t\right) \leq a_{s_t}^i\left(s^{t-1}\right) + p_t\left(s^t\right)\left[m_{t-1}^i\left(s^{t-1}\right) - \frac{c_{t-1}^i\left(s^{t-1}\right)}{p_{t-1}\left(s^{t-1}\right)}\right] + T_t\left(s^t\right) \text{ if } s_{t-1} \neq i \tag{3}$$

a cash-in-advance constraint

$$c_t^i\left(s^t\right) \leq p_t\left(s^t\right) m_t^i\left(s^t\right) \tag{4}$$

non-negativity constraints on consumption, labor and money holds, and $l^i\left(s^t\right) = 0$ if $s_t \neq i$. Here $p_t\left(s^t\right)$ is the goods price of money, $q_{s_{t+1}}\left(s^t\right)$ is the price of a security purchased in s^t which pays 1 unit of the consumption good in state s^{t+1} and $T_t\left(s^t\right)$ is a uniform lump-sum tax imposed by government.

A competitive equilibrium given a money supply and transfer policy $\left\{M_t\left(s^t\right), T_t\left(s^t\right)\right\}_{t,s^t}$ is defined in the standard fashion. Next, I show that if the money supply satisfies certain conditions, the complete markets equilibrium outcome characterized above is also an equilibrium outcome in the environment with fiat money for some price sequence.

Proposition 1 *There exist prices $\left\{p_t\left(s^t\right)\right\}$ such that $c_t^i\left(s^t\right) = g\left(1\right)$ for all i, t, s^t and $R_t = \frac{1}{\beta}$*

1. $\liminf_{t\to\infty} M_t = 0$

2. $\inf_t M_t \beta^{-t} = \kappa > 0$

In such an equilibrium, the nominal interest rate is 1.

Proof. See Appendix. ■

The result serves as a warm-up for the analyses in later sections when agents face state contingent debt constraints. It is similar to one proved by Cole and Kocherlakota (1998) who focused on the kinds of money supply policies that can implement Pareto-optimal allocations in a growth model with cash-in-advance constraints. As I find here, money supply policies that deliver the

Friedman Rule achieve this. The basic idea is that the Friedman Rule removes the distortion from having to hold cash from one period to the next. I show that these policies are consistent with the price level $\frac{p_t(s^t)}{p_{t-1}(s^{t-1})} = \frac{1}{\beta}$ which implies that in equilibrium, the multiplier on the cash-advance constraint is 0. As a result it is easy to show that the equilibrium allocation computed in the environment without the CIA constraints satisfy the agent's first order conditions in the above problem. One only needs to check that the transversality conditions hold which are guaranteed by the assumption stated in the proposition.

In subsequent sections I analyze environments in which agents are subject to constraints on how much Arrow securities they can sell. Clearly, ex-ante welfare in these equilibria will be lower than the complete markets benchmark studied here. Moreover, in this setup, the complete markets risk-free rate provides an upper bound on the equilibrium risk free rate in any model with debt constraints. To see this notice that

$$\tilde{R}_t = \frac{1}{\beta}\frac{u'\left(\tilde{c}_t^i\left(s^{t-1},i\right)\right)}{\lambda u'\left(\tilde{c}_{t+1}^i\left(s^t,j\right)\right)+(1-\lambda)\,u'\left(\tilde{c}_{t+1}^i\left(s^t,i\right)\right)} = \frac{1}{\beta}\frac{1}{\lambda u'\left(\tilde{c}_{t+1}^i\left(s^t,j\right)\right)+(1-\lambda)} \leq \frac{1}{\beta}$$

where $\tilde{R}$ is the equilibrium interest rate in a model with debt constraints. In equilibrium, the interest rate is determined by the unconstrained agents i.e. the savers who in this case are the productive agents. The quasi-linearity assumption implies that for any agent, his marginal utility of consumption in the productive state is always 1. The fact that the agent is constrained implies that in the unproductive state his marginal utility is less than 1. This implies that $\tilde{R}_t \leq \frac{1}{\beta}$ which is the interest rate in the complete markets case.

2.2 Equilibrium with Debt Constraints and Fiat Money

Next, I consider an environment in which agents must purchase all consumption goods with fiat money and are subject to state contingent debt constraints that are determined in equilibrium. The timing within the period is as follows. After s_t is realized, agents can choose to default on debt. Default is publicly observable and after default, an agent is barred from trading Arrow securities in all future periods. However, he can continue to hold fiat money. Next, the markets for Arrow securities and money open and agents trade with each other after which the agent splits into a shopper and worker. As before, the shopper must purchase consumption goods with money and the worker if productive receives cash in return for labor services. Consumption takes place at the end of the period.

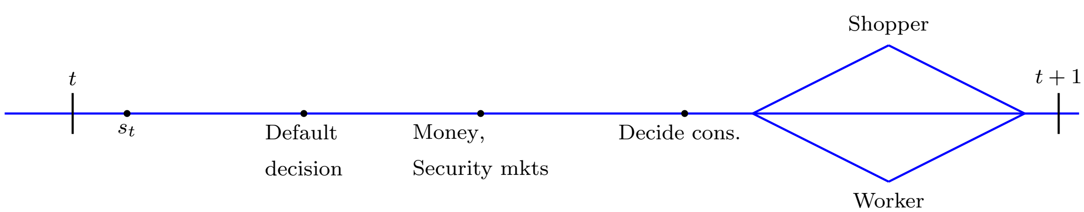

Figure 1: Timing.

Before analyzing the competitive equilibrium, a quick note on the default assumption. The literature on limited commitment in financial markets has made various assumptions on the consequences of default for agents. Some of these include permanent autarky (Kehoe and Levine (1993), Alvarez and Jermann (2000)), autarky with a positive probability of being able to regain entry into financial markets (Azariadis and Kaas (2012)), and only being able to save in Arrow securities (Hellwig and Lorenzoni (2009)). I find the introduction of fiat money convenient as allows us to make a natural assumption of allowing agents to hold money after default. As a result, agents can continue to achieve some consumption smoothing even after they default. Furthermore, the assumption of money being "hidden" from any legal authority that can seize financial assets of anyone defaulting is also a reasonable one.

Agent i chooses sequences of consumption, labor, money holdings and Arrow securities $\left\{c_t^i\left(s^t\right), l_t^i\left(s^t\right), m_t^i\left(s^t\right), \left\{a_{s_{t+1}}^i\left(s^t\right)\right\}_{s_{t+1}\in S}\right\}_{s^t\in S^t, t\geq 0}$ to maximize (1) subject to budget constraints (2), (3), state contingent debt constraints

$$a_{s'}^i\left(s^t\right) \geq \phi_{s'}^i\left(s^t\right) \text{ for all } s^t, s' \in S, \tag{5}$$

cash-in-advance constraints (4) and non-negativity constraints on consumption, labor and money. The debt constraints can be agent and state specific and limit the amount of each Arrow security that an agent can sell. We can define the agent's problem starting at date t, s^t if he hasn't defaulted in the past. Denote the value of this problem as $V_t^{i,c}\left(s^t, a_{s_t}^i\left(s^{t-1}\right); \Phi_t^i(s^t)\right)$ where $\Phi_t^i(s^t) = \left\{\phi_{t'}^i\left(s^{t'}\right)\right\}_{s^{t'}\geq s^t, t'\geq t}$ is the sequence of debt constraints in all future dates and states. As described earlier, the cash-in-advance constraint requires that all purchases of the consumption good must be made with previously accumulated money. As is standard, the CIA constraint will bind in equilibrium if the real risk-free rate is larger than 1.

Given a date t and state s^t, an agent who has defaulted chooses a sequence $\left\{c_{t'}^i\left(s^{t'}\right), \ l_{t'}^i\left(s^{t'}\right), \ \tilde{m}_{t+1}^i\left(s^t\right)\right\}_{s^{t'}\succeq s^t, t'\geq t}$ to maximize

$$\sum_{t'\geq t}\sum_{s^{t'}\succeq s^t}\beta^{t'-t}\pi\left(s^{t'} \mid s^t\right)\left[u\left(c_{t'}^i\left(s^{t'}\right)\right) - l_{t'}^i\left(s^{t'}\right)\right]$$

subject to budget constraints,

$$\begin{aligned} & p_t\left(s^t\right)\tilde{m}_t^i\left(s^t\right) \\ & \leq \frac{p_t\left(s^t\right)}{p_{t-1}\left(s^{t-1}\right)} l_{t-1}^i\left(s^{t-1}\right) + p_t\left(s^t\right)\left[\tilde{m}_{t-1}^i\left(s^{t-1}\right) - \frac{c_{t-1}^i\left(s^{t-1}\right)}{p_{t-1}\left(s^{t-1}\right)}\right] + T_t\left(s^t\right), \text{ if } s_{t-1} = i, \end{aligned} \tag{6}$$

cash-in-advance constraints (4) and non-negativity constraints on consumption, labor and money. As before $\tilde{m}_t^i\left(s^t\right)$ denotes the agent's money purchases and $p_t\left(s^t\right)$ is the goods price of money. Denote the value of default in t, s^t for agent i by $V_t^{i,d}(s^t; \mathbf{p}_t)$ where $\mathbf{p}_t = \left\{p_{t'}\left(s^{t'}\right)\right\}_{s^{t'}\geq s^t, t'\geq t}$.

Definition 1 *A Competitive Equilibrium given a money supply policy* $\{M,T\}_0$ *consists of prices* $\left\{p_t\left(s^t\right),\left(q_{s_{t+1}}\left(s^t\right)\right)_{s_{t+1}}\right\}_{t,s^t}$, *debt constraints* $\left\{\phi^i_{s'}\left(s^t\right)\right\}_{i\in I,s^t,s'}$ *and allocations* $\left\{\left(c^i_t\left(s^t\right),l^i_t\left(s^t\right),m^i_t\left(s^t\right),\left\{a^i_{s'}\left(s^t\right)\right\}_{s'\in S}\right)_{i\in I}\right\}_{t,s^t}$ *such that*

- *Given prices and debt constraints, the allocations solve the agents problem*
- *Markets clear*

$$\sum_{i\in I} c^i_t\left(s^t\right)=\sum_{i\in I} l^i_t\left(s^t\right) \text{ for all } t,s^t$$
$$\sum_{i\in I} a^i_{s'}\left(s^t\right)=0 \text{ for all } t,s^t,s'\in S$$
$$\sum_{i\in I} m^i_t\left(s^t\right)=M_t\left(s^t\right) \text{ for all } t,s^t$$

- *Debt constraints are chosen to be not-too-tight; For all* $i\in I,\ t\geq 0,\ s^t$,

$$V^{i,c}_t\left(s^t,\phi^i_{s_t}\left(s^{t-1}\right);\Phi^i_t(s^t)\right)=V^{i,d}_t(s^t;\mathbf{p}_t)$$

As in Alvarez and Jermann (2000), debt constraints are equilibrium objects and are chosen for each agent i in each date and state to satisfy the above condition. The condition says that debt constraints are chosen so that an an agent who has borrowed up to the limit is indifferent between paying back his debt and defaulting. Loosely speaking, a way to interpret this condition is to consider the effects of weakening the equality to an inequality. If $V^{i,c}_t\left(s^t,\phi^i_{s_t}\left(s^{t-1}\right);\Phi^i_t(s^t)\right)<V^{i,d}_t\left(s^t\right)$, the agent will choose to default in that state as the value of default is strictly greater than repaying his debt. On the other hand, if $V^{i,c}_t\left(s^t,\phi^i_{s_t}\left(s^{t-1}\right);\Phi^i_t(s^t)\right)>V^{i,d}_t\left(s^t\right)$ the agent could borrow a little more and still would prefer not to default. I formalize this intuition in a later section where I consider an environment in which intermediaries offer insurance contracts subject to limited commitment frictions and show that the set of equilibria is identical to environment above.

This paper is interested in whether policy that depends on histories of endogenous events can aid the implementation of desirable outcomes in this model. To motivate the role for policy I first consider the Ramsey outcome when the space of policies are money supply rules and uniform lump sum taxes. In contrast to the preceding section with complete markets I show that the Ramsey policy departs from the Friedman rule. I then show that these policies are also consistent with another Pareto-inferior equilibrium in which debt constraints are zero an all dates and states.

2.2.1 Ramsey Outcome

The space of policies is given by

$$\mathcal{P}^R=\left\{\{M\}_0,\{T\}_0\mid\forall t,\ s^t,\ M_t\left(s^t\right)\in\mathbb{R}_+,\ T_t\left(s^t\right)\in\mathbb{R}\right\}$$

Given a policy $\pi \in \mathcal{P}^R$, let $CE(\pi)$ denote the set of competitive equilibria given this policy. Given an element $\psi \in CE(\pi)$ let $\left\{ \left(c^{i,\psi}, l^{i,\psi}, m^{i,\psi}, \left\{ a^{i,\psi}_{s'} \right\}_{s' \in S} \right)_{i \in I} \right\}_0$ be allocations corresponding to ψ. Define

$$U(\pi) = \max_{\psi \in CE(\pi)} \sum_{t=0}^{\infty} \sum_{s^t \in S^t} \sum_{i \in I} \beta^t \pi\left(s^t\right) \left(u\left(c^{i,\psi}_t \left(s^t\right) \right) - l^{i,\psi}_t \left(s^t\right) \right)$$

Definition 2 *A Ramsey policy is* $\pi^* \in \mathcal{P}^R$ *such that* $\pi^* \in \arg\max_{\pi \in \mathcal{P}^R} U(\pi)$.

I next characterize some properties of a Ramsey equilibrium. While a full analytical characterization is not possible, I derive what is perhaps the most surprising feature of optimal policy in this environment.

Proposition 2 *In any Ramsey policy,* π^* $\exists t,\ s^t,\ s^{t+1}$ *such that* $\frac{p_{t+1}\left(s^{t+1}\right)}{p_t\left(s^t\right)} \neq \frac{1}{\sum_{s_{t+1}} q_{s_{t+1}}\left(s^t\right)}$.

In other words, Ramsey policies feature departures from the Friedman rule. The key to understanding this result is to notice that unlike standard cash-in-advance models, the introduction of fiat money is not just a friction. This is because of the assumption that agents can hold money after default and so the return on money affects the value of defaulting. Since the value of default is increasing in the return on money, the Ramsey planner has an incentive to lower this rate since it makes default more costly and could sustain more risk-sharing. However, decreasing the return also worsens the cash-in-advance friction if the rate is different from the market interest rate. I prove that at the allocations and prices associated with the Friedman rule, the first effect dominates and the Ramsey planner does want to introduce a wedge in between market return and the return on money.

To prove this result, I first characterize the best equilibrium consistent with the Friedman Rule. As in Cole and Kocherlakota (1998) I show that this corresponds to the best equilibrium without cash-in-advance constraints. As this equilibrium is stationary, I consider the set of stationary equilibria and prove that moving away from the Friedman rule yields an equilibrium with strictly higher welfare for the planner. As a result the Ramsey policy must feature departures from the Friedman Rule.

The environment without cash-in-advance constraints is identical to the previous one except that now there is no fiat money in the environment. Agents can directly consume their labor output and can continue to trade Arrow securities. I assume that after default, agents can save in a risk-free security whose interest rate is determined by the market.

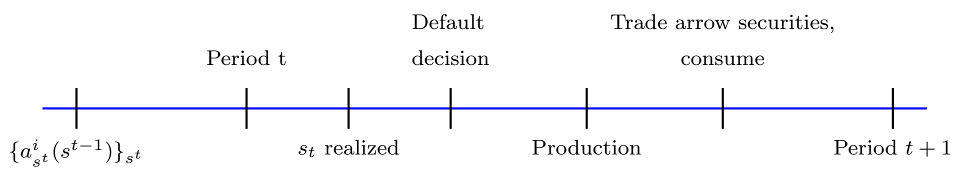

Figure 2: Timing

Agent i chooses sequences of consumption, labor, and Arrow securities $\left\{c_t^i\left(s^t\right), l_t^i\left(s^t\right), \left\{a_{s_{t+1}}^i\left(s^t\right)\right\}_{s_{t+1}\in S}\right\}_{s^t\in S^t, t\geq 0}$ to maximize (1) subject to budget constraints,

$$c_t^i\left(s^t\right) + \sum_{s^{t+1}} q_{s_{t+1}}\left(s^t\right) a_{s_{t+1}}^i\left(s^t\right) \leq l^i\left(s^t\right) + a_i^i\left(s^{t-1}\right) \text{ if } s_t = i,$$
$$c_t^i\left(s^t\right) + \sum_{s^{t+1}} q_{s_{t+1}}\left(s^t\right) a_{s_{t+1}}^i\left(s^t\right) \leq a_j^i\left(s^{t-1}\right) \text{ if } s_t = j,$$

state contingent debt constraints (5) and the usual non-negativity constraints. As mentioned earlier, in each period an agent can default on his existing obligations consequently be barred from trading Arrow securities in all future periods but can continue to save in a risk-free bond. Given a date t and state s^t, an agent who has defaulted chooses a sequence $\left\{c_{t'}^i\left(s^{t'}\right),\ l_{t'}^i\left(s^{t'}\right),\ x_{t+1}^i\left(s^t\right)\right\}_{s^{t'}\succeq s^t, t'\geq t}$ to maximize

$$\sum_{t'\geq t}\sum_{s^{t'}\succeq s^t} \beta^{t'-t}\pi\left(s^{t'} \mid s^t\right)\left[u\left(c_{t'}^i\left(s^{t'}\right)\right) - l_{t'}^i\left(s^{t'}\right)\right]$$

subject to budget constraints,

$$c_{t'}^i\left(s^{t'}\right) + Q_{t'}(s^{t'})x_{t'+1}^i\left(s^{t'}\right) \leq l_{t'}^i\left(s^{t'}\right),$$

a constraint that the agent can only save in a risk-free bond,

$$x_{t'+1}^i(s^{t'}) \geq 0, \text{for all } t' \geq t$$

and other non-negativity constraints. Here $x_{t'}^i\left(s^{t'+1}\right)$ denotes the agent's holding of the risk free bond and $Q_{t'}(s^{t'})$ is the market price of a risk free bond. Denote the value of default in t, s^t for agent i by $V_t^{i,d}(s^t; \mathbf{Q}_t)$ where $\mathbf{Q}_t = \left\{Q_{t'}\left(s^{t'}\right)\right\}_{s^{t'}\geq s^t, t'\geq t}$.

Definition 3 *A Competitive equilibrium consists of prices* $\left\{q_i\left(s^t\right), q_j\left(s^t\right)\right\}_{t,s^t}$, *debt constraints* $\left\{\phi_{s'}^i\left(s^t\right)\right\}_{i,s^t,s'}$ *and allocations* $\left\{\left(c_t^i\left(s^t\right), l_t^i\left(s^t\right), \left\{a_{s'}^i\left(s^t\right)\right\}_{s'\in S}\right)_{i\in I}\right\}_{t,s^t}$ *such that*

- *Given prices and debt constraints, the allocations solve the agent's problem*
- *Markets clear*

$$\sum_{i\in I} c_t^i\left(s^t\right) = \sum_{i\in I} l_t^i\left(s^t\right) \textit{ for all } t, s^t$$
$$\sum_{i\in I} a_{s'}^i\left(s^t\right) = 0 \textit{ for all } t, s^t, s' \in S$$

- *Debt constraints are chosen to be not-too-tight; For all* $i \in I,\ t \geq 0,\ s^t$,

$$V_t^{i,c}\left(s^t, \phi_{s_t}^i\left(s^{t-1}\right); \Phi_t^i(s^t)\right) = V_t^{i,d}(s^t; \mathbf{Q}_t)$$

Next I characterize the best stationary equilibrium in this environment.

Proposition 3 *A stationary equilibrium exists with*

$$q^c + q^{nc} < 1$$

where

$$q^c = q_j\left(s^t, i\right) = q_i\left(s^t, j\right)$$
$$q^{nc} = q_i\left(s^t, i\right) = q_j\left(s^t, j\right)$$

Proof. See Appendix. ■

A sketch of the proof is as follows. I first conjecture the form of the stationary equilibrium and later confirm that an equilibrium of the form exists. The equilibrium I conjecture has constant prices for each of the two Arrow securities- one that pays off if the state next period is the same and another which pays if the state switches. In addition, consumption and labor is constant for the productive and unproductive types. The first step in the proof is to consider a setup in which after default, agents are allowed to save in Arrow securities. This is the assumption made by Hellwig and Lorenzoni (2009) in an endowment economy. I show that an equilibrium of the conjectured form exists in this environment as well with risk free rate equaling 1. This is similar to the low-interest rate results proved by Hellwig and Lorenzoni (2009). They find that in equilibrium, in order to support borrowing and lending, the present discounted value of endowments must be infinite (as is true when the real interest rate is 1). The reason why this result is useful in proving our result is that one can compute the agents' choices after default in closed form and use this to derive properties of the equilibrium when the agents can only save in a risk free bond. Since the value of default when agents can save in Arrow securities is strictly greater than if agents can only save in a risk free bond, we know that at the debt constraints associated with the former punishment $\tilde{\phi}$

$$V^i\left(i, \tilde{\phi}\right) > V^{i,d}(i)$$

where $V^i\left(i, \tilde{\phi}\right)$ is the value of not defaulting when agent i owes $\tilde{\phi}$ and $V^{i,d}(i)$ is the value of default when the agent can save in a risk free bond. One can use continuity arguments to show that when agents can only save in a risk free bond after default, we can sustain a higher level of debt than $\tilde{\phi}$. From an equilibrium relation between the debt constraints and the Arrow security prices I prove that the equilibrium interest rate $q^c + q^{nc}$ is strictly greater than 1.

A natural question that arises is whether the best competitive equilibrium is constrained efficient. In this case, one can just obtain the allocations corresponding to this equilibrium by solving the appropriate planning problem. However, as the next proposition shows, all competitive equilibria are inefficient. As a result we need to work directly with the decentralized environment in order to characterize the set of equilibria.

Proposition 4 *Any competitive equilibrium is constrained inefficient.*

Proof. See Appendix. ■

Next I demonstrate that policies which implement the Friedman rule in the model with cash-in-advance constraints are consistent with an equilibrium where the allocations and Arrow security prices coincide with those in equilibrium characterized above. As mentioned earlier, this is similar to the result in Chari and Kehoe (1990). However in sharp contrast to them, as Proposition 4 suggests, this equilibrium is not constrained efficient. This also suggests that the Friedman rule might not coincide with the Ramsey policy as the Ramsey planner might want use the return on money to introduce distortions in the agents' decisions as a planner solving the constrained efficient planner would. Let $\left\{c_t^i\left(s^t\right), l_t^i\left(s^t\right), a_i^i\left(s^t\right), a_j^i\left(s^t\right), \phi_i^i\left(s^t\right), \phi_j^i\left(s^t\right), q_t^i\left(s^t\right), q_t^j\left(s^t\right)\right\}_{s^t,i}$ correspond to the best equilibrium constructed above

Proposition 5 *There exist prices* $\{p_t\left(s^t\right)\}$ *such that* $\left\{c_t^i\left(s^t\right), l_t^i\left(s^t\right), a_i^i\left(s^t\right), a_j^i\left(s^t\right), \phi_i^i\left(s^t\right), \phi_j^i\left(s^t\right)\right\}_{s^t,i}$ *and prices* $\left\{p_t\left(s^t\right), q_t^i\left(s^t\right), q_t^j\left(s^t\right)\right\}$ *constitute a competitive equilibrium if*

1. $\liminf_{t\to\infty} M_t = 0$

2. $\inf_t M_t \beta^{-t} = \kappa > 0$

In such an equilibrium, the nominal interest rate is 1.

Proof. See Appendix. ■

This equilibrium can be implemented using a deterministic sequence $\{M_t\}_{t=0}^{\infty}$. The proof is similar to that of Proposition 1. I show that these policies are consistent with a price sequence that satisfies $\frac{p_t\left(s^t\right)}{p_{t-1}\left(s^{t-1}\right)} = \frac{1}{Q}$ where $Q = q^c + q^{nc}$ which are defined and computed in Proposition 3. This implies that in equilibrium, the multiplier on the cash-in-advance constraint is zero. The crucial point to note is that since in equilibrium the real return on money equals the return on risk free debt, the value of default if the agent can only hold money is identical to the case in which he can hold only a risk free bond. I can show that the allocation from the economy without money satisfies the agent's first order conditions given these prices and money supply policies. Finally, the Transversality condition for money holds if assumption 1 in the above proposition is satisfied. Assumption 2 guarantees that the money supply is large enough so that the CIA constraints can be satisfied.

I now prove that one can perturb this policy slightly and yield strictly higher welfare for the Ramsey planner. To do this I restrict the planner to choosing from a set of *stationary* equilibria. In a stationary equilibrium $p_t M_t = p_{t+1} M_{t+1}$ and so a money supply rule $\frac{M_{t+1}}{M_t} = \mu$ is equivalent to an interest rate rule $\frac{p_{t+1}}{p_t} = R^m$.

Lemma 2 *Given a stationary interest rate rule policy R^m, an allocation and price pair constitute a stationary competitive equilibrium if*

$$c_t^i\left(s^{t-1},i\right)=c_t^j\left(s^{t-1},j\right)=g\left(\frac{1}{R^mQ}\right)$$
$$c_t^i\left(s^{t-1},j\right)=c_t^j\left(s^{t-1},i\right)=g\left(\frac{q^c}{R^mQ\beta\lambda}\right)$$
$$l_t^i\left(s^{t-1},i\right)=l_t^j\left(s^{t-1},j\right)=g\left(\frac{1}{R^mQ}\right)+g\left(\frac{q^c}{R^mQ\beta\lambda}\right)$$
$$q^{nc}=\beta\left(1-\lambda\right)$$
$$\frac{\left(1-\beta\left(1-\lambda\right)\right)\left(u\left(g\left(\frac{1}{R^mQ}\right)\right)-\left[g\left(\frac{1}{R^mQ}\right)+g\left(\frac{q^c}{R^mQ\beta\lambda}\right)\right]\right)+\beta\lambda u\left(g\left(\frac{q^c}{R^mQ\beta\lambda}\right)\right)}{\left[1-\beta\left(1-\lambda\right)\right]^2-\beta^2\lambda^2}=V^{i,d}\left(s^t;R^m\right) \tag{7}$$

where

$$q^c=q_i\left(s^{t-1},j\right)=q_j\left(s^{t-1},i\right)$$
$$q^{nc}=q_j\left(s^{t-1},j\right)=q_i\left(s^{t-1},i\right)$$
$$Q=q^c+q^{nc}$$

The lemma highlights the distortions introduced by departing from the Friedman Rule. For example, under the Friedman rule ($R^mQ=1$), a productive agent always consumes $g\left(1\right)$ where $g=u'^{-1}$ while in this case it is $g\left(\frac{1}{R^mQ}\right)$. In addition, the money supply policy (which in this case equivalent to an interest rate policy) also affects the right hand side of the not-too-tight constraint (7).

Given our restriction to stationary equilibria, we can define the Ramsey policy as solving

$$\begin{aligned}&\max_{R^m}W\left(R^m,q^c\left(R^m\right)\right)\\&=\max_{R^m}u\left(g\left(\frac{1}{R^m\left[q^c\left(R^m\right)+q^{nc}\right]}\right)\right)-\left[g\left(\frac{1}{R^m\left[q^c\left(R^m\right)+q^{nc}\right]}\right)+g\left(\frac{q^c\left(R^m\right)}{R^m\left[q^c\left(R^m\right)+q^{nc}\right]\beta\lambda}\right)\right]\\&+u\left(g\left(\frac{q^c\left(R^m\right)}{R^m\left[q^c\left(R^m\right)+q^{nc}\right]\beta\lambda}\right)\right)\end{aligned} \tag{8}$$

where $q^c\left(R^m\right)$ solves (7). The next proposition is a restatement of Proposition 2.

Proposition 6 *In the best equilibrium corresponding to the Friedman rule*

$$\left.\frac{\partial}{\partial R^m}W\left(R^m,q^c\left(R^m\right)\right)\right|_{R^m=\frac{1}{Q}}<0$$

To understand this result notice that

$$\frac{\partial}{\partial R^m} W\left(R^m, q^c\left(R^m\right)\right) = W_1 + W_2 \cdot q^{c'}\left(R^m\right)$$

Consider the first term on the right hand side of the above equation. This is the value of lowering the return on money holding fixed the market prices. Since this worsens the frictions from the cash-in-advance constraint $W_1 < 0$. Next, the term $W_2 \cdot q^{c'}\left(R^m\right)$ measures the change in welfare given that the prices must adjust via (7) in order to constitute a not-too-tight equilibrium. Since lowering R^m reduces the value of default, the price of the arrow security q^c falls in order to satisfy this constraint. Since we are at a constrained equilibrium, lower q^c implies a greater degree of risk sharing and therefore $W_2 \cdot q^{c'}\left(\ R^m\right) > 0$. Finally, one can show that when $R^m = \frac{1}{q^c+q^{nc}}$, $W_1 + W_2 \cdot q^{c'}\left(R^m\right) > 0$ or that the second effect dominates. It is worth noting that if there were no debt constraints, we would not want to depart from the Friedman rule as in that case $q^c = \beta\lambda$ and $q^{c'}\left(\ R^m\right) = 0$. As result setting $R^m < \frac{1}{Q}$ only worsens the friction from the CIA constraints.

From the above analysis it is clear that in any Ramsey equilibrium there must be some trade of Arrow securities. In particular since we can always find policies (for example the Friedman Rule) such that they induce some borrowing and lending among agents, the Ramsey policy will also have this feature. In general however, the Ramsey policy π^* might not be stationary.

The next main result shows that in general, policy $\pi^*$is also consistent with another equilibrium in which there is financial autarky and no Arrow securities are traded. We turn to the characterization of this equilibrium next.

Proposition 7 *Given a Ramsey policy $\pi^* = \{M^*, T^*\}_0$ there exists another equilibrium in which*

$$\phi^i_{s'}\left(s^t\right) = 0 \text{ for all } i, s^t, s'$$

Proof. See Appendix. ■

To prove this result, I conjecture an equilibrium in which debt constraints are zero and no Arrow securities are traded in any date and state. In such an equilibrium agents use cash balances to achieve limited smoothing. The structure of the conjectured equilibrium is similar to the environment studied by Scheinkman and Weiss (1986) except that it may not be Markov if in general the money supply policies are history dependent. I prove that an equilibrium exists in which the relevant aggregate state variables are $\left(s^t, z^t\right)$ where $z_t = \frac{\left[m^1_{t-1}\left(s^{t-1}\right) - \frac{c^1_{t-1}\left(s^{t-1}\right)}{p_{t-1}\left(s^{t-1}\right)}\right]}{M_t\left(s^t\right)}$ is the fraction of the money supply held by agent i. While in general proving the existence of equilibria with aggregate uncertainty and incomplete markets is tricky, the assumptions on the state space S along with results from Miao (2006) allow us to prove existence in this setup. The idea is to first show that given continuous price functions for money $p_t\left(s^t, z^t\right)$, the agent's problem can be written as dynamic program and that a unique sequence of value functions and policy functions exist. The next step is to prove that such price functions do exists.

Finally to prove that the conjectured allocation in which securities are not traded constitutes an equilibrium, I show there exist Arrow security prices such that those along with this allocation and the associated money supply sequence constitute an equilibrium with $\phi^i_{s'}\left(s^t\right)=0$ for all i, s^t, s'. The prices of Arrow securities satisfy

$$q_{s'}\left(s^t, z^t\right)=\max_i\left\{\int_Z \frac{u'\left(c^i\left(s^{t+1}, z^{t+1}\right)\right)}{u'\left(c^i\left(s^t, z^t\right)\right)} Q_{t+1}\left(s', dz_{t+1}, s^t, z^t\right)\right\} \tag{9}$$

where Q_{t+1} is the joint distribution of the aggregate states and $q_{s'}\left(s^t, z^t\right)$ is the price of an Arrow security in state $\left(s^t, z^t\right)$ which pays one unit of the consumption good if the state next period s'. The consumption policy functions $c^i\left(s^{t+1}, z^{t+1}\right)$ are those computed in the environment without Arrow securities. To see why this constitutes a Competitive Equilibrium, notice that for security prices defined as in (9), no agent wishes to save in Arrow securities since for any agent i , $q_{s'}\left(s^t, z^t\right) \geq \int_Z \frac{u'\left(c^i\left(s^{t+1}, z^{t+1}\right)\right)}{u'\left(c^i\left(s^t, z^t\right)\right)} Q_{t+1}\left(s', dz_{t+1}, s^t, z^t\right)$. Agents would like to borrow, but are constrained from doing so by the zero borrowing limits. As a result, markets for Arrow securities clear and hence we have an equilibrium.

3 Policy

The key takeaway from the previous section is that policies consistent with the Ramsey outcome which only depend on the exogenous states of the world can only weakly implement the desired equilibrium in models with debt constraints. This motivates the study of Sophisticated Policies as in Atkeson, Chari, and Kehoe (2010) where policies can also depend on histories of private actions and can differ on and off the equilibrium path. The main result in this section is that there exist such policies that uniquely implement the desired equilibrium. A key requirement of Sophisticated Policies is that all continuation outcomes be continuation competitive equilibria. This ensures that the approach does not achieve implementation via non-existence. To understand the construction of these policies and how they achieve implementation, I first consider a related environment with the property that the set of competitive equilibria is identical to environment above. This environment is formulated in terms of competitive intermediaries offering insurance contracts to agents. The reason I consider implementation in this setup is that the equilibrium conditions determining debt constraints arise naturally from the best responses of intermediaries. Policies will target these best responses and make deviations unprofitable.

3.1 Contracting framework

Consider a set J of $T-$period lived intermediary and I agents. Time $t=0,1,..$ is discrete and let S be the finite state space and S^t be set of histories till time t with typical element $s^t=\left(s_0, s_1, ..., s_t\right)$ (the timing will be described below). The state is known to all agents and the intermediary. The transition probabilities are given by a matrix Π, with the unconditional probabilities of histories denoted as $\pi\left(s^t\right)$ and the conditional probabilities by $\pi\left(s^{t+1} \mid s^t\right)$. The symbol $\succeq$ is used the denote the partial order on histories. For example, $s^{t'} \succeq s^t$ for $t' \geq t$ denotes a possible continuation of history s^t. Given a random variable x, I use the notation $\{x\}_{t'}$ to denote the stochastic process $\{x_t(s^t); \forall\ t' \leq t \leq \infty,\ s^t \in S^t\}$. Given a stochastic process for consumption $\{c^i\}_0$ and labor $\{n^i\}_0$, the utility for agent i is

$$\sum_{t=0}^{\infty} \sum_{s^t \in S^t} \beta^t \pi(s^t) \left[u\left(c_t^i(s^t)\right) - l^i(s^t)\right]$$

I assume that u^i is strictly increasing, strictly concave and C^1. We have $l^i(s^t) = 0$ if $s_t \neq i$.

Intermediaries are risk-neutral and offer insurance contracts to entrepreneurs. Formally, an intermediary j offers agent i in state s^t, a contract

$$C^{i,j}(s^t) = \left(\left(\left\{c_{t'}^i\left(s^{t'}\right), m_{t'}^i\left(s^{t'}\right), l_{t'}^i\left(s^{t'}\right), \zeta_{t'}^{i,j}\left(s^{t'}\right)\right\}_{s^{t'} \in S^{t'}}\right)_{t' \in \{t, \ldots t+T-1\}}\right)$$

where $\zeta_{t'}^{i,j}\left(s^{t'}\right)$, is the state contingent insurance offered by the intermediary j in state $s^{t'}$.Along with transfers, the contract specifies a set of allocations to the agent. While intermediaries can observe the aggregate state (the history of agent's productivity shocks) and the actions of all other intermediaries, an important friction in the contracting environment is that the entrepreneurs' actions are unobservable. In particular the intermediary cannot observe the agent's consumption and money holding. Define $C^j \equiv \left(C^{i,j}\right)_{i \in I}$. Let $\mathcal{C}$ denote the space of contracts. $\mathcal{C}$ with the sup norm is a Banach space. Let $\mathcal{M}(C)$ denote the space of finite Borel measures over $\mathcal{C}$. Given a contract $C \in \mathcal{C}$, let $x_t^{C,i}(s^t)$ denote the value at t, s^t of the stochastic process x as specified by contract C.

The timing in the last period of a $T-$period contract is as follows[1]: At the beginning of period t, after s_t is known, agents decide whether to default on payments owed to the intermediary. Next, intermediaries transfer $\zeta_t^{i,j}(s^t)$ to agents who have not defaulted in the past following which markets for money open. After this, agents split into a worker and shopper as in the previous section following which (unobservable) consumption takes place. If agents choose to default, they longer receive any insurance from the intermediary they are currently signed to and cannot sign with any other intermediary in the future. Given past beliefs about actions and taking the actions

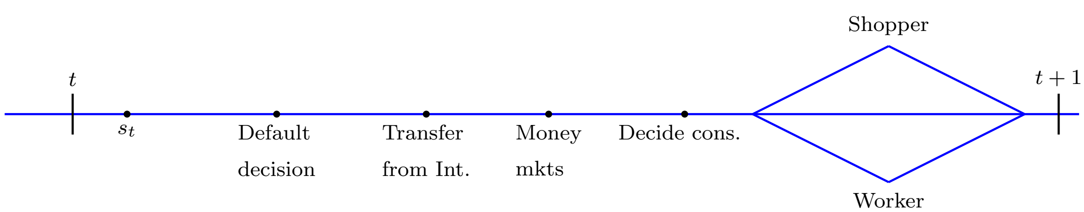

Figure 3: Timing.

of all other intermediaries' as given, any feasible contract $C^{i,j}(s^t)$ that an intermediary j can offer

[1]The other periods are identical, except agents dont sign with new intermediaries the following period

in s^t must satisfy budget constraints for all agents for all $t' \in \{t+1, ..., T-1\}$

$$p_{t'}\left(s^{t'}\right) m^i_{t'}\left(s^{t'}\right) = \frac{p_{t'}\left(s^{t'}\right)}{p_{t'-1}\left(s^{t'-1}\right)} l^i_{t'-1}\left(s^{t'-1}\right) + p_{t'}\left(s^{t'}\right)\left[m^i_{t'-1}\left(s^{t'-1}\right) - \frac{c^i_{t'-1}\left(s^{t'-1}\right)}{p_{t'-1}\left(s^{t'-1}\right)}\right] + \zeta^{j,i}_{t'}\left(s^t\right) \tag{10}$$

cash in advance constraints

$$c^i_{t'}\left(s^{t'}\right) \leq p_{t'}\left(s^{t'}\right) m^i_{t'}\left(s^{t'}\right) \text{ for all } i, t', s^{t'}$$

a participation constraint

$$\sum_{t'=t}^{\infty} \sum_{s^{t'} \succeq s^t} \beta^{t'-t}\left[u\left(c^i_{t'}\left(s^{t'}\right)\right) - l^i_{t'}\left(s^{t'}\right)\right] \geq \bar{V}^i_t\left(s^t\right)$$

and incentive compatibility constraints,

$$\sum_{t'=t}^{\infty} \sum_{s^{t'} \succeq s^t} \beta^{t'-t}\left[u\left(c^i_{t'}\left(s^{t'}\right)\right) - l^i_{t'}\left(s^{t'}\right)\right] \geq \hat{V}^i_t\left(s^t, \boldsymbol{\zeta}^{j,i}; \mathbf{p}_t\right) \tag{11}$$

where $\boldsymbol{\zeta}^{i,j}_t\left(s^t\right) = \left\{\left\{\zeta^{i,j}_{t'}\left(s^{t'}\right)\right\}_{t \leq t' \leq T-1, s^t \preccurlyeq s^{t'} \preccurlyeq s^{T-1}}, \boldsymbol{\zeta}^{i,j'}_{T-1}\left(s^{T-1}\right)\right\}$, $\mathbf{p}_t = \left\{p_{t'}\left(s^{t'}\right)\right\}_{s^{t'} \succeq s^t, t' \geq t}$.

$\hat{V}^i_t\left(s^t, \boldsymbol{\zeta}^{i,j}\left(s^t\right); \mathbf{p}_t\right)$ represents the best deviation an agent can undertake given the insurance contract offered by the intermediary and is the solution to

$$\max_{\{\tilde{c}_t, \tilde{m}_t, \Delta\}} \left(1 - \Delta^i\left(s^t\right)\right) \left[\begin{array}{c} \sum_{t'=t}^{T-3} \beta^{t'-1} \prod_{\hat{t}=t+1}^{t'-1}\left(1 - \Delta^i\left(s^{\hat{t}}\right)\right)\left[\begin{array}{c} \left(1 - \Delta^i\left(s^{t'}\right)\right)\left[u\left(\tilde{c}^i_{t'}\left(s^{t'}\right)\right) - \tilde{l}^i_{t'}\left(s^{t'}\right)\right] \\ +\Delta^i\left(s^{t'}\right) V^{i,d}_{t'}\left(s^{t'}; \mathbf{p}_{t'}\right) \end{array}\right] \\ \prod_{\hat{t}=t+1}^{T-2}\left(1 - \Delta^i\left(s^{\hat{t}}\right)\right)\left[\begin{array}{c} \left(1 - \Delta^i\left(s^{T-1}\right)\right) \hat{V}^i_{T-1}\left(s^{T-1}, \boldsymbol{\zeta}^{i,j'}_{T-1}\left(s^{T-1}\right); \mathbf{p}_{T-1}\right) \\ +\Delta^i\left(s^{T-1}\right) V^{i,d}_{T-1}\left(s^{T-1}; \mathbf{p}_{T-1}\right) \end{array}\right] \end{array}\right]$$
$$+ \Delta^i\left(s^t\right) V^{i,d}_t\left(s^t; \mathbf{p}_t\right) \tag{12}$$

subject to budget constraints

$$p_{t'}\left(s^{t'}\right) \tilde{m}^i_{t'}\left(s^{t'}\right) = \left[\begin{array}{c} \frac{p_t\left(s^{t'}\right)}{p_{t-1}\left(s^{t'-1}\right)} \tilde{l}^i_{t'-1}\left(s^{t'-1}\right) + p_{t'}\left(s^{t'}\right)\left[\tilde{m}^i_{t'-1}\left(s^{t'-1}\right) - \frac{\tilde{c}^i_{t-1}\left(s^{t'-1}\right)}{p_{t-1}\left(s^{t'-1}\right)}\right] \\ + \prod_{j=0}^{t'}\left[1 - \Delta\left(s^j\right)\right] \zeta^{i,j}_{t'}\left(s^{t'}\right) \end{array}\right], \; t' = t+1, ..., T-1$$

and CIA constraints

$$\tilde{c}^i_{t'}\left(s^{t'}\right) \leq p_{t'}\left(s^{t'}\right) \tilde{m}^i_{t'}\left(s^{t'}\right) \text{ for all } i, t, s^t$$

Here $\Delta^i\left(s^{t'}\right) \in \{0, 1\}$ are the agent's default decision in state $s^{t'}$, $\hat{V}^i_{T-1}\left(s^{T-1}, \boldsymbol{\zeta}^{i,j'}_{T-1}\left(s^{T-1}\right); \mathbf{p}_{T-1}\right)$

is his continuation value of the best deviation if the agent chooses to not default in s^{T-1} and sign with some intermediary j', and $V_{t'}^{i,d}\left(s^{t'};\mathbf{p}_{t'}\right)$ is the value of defaulting in t' and not being able to sign with an intermediary in the future. Given any t, s^t, $V_{t'}^{i,d}\left(s^{t'};\mathbf{p}_{t'}\right)$ is defined as the solution to maximizing

$$\sum_{t'=t}^{\infty}\sum_{s^{t'}\succeq s^t}\beta^{t'-t}\left[u\left(c_{t'}^i\left(s^{t'}\right)\right)-l_{t'}^i\left(s^{t'}\right)\right] \tag{13}$$

subject to

$$\begin{aligned}
&p_t\left(s^t\right)m_t^i\left(s^t\right)\leq\frac{p_t\left(s^t\right)}{p_{t-1}\left(s^{t-1}\right)}l_{t-1}^i\left(s^{t-1}\right)+p_t\left(s^t\right)\left[m_{t-1}^i\left(s^{t-1}\right)-\frac{c_{t-1}^i\left(s^{t-1}\right)}{p_{t-1}\left(s^{t-1}\right)}\right]\\
&c_t^i\left(s^t\right)\leq p_t\left(s^t\right)m_t^i\left(s^t\right)\\
&l^i\left(s^t\right)=0\text{ if }s_t\neq i,\ l^i\left(s^t\right)\geq 0,\ c^i\left(s^t\right)\geq 0, m_t^i\left(s^t\right)\geq 0
\end{aligned} \tag{14}$$

As earlier, if the entrepreneur defaults he can continue to hold money after default but is barred from signing with intermediaries in all future periods.

At the end of the current T period contract, if the agent hasn't defaulted in the past, he can sign with a new intermediary who offers him an insurance contract along with recommended allocations. Since actions are unobservable, the agent can choose different levels of consumption and money holdings. The term $\prod_{j=0}^{t}\left[1-\Delta\left(s^j\right)\right]\left[\zeta_t^{i,j}\left(s^t\right)\right]$ captures whether the agent has defaulted in the past or not. Notice that this formulation allows for a rich set of deviations an agent can undertake. For example, the agent can engage in a "double" deviation where he chooses to hold a different amount of money in s^t, and default the following period. The profit maximizing contract must prevent such deviations.

Intermediary j is risk neutral and maximizes profits

$$-\sum_{i\in I}\sum_{t'=t}^{T}\sum_{s^{t'}\in S^{t'}}\left[\prod_{\hat{t}=t}^{t'}q_{s^{\hat{t}}}\left(s^{\hat{t}-1}\right)\right]\zeta_{t'}^{i,j}\left(s^{t'}\right)$$

Intermediaries can borrow and lend amongst each other at market determined state contingent prices $q_{s_{t+1}}\left(s^t\right)$.

Definition 4 *Given a sequence of money supplies* $\left\{M_t\left(s^t\right)\right\}_{t,s^t}$ *,a competitive equilibrium in the contracting environment consists of prices* $\left\{q_{s_{t+1}}\left(s^t\right),p_t\left(s^t\right)\right\}_{t,s^t}$ *, allocations for each intermediary* $\left(C^{i,j}\left(s^t\right),\bar{V}_t^i\left(s^t\right)\right)_{i,j}$ *and a sequence of measures* $\left\{\left(\mu_{s^t}^{i*}\right)_{i\in I}\right\}_{t,s^t}$ *such that*

- *Given prices and the actions of other intermediaries, the contract offered by intermediary* j

solves the problem,

$$\max_{\{c,k,l,\zeta\}} -\sum_{i\in I}\sum_{t'=t}^{T}\sum_{s^{t'}\in S^{t'}}\left[\prod_{\hat{t}=t}^{t'} q_{s^{\hat{t}}}\left(s^{\hat{t}-1}\right)\right]\zeta_{t'}^{i,j}\left(s^{t'}\right)$$

subject to

$$p_{t'}\left(s^{t'}\right)m_{t'}^{i}\left(s^{t'}\right)=\frac{p_{t'}\left(s^{t'}\right)}{p_{t'-1}\left(s^{t'-1}\right)}l_{t'-1}^{i}\left(s^{t'-1}\right)+p_{t'}\left(s^{t'}\right)\left[m_{t'-1}^{i}\left(s^{t'-1}\right)-\frac{c_{t'-1}^{i}\left(s^{t'-1}\right)}{p_{t'-1}\left(s^{t'-1}\right)}\right]+\zeta_{t'}^{i,j}\left(s^{t}\right)$$

$$c_{t}^{i}\left(s^{t}\right)\leq p_{t}\left(s^{t}\right)m_{t}^{i}\left(s^{t}\right)$$

$$\sum_{t'=t}^{\infty}\sum_{s^{t'}\succeq s^{t}}\beta^{t'-t}\left[u\left(c_{t'}^{i}\left(s^{t'}\right)-l_{t'}^{i}\left(s^{t'}\right)\right)\right]\geq\bar{V}_{t}^{i}\left(s^{t}\right)$$

$$\sum_{t'=t}^{\infty}\sum_{s^{t'}\succeq s^{t}}\beta^{t'-t}\left[u\left(c_{t'}^{i}\left(s^{t'}\right)-l_{t'}^{i}\left(s^{t'}\right)\right)\right]\geq\hat{V}_{t}^{i}\left(s^{t},\boldsymbol{\zeta}^{i,j}\left(s^{t}\right);\mathbf{p}_{t}\right)$$

- *Intermediaries make zero profits*
- *Markets clear*

$$\sum_{i\in I}\int_{\xi\in\mathcal{C}}c_{t}^{\xi,i}\left(s^{t}\right)d\mu_{s^{t}}^{i*}\left(\xi\right)=\sum_{i\in I}\int_{\xi\in\mathcal{C}}l_{t}^{\xi,i}\left(s^{t}\right)d\mu_{s^{t}}^{i*}\left(\xi\right)$$

$$\sum_{i\in I}\int_{\xi\in\mathcal{C}}\zeta_{t}^{\xi,i}\left(s^{t}\right)d\mu_{s^{t}}^{i*}\left(\xi\right)=0$$

$$\sum_{i\in I}\int_{\xi\in\mathcal{C}}m_{t}^{\xi,i}\left(s^{t}\right)d\mu_{s^{t}}^{i*}\left(\xi\right)=M_{t}^{i}\left(s^{t}\right)$$

In the definition, the measure $\mu_{s^t}^{i*}(\xi)$ denotes the measure of agents of type i in equilibrium who are signed to contract ξ. One main result of this paper proves an equivalence between the sets of equilibria defined in the previous two sections.

Theorem 1 *1. Given an equilibrium of the not-too-tight debt constraint economy,* $\left(\{q_{s'},p\}_0\ ,\left\{\left(\phi^i\right)_{i\in I}\right\}_0,\left\{\left(c^i,l^i,m^i,a^i_{s'}\right)_{i\in I}\right\}_0\right)$ *there exist,* $\left(\{\zeta^{i,j}\}_0,\{\bar{V}^i\}_0\right)_{i\in I},\left\{\left(\mu_s^{i*}\right)_{i\in I}\right\}_0$ *such that*

$$\left(\{q_{s'},p\}_0,C^{i,j}\left(s^t\right)=\left(\left(\left\{c_{t'}^{i}\left(s^{t'}\right),m_{t'}^{i}\left(s^{t'}\right),l_{t'}^{i}\left(s^{t'}\right),\zeta_{t'}^{i,j}\left(s^{t'}\right)\right\}_{s^{t'}\in S^{t'}}\right)_{t'\in\{t,\ldots t+T-1\}}\right),\{\bar{V}^i\}_0,\left\{\left(\mu_{s^t}^{i*}\right)_{i\in I}\right\}_0\right)$$

constitute an equilibrium in the contracting environment.

2. *Given an equilibrium of the contracting environment*

$$\left(\{q_{s'},p\}_0,C^{i,j}\left(s^t\right)=\left(\left(\left\{c_{t'}^{i}\left(s^{t'}\right),m_{t'}^{i}\left(s^{t'}\right),l_{t'}^{i}\left(s^{t'}\right),\zeta_{t'}^{i,j}\left(s^{t'}\right)\right\}_{s^{t'}\in S^{t'}}\right)_{t'\in\{t,\ldots t+T-1\}}\right),\bar{V}_t^i\left(s^t\right),\left\{\left(\mu_{s^t}^{i*}\right)_{i\in I}\right\}_0\right),$$

there exist debt constraints $\left\{\left(\phi^i\right)_{i\in I}\right\}_0$, *such that*

$\left(\{q_{s'},p\}_0\ ,\left\{\left(\phi^i\right)_{i\in I}\right\}_0,\left\{\left(c^i,l^i,m^i,a^i_{s'}\right)_{i\in I}\right\}_0\right)$ constitute an equilibrium with not-too-tight debt constraints.

The full proof is in the appendix. Here I give an overview and some intuition for the result. Consider part 1 of the result. Suppose we have an equilibrium allocation and price sequence from the debt constrained problem. To show that these constitute an equilibrium in the contracting environment, we first need to construct a sequence of transfers $\left\{\zeta^{i,j}_{t'}\left(s^{t'}\right)\right\}$. We can do this by setting $\zeta^{i,j}_t\left(s^t\right)=a^i_{s_t}\left(s^{t-1}\right)-\sum_{s_{t+1}}a^i_{s_{t+1}}\left(s^t\right)$, for each j, which correspond the agent's net asset holdings at each date and state. Notice that the contract consisting of these transfers along with the allocations from the debt constrained competitive equilibrium is incentive compatible given prices in the contracting problem, since if the agent strictly preferred to default or choose some deviating allocation, he would done so in the contracting problem. Since debt constraints are chosen to prevent default in equilibrium, this choice is incentive compatible. The only concern is if an intermediary could offer a different contract with slightly more insurance that would make him and some agent strictly better off while respecting all other constraints. However, as I show in the proof, if there existed such a contract, then the current debt limits from the debt constrained equilibrium could not have been not-too-tight thus contradicting the assumption that it was.

The key step in proving the converse is to show that in any equilibrium contract satisfies

$$q_{s_{t+1}}\left(s^t\right)\geq\max_{i\in I}\left\{\frac{\beta\pi\left(s^{t+1}\mid s^t\right)u'\left(c^i_{t+1}\left(s^{t+1}\right)\right)}{u'\left(c^i_t\left(s^t\right)\right)}\right\}\text{ for all }t,s^t,s_{t+1}$$

The reason this is important is that we know from the literature on not-too-tight debt constraints that any equilibrium and price schedule must satisfy this condition. What its says is that the price of a Arrow security is determined by the unconstrained buyers of the security, i.e. those with the maximum marginal rates of substitutions. It is exactly for those agents, who are unconstrained, that the first order condition for the security holds with equality. For those that are constrained (sellers), the condition holds with an inequality. To prove that this condition holds in the contracting environment I show that if it did not hold, an intermediary could offer a different contract that would make both the agent and him strictly better off. To see why, consider the case in which $q_{s_{t+1}}\left(s^t\right)>\max_{i\in I}\left\{\frac{\beta\pi\left(s^{t+1}|s^t\right)u'\left(c^i_{t+1}\left(s^{t+1}\right)\right)}{u'\left(c^i_t\left(s^t\right)\right)}\right\}$ and some insurance is being offered. Since insurance is being offered there must exist some agent (receiving a positive transfer) who in s^{t+1} strictly prefers to stay in the contract rather than default. The intermediary can offer this agent a little more transfer $q\left(s^{t+1}\mid s^t\right)\varepsilon$ today at the cost of reducing the transfer to him in s^{t+1} by ε. We can approximate his change in utility using a Taylor expansion

$$\Delta u=\left[q_{s_{t+1}}\left(s^t\right)u'\left(c^i_t\left(s^t\right)\right)-\beta\pi\left(s^{t+1}\mid s^t\right)u'\left(c^i_{t+1}\left(s^{t+1}\right)\right)\right]\varepsilon$$

which is greater than zero since $q_{s_{t+1}}\left(s^t\right)$ is greater than the entrepreneur's marginal rate of substitution. Moreover, given that intermediaries can borrow and lend at prices $q_{s_{t+1}}\left(s^t\right)$, such a contract is payoff neutral for the intermediary. Similarly, one can construct a deviating contract that makes both the intermediary and the agent strictly better off. This violates the zero profit condition and so is a contradiction. A similar argument applies for the reverse inequality.

The next step in the proof is to show that in any competitive equilibrium of the contracting environment in which full insurance is not being provided, for any consecutive states $\left(s^t, s_{t+1}\right)$, there exists some agent for whom

$$\hat{V}_t^i\left(s^t, \boldsymbol{\zeta}^{i,j}\left(s^t\right); \mathbf{p}_t \mid \delta\left(s^{t+1}\right) = 0\right) = V_t^{i,d}\left(s^t; \mathbf{p}_t\right)$$

The first term in the equality represents the agent's best deviation conditional on not defaulting[2], and the second, is the value of defaulting. The idea is that if this were not the case, the intermediary could increase the amount of insurance being offered while continuing to respect the incentive compatibility constraints. This will be useful in our construction of not-too-tight debt limits. The construction of equilibrium debt constraints relies on a limiting argument due to Fudenberg and Levine (1983). The idea is to construct truncated allocations of the debt constrained environment, the limit of which converges to an equilibrium with not-too-tight debt constraints. I briefly summarize the construction here. Suppose we have an equilibrium allocation and price sequence from the contracting environment. We can construct a sequence of truncated Arrow security holdings as follows; for each T

$$a_{s_T}^{T,i}\left(s^{T-1}\right) = \zeta_T^{i,j}\left(s^T\right) \tag{15}$$

$$a_{s_t}^{T,i}\left(s^{t-1}\right) - \sum_{s^{t+1}} q_{s_{t+1}}\left(s^t\right) a_{s_{t+1}}^{T,i}\left(s^t\right) = \zeta_t^{i,j}\left(s^t\right) \text{ for all } t < T \tag{16}$$

Next, using the previous result I construct debt constraints equal to the asset holding of the agent i for whom $\hat{V}_t^i\left(s^t, \boldsymbol{\zeta}^{i,j}\left(s^t\right); \mathbf{p}_t \mid \delta\left(s^{t+1}\right) = 0\right) = V_t^{i,d}\left(s^t; \mathbf{p}_t\right)$. For $t > T$ define, the truncated debt constraints and asset holdings to 0 for all agents. To complete the construction of the truncated allocations for $t \leq T$, let the consumption, labor and money holdings correspond to those from the Competitive equilibrium, while after T, correspond to the best allocation given 0 debt constraints. Clearly the above allocation does not constitute a competitive equilibrium with not-too-tight constraints. The rest of the proof involves showing that the limit of these truncated allocations constitute an equilibrium with not-too-tight constraints (given prices and money supply policies). To do this I show that the best deviation by any agent from these truncated allocations is bounded and that the maximal deviation possible converges to 0 as $T \to \infty$. We can then use results from Fudenberg and Levine (1983) to prove this allocation converge to an equilibrium.

We have established that when the space of policies are simply money supply rules, the set of equilibria from the two environments coincide. I now consider a larger space of policies and first define a competitive equilibrium when these policies can only depend on the exogenous state

[2] Recall, that the agent's actions are hidden and so a deviation might involve choosing allocation different from those recommend and not necessarily defaulting.

of the world. Next, I prove that set of equilibria is identical to the case in which policies only consist of money supply rules. This result suggests on the surface that expanding the set of policies in this fashion has no bite. However, when I allow these policies to also depend on histories of private actions, I prove that given a competitive equilibrium there exist such policies that uniquely implement it. Formally, let the space of policies be

$$\mathcal{P} = \left\{ \{M\}_0 , \{\chi\}_0 , \{R^g\}_0 , \{\tau\}_0 \mid \forall t,\ s^t,\ \left(M_t \left(s^t\right), R^g_t \left(s^t\right), \tau_t \left(s^t\right) \right) \in \mathbb{R}^3_+,\ \chi_{s^t} : \mathcal{C} \to \{0,1\} \right\}$$

As before $\{M\}_0$ corresponds to the sequence of money supply policies $\left\{M_t\left(s^t\right)\right\}_{t,s^t}$. $\{\chi\}_0$ denotes a sequence of indicator functions that determine whether a particular contract is eligible for the following scheme: all agents signed to this contract who have not defaulted in the past, can deposit the cash received from selling consumption goods (when productive) with the government overnight and receive a return $R^g_{t+1}\left(s^t\right)$ the following period. $\{\tau\}_0$ denotes the sequence of lump sum tax rates that the government can impose on all agents. Given a policy $\pi \in \mathcal{P}$, the intermediaries and agents' problems is almost identical to that described above. A contract which eligible for this savings scheme also stipulates a choice of whether the agent should deposit his labor income with the government. Formally we denote this choice by a sequence indicator function $\{\Sigma\}_0$ where $\Sigma_{s^t} : \mathcal{P} \to \{0,1\}$ and a choice of 1 corresponds to saving with the government. All other agents have no such choice and as a result their decision problems are as above. An eligible agent's budget constraint is

$$p_{t'}\left(s^{t'}\right) m^i_{t'}\left(s^{t'}\right) = \mathcal{R}\left(l^i_{t'-1}\left(s^{t'-1}\right)\right) + p_{t'}\left(s^{t'}\right)\left[m^i_{t'-1}\left(s^{t'-1}\right) - \frac{c^i_{t'-1}\left(s^{t'-1}\right)}{p_{t'-1}\left(s^{t'-1}\right)}\right] + \zeta^{i,j}_{t'}\left(s^t\right)$$

where $\mathcal{R}\left(l^i_{t'-1}\left(s^{t'-1}\right)\right) = R^g_t\left(s^{t-1}\right)$ if $\Sigma_{s^{t-1}} = 1$ and $\mathcal{R}\left(l^i_{t'-1}\left(s^{t'-1}\right)\right) = \frac{p_t\left(s^t\right)}{p_{t-1}\left(s^{t-1}\right)}$ if $\Sigma_{s^{t-1}} = 0$. All agents not eligible for the scheme have budget constraints as in (10).

Definition 5 *Given a policy rule $\pi \in \mathcal{P}$, a competitive equilibrium in this contracting environment consists of prices $\left\{q_{s_{t+1}}\left(s^t\right), p_t\left(s^t\right)\right\}_{t,s^t}$, allocations for each intermediary $\left(C^{i,j}\left(s^t\right), \bar{V}^i_t\left(s^t\right)\right)_{i,j}$ and a sequence of measures $\left\{\left(\mu^{i*}_{s^t}\right)_{i\in I}\right\}_{t,s^t}$ such that*

- *Given prices and the actions of other intermediaries, the allocation for the s^t intermediary solves its problem,*
- *Intermediaries make zero profits*

- *Markets clear*

$$\sum_{i\in I}\int_{\xi\in\mathcal{C}} c_t^{\xi,i}\left(s^t\right)d\mu_{s^t}^{i*}\left(\xi\right)=\sum_{i\in I}\int_{\xi\in\mathcal{C}} l_t^{\xi,i}\left(s^t\right)d\mu_{s^t}^{i*}\left(\xi\right)$$
$$\sum_{i\in I}\int_{\xi\in\mathcal{C}} \zeta_t^{\xi,i}\left(s^t\right)d\mu_{s^t}^{i*}\left(\xi\right)=0$$
$$\sum_{i\in I}\int_{\xi\in\mathcal{C}} m_t^{\xi,i}\left(s^t\right)d\mu_{s^t}^{i*}\left(\xi\right)=M_t^i\left(s^t\right)$$
$$\sum_{i\in I}\int_{\xi\in\mathcal{C}} \chi_{s^t}\left(\xi\right)\left[R_t^g\left(s^t\right)-\frac{p_t\left(s^t\right)}{p_{t-1}\left(s^{t-1}\right)}\right]l_{t-1}^{\xi,i}\left(s^{t-1}\right)d\mu_{s^t}^{i*}\left(\xi\right)=2\tau_t\left(s^t\right)$$

As before, the measure $\mu_{s^t}^{i*}\left(\xi\right)$ denotes the measure of agents of type i in equilibrium who are signed to contract ξ. The next important result states that in equilibrium if any of the contracts are eligible for the scheme, the rate of return offered by the government must coincide with the market return on money.

Proposition 8 *In any competitive equilibrium if $\chi\left(C\right)=1$ for any C such that $\mu^{i*}\left(C\right)>0$ for some i, $\tau_t\left(s^t\right)=0$ for all t, s^t and $R_t^g\left(s^t\right)=\frac{p_t\left(s^t\right)}{p_{t-1}\left(s^t\right)}$ for all t, s^{t-1}, s^t.*

Proof. See Appendix. ■

The reason for this is a simple feasibility argument. Consider the case in which all contracts offered in equilibrium are eligible for the scheme and that $R_t^g\left(s^t\right)>\frac{p_t\left(s^t\right)}{p_{t-1}\left(s^t\right)}$ for some date and state. Then it is easy to see that such a policy can never satisfy the government budget constraint since the total amount owed by the government exceeds the total output in the economy. On the other hand, if only some contracts are eligible then it must be these offer more insurance and as a result will be chosen by all agents and as a result, the same feasibility argument applies.

The result states that the set of equilibria with the expanded set of policy instruments is identical to the set with money supply rules as studied previously. Importantly, it says that these policies do not give the government the ability to implement equilibria better than the best equilibrium with money supply rules. As I will demonstrate in the next section such policies are useful because they help *uniquely* implement particular equilibria rather than expand the equilibrium set.

3.2 Sophisticated Policies

From the equivalence result proved in the previous section, we know that there are multiple equilibria in the contracting environment given the set of instruments available to the government. In particular, there is an equilibrium in which no intermediary offers any insurance and agents only use money balances to smooth consumption. The goal of this section is to construct policies that uniquely implement the desired equilibrium. I consider Sophisticated Policies as in Atkeson, Chari, and Kehoe (2010) which allow policy to depend on histories of private actions and to differ on and off the equilibrium path and require that all continuation outcomes be continuation competitive equilibria. There are two key requirements that any Sophisticated policy must satisfy

1. Controllability: A necessary condition for a set of contracts $\left(\tilde{C}^j, \tilde{C}^{-j}\right)$, where $C^{-j} = \left(C^{j'}\right)_{j' \neq j}$ is that $\tilde{C}^j \in BR^j\left(\tilde{C}^{-j}; \Phi^g\right)$ where $BR^j\left(\tilde{C}^{-j}; \Phi^g\right)$ denotes the best response correspondence of intermediary j when other intermediaries are offering $\tilde{C}^{-j}$, conditional on prices, policy (Φ^g) and past histories. *We say that a Sophisticated policy is controllable if* $\tilde{C}^j \notin BR^j\left(\tilde{C}^{-j}; \Phi^g\right)$ *for* $\left(\tilde{C}^j, \tilde{C}^{-j}\right) \neq \left(C^{j*}, \tilde{C}^{-j*}\right)$ *and* $C^{j*} \in BR^j\left(\tilde{C}^{-j*}; \Phi^g\right)$.

2. *For all histories (including deviations), the continuation outcomes constitute a continuation competitive equilibrium.*

An overview of the construction is as follows. We now allow the policy instruments defined in the previous section to also depend on endogenous histories. The policy which implements the desired equilibrium has the government intervening after certain histories and allowing certain contracts to be eligible for the risk free savings scheme. To illustrate why history dependence is important suppose we want to uniquely implement the equilibrium in which all intermediaries offer the contract associated with the best equilibrium C^* and none of the agents default. We know that another equilibrium exists in which all intermediaries offer some contract $\tilde{C} \neq C^*$. Consider a history in which all but a positive measure of intermediaries offer $\tilde{C}$ while the small measure offer a contract $\hat{C}$ with the property that $\hat{C}$ offers more insurance than $\tilde{C}$. After such a history, the government allows only $\hat{C}$ to be eligible for the scheme. The role of the scheme is that given current prices and transfers, it raises the value of not defaulting relative to defaulting. As a result, a contract like $\hat{C}$ is incentive compatible and can be constructed to give the intermediaries positive profit. Therefore an allocation in which all intermediaries offer $\tilde{C}$ cannot constitute a competitive equilibrium. While there are other types of deviating allocations, the argument is similar. We formalize this argument below.

The timing of the game is follows (see Figure 4). After s_t realized, agents make default decisions, then intermediaries make transfers to agents following which markets for money open. Next, the government announces which contracts are eligible for the scheme after which the agent splits into a shopper and worker. If eligible, the worker can hold the income from production with the government overnight and finally consumption takes place.

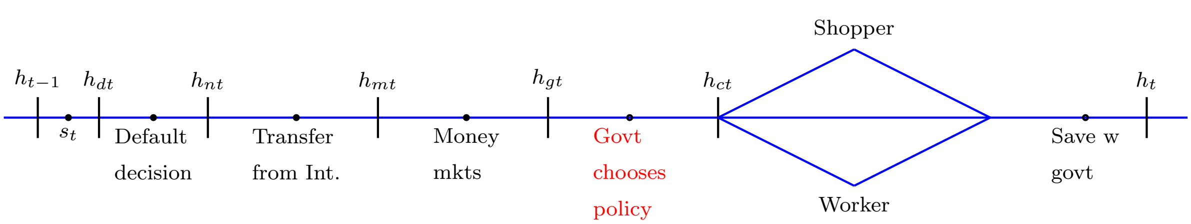

Figure 4: Timing

Let $\iota_t = \left(s_t, \left\{\Delta_t^i\right\}_i, \left\{q_{s_{t+1}}\right\}_{s_{t+1}}, \left\{C^{i,j}\right\}_{i,j},\ p_t, \Phi_t^g, \tau_t\right)$ where Δ_t^i is agent $i's$ default decision, $C^{i,j}$ is the current contract agent between agent i and intermediary j, $\Phi_t^g = \left(M_{t+1}, \chi_t, R_{t+1}^g, \tau_t\right)$ denotes the government policy and $h_t = (h_{t-1}, \iota_t)$ be the public history after period t. The public history faced by agents making their default decisions is $h_{dt}\,(h_{t-1}, s_t)$ and the default strategy is given by $\sigma_d^i\,(h_{dt}) = \Delta_t^i\,(h_{dt})$. Consequently, the history faced by intermediaries when offering contracts is $h_{nt} = \left(h_{t-1}, s_t, \left\{\Delta_t^i\right\}, \left\{q_{s_{t+1}}\right\}_{s_{t+1}}\right)$. Denote the strategy for intermediary j as $\sigma_n^j = \left\{C^{i,j}\,(h_{nt})\right\}_i$. Next, the history faced by agents when purchasing money in private markets is $h_{mt} = (h_{t-1}, s_t, h_{nt})$. We denote the strategy for agent i as $\sigma_a^i = m_t^i\,(h_{mt})\,(\cdot)$ where $m_t\,(h_{mt})\,(\cdot)$ is a demand schedule for money

$$m_t^i\,(h_{mt}) : R_+ \to R$$

Agents submit demand schedules to a Walrasian auctioneer who chooses a price p_t to clear the money markets. A price rule as determined by the auctioneer is given by $\sigma_p = \{p\,(h_{mt})\}$. The public history faced by the government when setting policy $h_{gt} = \left(h_{t-1}, s_t, \left\{\Delta_t^i\right\}_i, \left\{q_t^{s_{t+1}}\right\}_{s_{t+1}}, \left\{C^i\right\}_i, p_t\right)$. Note that the definition of Sophisticated equilibrium will not require that the government sets its policy optimally. Denote the government strategy by $\sigma_g = \left\{M_{t+1}\,(h_{gt}), \chi_t\,(h_{gt}), R_{t+1}^g\,(h_{gt}), \tau_t\,(h_{gt})\right\}$. Next, before the worker and shopper split up, the agent decides how much consumption goods the shopper should purchase (subject to CIA constraints). Let $h_{ct} = (h_{t-1}, h_{gt})$. Denote the consumption strategy by $\sigma_c^i = \left\{c_t^i\,(h_{ct})\right\}$. Finally if eligible, the worker can hold the cash earned from producing consumptions goods with the government overnight. This strategy is given by $\sigma_\Sigma\,(h_{ct}) = \Sigma_t^i\,(h_{ct})$.

Strategies induce continuation outcomes $\{o_r\,(s^r \mid h_{t-1}; \sigma)\}$ as follows. Agents' default policy is given by $\Delta^i\left(s^t \mid h_{dt}; \sigma\right) = \Delta_t^i\,(h_{dt})$ which is obtained from σ_d^i, the intermediary's contract choice is given by $C^{i,j}\left(s^t \mid h_{nt}; \sigma\right) = C^{i,j}\,(h_{nt})$ obtained from σ_n^j and the agent's choice of money holdings is given by $m_t^i\left(s^t \mid h_{t-1}; \sigma\right) = m_t^i\,(h_{at})$ obtained from σ_a^i. The government's policy is determined by $\chi_t\left(s^t \mid h_{gt}; \sigma\right) = \chi_t\,(h_{gt})$, $R_{t+1}^g\left(s^t \mid h_{gt}; \sigma\right) = R_{t+1}^g\,(h_{gt})$ and $\tau_t\left(s^t \mid h_{gt}; \sigma\right) = \tau_t\,(h_{gt})$ obtained from σ_g. Finally, the agent's decision of how much to consume is given by $c_t^i\left(s^t \mid h_{ct}; \sigma\right) = c_t^i\,(h_{ct})$ obtained from σ_c^i.

We can define the concept of a continuation competitive equilibrium as follows

Definition 6 *A continuation competitive equilibrium given a history h_{gt} is a collection of allocations, prices and policies that satisfy*

1. *Agent optimality conditions*
2. *Government budget feasibility*
3. *Market Clearing*
4. *Debt constraints are chosen to be NTT*

Using the previous definition, we can define a Sophisticated Equilibrium.

Definition 7 *A Sophisticated equilibrium given policies is a collection of strategies for each agent* $i \in I$, $\left(\sigma_d^i, \sigma_a^i, \sigma_c^i\right)$, *intermediary* $j \in J$, σ_n^j, *the government* σ^g *and price rules* σ_p *such that*

1. *Given any history* h_{t-1}, *the continuation outcomes* $\{o_r\left(s^r \mid h_{t-1}; \sigma\right)\}$ *induced by* σ *constitute a competitive equilibrium and*

2. *Given any history* h_{gt} *the continuation outcomes* $\{o_r\left(s^r \mid h_{gt}; \sigma\right)\}$ *constitute a continuation competitive equilibrium*

As mentioned earlier, the key requirement of a Sophisticated equilibrium is that continuation outcomes be continuation competitive equilibria. Competitive equilibria must exist after all histories, and as a result we rule out policies that implement equilibria via non-existence. For example, policies that only feasible on path but are not off path can never be part of a competitive equilibrium.

Definition 8 *A policy* σ_g^*, *uniquely implements a desired competitive equilibrium* $\{o_r^*\}$ *if the Sophisticated outcome associated with any Sophisticated equilibrium of the form* $\left(\sigma_g^*, \sigma_d, \sigma_n, \sigma_a, \sigma_g, \sigma_{a2}, \sigma_p, \sigma_c\right)$ *coincides with the desired competitive equilibrium.*

We now turn to the main result in the section which proves that there exists Sophisticated policies that uniquely implement the desired equilibrium.

Theorem 2 *Given any symmetric competitive equilibrium, there exist Sophisticated policies that uniquely implement it.*

The policies I consider will allow the instruments defined in the previous section to now depend on histories h_{gt}. In particular, after such a history some agents will be allowed to hold money overnight with the government at attractive rates. Recall that an agent who is productive at date t and state s^t and who works an amount $l_t\left(s^t\right)$ receives his income $\frac{p_{t+1}\left(s^{t+1}\right)}{p_t\left(s^t\right)} l_t\left(s^t\right)$ the following period. Under this scheme, the agent will be able to hold the income overnight with the government and receive $R_{t+1}^g l_t\left(s^t\right)$ the following period with $R_{t+1}^g \geq \frac{p_{t+1}\left(s^{t+1}\right)}{p_t\left(s^t\right)}$. Let $\mathbf{R}_t^g = \left\{R_{t'}^g\right\}_{t' \geq t}$ denote a sequence of such interest rates. Define $V_t^{i,g}\left(s^t; \mathbf{R}_t^g, \mathbf{p}_t\right)$ to be the value for agent i of the following problem; he maximizes (13) subject to budget constraints,

$$p_t\left(s^t\right) m_t^i\left(s^t\right) \leq \mathcal{R}\left(l_{t'-1}^i\left(s^{t'-1}\right)\right) + x_t^i + p_t\left(s^t\right)\left[m_{t-1}^i\left(s^{t-1}\right) - \frac{c_{t-1}^i\left(s^{t-1}\right)}{p_{t-1}\left(s^{t-1}\right)}\right], \text{ if } s_{t-1} = i,$$

$$p_t\left(s^t\right) m_t^i\left(s^t\right) \leq p_t\left(s^t\right)\left[m_{t-1}^i\left(s^{t-1}\right) - \frac{c_{t-1}^i\left(s^{t-1}\right)}{p_{t-1}\left(s^{t-1}\right)}\right] \text{ if } s_{t-1} \neq i$$

cash in advance constraints (4) and non-negativity constraints $m_{t+1}^i\left(s^t\right) \geq 0$. Note that this is the value to agent i of not being able to trade arrow securities but have access to the overnight scheme. The following lemma will be useful for proving the result.

Lemma 3 *For all t, s^t, there exists a sequence of risk free rates $\mathbf{R}_t^g = \left\{R_{t'}^g\right\}_{t' \geq t}$ with the property that for any agent i,*

$$V_{t+1}^{i,g}\left(s^{t+1}; \mathbf{R}_{t+1}^g, \mathbf{p}_{t+1}\right) > V_{t+1}^{i,d}\left(s^{t+1}; \mathbf{p}_{t+1}\right)$$

Proof. Recall that $V_{t+1}^{i,d}\left(s^{t+1}; \mathbf{p}_{t+1}\right)$ is the value of not participating in financial markets in all future periods for agent i in s^{t+1} and has been defined previously. While in financial autarky, agents can continue to hold money, thereby having access to a savings technology with interest rate $\frac{p_{t+1}}{p_t}$. Notice that under the government scheme the agent's budget constraints are identical except that they can hold (if they choose) their labor income with the government overnight. Clearly if $R_{t+1}^g \geq \frac{p_{t+1}}{p_t}$

$$V_{t+1}^{i,g}\left(s^{t+1}; \mathbf{R}_{t+1}^g, \mathbf{p}_{t+1}\right) \geq V_{t+1}^{i,d}\left(s^{t+1}; \mathbf{p}_{t+1}\right)$$

with this inequality being strict if $R_{t+1} > \frac{p_{t+1}}{p_t}$. ■

Proof of Theorem 2. Let the histories associated with the desired equilibrium allocations/prices be denoted by *. We need to consider all possible deviations from the desired equilibrium and construct policy with the property that going along with these deviations is not in the best interest of an individual agent and that the continuation outcomes constitute continuation competitive equilibria.

We first specify government policy after all continuation histories h_{gt} :

1. Consider a history $\tilde{h}_{gt} = \left(h_{t-1}^*, s_t, \left\{\Delta_t^i\right\}_i, \tilde{p}_t, \left\{\tilde{q}_t^{s_{t+1}}\right\}_{s_{t+1}}, \left\{\tilde{C}^{i,j}\right\}_{j \in J \setminus J^d}, \left\{C^{i,j}\right\}_{j \in J^d}, \left\{\tilde{m}^i\left(s^t\right)\right\}_{i \in I^d}, \left\{m^i\left(s^t\right)\right\}_{i \in I \setminus I^d}\right)$. In such a history, there is no default, but the equilibrium prices and contracts do not correspond to those of the desired equilibrium. Moreover, all intermediaries besides those in J^d , offer contracts different from the desired ones while all intermediaries $j \in J^d$ offer a contract that with the property that some agent at some date receives strictly more insurance than $\tilde{C}$. These contracts satisfy

$$\frac{\beta\pi\left(s^{t+1} \mid s^t\right) u'\left(c_{t+1}^i\left(s^{t+1}\right)\right)}{u'\left(c_t^i\left(s^t\right)\right)} = \frac{\beta\pi\left(s^{t+1} \mid s^t\right) u'\left(\tilde{c}_{t+1}^i\left(s^{t+1}\right)\right)}{u'\left(\tilde{c}_t^i\left(s^t\right)\right)} \quad \text{for all } t \neq t', \left(s^t, s^{t+1}\right) \neq \left(s^{t'}, s^{t'+1}\right), i \neq i'$$

$$\frac{\beta\pi\left(s^{t+1} \mid s^t\right) u'\left(c_{t'+1}^{i'}\left(s^{t'+1}\right)\right)}{u'\left(c_{t'}^i\left(s^{t'}\right)\right)} > \frac{\beta\pi\left(s^{t'+1} \mid s^{t'}\right) u'\left(\tilde{c}_{t'+1}^i\left(s^{t'+1}\right)\right)}{u'\left(\tilde{c}_{t'}^i\left(s^{t'}\right)\right)} \text{ for } i$$

After such a history, the government sets $\chi_t\left(\tilde{h}_{gt}\right)(C) = 1$ and $\chi_t\left(\tilde{h}_{gt}\right)\left(\tilde{C}\right) = 0$ and so offers agents who have signed with intermediaries $j \in J^d$ and who haven't defaulted in the past, the ability hold their income overnight with the government at interest rates $\mathbf{R}_t^g\left(\tilde{h}_{gt}\right)$ in all future periods chosen so that

$$V_{t+1}^{i,g}\left(s^{t+1}; \mathbf{R}_{t+1}^g\left(\tilde{h}_{gt+1}\right), \mathbf{p}_{t+1}\right) > V_{t+1}^{i,d}\left(s^{t+1}; \mathbf{p}_{t+1}\right)$$

The rate of return on the government scheme is greater than the return on money. We know that such a policy exists from the previous lemma. This is financed by a lumpsum tax on all agents. In order for the scheme to be feasible,

$$\begin{aligned}&\sum_{i\in I}\int_{\xi\in\mathcal{C}}\chi_{s^t}(\xi)\left[R_t^g\left(s^t\right)-\frac{p_t\left(s^t\right)}{p_{t-1}\left(s^{t-1}\right)}\right]l_{t-1}^{\xi,i}\left(s^{t-1}\right)d\mu_{s^t}^{i*}(\xi)\\&=\sum_{i\in I}\left[R_t^g\left(s^t\right)-\frac{p_t\left(s^t\right)}{p_{t-1}\left(s^{t-1}\right)}\right]l_{t-1}^{C,i}\left(s^{t-1}\right)\mu_{s^t}^{i*}(C)=2\tau_t\left(s^t\right)\end{aligned}$$

Therefore for $\mu_{s^t}^{i*}(C)$ small enough, the policy is always feasible. Since we need only consider infinitesimal deviations, the set J^d can always be made small enough so that feasibility is satisfied.

2. Consider a history $\tilde{h}_{gt}=\left(h_{t-1}^*,s_t,\left\{\tilde{\Delta}_t^i\right\}_i,...\right)$, i.e. one in which all agents but a positive measure $\hat{\mu}^d$ of agents default on their obligations to the intermediaries. After such a history the government sets $\chi\left(\tilde{h}_{gt}\right)(C)=1$ for all contracts offered in equilibrium thereby offering all agents who haven't defaulted access to the savings scheme with high interest rates as described above. As $\hat{\mu}^d\to 0$, the amount owed by the government $\sum_{i\in I}\int_{\xi\in\mathcal{C}}\chi_{s^t}(\xi)\left[R_t^g\left(s^t\right)-\frac{p_t\left(s^t\right)}{p_{t-1}\left(s^{t-1}\right)}\right]l_{t-1}^{\xi,i}\left(s^{t-1}\right)d\mu_{s^t}^{i*}(\xi)\to 0$, since even though all contracts are eligible, most agents are choosing to default and hence cannot avail the scheme.

3. After all other histories, the government chooses $\chi(h_{gt})$ so that none of the contracts are eligible for the scheme.

4. Money supply policies are chosen after all histories to be consistent with the Ramsey policy (which departs from the Friedman Rule).

   Next, I show how such a policy uniquely implements the desired equilibrium. We need to consider the set of competitive equilibria different from the desired one in the environment without government intervention and show that they cease to be equilibria given the above policy. There are two relevant deviations to consider. First consider the case in which in some period τ, the government observes a history $\tilde{h}_{g\tau}$ with no history of default, but with $\tilde{h}_{g\tau}\neq h_{g\tau}^*$. For example, it could observe the prices and allocations corresponding to the worst competitive equilibrium (financial autarky). If we consider symmetric equilibria, then after such a history, the government does not intervene. Now let us check if these allocations can actually constitute a CE. Even though all other intermediaries are offering contract $\tilde{C}$, consider the incentive of an intermediary j to go along with this equilibrium. He knows by offering some agent slightly more insurance, his agents will have access to the savings scheme. As we will see, this allows him to offer some agent a little more insurance and allow him to make strictly positive profits.

Given that full insurance is not being provided, there is some agent i, states s^{τ}, $s^{\tau+1}$ such that

$$\tilde{q}_{s_{\tau+1}}\left(s^{\tau}\right) > \frac{\beta\pi\left(s^{\tau+1} \mid s^{\tau}\right) u'\left(\tilde{c}^{i}_{\tau+1}\left(s^{\tau+1}\right)\right)}{u'\left(\tilde{c}^{i}_{\tau}\left(s^{\tau}\right)\right)}$$

Then an intermediary j at s^{τ} can offer the following contract to agent i

$$\hat{\zeta}^{i,j}_{\tau}\left(s^{\tau}\right) = \tilde{\zeta}^{s^t,i}_{\tau}\left(s^{\tau}\right) + \tilde{q}_{s_{\tau+1}}\left(s^{\tau}\right)\varepsilon_1$$
$$\hat{\zeta}^{i,j}_{\tau+1}\left(s^{\tau+1}\right) = \tilde{\zeta}^{i,j}_{\tau+1}\left(s^{\tau+1}\right) - \varepsilon_2$$

with the rest of the contract unchanged. For $\varepsilon_1 = \varepsilon_2 = \varepsilon$ small the change in welfare is

$$\left[\tilde{q}_{s_{\tau+1}}\left(s^{\tau}\right) u'\left(\hat{c}^{i}_{\tau}\left(s^{\tau+1}\right)\right) - \beta\pi\left(s^{\tau+1} \mid s^{\tau}\right) u'\left(\hat{c}^{i}_{\tau+1}\left(s^{\tau+1}\right)\right)\right]\varepsilon > 0$$

For ε small enough this contract gives greater utility to the agent while leaving the intermediary equally well off. In particular we can find $\varepsilon_1, \varepsilon_2$ such that the agent is made strictly better off and

$$-q_{s_{\tau+1}}\left(s^{\tau}\right)\varepsilon_1 + q_{s_{\tau+1}}\left(s^{\tau}\right)\varepsilon_2 > 0$$

Recall that since the tilde allocations correspond to a CE for this agent i

$$\sum_{t'=\tau+1}^{\infty}\sum_{s^{t'}\succeq s^{\tau}} \beta^{t'-\tau}\left[u\left(\tilde{c}^{i}_{t'}\left(s^{t'}\right) - \tilde{l}^{i}_{t'}\left(s^{t'}\right)\right)\right] = \hat{V}^{i}_{\tau+1}\left(s^{\tau+1}, \tilde{\boldsymbol{\zeta}}^{i,j}\left(s^{\tau+1}\right); \tilde{\mathbf{p}}_{\tau+1}\right)$$

and so if there were no government. intervention after such a history

$$\sum_{t'=\tau+1}^{\infty}\sum_{s^{t'}\succeq s^{\tau}} \beta^{t'-\tau}\left[u\left(\hat{c}^{i}_{t'}\left(s^{t'}\right) - \hat{l}^{i}_{t'}\left(s^{t'}\right)\right)\right] < \hat{V}^{i}_{\tau+1}\left(s^{\tau+1}, \tilde{\boldsymbol{\zeta}}^{i,j}\left(s^{\tau+1}\right); \tilde{\mathbf{p}}_{\tau+1}\right)$$

since the agent will strictly prefer to default and live in autarky forever. However, given government policy after such a history, the intermediary can choose some $\varepsilon_1, \varepsilon_2 \neq 0$ so that incentive compatibility still holds.

To see that all the incentive compatibility constraints are still satisfied notice that as long as $\hat{\zeta}^{s^t,i}_{t+1}\left(s^{t+1}\right)$ is chosen so that

$$u\left(\hat{c}^{i}_{\tau+1}\left(s^{\tau+1}\right)\right) + \beta E_{\tau+1}\left[V^{i,g}_{\tau+2}\left(s^{\tau+2}; \mathbf{R}^{g}_{\tau+2}\left(\tilde{h}_{g\tau+2}\right), \mathbf{p}_{\tau+2}\right)\right] \geq V^{i,d}_{t+1}\left(s^{t+1}; \mathbf{p}_{t+1}\right)$$

the incentive compatibility constraints are satisfied since

$$\begin{aligned}
&\sum_{t'=\tau+1}^{\infty} \sum_{s^{t'} \succeq s^{\tau}} \beta^{t'-\tau} \left[u\left(\tilde{c}^i_{t'}\left(s^{t'}\right) - \tilde{l}^i_{t'}\left(s^{t'}\right)\right)\right] \\
&\geq u\left(\hat{c}^i_{\tau+1}\left(s^{\tau+1}\right)\right) + \beta E_{\tau+1}\left[\hat{V}^i_{\tau+1}\left(s^{\tau+1}, \boldsymbol{\zeta}^{i,j}\left(s^{\tau+1}\right); \mathbf{p}_{\tau+1}, \mathbf{R}^g_{\tau+1}, \Delta\left(s^{\tau+1}\right)=0\right)\right] \\
&\geq u\left(\hat{c}^i_{\tau+1}\left(s^{\tau+1}\right)\right) + \beta E_{\tau+1}\left[V^{i,g}_{\tau+2}\left(s^{\tau+2}; \mathbf{R}^g_{\tau+2}\left(\tilde{h}_{g\tau+2}\right), \mathbf{p}_{\tau+2}\right)\right] \\
&\geq V^{i,d}_{\tau+1}\left(s^{\tau+1}; \mathbf{p}_{\tau+1}\right) \\
&= \hat{V}^i_{\tau+1}\left(s^{\tau+1}, \boldsymbol{\zeta}^{i,j}\left(s^{\tau+1}\right); \mathbf{p}_{\tau+1}, \mathbf{R}^g_{\tau+1}, \Delta\left(s^{\tau+1}\right)=1\right)
\end{aligned}$$

As a result, $\tilde{h}_{g\tau}$ can never be part of a competitive equilibrium.

The second kind of equilibria we need to rule out are sunspot equilibria in which agents default due to their beliefs about the types of contracts intermediaries will be willing to offer them in the future. Consider the case in which at date zero we start off with existing contracts in which some agents owe payments to the intermediaries. At date 0, there is a sunspot shock $\iota_0 \in \{0, 1\}$ with $\Pr[\iota_0 = 0] = \psi$ which determines if agents stay in the prescribed equilibrium ($\iota_0 = 1$) or if autarky will be played in all future dates in states. If the sunspot is realized, agents who owe payments to the intermediary will strictly prefer to default and live in autarky forever. Moreover, given this no intermediary is willing to lend to agents in the current period As above, we need to consider government. policy after such a history which helps rule it out. Consider the policy after a history in which all agents besides a positive measure have defaulted. Then as described, the government. offers these agents access to the overnight savings scheme with interest rates chosen so that

$$V^{i,g}_{t+1}\left(s^{t+1}; \mathbf{R}^g_{t+1}\left(\tilde{h}_{gt+1}\right), \mathbf{p}_{t+1}\right) > V^{i,d}_{t}\left(s^{t+1}; \mathbf{p}_{t+1}\right)$$

As a result, even if all other agents are defaulting and no intermediary he is still willing to pay the intermediary back in order to be eligible for this government. scheme. Therefore, given that agents are not defaulting, an individual intermediary as an incentive to offer some insurance and so a history consisting of default by all agents and no intermediary offering insurance can never be part of a CE.

Finally we need to check that all continuation histories constitute continuation competitive equilibria. The feasibility of the government policy was shown earlier. To show that all continuation histories constitute continuation competitive equilibria, it suffices to note that since the government policy is feasible, and the prices already constitute a part of a CE, the continuation histories constitute a continuation CE. ■

The proof constructs policies such that given these, there exists a unique symmetric competitive equilibrium corresponding to the desired one. To show this, I consider any other allocation-price pair and show that this cannot be an equilibrium given these policies. There are two types of allocations to consider. The first is one in which there is no default, but all intermediaries offer contracts different from the desired one C^*. The second type is one in which there is default, and intermediaries offer some contract. Sophisticated policies must be specified after each history. After a history of no default and intermediaries offering contracts corresponding to the desired ones, the government continues its money supply policy consistent with the Ramsey policy. Next, consider a history in which all intermediaries besides those in some set J^d offer $\tilde{C}$ different from the desired one, while intermediaries in J^d offer contracts $\hat{C}$ which are identical to $\tilde{C}$ except that it offers some agent slightly more insurance. After such a history, the government offers an overnight risk free savings scheme to all agents signed with any of the intermediaries in J^d. Next consider a history in which all agents besides those in I^d have defaulted on their obligations. After such a history, the government offers only agents in I^d access to a similar overnight risk free savings technology as described above. After any other history not encompassing the ones specified above, the government continues with the usual Ramsey policy.

To see why how this policy rules out the types of equilibria specified above, first consider the allocation in which there is no default but intermediaries offer contracts $\tilde{C} \neq C^*$. Consider the best response of a particular intermediary j, when all other intermediaries are offering contracts $\tilde{C}$ given the Sophisticated policy constructed above. Intermediary j has an incentive to offer some agent slightly more insurance than $\tilde{C}$ which would make both him and the agent strictly better off. This perturbation is incentive compatible since it is exactly after such histories that the government chooses to intervene and offer its overnight scheme. The agent will not want to default if this savings technology is particularly attractive. As a result, even though all other intermediaries are offering contracts $\tilde{C}$, intermediary j has an incentive to deviate and offer a different contract thereby implying that an equilibrium with contracts $\tilde{C}$ cannot exist. A similar argument works for an allocation with default. This includes sunspot allocations in which each period there is a positive probability that in all future periods, no intermediary will be willing to lend to agents. In such a state all agents who owe payments to intermediaries will strictly prefer to default. However as above, even if all other agents are defaulting and no intermediary in the future is willing to lend, a particular intermediary j will have an incentive to offer some insurance to an agent making both strictly better off. The agent will not want to default if the return on the risk free savings technology offered by the government is sufficiently attractive. As a result, these sunspot outcomes will cease to be equilibria given these policies.

An important point to note is that the government cannot use these Sophisticated policies to achieve outcomes that welfare dominate all competitive equilibria. This is clear from proposition 8 which states that the government can never offer such a savings scheme on the equilibrium path. As a result the set of implementable equilibria correspond to the ones with simple money supply policies.

4 Discussion

In this section, I discuss some of assumptions and consider some generalizations of the results presented in this paper.

4.1 Generalizations

The model presented in the earlier sections was deliberately chosen to be simple. The assumptions allowed us to obtain sharp characterizations of the multiplicity and the effect of simple policies. The implementation argument presented in the last section did use any of these simplifying assumptions, for example that there were two types or the quasi-linearity of utility function. If we increase the number of types and assume more general utility functions, very similar policies will work. More generally, such policies will work in environments where the agents can achieve limited consumption smoothing after default.

4.2 Equivalence Theorem

The results in the previous sections illustrate why the equivalence result is useful. In particular, one way to interpret not-too-tight debt constraints is that they are constraints on debt derived from the profit maximizing incentive compatible contracts offered by intermediaries when the agents have limited commitment and their actions are unobservable. Consider using similar Sophisticated Policies to implement the desired equilibrium in the debt constrained environment. Suppose the government committed to offering a similar risk free savings scheme; after appropriately chosen histories it allowed a subset of agents to save with it at attractive rates. The reason this implemented the desired equilibrium in the contracting environment was that even though no other intermediary was willing to lend to the agent in the future, an individual intermediary was willing to offer some insurance to an agent. The reason the same logic does not quite apply to the debt constrained environment is that in any equilibrium

$$q_{s_{t+1}}\left(s^t\right) = \max_{i \in I}\left\{\frac{\beta\pi\left(s^{t+1} \mid s^t\right) u'\left(c^i_{t+1}\left(s^{t+1}\right)\right)}{u'\left(c^i_t\left(s^t\right)\right)}\right\}$$

In particular, since the Arrow prices equal the unconstrained agent's marginal rate of substitution (i.e. the savers), they have no incentive to lend to the constrained agents at these prices no matter what the interest rate on the savings scheme is. As a result, even though the government policy raises the value of not defaulting relative to default, at the market prices, the unconstrained agent is not willing to lend. Does this imply that these policies fail to implement the desired equilibrium? The answer to this question is no, since the definition of an equilibrium also requires these constraints to be not-too-tight. However since this neither an optimality nor a market clearing constraint in this environment it is unclear how they are affected by policy. This why the contracting environment is particularly suited to understanding implementation. The not-too-tight constraints now arise from the best responses of intermediaries. In that environment, the savings policy by the government achieves unique implementation because *offering no insurance is not a best response of a particular intermediary* even though all other intermediaries are not offering any insurance.

4.3 Using a Monetary Model

There are a few reasons why considering a monetary economy is attractive. The first is that it allows us to impose a reasonable assumption on what the consequences of default are. As mentioned in the introduction the literature has made various assumptions on what these can be, for example, autarky and being only allowed to save in Arrow securities. Here I assume that while defaulting agents cannot trade Arrow securities, they can continue to hold money. The interpretation is that money is hidden from the eyes of any regulatory authority and thus cannot be seized. As a result, defaulting agents can achieve some level of consumption smoothing. The second attractive feature of introducing money, is that in the bad equilibrium, there is still some consumption smoothing as opposed to pure autarky which is true in the other models with endogenous default.

The third attractive feature is that there is a literature on optimal policy in models with fiat money. This allows us to consider existing policy proposals that were deemed to implement desirable outcomes and see whether such policies continued to do so when we included financial frictions in these models. For example, policies consistent with the Friedman Rule are widely known to be desirable in monetary models. As I showed in previous sections, this is not the case here and moreover, there are also other undesirable equilibria when we include financial frictions. Notice that these equilibria are not the ones we are generally familiar with in monetary models in which there is an indeterminacy of the price level. The multiplicity arises because the endogenous debt constraints induce a strategic complementarity between the actions of agents. This motivates the study alternate policies and particular those that depend also on the histories of private actions.

5 Conclusion

Any reasonable discussion of government intervention in financial markets cannot be had using models that exogenously impose the form and tightness of financial constraints. In this paper, I provide a framework to think about the role policy can play in mitigating undesirable outcomes that occur in models with financial frictions. To do this I study a financial frictions model in which debt constraints are an equilibrium object. This model has multiple equilibria that differ in how well financial markets work. The main result in the paper shows that in models with endogenous debt constraints, policies that offer attractive savings opportunities after certain histories uniquely implement the best competitive equilibrium. There are two main results that are key to understanding the implementation argument. The first is that the set of equilibria in models of endogenous debt constraints is identical to the set in a decentralized contracting problem in which intermediaries offer insurance contracts to agents whose actions are hidden. The second result is that policies that offer such savings schemes after certain histories raise the value of not defaulting relative to that of defaulting for certain agent. As a result, even though all other intermediaries are offering contracts associated with an undesirable equilibrium, an individual intermediary strictly prefers to offer a different contract which offers more insurance to some agent. As a result, this ceases to be competitive equilibrium.

The environment was deliberately chosen to be simple in order to characterize in closed form the properties of the different equilibria. The implementation argument is very general and can apply to a wider variety of environments. For example, in ongoing work, I study a similar model with capital accumulation and heterogeneous entrepreneurs who has who run production technologies with stochastic productivities. As in this paper, I assume that these entrepreneurs are subject to state contingent debt limits which are determined to be not-too-tight. I show that a similar equivalence result holds for that environment as well and that are multiple equilibria. As a result, similar Sophisticated Policies can implement desirable outcomes in that setup as well.

Appendix A

Proof of Lemma 1. The first order conditions from the agent's problem imply

$$\beta^t \pi\left(s^t\right) u'\left(c_t^i\left(s^t\right)\right) = Q_t\left(s^t\right) \mu^i$$

If $s_t = i$,

$$\begin{aligned} \beta^t \pi\left(s^t\right) &= Q_t\left(s^t\right) \mu^i \\ &\Rightarrow u'\left(c_t^i\left(s^t\right)\right) = 1 \end{aligned}$$

Since markets are complete, marginal rates of substitutions are equalized across agents and states. Therefore

$$\begin{aligned} \frac{\beta^{t'} \pi\left(s^{t'}\right) u'\left(c_{t'}^i\left(s^{t'}\right)\right)}{\beta^t \pi\left(s^t\right) u'\left(c_t^i\left(s^t\right)\right)} &= \frac{\beta^{t'} \pi\left(s^{t'}\right) u'\left(c_{t'}^j\left(s^{t'}\right)\right)}{\beta^t \pi\left(s^t\right) u'\left(c_t^j\left(s^t\right)\right)} \\ &\Rightarrow \frac{u'\left(c_{t'}^i\left(s^{t'}\right)\right)}{u'\left(c_t^i\left(s^t\right)\right)} = \frac{u'\left(c_{t'}^j\left(s^{t'}\right)\right)}{u'\left(c_t^j\left(s^t\right)\right)} \\ &\Rightarrow u'\left(c_{t'}^i\left(s^{t'}\right)\right) = \frac{1}{u'\left(c_t^j\left(s^t\right)\right)} \end{aligned}$$

So in a symmetric competitive equilibrium $u'\left(c_{t'}^i\left(s^{t'}\right)\right) = u'\left(c_t^j\left(s^t\right)\right) = 1$. And so $c^i\left(s^t\right) = g\left(1\right)$ for all s^t. Also

$$\frac{Q\left(s^{t+1}\right)}{Q\left(s^t\right)} = \beta\pi\left(s^{t+1} \mid s^t\right)$$

And so the price of a risk free bond is

$$\sum_{s^{t+1}} \frac{Q\left(s^{t+1}\right)}{Q\left(s^t\right)} = \beta$$

and the risk free rate is $\frac{1}{\beta}$. ■

Proof of Proposition 1. The first order conditions from the agent i's problem are

$$\beta^t \pi\left(s^t\right) u'\left(c_t^i\left(s^t\right)\right) = \mu_t^i\left(s^t\right) + \sum_{s^{t+1}} \lambda_{t+1}^i\left(s^t\right) \frac{p_{t+1}\left(s^{t+1}\right)}{p_t\left(s^t\right)}$$

$$0 = \lambda_t^i\left(s^t\right) q_{s_{t+1}}\left(s^t\right) - \lambda_{t+1}^i\left(s^{t+1}\right)$$

$$0 = \lambda_t^i\left(s^t\right) p_t\left(s^t\right) - \mu_t^i\left(s^t\right) p_t\left(s^t\right) - \sum_{s^{t+1}} \lambda_{t+1}^i\left(s^{t+1}\right) p_{t+1}\left(s^{t+1}\right)$$

$$\beta^t \pi\left(s^t\right) = \sum_{s^{t+1}} \lambda_{t+1}^i\left(s^{t+1}\right) \frac{p_{t+1}\left(s^{t+1}\right)}{p_t\left(s^t\right)} \text{if } s_t = i$$

Combining, we get

$$\beta^t \pi\left(s^t\right) u'\left(c_t^i\left(s^t\right)\right) = \mu_t^i\left(s^t\right) + \sum_{s^{t+1}} \lambda_{t+1}^i\left(s^t\right) \frac{p_{t+1}\left(s^{t+1}\right)}{p_t\left(s^t\right)} = \lambda_t^i\left(s^t\right)$$

Define

$$\frac{p_{t+1}\left(s^{t+1}\right)}{p_t\left(s^t\right)} = \frac{1}{\beta} \text{ for all } s^t, s^{t+1}$$

and so

$$p_t = p_0 \left(\frac{1}{\beta}\right)^t$$

for some constant p_0 to be determined.

Conjecture that $c_t^i\left(s^t\right) = g\left(1\right)$ for all i, t, s^t. Then

$$\begin{aligned}
\mu_t^i\left(s^t\right) &= \beta^t \pi\left(s^t\right) u'\left(c_t\left(s^t\right)\right) - \sum_{s^{t+1}} \beta^{t+1} \pi\left(s^{t+1}\right) \frac{p_{t+1}\left(s^{t+1}\right)}{p_t\left(s^t\right)} u'\left(c_{t+1}\left(s^{t+1}\right)\right) \\
&= \beta^t \pi\left(s^t\right) \left[u'\left(c_t\left(s^t\right)\right) - \sum_{s^{t+1}} \beta \pi\left(s^{t+1} \mid s^t\right) \frac{p_{t+1}\left(s^{t+1}\right)}{p_t\left(s^t\right)} u'\left(c_{t+1}\left(s^{t+1}\right)\right)\right] \\
&= \beta^t \pi\left(s^t\right) \left[1 - \beta \frac{p_{t+1}}{p_t}\right] \\
&= 0
\end{aligned}$$

and so the multiplier on the cash-in-advance constraint is 0. Given this, notice that conjectured allocation satisfies the agent's first order conditions.

Finally, we need to verify that the transversality conditions holds. The one for state contingent assets follows directly. Notice from our construction of prices

$$
\begin{aligned}
&\lim\inf_{t\to\infty} E_{t-1}\left[\beta^t u'\left(c_t^i\left(s^t\right)\right) p_t\left(s^t\right) m_t^i\left(s^t\right)\right] \\
&= \lim\inf_{t\to\infty} E_{t-1}\left[\beta^t p_0 \left(\frac{1}{\beta}\right)^t m_t^i\left(s^t\right)\right] \\
&= p_0 \lim\inf_{t\to\infty} E_{t-1}\left[m_t^i\left(s^t\right)\right] \\
&\le p_0 \lim\inf_{t\to\infty} M_t \\
&= 0
\end{aligned}
$$

Choose p_0 so that

$$\frac{2g(1)}{\kappa} \ge p_0$$

Then

$$
\begin{aligned}
&\sum_i c_t^i\left(s^t\right) \le p_t\left(s^t\right) M_t \\
&\Leftrightarrow M_t \ge \frac{\sum_i c_t^i\left(s^t\right)}{p_t\left(s^t\right)} \Leftrightarrow M_t \ge \frac{2g(1)}{p_0\beta^{-t}} \Leftrightarrow \beta^{-t} M_t \ge \frac{2g(1)}{p_0} \ge \kappa
\end{aligned}
$$

Since $\inf M_t\beta^{-t} = \kappa > 0$ the CIA constraints are always satisfied. ■

Proofs From Section 2.2.2

Proof of Proposition 3

I conjecture that a stationary equilibrium of the following form exists

$$
\begin{aligned}
q^j\left(s^{t-1}, i\right) = q^i\left(s^{t-1}, j\right) &= q^c \\
q^j\left(s^{t-1}, j\right) = q^i\left(s^{t-1}, i\right) &= q^{nc} \\
c^i\left(s^{t-1}, i\right) = c^h &= 1 \\
c^i\left(s^{t-1}, j\right) = c^l &= g\left(\frac{q^c}{\beta\lambda}\right) \\
\phi_{s'}^i\left(s^t\right) &= \phi \\
a_j^i\left(s^{t-1}, i\right) &= \phi \\
a_i^i\left(s^{t-1}, i\right) &= -\phi
\end{aligned}
$$

It suffices to check that given these prices, the allocations satisfy the agent's first order conditions, clear markets and the debt limits are not-too-tight. Notice from the budget constraints of the agents

$$
\begin{aligned}
c^h + q^c\phi - q^{nc}\phi &= l - \phi \\
c^l - q^c\phi + q^{nc}\phi &= \phi
\end{aligned}
$$

we have

$$\phi = \frac{c^l}{[1+q^c-q^{nc}]} = \frac{g\left(\frac{q^c}{\beta\lambda}\right)}{[1+q^c-q^{nc}]} \tag{17}$$

Therefore

$$c^h + [q^c - q^{nc} + 1]\,\phi = l$$
$$\Rightarrow c^h + [q^c - q^{nc} + 1]\,\frac{g\left(\frac{q^c}{\beta\lambda}\right)}{[1+q^c-q^{nc}]} = l$$
$$\Rightarrow c^h + g\left(\frac{q^c}{\beta\lambda}\right) = l$$

and so given this level of l, the allocation is feasible for the agents. Given prices, it is easy to see that the allocation satisfies the agent's first order conditions

$$u'\left(c_t^i\left(s^{t-1}, i\right)\right) = 1$$
$$u'\left(c_t^i\left(s^t, j\right)\right) = \frac{q_t^j\left(s^t\right)}{\beta\lambda}$$

The only thing to be checked is that the debt limits satisfy the not-too-tight property. The following lemma will be useful in proving the main result.

Lemma 4 *If after default, agents can purchase non-negative quantities of Arrow securities, an equilibrium of the conjectured form exists with*

$$q^c + q^{nc} = 1$$

Proof of Lemma 4. In our conjectured equilibrium we need only consider the productive agent's incentives to default. In particular, ϕ will be chosen so that the productive agent who ϕ will be indifferent between defaulting and not. To characterize ϕ,we want to consider strategies that would replicate the agents allocation had he not defaulted when he is only allowed to save in Arrow securities. The agent's problem after default is

$$V_t^{i,d}(s^t) = \max \sum_{t' \geq t} \sum_{s^{t'} \succeq s^t} \beta^{t'-t} \pi\left(s^{t'} \mid s^t\right)\left[u\left(c_{t'}^i\left(s^{t'}\right)\right) - l_{t'}^i\left(s^{t'}\right)\right]$$

subject to

$$c_t^i\left(s^t\right) + q_i\left(s^t\right) a_i^{i,d}\left(s^t\right) + q_j\left(s^t\right) a_j^{i,d,}\left(s^t\right) \leq l^i\left(s^t\right)$$
$$c_{t'}^i\left(s^{t'}\right) + q_i\left(s^{t'}\right) a_i^i\left(s^{t'}\right) + q_j\left(s^{t'}\right) a_j^i\left(s^{t'}\right) \leq l^i\left(s^{t'}\right) + a_i^i\left(s^{t'-1}\right)$$
$$a_{s'}^{i,d}\left(s^{t'}\right) \geq 0, \text{for all } t' \geq t, s'$$
$$l^i\left(s^{t'}\right) = 0 \text{ if } s_{t'} \neq i$$

Consider the following strategies

$$c^i\left(s^t,i\right)=c^{h,d}$$
$$c^i\left(s^t,j\right)=c^{l,d}$$
$$a_j^{i,d}\left(s^{t-1},i\right)=a_j^{i,d}\left(s^{t-1},j\right)=x$$
$$a_i^{i,d}\left(s^{t-1},i\right)=a_i^{i,d}\left(s^{t-1},j\right)=0$$

for some $x>0$. For the strategy to be feasible and optimal, it must be that

$$c^{h,d}+q^cx=l$$
$$c^{l,d}+q^{nc}x=x$$

and

$$u'\left(c^{h,d}\right)=1,\ c^{l,d}=g\left(\frac{q^c}{\beta\lambda}\right)$$

Therefore

$$x=\frac{g\left(\frac{q^c}{\beta\lambda}\right)}{(1-q^{nc})}$$

and so

$$c^{h,d}+q^c\frac{g\left(\frac{q^c}{\beta\lambda}\right)}{(1-q^{nc})}=l$$
$$\Rightarrow l=c^{h,d}+q^c\frac{g\left(\frac{q^c}{\beta\lambda}\right)}{(1-q^{nc})}$$

Notice that $c^{h,d}=c^h$, $c^{l,d}=c^l$ and

$$l^d=g\left(1\right)+q^c\frac{g\left(\frac{q^c}{\beta\lambda}\right)}{(1-q^{nc})}$$

Therefore, the value of default us

$$V^{i,d}\left(i\right)=\frac{\left(1-\beta\left(1-\lambda\right)\right)\left[u\left(g\left(1\right)\right)-g\left(1\right)-q^c\frac{g\left(\frac{q^c}{\beta\lambda}\right)}{(1-q^{nc})}\right]+\beta\lambda u\left(g\left(\frac{q^c}{\beta\lambda}\right)\right)}{\left[1-\beta\left(1-\lambda\right)\right]^2-\beta^2\lambda^2}$$

while the value of not defaulting is given by

$$V^i\left(i,\phi\right)=\frac{\left(1-\beta\left(1-\lambda\right)\right)\left[u\left(g\left(1\right)\right)-g\left(1\right)-g\left(\frac{q^c}{\beta\lambda}\right)\right]+\beta\lambda u\left(g\left(\frac{q^c}{\beta\lambda}\right)\right)}{\left[1-\beta\left(1-\lambda\right)\right]^2-\beta^2\lambda^2}$$

We see that if $q^c=1-q^{nc}$

$$V^i\left(i,\phi\right)=V^{i,d}\left(i\right)$$

Finally, we construct debt constraints that are not-too-tight using (17).
Since the first order conditions and budget constraints are all satisfied, we conclude that the an

equilibrium of the conjectured form exists with not-too-tight debt constraints. ■

Proof of Proposition 3. Consider the agent's problem after default

$$\max E\left[\sum_{t=0}^{\infty}\sum_{s^t}\beta^t\pi\left(s^t\right)\left(u\left(c_t^{i,d}\left(s^t\right)\right)-l_t^{i,d}\left(s^t\right)\right)\right]$$

$$\begin{aligned}&s.t.\\&c_t^{i,d}\left(s^t\right)+Qx_{t+1}^{i,d}\left(s^t\right)=l_t^{i,d}\left(s^t\right)+x_t^{i,d}\left(s^{t-1}\right)\ \text{ if } s_t=i\\&c_t^{i,d}\left(s^t\right)+Qx_{t+1}^{i,d}\left(s^t\right)=x_t^{i,d}\left(s^{t-1}\right)\ \text{ if } s_t\neq i\\&x_{t+1}^{i}\geq 0\end{aligned}$$

where $Q=q^c+q^{nc}$. We can write the problem recursively and denote the value by $V^{i,d}\left(s,x;q^c\right)$. As earlier we are looking for equilibria that are not-too-tight, i.e.

$$V^i\left(i,\phi\right)=V^{i,d}\left(i;q^c\right)$$

which implies

$$\begin{aligned}V^i\left(i,\phi\right)&=\frac{\left(1-\beta\left(1-\lambda\right)\right)\left(u\left(c^h\right)-l\right)+\beta\lambda u\left(c^l\right)}{\left[1-\beta\left(1-\lambda\right)\right]^2-\beta^2\lambda^2}\\&=\frac{\left(1-\beta\left(1-\lambda\right)\right)\left[u\left(g\left(1\right)\right)-g\left(1\right)-g\left(\frac{q^c}{\beta\lambda}\right)\right]+\beta\lambda u\left(g\left(\frac{q^c}{\beta\lambda}\right)\right)}{\left[1-\beta\left(1-\lambda\right)\right]^2-\beta^2\lambda^2}\end{aligned}$$

Let $z=\frac{q^c}{\beta\lambda}$. Then we have that

$$\begin{aligned}\frac{\partial}{\partial z}V^i\left(i,\phi\right)&=\frac{\partial}{\partial z}\left[\frac{\left(1-\beta\left(1-\lambda\right)\right)\left[u\left(g\left(1\right)\right)-g\left(1\right)-g\left(z\right)\right]+\beta\lambda u\left(g\left(z\right)\right)}{\left[1-\beta\left(1-\lambda\right)\right]^2-\beta^2\lambda^2}\right]\\&=\frac{-\left(1-\beta\left(1-\lambda\right)\right)g'\left(\frac{q^c}{\beta\lambda}\right)+\beta\lambda u'\left(g\left(\frac{q^c}{\beta\lambda}\right)\right)g'\left(\frac{q^c}{\beta\lambda}\right)}{\left[1-\beta\left(1-\lambda\right)\right]^2-\beta^2\lambda^2}\\&=\frac{\left[-\left(1-\beta\left(1-\lambda\right)\right)+q^c\right]g'\left(\frac{q^c}{\beta\lambda}\right)}{\left[1-\beta\left(1-\lambda\right)\right]^2-\beta^2\lambda^2}\end{aligned}$$

Since $g'\left(\frac{q^c}{\beta\lambda}\right)<0$ we have

$$\begin{aligned}\frac{\partial}{\partial z}V^i\left(i,\phi\right)|_{q^c>1-\beta(1-\lambda)}&>0\\\frac{\partial}{\partial z}V^i\left(i,\phi\right)|_{q^c=1-\beta(1-\lambda)}&=0\\\frac{\partial}{\partial z}V^i\left(i,\phi\right)|_{q^c<1-\beta(1-\lambda)}&<0\end{aligned}$$

To see how $V^{i,d}\left(s,0;q^c\right)$ changes with q^c notice that after default, we can compute the agent's consumption in states in which he is unproductive from the Euler equations. Suppose he defaults in period t,when productive. Then his consumption the following period if the state is j is given by

$$1 = \beta R \left[\lambda u' \left(c_{t+1}^{i,d} \left(s^t, j \right) \right) + (1 - \lambda) \right]$$
$$\Rightarrow c_{t+1}^{i,d} \left(s^t, j \right) = g \left(\frac{1 - \beta R (1 - \lambda)}{\beta R \lambda} \right)$$

where $\frac{1}{R} = Q = q^c + q^{nc}$.

Similarly

$$u' \left(c_{t+1}^{i,d} \left(s^t, j \right) \right) = \beta R \left[\lambda + (1 - \lambda) u' \left(c_{t+2}^{i,d} \left(s^{t+1}, j \right) \right) \right]$$
$$\Rightarrow c_{t+2}^{i,d} \left(s^{t+1}, j \right) = g \left(\frac{\frac{1-\beta R(1-\lambda)}{\beta R \lambda} - \beta R \lambda}{\beta R (1 - \lambda)} \right)$$

Clearly $\frac{\partial c_{t+n}^{i,d}\left(s^{t+n-1}, j\right)}{\partial q^c} < 0$ for all n. Since $c_{t+n}^{i,d} \left(s^{t+n-1}, i \right) = g(1)$, we have that $\frac{\partial V^{i,d}(s,0;q^c)}{\partial q^c} < 0$ for all agents i.

Let $\tilde{V}^{i,d}(s, 0; q^c)$ denote the value of default when agents can save in Arrow securities.

Now since

$$V^i (i, \phi) = \tilde{V}^{i,d} (s, 0; 1 - \beta (1 - \lambda)) > V^{i,d} (s, 0; 1 - \beta (1 - \lambda))$$

and

$$\frac{\partial V^{i,d} (s, 0; q^c)}{\partial q^c} < 0$$
$$\frac{\partial V^i (i, \phi)}{\partial q^c} \;|_{q^c = (1-\beta(1-\lambda))} = 0$$

any q^{c*} such that

$$V^i (i, \phi) = V^{i,d} (s; q^{c*})$$

must have the property that

$$q^{c*} < 1 - \beta (1 - \lambda)$$

Therefore

$$q^{nc} + q^{*c} < 1$$

Since $V^i (i, \phi) - V^{i,d} (s; q^c) > 0$ for $q^c = 1 - \beta (1 - \lambda)$ and $V^i (i, \phi) - V^{i,d} (s; q^c) < 0$ for q^c small enough, we know that such a q^{c*} exists.

Finally, similar to the previous lemma, we can construct not-too-tight debt limits using (17). ■

Remark 1 *How do we show that $V^i (i, \phi) - V^{i,d} (s, 0; q^c) < 0$ for q^c small enough? Consider supporting the complete markets allocation as an equilibrium. Then $\left(c^h, c^l, l\right) = (g(1), g(1), 2g(1))$ and $R = \frac{1}{\beta}$. Consider the value of default when productive. Had the agent not defaulted he would have made a payment of $g(1)$ to the unproductive agent. Consider the following strategy when he defaults; he saves $g(1)$ in the risk free bond. As a result his return next period is $\frac{g(1)}{\beta}$. In this period no matter what the state, he consumes $g(1)$ and does not work and saves $\frac{g(1)}{\beta} - g(1) = \frac{g(1)(1-\beta)}{\beta}$. The following period he does the same consuming $g(1)$ and saving $\frac{g(1)(1-\beta)}{\beta^2} - g(1) = \frac{g(1)\left(1-\beta-\beta^2\right)}{\beta^2}$ and so* *on.. Notice that as long as* $\frac{g(1)\left(1-\beta-\beta^2-\beta^3-..\beta^t\right)}{\beta^t} \geq 0$*, the strategy is feasible and the agent does not have to work in any future period after the first. Since* $1-\beta-\beta^2-... = \frac{1-2\beta}{\beta}$*, if* $\beta \leq \frac{1}{2}$*, the value of default is strictly greater than paying for the productive agent. As a result a sufficient condition for* $V^i(i,\phi) - V^{i,d}(s,0;q^c) < 0$ *for* q^c *low enough is* $\beta \leq \frac{1}{2}$*. Recall that the complete markets benchmark had* $\left(c^h, c^l, l\right) = (g(1), g(1), 2g(1))$*. Here* $\left(c^h, c^l, l\right) = \left(1, g\left(\frac{q^c}{\beta\lambda}\right), g(1) + g\left(\frac{q^c}{\beta\lambda}\right)\right)$*. Notice that* $g\left(\frac{q^c}{\beta\lambda}\right) < g(1)$ *since* $q^c > \beta\lambda$*. As a result, the above equilibrium is the best stationary equilibrium since any equilibrium with higher ex-ante welfare, must have a lower* q^c *which would violate the not-too-tight constraint. It is also worth noting that in any equilibrium* $Q = q^c + q^{nc} = \frac{\beta\lambda}{c^l} + \beta(1-\lambda) \geq \beta$ *since* $c^l \leq 1$*.* $c^l = 1$ *is the solution to the complete markets problem.*

Proof of Proposition 4. As in Golosov and Tsyvinski (2007) and Kehoe and Levine (1993) we can write constrained efficient problem as choosing sequences consumption, labor and price sequences $\left\{\left(c_t^i(s^t), l_t^i(s^t)\right)_{i\in I}, \{q_{s_{t+1}}(s^t)\}_{s_{t+1}\in S}\right\}_{t\geq 0, s^t\in S^t}$ to maximize

$$\sum_{t=0}^{\infty}\sum_{s^t}\sum_{i\in I}\beta^t\pi\left(s^t\right)\left[u\left(c_t^i\left(s^t\right)\right) - l_t^i\left(s^t\right)\right]$$

subject to resource feasibility,

$$\sum_{i\in I} c_t^i\left(s^t\right) = \sum_{i\in I} l_t^i\left(s^t\right)$$

voluntary participation constraints

$$\sum_{t'=t}^{\infty}\sum_{s^{t'}\succeq s^t}\beta^{t'-t}\left[u\left(c_{t'}^i\left(s^{t'}\right)\right) - l_{t'}^i\left(s^{t'}\right)\right] \geq V_t^{i,d}\left(s^t, \mathbf{q}_t\right)$$

where $\mathbf{q}_t = \left\{q_{s_{t'+1}}\left(s^{t'}\right)\right\}_{t'\geq t, s^{t'}\geq s^t}$ is defined by

$$q_{s_{t'+1}}\left(s^{t'}\right) = \max_{i\in I}\left\{\beta\pi\left(s^{t'+1} \mid s^{t'}\right)\frac{u'\left(c_{t'+1}^i\left(s^{t'+1}\right)\right)}{u'\left(c_{t'}^i\left(s^{t'}\right)\right)}\right\}$$

Let $\mu\left(s^t\right)$ be the multiplier on the resource constraint, $\kappa_{s_{t+1}}\left(s^t\right)$ on the pricing equation and $\eta^i\left(s^t\right)$ on the voluntary participation constraint. The first order condition with respect to $c_t^i\left(s^t\right)$ yields

$$\begin{aligned}
\beta^t\pi\left(s^t\right)u'\left(c_t^i\left(s^t\right)\right) &= \mu\left(s^t\right) \\
&-\sum_{s_{t+1}}\kappa_{s_{t+1}}\left(s^t\right)\mathbf{1}_{q_{s_{t+1}}(s^t)=\beta\pi(s^{t+1}|s^t)\frac{u'\left(c_{t+1}^i(s^{t+1})\right)}{u'\left(c_t^i(s^t)\right)}}\beta\pi\left(s^{t+1}\mid s^t\right)\frac{u'\left(c_{t+1}^i\left(s^{t+1}\right)\right)}{\left(u'\left(c_t^i\left(s^t\right)\right)\right)^2}u''\left(c_t^i\left(s^t\right)\right) \\
&-\sum_{t'\leq t}\eta^i\left(s^{t'}\right)\pi\left(s^t \mid s^{t'}\right)u'\left(c_t^i\left(s^t\right)\right)
\end{aligned}$$

and with respect to $l_t^i\left(s^t\right)$

$$\beta^t \pi\left(s^t\right) = \mu\left(s^t\right) - \sum_{t' \leq t} \eta^i\left(s^{t'}\right) \pi\left(s^t \mid s^{t'}\right)$$

We can combine these equations to obtain

$$\left[\beta^t \pi\left(s^t\right) + \sum_{t' \leq t} \eta^i\left(s^{t'}\right) \pi\left(s^t \mid s^{t'}\right)\right] = \mu\left(s^t\right)$$

and

$$u'\left(c_t^i\left(s^t\right)\right) = 1 - \frac{1}{\mu\left(s^t\right)} \sum_{s_{t+1}} \kappa_{s_{t+1}}\left(s^t\right) \mathbf{1}_{q_{s_{t+1}}\left(s^t\right) = \beta\pi\left(s^{t+1} \mid s^t\right) \frac{u'\left(c_{t+1}^i\left(s^{t+1}\right)\right)}{u'\left(c_t^i\left(s^t\right)\right)}} \beta\pi\left(s^{t+1} \mid s^t\right) \frac{u'\left(c_{t+1}^i\left(s^{t+1}\right)\right)}{\left(u'\left(c_t^i\left(s^t\right)\right)\right)^2} u''\left(c_t^i\left(s^t\right)\right)$$

Notice that if $s_t = i$, $\kappa_{s_{t+1}}\left(s^t\right) \mathbf{1}_{q_{s_{t+1}}\left(s^t\right) = \beta\pi\left(s^{t+1} \mid s^t\right) \frac{u'\left(c_{t+1}^i\left(s^{t+1}\right)\right)}{u'\left(c_t^i\left(s^t\right)\right)}} > 0$ for $s_{t+1} = j$ since

$$\beta\pi\left(s^{t+1} \mid s^t\right) \frac{u'\left(c_{t+1}^i\left(s^t, j\right)\right)}{u'\left(c_t^i\left(s^{t-1}, i\right)\right)} \geq \beta\pi\left(s^{t+1} \mid s^t\right) \frac{u'\left(c_{t+1}^j\left(s^t, j\right)\right)}{u'\left(c_t^j\left(s^{t-1}, i\right)\right)}$$

However, in any competitive equilibrium if $s_t = i$, $u'\left(c_t^i\left(s^t\right)\right) = 1$. As a result, any competitive equilibrium is constrained-inefficient. ■

Proof of Proposition 5. The first order conditions from the agent i's problem are

$$\begin{aligned}
l^i\left(s^t\right) &: \beta^t \pi\left(s^t\right) = \sum_{s^{t+1}} \frac{p_{t+1}\left(s^{t+1}\right)}{p_t\left(s^t\right)} \lambda^i\left(s^{t+1}\right) \\
c_t^i\left(s^t\right) &: \beta^t \pi\left(s^t\right) u'\left(c_t^i\left(s^t\right)\right) = \mu^i\left(s^t\right) + \sum_{s^{t+1}} \frac{p_{t+1}\left(s^{t+1}\right)}{p_t\left(s^t\right)} \lambda^i\left(s^{t+1}\right) \\
c_{t+1}^i\left(s^{t+1}\right) &: \beta^{t+1} \pi\left(s^{t+1}\right) u'\left(c_{t+1}^i\left(s^{t+1}\right)\right) = \mu^i\left(s^{t+1}\right) + \sum_{s^{t+2}} \frac{p_{t+1}\left(s^{t+2}\right)}{p_t\left(s^{t+1}\right)} \lambda^i\left(s^{t+2}\right) \\
a_i^i\left(s^t\right) &: 0 = \lambda^i\left(s^t\right) q_t^i\left(s^t\right) - \eta_i^i\left(s^t\right) - \lambda^i\left(s^{t+1}\right) \\
m_t^i\left(s^t\right) &: 0 = p_t\left(s^t\right) \lambda^i\left(s^t\right) - p_t\left(s^t\right) \mu^i\left(s^t\right) - \sum_{s^{t+1}} p_{t+1}\left(s^{t+1}\right) \lambda^i\left(s^{t+1}\right)
\end{aligned}$$

Therefore

$$\lambda^i\left(s^t\right) = \mu^i\left(s^t\right) + \sum_{s^{t+1}} \frac{p_{t+1}\left(s^{t+1}\right)}{p_t\left(s^t\right)} \lambda^i\left(s^{t+1}\right)$$

which implies

$$\beta^t \pi\left(s^t\right) u'\left(c_t^i\left(s^t\right)\right) = \lambda^i\left(s^t\right)$$
$$\beta^{t+1} \pi\left(s^{t+1}\right) u'\left(c_{t+1}^i\left(s^{t+1}\right)\right) = \lambda^i\left(s^{t+1}\right)$$
$$\Rightarrow \pi\left(s^t\right) u'\left(c_t^i\left(s^t\right)\right) q_t^i\left(s^t\right) - \eta_i^i\left(s^t\right) = \beta\pi\left(s^{t+1}\right) u'\left(c_{t+1}^i\left(s^{t+1}\right)\right)$$
$$\Rightarrow q_t^i\left(s^t\right) u'\left(c_t^i\left(s^t\right)\right) = \beta\pi\left(s^{t+1} \mid s^t\right) u'\left(c_{t+1}^i\left(s^{t+1}\right)\right) + \frac{\eta_i^i\left(s^t\right)}{\beta^t \pi\left(s^t\right)}$$

Consider the allocation $\left\{c_t^i\left(s^t\right), l_t^i\left(s^t\right), a_i^i\left(s^t\right), a_j^i\left(s^t\right), \phi_i^i\left(s^t\right), \phi_j^i\left(s^t\right)\right\}_{s^t,i}$ and prices $\left\{q_t^i\left(s^t\right), q_t^j\left(s^t\right)\right\}_{s^t}$ from the above equilibrium. We want to show that there exist prices $\left\{p_t\left(s^t\right)\right\}$ such that these constitute an equilibrium for the problem above. It suffices to show that the allocations and prices satisfy the above first order conditions along with the TVCs

$$\lim \inf_{t\to\infty} E_{t-1}\left[\beta^t u'\left(c_t^i\left(s^t\right)\right) p_t\left(s^t\right) m_t^i\left(s^t\right)\right] = 0 \text{ for all } i$$
$$\lim \inf_{t\to\infty} E_{t-1}\left[\beta^t u'\left(c_t^i\left(s^t\right)\right)\left[a_{s'}^i\left(s^t\right) - \phi_{s'}^i\left(s^t\right)\right]\right] = 0 \text{ for all } i \text{ and } s'$$

Define

$$\frac{p_{t+1}\left(s^{t+1}\right)}{p_t\left(s^t\right)} = \frac{p_{t+1}}{p_t}\frac{1}{Q_t} = \frac{1}{Q} \text{ for all } s^t, s^{t+1}$$

where

$$Q = q^c + q^{nc} = \beta\lambda u'\left(c^l\right) + \beta\left(1-\lambda\right)$$

and so

$$p_t = p_0\left(\frac{1}{Q}\right)^t$$

for some constant p_0 to be determined.

Then

$$\begin{aligned}
\mu^i\left(i\right) &= \beta^t \pi\left(s^t\right) u'\left(c_t\left(s^t\right)\right) - \sum_{s^{t+1}} \beta^{t+1}\pi\left(s^{t+1}\right)\frac{p_{t+1}\left(s^{t+1}\right)}{p_t\left(s^t\right)} u'\left(c_{t+1}\left(s^{t+1}\right)\right) \\
&= \beta^t \pi\left(s^t\right) u'\left(c_t\left(s^t\right)\right) - \sum_{s^{t+1}} \beta^{t+1}\pi\left(s^{t+1}\right)\frac{p_{t+1}\left(s^{t+1}\right)}{p_t\left(s^t\right)} u'\left(c_{t+1}\left(s^{t+1}\right)\right) \\
&= \beta^t \pi\left(s^t\right)\left[u'\left(c_t\left(s^t\right)\right) - \sum_{s^{t+1}} \beta\pi\left(s^{t+1} \mid s^t\right)\frac{p_{t+1}\left(s^{t+1}\right)}{p_t\left(s^t\right)} u'\left(c_{t+1}\left(s^{t+1}\right)\right)\right] \\
&= \beta^t \pi\left(s^t\right)\left[1 - \beta\lambda\frac{p_{t+1}}{p_t}u'\left(c^l\right) - \beta\left(1-\lambda\right)\frac{p_{t+1}}{p_t}\right] \\
&= \beta^t \pi\left(s^t\right)\left[1 - \frac{p_{t+1}}{p_t}\left(\beta\lambda u'\left(c^l\right) + \beta\left(1-\lambda\right)\right)\right] \\
&= \beta^t \pi\left(s^t\right)\left[1 - \frac{p_{t+1}}{p_t}Q\right] \\
&= 0
\end{aligned}$$

and so the multiplier on the cash-in-advance constraint is 0.

As a result, the choices for c and l satisfy the first order conditions of the problem. Similarly the first order conditions for Arrow securities are satisfied with $\eta_i^i(i) = \eta_i^i(j) > 0$ and $\eta_i^j(i) = \eta_i^j(j) = 0$.

Finally, we need to verify that the transversality conditions hold. The one for state contingent assets follows directly. Notice from our construction of prices

$$\begin{aligned}
&\lim\inf_{t\to\infty} E_{t-1}\left[\beta^t u'\left(c_t^i\left(s^t\right)\right) p_t\left(s^t\right) m_t^i\left(s^t\right)\right] \\
&= \lim\inf_{t\to\infty} \left[\beta^t \lambda u'\left(c^i(j)\right) p_t(j)\, m_t^i(j) + \beta^t (1-\lambda)\, p_t(i)\, m_t^i(i)\right] \\
&\leq \lim\inf_{t\to\infty} \beta^t p_t \left[\lambda u'\left(c^i(j)\right) + (1-\lambda)\right] M_t \\
&= \lim\inf_{t\to\infty} \beta^{t-1} p_t \left[\beta\lambda u'\left(c^i(j)\right) + \beta(1-\lambda)\right] M_t \\
&= \lim\inf_{t\to\infty} \beta^{t-1} \frac{1}{Q^t} Q M_t \\
&= \lim\inf_{t\to\infty} \left(\frac{\beta}{Q}\right)^t M_t \\
&\leq \lim\inf_{t\to\infty} M_t
\end{aligned}$$

Since $Q \geq \beta$. Choose p_0 so that

$$\frac{\left(1+c^l\right)}{\kappa} \geq p_0$$

Then

$$\begin{aligned}
&\sum_i c_t^i\left(s^t\right) \leq p_t\left(s^t\right) M_t \\
&\Leftrightarrow M_t \geq \frac{\sum_i c_t^i\left(s^t\right)}{p_t\left(s^t\right)} \\
&\Leftrightarrow M_t \geq \frac{1+c^l}{p_0\left(\frac{1}{Q}\right)^t} \geq \frac{1+c^l}{p_0\beta^{-t}} \\
&\Leftrightarrow \beta^{-t} M_t \geq \frac{1+c^l}{p_0} \geq \kappa
\end{aligned}$$

As a result, the second assumption ensures that the CIA constraints are always satisfied. ■

Proof of Lemma 2. The first order conditions from the agents' problem are

$$l^i\left(s^t\right):\beta^t\pi\left(s^t\right)=\sum_{s^{t+1}}\frac{p_{t+1}\left(s^{t+1}\right)}{p_t\left(s^t\right)}\lambda^i\left(s^{t+1}\right)$$

$$c_t^i\left(s^t\right):\beta^t\pi\left(s^t\right)u'\left(c_t^i\left(s^t\right)\right)=\mu^i\left(s^t\right)+\sum_{s^{t+1}}\frac{p_{t+1}\left(s^{t+1}\right)}{p_t\left(s^t\right)}\lambda^i\left(s^{t+1}\right)$$

$$c_{t+1}^i\left(s^{t+1}\right):\beta^{t+1}\pi\left(s^{t+1}\right)u'\left(c_{t+1}^i\left(s^{t+1}\right)\right)=\mu^i\left(s^{t+1}\right)+\sum_{s^{t+2}}\frac{p_{t+1}\left(s^{t+2}\right)}{p_t\left(s^{t+1}\right)}\lambda^i\left(s^{t+2}\right)$$

$$a_i^i\left(s^t\right):0=\lambda^i\left(s^t\right)q_t^i\left(s^t\right)-\eta_i^i\left(s^t\right)-\lambda^i\left(s^{t+1}\right)$$

$$m_t^i\left(s^t\right):0=p_t\left(s^t\right)\lambda^i\left(s^t\right)-p_t\left(s^t\right)\mu^i\left(s^t\right)-\sum_{s^{t+1}}p_{t+1}\left(s^{t+1}\right)\lambda^i\left(s^{t+1}\right)$$

$$\begin{aligned}\lambda^i\left(s^t\right)&=\mu^i\left(s^t\right)+\sum_{s^{t+1}}\frac{p_{t+1}\left(s^{t+1}\right)}{p_t\left(s^t\right)}\lambda^i\left(s^{t+1}\right)\\&\Rightarrow\mu^i\left(s^t\right)=\beta^t\pi\left(s^t\right)u'\left(c_t^i\left(s^t\right)\right)-\sum_{s^{t+1}}\beta^{t+1}\pi\left(s^{t+1}\right)\frac{p_{t+1}\left(s^{t+1}\right)}{p_t\left(s^t\right)}u'\left(c_{t+1}^i\left(s^{t+1}\right)\right)\end{aligned}$$

Combining

$$\begin{aligned}&\frac{1}{u'\left(c_t^i\left(s^t\right)\right)}\\&=\frac{\sum_{s^{t+1}}\frac{p_{t+1}\left(s^{t+1}\right)}{p_t\left(s^t\right)}\lambda^i\left(s^{t+1}\right)}{\mu^i\left(s^t\right)+\sum_{s^{t+1}}\frac{p_{t+1}\left(s^{t+1}\right)}{p_t\left(s^t\right)}\lambda^i\left(s^{t+1}\right)}\\&=\frac{\sum_{s^{t+1}}\frac{p_{t+1}\left(s^{t+1}\right)}{p_t\left(s^t\right)}\beta^{t+1}\pi\left(s^{t+1}\right)u'\left(c_{t+1}^i\left(s^{t+1}\right)\right)}{\begin{array}{c}\beta^t\pi\left(s^t\right)u'\left(c_t^i\left(s^t\right)\right)-\sum_{s^{t+1}}\beta^{t+1}\pi\left(s^{t+1}\right)\frac{p_{t+1}\left(s^{t+1}\right)}{p_t\left(s^t\right)}u'\left(c_{t+1}^i\left(s^{t+1}\right)\right)\\+\sum_{s^{t+1}}\frac{p_{t+1}\left(s^{t+1}\right)}{p_t\left(s^t\right)}\beta^{t+1}\pi\left(s^{t+1}\right)u'\left(c_{t+1}^i\left(s^{t+1}\right)\right)\end{array}}\\&=\frac{p_{t+1}}{p_t}\sum_{s^{t+1}}\beta\pi\left(s^{t+1}\mid s^t\right)\frac{u'\left(c_{t+1}^i\left(s^{t+1}\right)\right)}{u'\left(c_t^i\left(s^t\right)\right)}\end{aligned}$$

In this model when $s_t=i$,

$$\frac{p_{t+1}}{p_t}\sum_{s^{t+1}}\beta^{t+1}\pi\left(s^{t+1}\mid s^t\right)\frac{u'\left(c_{t+1}^i\left(s^{t+1}\right)\right)}{u'\left(c_t^i\left(s^t\right)\right)}=\frac{p_{t+1}}{p_t}Q\left(s^t\right)$$

Therefore in a stationary equilibrium

$$\begin{aligned}u'\left(c_t^i\left(s^t\right)\right)&=\frac{1}{R^mQ}\\\Rightarrow c^i\left(s^t,i\right)&=c^j\left(s^t,j\right)=g\left(\frac{1}{R^mQ}\right)\end{aligned}$$

Then given our conjecture

$$
\begin{aligned}
& q^c \frac{1}{R^m Q} = \beta\lambda u'\left(c_{t+1}^i\left(s^t, j\right)\right) = \beta\lambda u'\left(c_{t+1}^j\left(s^t, i\right)\right) \\
& \Rightarrow c_{t+1}^i\left(s^t, j\right) = c_{t+1}^j\left(s^t, i\right) = g\left(\frac{q^c}{R^m Q\beta\lambda}\right) \\
& \Rightarrow l = g\left(\frac{1}{R^m Q}\right) + g\left(\frac{q^c}{R^m Q\beta\lambda}\right)
\end{aligned}
$$

Clearly $q^{nc} = \beta(1-\lambda)$ and finally q^c is determined from

$$
\frac{(1-\beta(1-\lambda))\left(u\left(g\left(\frac{1}{R^m Q}\right)\right) - \left[g\left(\frac{1}{R^m Q}\right) + g\left(\frac{q^c}{R^m Q\beta\lambda}\right)\right]\right) + \beta\lambda u\left(g\left(\frac{q^c}{R^m Q\beta\lambda}\right)\right)}{[1-\beta(1-\lambda)]^2 - \beta^2\lambda^2} = V^{i,d}\left(s^t; R^m\right)
$$

■

Proof of Proposition 6. Differentiating (8) wrt R^m

$$
\begin{aligned}
& \frac{\partial}{\partial R^m} W\left(R^m, q^c\left(R^m\right)\right) \\
= \; & q^{c'}\left(R^m\right)\left[g'\left(\tfrac{1}{R^m Q}\right)\left(-\tfrac{1}{R^m Q^2}\right)\left[\tfrac{1}{R^m Q} - 1\right] + g'\left(\tfrac{q^c}{R^m Q\beta\lambda}\right)\left(\tfrac{1}{R^m\beta\lambda}\tfrac{Q-q^c}{Q^2}\right)\left[\tfrac{q^c}{\beta\lambda R^m Q} - 1\right]\right] \\
& + \left[g'\left(\tfrac{1}{R^m Q}\right)\left(-\tfrac{1}{R^{m2} Q}\right)\left[\tfrac{1}{R^m Q} - 1\right] + g'\left(\tfrac{q^c}{R^m Q\beta\lambda}\right)\left[-\tfrac{q^c}{R^{m2} Q\beta\lambda}\right]\left[\tfrac{q^c}{\beta\lambda R^m Q} - 1\right]\right] \\
= \; & q^{c'}\left(R^m\right)\left[g'\left(\tfrac{1}{R^m Q}\right)\left(\tfrac{1}{R^m Q^2}\right)\left[1 - \tfrac{1}{R^m Q}\right] + g'\left(\tfrac{q^c}{R^m Q\beta\lambda}\right)\left(\tfrac{1}{R^m\beta\lambda}\tfrac{Q-q^c}{Q^2}\right)\left[\tfrac{q^c}{\beta\lambda R^m Q} - 1\right]\right] \\
& + \left[g'\left(\tfrac{1}{R^m Q}\right)\left(\tfrac{1}{R^m Q^2}\right)\left[1 - \tfrac{1}{R^m Q}\right] + g'\left(\tfrac{q^c}{R^m Q\beta\lambda}\right)\left[\tfrac{q^c}{R^{m2} Q\beta\lambda}\right]\left[1 - \tfrac{q^c}{\beta\lambda R^m Q}\right]\right]
\end{aligned}
$$

Therefore, under Friedman Rule i.e. $R^m = \frac{1}{Q}$

$$
\begin{aligned}
& \left.\frac{\partial}{\partial R^m} W\left(R^m, q^c\left(R^m\right)\right)\right|_{R^m = \frac{1}{Q}} \\
& = q^{c'}\left(R^m\right)\left[g'\left(\frac{q^c}{\beta\lambda}\right)\left(\frac{1}{\beta\lambda}\frac{Q-q^c}{Q}\right)\left[\frac{q^c}{\beta\lambda} - 1\right]\right] + \left[g'\left(\frac{q^c}{\beta\lambda}\right)\left[\frac{q^c}{R^m\beta\lambda}\right]\left[1 - \frac{q^c}{\beta\lambda}\right]\right]
\end{aligned}
$$

Define

$$
\begin{aligned}
A &= \left[g'\left(\frac{q^c}{\beta\lambda}\right)\left(\frac{1}{\beta\lambda}\frac{Q-q^c}{Q}\right)\left[\frac{q^c}{\beta\lambda} - 1\right]\right] \\
B &= \left[g'\left(\frac{q^c}{\beta\lambda}\right)\left[\frac{q^c}{R^m\beta\lambda}\right]\left[1 - \frac{q^c}{\beta\lambda}\right]\right]
\end{aligned}
$$

In any constrained equilibrium $q^c > \beta\lambda$ and therefore (note that $g'\left(\frac{q^c}{\beta\lambda}\right) < 0$) $A < 0$ and $B > 0$

Next we need to compute and sign $q^{c'}\left(R^m\right)$. To do this totally differentiate the not-too-tight constraint constraint

$$\frac{\partial}{\partial R^m}\frac{\left(1-\beta\left(1-\lambda\right)\right)\left(u\left(g\left(\frac{1}{R^mQ}\right)\right)-\left[g\left(\frac{1}{R^mQ}\right)+g\left(\frac{q^c}{R^mQ\beta\lambda}\right)\right]\right)+\beta\lambda u\left(g\left(\frac{q^c}{R^mQ\beta\lambda}\right)\right)}{\left[1-\beta\left(1-\lambda\right)\right]^2-\beta^2\lambda^2}=\frac{\partial}{\partial R^m}V^{i.d}\left(s^t;R^m\right)$$

$$\Rightarrow\frac{q^{c'}\left(R^m\right)C+D}{\left[1-\beta\left(1-\lambda\right)\right]^2-\beta^2\lambda^2}=\frac{\partial}{\partial R^m}V^{i.d}\left(s^t;R^m\right)$$

where

$$C=\left[\begin{array}{c}\left(1-\beta\left(1-\lambda\right)\right)g'\left(\frac{1}{R^mQ}\right)\left(-\frac{1}{R^mQ^2}\right)\left[\frac{1}{R^mQ}-1\right]\\+g'\left(\frac{q^c}{R^mQ\beta\lambda}\right)\left(\frac{1}{R^m\beta\lambda}\frac{Q-q^c}{Q^2}\right)\left[\frac{q^c}{R^mQ}-\left(1-\beta\left(1-\lambda\right)\right)\right]\end{array}\right]$$

$$D=\left[\begin{array}{c}\left(1-\beta\left(1-\lambda\right)\right)g'\left(\frac{1}{R^mQ}\right)\left(-\frac{1}{R^{m2}Q}\right)\left[\frac{1}{R^mQ}-1\right]\\+g'\left(\frac{q^c}{R^mQ\beta\lambda}\right)\left[-\frac{q^c}{R^{m2}Q\beta\lambda}\right]\left[\frac{q^c}{R^mQ}-\left(1-\beta\left(1-\lambda\right)\right)\right]\end{array}\right]$$

Therefore

$$q^{c'}\left(R^m\right)=\frac{\frac{\partial}{\partial R^m}V^{i.d}\left(s^t;R^m\right)-\frac{D}{\left[1-\beta(1-\lambda)\right]^2-\beta^2\lambda^2}}{\frac{C}{\left[1-\beta(1-\lambda)\right]^2-\beta^2\lambda^2}}$$

At the Friedman Rule

$$q^{c'}\left(R^m\right)\Big|_{R^m=\frac{1}{Q}}=\frac{\frac{\partial}{\partial R^m}V^{i.d}\left(s^t;R^m\right)-\frac{\left[g'\left(\frac{q^c}{\beta\lambda}\right)\left[-\frac{q^c}{Q\beta\lambda}\right]\left[q^c-(1-\beta(1-\lambda))\right]\right]}{1-\beta^2}}{\frac{g'\left(\frac{q^c}{\beta\lambda}\right)\left(\frac{1}{\beta\lambda}\frac{Q-q^c}{Q}\right)\left[q^c-(1-\beta(1-\lambda))\right]}{1-\beta^2}}$$

Signing the terms

$$q^{c'}\left(R^m\right)=\frac{\left(1-\beta^2\right)\underbrace{\frac{\partial}{\partial R^m}V^{i.d}\left(s^t;R^m\right)}_{>0}-\underbrace{\left[g'\left(\frac{q^c}{\beta\lambda}\right)\left[\frac{q^c}{R^m\beta\lambda}\right]\left[\left(1-\beta\left(1-\lambda\right)\right)-q^c\right]\right]}_{<0}}{\underbrace{\left[g'\left(\frac{q^c}{\beta\lambda}\right)\left(\frac{1}{\beta\lambda}\frac{Q-q^c}{Q}\right)\left[q^c-\left(1-\beta\left(1-\lambda\right)\right)\right]\right]}_{>0}}$$

and so $q^{c'}\left(R^m\right)>0$.
Given this we now know that

$$\begin{aligned}&\underbrace{q^{c'}\left(R^m\right)A}_{<0}+\underbrace{B}_{>0}\\&=g'\left(\frac{q^c}{\beta\lambda}\right)\left[\frac{q^c}{\beta\lambda}-1\right]\left[q^{c'}\left(R^m\right)\left(\frac{1}{\beta\lambda}\frac{Q-q^c}{Q}\right)-\frac{q^c}{R^m\beta\lambda}\right]\end{aligned}\qquad(18)$$

Need to sign $\left[q^{c'}\left(R^m\right)\left(\frac{1}{\beta\lambda}\frac{Q-q^c}{Q}\right)-\frac{q^c}{R^m\beta\lambda}\right]$. Substituting (18) we get

$$\left[q^{c'}\left(R^m\right)\left(\frac{1}{\beta\lambda}\frac{Q-q^c}{Q}\right)-\frac{q^c}{R^m\beta\lambda}\right]$$

$$=\left[\frac{\left[1-\beta\left(1-\lambda\right)\right]^2-\beta^2\lambda^2\frac{\partial}{\partial R^m}V^{i.d}\left(s^t;R^m\right)-\left[g'\left(\frac{q^c}{\beta\lambda}\right)\left[\frac{q^c}{R^m\beta\lambda}\right]\left[\left(1-\beta\left(1-\lambda\right)\right)-q^c\right]\right]}{\left[g'\left(\frac{q^c}{\beta\lambda}\right)\left(\frac{1}{\beta\lambda}\frac{Q-q^c}{Q}\right)\left[q^c-\left(1-\beta\left(1-\lambda\right)\right)\right]\right]}\left(\frac{1}{\beta\lambda}\frac{Q-q^c}{Q}\right)-\frac{q^c}{R^m\beta\lambda}\right]$$

$$>\left[\frac{\left[g'\left(\frac{q^c}{\beta\lambda}\right)\left[\frac{q^c}{R^m\beta\lambda}\right]\left[q^c-\left(1-\beta\left(1-\lambda\right)\right)\right]\right]}{\left[g'\left(\frac{q^c}{\beta\lambda}\right)\left(\frac{1}{\beta\lambda}\frac{Q-q^c}{Q}\right)\left[q^c-\left(1-\beta\left(1-\lambda\right)\right)\right]\right]}\left(\frac{1}{\beta\lambda}\frac{Q-q^c}{Q}\right)-\frac{q^c}{R^m\beta\lambda}\right]$$

$$=\left[\frac{q^c}{R^m\beta\lambda}-\frac{q^c}{R^m\beta\lambda}\right]=0$$

Therefore

$$\underbrace{q^{c'}\left(R^m\right)\left[g'\left(\frac{q^c}{\beta\lambda}\right)\left(\frac{1}{\beta\lambda}\frac{Q-q^c}{Q}\right)\left[\frac{q^c}{\beta\lambda}-1\right]\right]}_{<0}+\underbrace{\left[g'\left(\frac{q^c}{\beta\lambda}\right)\left[\frac{q^c}{R^m\beta\lambda}\right]\left[1-\frac{q^c}{\beta\lambda}\right]\right]}_{>0}>0$$

and

$$\left.\frac{\partial}{\partial R^m}W\left(R^m,q^c\left(R^m\right)\right)\right|_{R^m=\frac{1}{Q}}<0$$

■

Proof of Proposition 7

As earlier, I conjecture the form of the equilibrium and then verify that all the equilibrium conditions hold. To construct the guess, I consider an environment which only money can be held and traded. In this setup agent i maximizes (1) subject to budget constraints

$$p_t\left(s^t\right)m_t^i\left(s^t\right)\le\frac{p_t\left(s^t\right)}{p_{t-1}\left(s^{t-1}\right)}l_{t-1}^i\left(s^{t-1}\right)+p_t\left(s^t\right)\left[m_{t-1}^i\left(s^{t-1}\right)-\frac{c_{t-1}^i\left(s^{t-1}\right)}{p_{t-1}\left(s^{t-1}\right)}\right]+T_t\left(s^t\right)\text{ if }s_{t-1}=i$$

$$p_t\left(s^t\right)m_t^i\left(s^t\right)\le p_t\left(s^t\right)\left[m_{t-1}^i\left(s^{t-1}\right)-\frac{c_{t-1}^i\left(s^{t-1}\right)}{p_{t-1}\left(s^{t-1}\right)}\right]+T_t\left(s^t\right)\text{ if }s_{t-1}\neq i,$$

cash-in-advance constraints (4) and non-negativity constraints. This equilibrium is similar to the one characterized by Scheinkman and Weiss (1986) except that since in general the money supply policy $\left\{M_t\left(s^t\right)\right\}_{t,s^t}$ might be history dependent the equilibrium need not be Markovian. Define $z_t=\frac{\left[m_{t-1}^1\left(s^{t-1}\right)-\frac{c_{t-1}^1\left(s^{t-1}\right)}{p_{t-1}\left(s^{t-1}\right)}\right]}{M_{t-1}\left(s^{t-1}\right)}$ to be the fraction of fiat money held by type i in s^t. Notice that this problem is an incomplete markets environment with aggregate uncertainty due the government policy. I prove that there exists an equilibrium in which the aggregate state variables are $\left(s^t,z^t\right)\in S^t\times Z^t$ where S is the exogenous state space as defined before endowed with the discrete topology and $Z=[0,1]$.

Given any agent i, let the individual state variables be $\left(s^t,\xi_t^i,z^t\right)$ where ξ_t^i is the amount of fiat money agent i brings over from the previous period. We can analyze the agent's consumption and money holdings using the following dynamic program

$$V_t^i\left(s^t,\xi_t^i,z_t\right)=\sup_{l_t^i,\ \xi_{t+1}^i} u\left(l_t^i+p_t\left(s^t,z_t\right)\xi_t^i-p_t\left(s^t,z_t\right)\xi_{t+1}^j-T_t\right)-l_t^i$$
$$+\beta\int_{S\times Z}V_{t+1}\left(s^{t+1},\xi_{t+1}^i,z^{t+1}\right)Q_{t+1}\left(ds_{t+1},dz_{t+1},s^t,z^t\right)$$

where Q_{t+1} is joint distribution of s and z. Notice that

$$z_{t+1}\left(s^t,z^t\right)=\frac{M_t\left(s^t\right)-\xi_{t+1}^j}{M_{t+1}\left(s^{t+1}\right)}$$

and as a result Q_{t+1} computed using the distribution of the money supply policies $\left\{M_t\left(s^t\right)\right\}$. I assume that the Ramsey policy induces a joint distribution Q over S and Z such that it is appropriately measurable and satisfies the Feller property.[3]

Given our assumptions on state space S we can simplify the structure of this problem as follows; If $s_t=i$ then we know from the agents first order conditions $c_t^i\left(s^t\right)=g\left(1\right)$. Therefore

$$l_t^i\left(s^t,\xi_t^i,z_t\right)=g\left(1\right)+\left[p_t\left(s^t,z_t\right)\xi_t^j-p_t\left(s^t,z_t\right)\xi_{t+1}^j\right]+2T_t\left(s^t\right)$$

and so

$$V_t^i\left(s^t,\xi_t^i,z_t\right)=\sup u\left(g\left(1\right)\right)-\left[g\left(1\right)+p_t\left(s^t,z_t\right)\xi_t^j-p_t\left(s^t,z_t\right)\xi_{t+1}^j+2T_t\left(s^t\right)\right]$$
$$+\beta\int_{S\times Z}V_{t+1}\left(s^{t+1},\xi_{t+1}^i,z_{t+1}\left(s^t,z_t\right)\right)Q_{t+1}\left(ds_{t+1},dz_{t+1},s^t,z^t\right)$$

$$V_t^j\left(s^t,\xi_t^j,z_t\right)=\sup_{\xi_{t+1}^j\in\left[0,p_t\left(s^t,z^t\right)\xi_t^j\right]} u\left(p_t\left(s^t,z^t\right)\xi_t^j-p_t\left(s^t,z^t\right)\xi_{t+1}^j-T_t\left(s^t\right)\right)$$
$$+\beta\int_{S\times Z}V_{t+1}\left(s^{t+1},\xi_t^j,z_{t+1}\left(s^t,z_t\right)\right)Q_{t+1}\left(ds_{t+1},dz_{t+1},s^t,z^t\right)$$

The policy correspondence is given by $\gamma^\xi\left(s^t,\xi_t^j,z_t\right)$. The first step of the proof is to show that given continuous pricing functions $p_t\left(s^t,z^t\right)$, there exists a unique sequence of value functions $\left\{V^i\right\}_0$, $\left\{V^j\right\}_0$ and policy functions $\gamma^\xi\left(s^t,\xi_t^j,z_t\right)$ that solve the individual agents' problems. This part of the proof uses arguments developed in Miao (2006).

Let $\mathbb{V}$ denote the set of uniformly bounded and continuous real valued functions on $\mathbb{S}^t\times\mathbb{A}\times\mathbb{Z}^t$ and let $\mathbb{V}^\infty$ denote the set of sequences $v=(v_0,v_1,..)$ of such functions. $\mathbb{V}^\infty$ is a complete metric space with the norm

$$||v||=\sup_{(t.s^t,\xi_t,z^t)}\left|v_t\left(s^t,\xi_t,z^t\right)\right|$$

Define operator $\mathbb{T}$ as follows

[3] Given any continuous function f, $\int f\left(s',z'\right)Q_{t+1}\left(ds',dz',\cdot\right)$ is a continuous function on $S\times Z$.

$$(\mathbb{T}v)_t\left(s^t,\xi_t,z^t\right)=\max_{l_t,\ \xi_{t+1}} u\left(l_t+p_t\left(s^t,z_t\right)\xi_t-p_t\left(s^t,z_t\right)\xi_{t+1}-T_t\right)-l_t$$
$$+\beta\int_{S\times Z}V_{t+1}\left(s^{t+1},\xi^i_{t+1},z^{t+1}\right)Q_{t+1}\left(ds_{t+1},dz_{t+1},s^t,z^t\right)$$

In order to apply the contraction mapping theorem I first show $\mathbb{T}v\in\mathbb{V}^{\infty}$. Boundedness follows. To show continuity, consider a sequence $\left(s^t,\xi_t,z_t,\xi_{t+1}\right)^n\to\left(s^t,\xi_t,z_t,\xi_{t+1}\right)$. Since S is countable with the discrete topology, $\left(s^t\right)^n=s^t$ for n sufficiently large. Given our restriction to continuous pricing functions $p_t\left(s^t,z^t\right)^n\to p_t\left(s^t,z^t\right)$. As a result correspondence Γ is continuous. Then first term on the right hand side of the above dynamic program is continuous since u is continuous. Consider second term. For n sufficiently large

$$\int_{S\times Z}v_{t+1}\left(\left(s^{t+1}\right)^n,\left(\xi_{t+1}\right)^n,z^n_{t+1}\right)Q_{t+1}\left(ds_{t+1},dz_{t+1},\left(s^t\right)^n,z^n_t\right)$$
$$=\int_{S\times Z}v^i_{t+1}\left(\left(s^{t+1}\right)^n,\left(\xi_{t+1}\right)^n,z^n_{t+1}\right)Q_{t+1}\left(ds_{t+1},dz_{t+1},s^t,z^n_t\right)$$

Notice that

$$\left|\begin{array}{c}\int_{S\times Z}v^i_{t+1}\left(\left(s^{t+1}\right)^n,\left(\xi_{t+1}\right)^n,z^n_{t+1}\right)Q_{t+1}\left(ds_{t+1},dz_{t+1},s^t,z^n_t\right)\\-\int_{S\times Z}v^i_{t+1}\left(s^{t+1},\xi_{t+1},z_{t+1}\right)Q_{t+1}\left(ds_{t+1},dz_{t+1},s^t,z_t\right)\end{array}\right|$$
$$\leq\left|\begin{array}{c}\int_{S\times Z}v^i_{t+1}\left(\left(s^{t+1}\right)^n,\left(\xi_{t+1}\right)^n,z^n_{t+1}\right)Q_{t+1}\left(ds_{t+1},dz_{t+1},s^t,z^n_t\right)\\-\int_{S\times Z}v^i_{t+1}\left(s^{t+1},\xi_{t+1},z_{t+1}\right)Q_{t+1}\left(ds_{t+1},dz_{t+1},s^t,z^n_t\right)\end{array}\right|$$
$$+\left|\begin{array}{c}\int_{S\times Z}v^i_{t+1}\left(s^{t+1},\xi_{t+1},z_{t+1}\right)Q_{t+1}\left(ds_{t+1},dz_{t+1},s^t,z^n_t\right)\\-\int_{S\times Z}v^i_{t+1}\left(s^{t+1},\xi_{t+1},z_{t+1}\right)Q_{t+1}\left(ds_{t+1},dz_{t+1},s^t,z_t\right)\end{array}\right|$$

Since $\mathbb{S}^t\times\mathbb{A}\times\mathbb{Z}^t$ is compact by Tychonoff's theorem, v_{t+1} is uniformly continuous and as a result $v^i_{t+1}\left(\left(s^{t+1}\right)^n,\left(\xi_{t+1}\right)^n,\left(z^{t+1}\right)^n\right)\to v^i_{t+1}\left(s^{t+1},\xi_{t+1},z^{t+1}\right)$ uniformly. As a result we can interchange the limit and integrals. Next, the Feller property ensures that second term goes to 0 and $n\to\infty$. Therefore by Maximum theorem, $\mathbb{T}v$ is also continuous and hence $\mathbb{T}v\in\mathbb{V}^{\infty}$. It is easy to see that the operator satisfies Blackwell's sufficiency conditions. As a result operator $\mathbb{T}$ have a contraction and so by CMT we have unique sequence of functions v^* corresponding policy functions $\gamma^{\xi*}$

The next step is to prove the existence of these pricing functions. Define

$$\Lambda_t=p_t\left(s^t,z_t\right)u'\left(c^i_t\left(\xi_{t+1}\left(s^t,\xi^i_t,z_t\right)\right)\right)$$

Notice that

$$c^i_t\left(\xi_{t+1}\left(s^t,\xi^i_t,z_t\right)\right)=g\left(1\right)\text{ if }s_t=i$$
$$c^i_t\left(\xi_{t+1}\left(s^t,\xi^i_t,z_t\right)\right)=\left[p_t\left(s^t,z_t\right)\xi^j_t-p_t\left(s^t,z_t\right)\xi^j_{t+1}\left(s^t,\xi^j_t,z_t\right)\right]$$

We can write

$$\xi^j_t=M_{t-1}\left(s^{t-1}\right)-M_t\left(s^t\right)z_t\left(s^t\right)$$

Therefore Λ_t depends only on $\left(s^t, z_t\right)$. Agent i's Euler condition implies that

$$u'\left(c^i\left(s^t, z\right)\right) p_t\left(s^t, z^t\right) = \beta \int_{S\times Z} u'\left(c^i\left(s^{t+1}, z_{t+1}\right)\right) p\left(s^{t+1}, z^t\right) Q_{t+1}\left(ds_{t+1}, dz_{t+1}, s^t, z^t\right)$$
$$\Rightarrow \Lambda_t\left(s^t, z^t\right) = \beta \int_{S\times Z} \Lambda_{t+1}\left(s^{t+1}, z^{t+1}\right) Q_{t+1}\left(ds_{t+1}, dz_{t+1}, s^t, z^t\right)$$

Let $\mathbb{V}^p$ denote the set of uniformly bounded and continuous real valued functions on $\mathbb{S}^t \times \mathbb{Z}^t$ and let $(\mathbb{V}^p)^\infty$ denote the set of sequences $v = (v_0, v_1, ..)$ of such functions. $(\mathbb{V}^p)^\infty$ is a complete metric space with the norm

$$||v|| = \sup_{(t.s^t,\xi_t,z_t)} \left|v_t\left(s^t, z^t\right)\right|$$

Given $\Lambda \in (\mathbb{V}^p)^\infty$ define operator $\mathbb{K}$

$$(\mathbb{K}\Lambda)_t\left(s^t, z^t\right) = \beta \int_{S\times Z} \Lambda_{t+1}\left(s^{t+1}, z^{t+1}\right) Q_{t+1}\left(ds_{t+1}, dz_{t+1}, s^t, z^t\right)$$

Since Λ is bounded $\mathbb{K}\Lambda$ is also bounded. To show continuity consider a sequence $\left(s^{t+1}, z^{t+1}\right)^n \to \left(s^{t+1}, z^{t+1}\right)$. For n large enough have$\left(s^{t+1}, \left(z^{t+1}\right)^n\right)$. Then

$$\left|\int_{S\times Z} \Lambda_{t+1}\left(s^{t+1}, z^{t+1}\right)^n Q_{t+1}\left(ds_{t+1}, dz_{t+1}, \left(s^t, z^t\right)^n\right) - \int_{S\times Z} \Lambda_{t+1}\left(s^{t+1}, z^{t+1}\right) Q_{t+1}\left(ds_{t+1}, dz_{t+1}, s^t, z^t\right)\right|$$
$$\leq \left|\int_{S\times Z} \Lambda_{t+1}\left(s^{t+1}, \left(z^{t+1}\right)^n\right) Q_{t+1}\left(ds_{t+1}, dz_{t+1}, \left(s^t, z^t\right)^n\right) - \int_{S\times Z} \Lambda_{t+1}\left(s^{t+1}, z^{t+1}\right) Q_{t+1}\left(ds_{t+1}, dz_{t+1}, \left(s^t, z^t\right)^n\right)\right|$$
$$+ \left|\int_{S\times Z} \Lambda_{t+1}\left(s^{t+1}, z^{t+1}\right) Q_{t+1}\left(ds_{t+1}, dz_{t+1}, \left(s^t, z^t\right)^n\right) - \int_{S\times Z} \Lambda_{t+1}\left(s^{t+1}, z^{t+1}\right) Q_{t+1}\left(ds_{t+1}, dz_{t+1}, s^t, z^t\right)\right|$$

As before, the first term vanishes since Λ is uniformly continuous and second term by the Feller property. Thus $\mathbb{K}\Lambda$ is continuous. Since the operator satisfies Blackwell's sufficiency conditions it is a contraction. hence there exists unique fixed point Λ^*. Then given early result of existence of policy functions can compute pricing functions

$$p\left(s^t, z_t\right) = \frac{\Lambda_t^*\left(s^t, z_t\right)}{u'\left(c\left(s^t, z\right)\right)}$$

The last step in the proof is to show that there exist price functions for Arrow securities such that those along with the above policy and price functions constitute a not-too-tight equilibrium when debt constraints are zero. In this case each agent solves the same problem as described in Section 2.2.2 with

$$a_{s'}^i\left(s^t\right) \geq 0 \text{ for all } s^t, s' \in S$$

Here, an agent can only save in Arrow securities. To have an equilibrium with this property, market clearing dictates that no agent must wish to save in these securities. Therefore, we construct Arrow security prices as follows; for any $\left(s^t, s'\right)$

$$q_{s'}\left(s^t, z^t\right) = \max_i \left\{\int_{S\times Z} \frac{u'\left(c^i\left(s^{t+1}, z^{t+1}\right)\right)}{u'\left(c^i\left(s^t, z^t\right)\right)} Q_{t+1}\left(ds_{t+1}, dz_{t+1}, s^t, z^t\right)\right\}$$

At these prices, no agent will strictly prefers to save in Arrow securities since for all agents

$$q_{s'}\left(s^t,z^t\right)\geq\int_Z\frac{u'\left(c^i\left(s^{t+1},z^{t+1}\right)\right)}{u'\left(c^i\left(s^t,z^t\right)\right)}Q_{t+1}\left(s',dz_{t+1},s^t,z^t\right)$$

Agents would like to borrow but are constrained from doing so. As a result, markets clear which verifies that our guess is indeed an equilibrium. These debt constraints are trivially not-too-tight since in equilibrium there are no Arrow securities being traded.

5.1 Proofs From Section 3.1

5.1.1 Proof of Theorem 1

Proof of Part 1. Consider a competitive equilibrium of the debt constrained problem. Define

$$\zeta_t^{i,j}\left(s_t\right)=a^i\left(s_t\right)-\sum_{s_{t+1}}q_{s_{t+1}}\left(s^t\right)a^i_{s_{t+1}}\left(s^t\right)\text{ for all }j$$

Now given our proposed $\zeta^{i,j}$ along with the allocation $\left\{\left(c_t^i\left(s^t\right),l_t^i\left(s^t\right),m_t^i\left(s^t\right)\right)_{i\in I}\right\}_{t,s^t}$ from the debt constrained competitive equilibrium it must be that

$$\hat{V}_t^i\left(s^t,\boldsymbol{\zeta}^{i,j}\left(s^t\right);\mathbf{p}_t\right)=V_t^{i,c}\left(s^t,a^i_{s_t}\left(s^{t-1}\right);\Phi^i\left(s^t\right)\right)$$

where the term on the left hand side is defined using (12) and the term on the right hand side using (**??**) . This is because given the construction of $\zeta^{i,j}$ and the fact that the allocation satisfies the agent's optimality conditions in the debt constrained environment, it must be that the best the agent can do given the insurance contract is what she does in the debt constrained environment. Since the debt limits are chosen to be not-too-tight the allocation $\left\{\left(c_t^i\left(s^t\right),l_t^i\left(s^t\right),m_t^i\left(s^t\right)\right)_{i\in I}\right\}_{t,s^t}$ along with $\zeta_t^{i,j}$ constructed above satisfies incentive compatibility and so is feasible for the intermediary given the prices $\left\{q_{s_{t+1}}\left(s^t\right)\right\}_{s_{t+1},s^t}$.

Suppose that there exists an allocation that is feasible and gives the intermediary strictly higher profit. Note that perfect competition implies that such an allocation must increases both the ex-ante welfare of the firm and the agent. The only way an intermediary can do is to increase the amount of insurance provided to the agent. Consider any such contract with greater insurance that satisfies incentive compatibility.

Then it must be that for some i,s^t,s^{t+1},

$$q_{s_{t+1}}\left(s^t\right)>\frac{\beta\pi\left(s^{t+1}\mid s^t\right)u'\left(c^i_{t+1}\left(s^{t+1}\right)\right)}{u'\left(c^i_t\left(s^t\right)\right)}\tag{19}$$

and

$$V_t^{i,c}\left(s^{t+1},a^i_{s_{t+1}}\left(s^t\right);\Phi^i\left(s^{t+1}\right)\right)>V_{t+1}^{i,d}\left(s^{t+1}\right)\tag{20}$$

To see why notice that if (19) held with an equality, marginal rates of substitution would be equated amongst all agents and as a result an intermediary cannot offer a contract with greater ex-ante insurance. On the other hand if (19) held and (20) held with equality increasing the amount of insurance between those states would result in the agent strictly preferring to default in s_{t+1} and hence would violate incentive compatibility. However equations (19) and (20) contradict the not-too-tightness requirement of debt constraints.

Given this constructed contract, set $\mu^{*i}(C) = \delta_C$ and let all intermediaries j offer only this contract. ■

Next, I prove the converse. Assume first that in the equilibrium $\mu^{*i}(C) = \delta_C$ for some $C \in \mathcal{C}$. The following lemmas will be useful

Lemma 5 *Consider an equilibrium of the contracting environment. Then*

$$q_{s_{t+1}}\left(s^t\right) \geq \max_{i\in I}\left\{\frac{\beta\pi\left(s^{t+1} \mid s^t\right) u'\left(c^i_{t+1}\left(s^{t+1}\right)\right)}{u'\left(c^i_t\left(s^t\right)\right)}\right\} \text{ for all } s_{t+1} \tag{21}$$

with equality if there is some insurance being offered by the intermediary.

Proof of Lemma 5. I prove this by contradiction. Suppose that in the competitive equilibrium, for some i, s^t, s^{t+1}

$$q_{s_{t+1}}\left(s^t\right) < \frac{\beta\pi\left(s^{t+1} \mid s^t\right) u'\left(c^i_{t+1}\left(s^{t+1}\right)\right)}{u'\left(c^i_t\left(s^t\right)\right)} \tag{22}$$

Consider the following contract intermediary s^t can offer to this agent

$$\tilde{\zeta}^{i,j}_t\left(s^t\right) = \zeta^{i,j}_t\left(s^t\right) - q_{s_{t+1}}\left(s^t\right)\varepsilon$$
$$\tilde{\zeta}^{i,j}_{t+1}\left(s^{t+1}\right) = m^{s^t,i}_t\left(s^{t+1}\right) + \varepsilon$$

where $\varepsilon > 0$ and with the rest of the contract being unchanged. For ε small but positive the change in welfare to agent i is

$$\left[-q_{s_{t+1}}\left(s^t\right) u'\left(c^i_t\left(s^{t+1}\right)\right) + \beta\pi\left(s^{t+1} \mid s^t\right) u'\left(c^i_{t+1}\left(s^{t+1}\right)\right)\right]\varepsilon > 0$$

because of (22), while the change in intermediary welfare is

$$q_{s_{t+1}}\left(s^t\right)\varepsilon - q_{s_{t+1}}\left(s^t\right)\varepsilon = 0$$

As a result an intermediary can offer a deviating contract and make strictly positive profit which contradicts the definition of a competitive equilibrium.

Therefore,

$$q_{s_{t+1}}\left(s^t\right) \geq \max_{i\in I}\left\{\frac{\beta\pi\left(s^{t+1} \mid s^t\right) u'\left(c^i_{t+1}\left(s^{t+1}\right)\right)}{u'\left(c^i_t\left(s^t\right)\right)}\right\}$$

Now consider an equilibrium in which

$$q_{s_{t+1}}\left(s^t\right) > \max_{i\in I}\left\{\frac{\beta\pi\left(s^{t+1} \mid s^t\right) u'\left(c^i_{t+1}\left(s^{t+1}\right)\right)}{u'\left(c^i_t\left(s^t\right)\right)}\right\}$$

and some insurance is being offered between periods t and $t+1$. Since insurance is being offered there must exist some agent j such that

$$\hat{V}^i_t\left(s^t, \boldsymbol{\zeta}^{i,j}\left(s^t\right); \mathbf{p}_t, \delta\left(s^{t+1}\right) = 0\right) > \hat{V}^i_t\left(s^t, \boldsymbol{\zeta}^{i,j}\left(s^t\right); \mathbf{p}_t, \delta\left(s^{t+1}\right) = 1\right) \tag{23}$$

where $\hat{V}^i_t\left(s^t, \boldsymbol{\zeta}^{i,j}\left(s^t\right); \mathbf{p}_t, \delta\left(s^{t+1}\right) = 0\right)$ is the agent's best deviation conditional on not defaulting while the term on the right hand side is the best deviation conditional on defaulting. Inequality (23) says that the value for some agent j (who receives positive transfers) staying in the contract is strictly greater than defaulting. Consider the following contract an intermediary could offer agent j

$$\begin{aligned}\tilde{\zeta}_t^{i,j}\left(s^t\right) &= \zeta_t^{i,j}\left(s^t\right) + q_{s_{t+1}}\left(s^t\right)\varepsilon \\ \tilde{\zeta}_{t+1}^{i,j}\left(s^{t+1}\right) &= m_t^{s^t,i}\left(s^{t+1}\right) - \varepsilon\end{aligned}$$

For ε small the change in welfare is

$$\left[q_{s_{t+1}}\left(s^t\right)u'\left(c_t^i\left(s^{t+1}\right)\right) - \beta\pi\left(s^{t+1}\mid s^t\right)u'\left(c_{t+1}^i\left(s^{t+1}\right)\right)\right]\varepsilon > 0$$

while

$$\hat{V}_t^i\left(s^t,\boldsymbol{\zeta}^{i,j}\left(s^t\right);\mathbf{p}_t,\delta\left(s^{t+1}\right)=0\right) \geq \hat{V}_t^i\left(s^t,\boldsymbol{\zeta}^{i,j}\left(s^t\right);\mathbf{p}_t,\delta\left(s^{t+1}\right)=1\right)$$

And so in period t and state s^t, the agent is strictly better off under this contract. By a similar argument to above, since such a perturbation is welfare neutral to the intermediary he can offer a contract and make strictly positive profits contradiction the competitive equilibrium assumption. This proves the claim. ■

Lemma 6 *In the competitive equilibrium, for any* $\left(s^t,s_{t+1}\right)$, *if full insurance is not being provided by the intermediary, then for some* i

$$\hat{V}_t^i\left(s^t,\boldsymbol{\zeta}^{i,j}\left(s^t\right);\mathbf{p}_t,\delta\left(s^{t+1}\right)=0\right) = \hat{V}_t^i\left(s^t,\boldsymbol{\zeta}^{i,j}\left(s^t\right);\mathbf{p}_t,\delta\left(s^{t+1}\right)=1\right)$$

Proof of Lemma 6. Suppose not and that full insurance is not being provided. Then there is some agent i, states s^t, s^{t+1} such that

$$q_{s_{t+1}}\left(s^t\right) > \frac{\beta\pi\left(s^{t+1}\mid s^t\right)u'\left(c_{t+1}^i\left(s^{t+1}\right)\right)}{u'\left(c_t^i\left(s^t\right)\right)}$$

and by assumption

$$\hat{V}_t^i\left(s^t,\boldsymbol{\zeta}^{i,j}\left(s^t\right);\mathbf{p}_t,\delta\left(s^{t+1}\right)=0\right) > \hat{V}_t^i\left(s^t,\boldsymbol{\zeta}^{i,j}\left(s^t\right);\mathbf{p}_t,\delta\left(s^{t+1}\right)=1\right)$$

Then an intermediary at s^t can offer the following contract to agent i

$$\begin{aligned}\tilde{\zeta}_t^{i,j}\left(s^t\right) &= \zeta_t^{i,j}\left(s^t\right) + q_{s_{t+1}}\left(s^t\right)\varepsilon \\ \tilde{\zeta}_{t+1}^{i,j}\left(s^{t+1}\right) &= m_t^{s^t,i}\left(s^{t+1}\right) - \varepsilon\end{aligned}$$

with the rest of the contract unchanged. For ε small enough this contract gives the agent greater utility to the agent while leaving the intermediary equally well off. To see that all the incentive compatibility constraints are still satisfied notice that since $\frac{\partial\hat{V}_{t+1}^i\left(s^{t+1},\boldsymbol{\zeta}^{i,j}\left(s^t\right);\mathbf{p}_t\right)}{\partial\zeta_{t+1}^{i,j}\left(s^{t+1}\right)} > 0$, and $\hat{V}_{t+1}^i\left(s^{t+1},\boldsymbol{\zeta}^{i,j}\left(s^t\right);\mathbf{p}_t\right)$ is continuous for ε small enough the voluntary participation constraint in s^{t+1} still holds while in s^t, the value of not defaulting increases. As a result there exists a contract with gives the intermediary positive profit which contradicts the allocation/price being a competitive equilibrium ■

The proof of the part 2 of the theorem relies on a limiting argument due to Fudenberg and Levine (1983). The idea is to construct truncated allocations of the debt

constrained environment, the limit of which converges to an equilibrium with not-too-tight debt constraints. I turn to this construction next.

Let $\left\{\left(c_t^i\left(s^t\right), m_t^i\left(s^t\right), l_t^i\left(s^t\right), \zeta_t^{i,j}\left(s^t\right), \bar{V}_t^i\left(s^t\right)\right)_{i\in I}\right\}_{t,s^t}$ be the allocation associated with the competitive equilibrium in the contracting environment. I first construct a truncated $T-$period allocation for the economy with debt constraints as follows: Define $\left\{a^{T,i}\left(s^t\right)\right\}$ using the equations

$$a_{s_T}^{T,i}\left(s^{T-1}\right) = \zeta_T^{i,j}\left(s^T\right) \tag{24}$$

$$a_{s_t}^{T,i}\left(s^{t-1}\right) - \sum_{s^{t+1}} q_{s_{t+1}}\left(s^t\right) a_{s_{t+1}}^{T,i}\left(s^t\right) = \zeta_t^{i,j}\left(s^t\right) \text{ for all } t < T \tag{25}$$

For all $t < T$, let

$$\phi_{s_{t+1}}^{T,i}\left(s^t\right) = a_{s_{t+1}}^{i,T}\left(s^t\right)$$

for i, s^{t+1} such that

$$\hat{V}_t^i\left(s^t, \boldsymbol{\zeta}^{i,j}\left(s^t\right); \mathbf{p}_t, \delta\left(s^{t+1}\right) = 0\right) = \hat{V}_t^i\left(s^t, \boldsymbol{\zeta}^{i,j}\left(s^t\right); \mathbf{p}_t, \delta\left(s^{t+1}\right) = 1\right) \tag{26}$$

where we know that there exists i for an s^t, s^{t+1} such that the above holds true from lemma 6 . For $t > T$ define

$$a_{s^t}^{T,i}\left(s^{t-1}\right) = 0,$$
$$\phi_{s^t}^{T,i}\left(s^{t-1}\right) = 0$$

Next, for all $t \leq T$, $c_t^{T,i}\left(s^t\right) = c_t^i\left(s^t\right)$, $l_t^{T,i}\left(s^t\right) = l_t^i\left(s^t\right)$, $m_t^{T,i}\left(s^t\right) = m_t^i\left(s^t\right)$ and for $t > T$, $c_t^{T,i}\left(s^t\right)$, $l_t^{T,i}\left(s^t\right)$, $m_t^{T,i}\left(s^t\right)$ be the best allocation that can be chosen by agent i given prices and the constructed debt constraints. Clearly the above allocation does not constitute a competitive equilibrium with not-too-tight constraints since in period T, given that there is no borrowing and lending the future, no agent will be willing to honor her debt. Let $\Delta^i\left(s^t\right) \in \{0,1\}$ denote the agent's default decision in any period and state. Define the sequence $\left\{\Delta^{T,i}\right\}_0$ where

$$\Delta^{T,i}\left(s^t\right) = 0 \text{ for } t \leq T$$
$$\Delta^{T,i}\left(s^t\right) = 1 \text{ for } t > T$$

For each entrepreneur/agent i, define

$$\Gamma^i(T) = \left\{ \begin{array}{c} \left\{c^i, m^i, l^i, a^i, \Delta\right\}_0 : \text{For all } t, s^t, \ \left(c_t^i\left(s^t\right), m_t^i\left(s^t\right), l_t^i\left(s^t\right), \left\{a_{s_{t+1}}^i\left(s^t\right)\right\}_{s_{t+1}}, \Delta^i\left(s^t\right)\right) \\ \text{satisfies equations (2), (3) and (5) and for } t' > T \\ \text{correspond to the best allocation given prices and debt constraints} \end{array} \right\}$$

and $\Gamma(T) = \prod_{i\in I} \Gamma^i(T)$. $\Gamma^i(T)$ consists of choices that are budget feasible for the agent and involve living in financial autarky after period T. $\Gamma(\infty)$ is the untruncated choice sets for the agent. Clearly, the truncated allocation constructed above in an element of this set for each i given the debt constraints.

Define $x^{i,T} = \left\{c^{T,i}, m^{T,i}, l^{T,i}, a^{i,T}, \Delta^{i,T}\right\}_0$ and $x^{i,*} = \left\{c^i, m^i, l^i, a^i, \Delta^i\right\}_0$ where $\Delta^i\left(s^t\right) = 0$ for all t and $\left\{c^i, m^i, l^i\right\}_0$ correspond to the competitive equilibrium allocation in the contracting problem and a^i is the limit as $T \to \infty$ of asset holdings constructed using (24) and (25).

For any i consider the best deviation in $\Gamma^i(T)$ from the truncated allocation constructed above given the debt constraints $\left\{\phi^{T.i}\right\}$. From lemma 5 we know that in any equilibrium of the contracting environment $q_{s_{t+1}}\left(s^t\right) \geq \max_{i\in I}\left\{\frac{\beta\pi\left(s^{t+1}|s^t\right)u'\left(c_{t+1}^i\left(s^{t+1}\right)\right)}{u'\left(c_t^i\left(s^t\right)\right)}\right\}$. As a result for all $t < T$, conditional on not defaulting the truncated allocation is optimal for each agent given the debt constraints constructed above. Therefore, the best possible deviation (if one exists) involves default by some agent i in some period $t \leq T$. Define $\varepsilon^{i,T} \geq 0$ to be the value of the best possible deviation for agent i in $\Gamma^i(T)$ for any $t \leq T$,

$$V_t^{i,d}\left(s^t\right) - V_t^{i,c}\left(s^t, a_{s_t}^i\left(s^{t-1}\right); \Phi^i\left(s^t\right)\right) = \varepsilon^{i,T}$$

and let $\varepsilon^T = \max_i \varepsilon^{i,T}$. In particular, ε^T corresponds to the best possible deviation from the truncated allocation that can be achieved by any player who chooses from choice set $\Gamma^i(T)$. Let

$$X_t = R_+ \times R_+ \times R_+ \times R_+ \times \{0,1\}$$

$X_{t'} = \prod_{t=0}^{t'} X_t$ and $X = \prod_{t=0}^{\infty} X_t$. I have $x^{i,T} \in X$ and $x^{i,*} \in X$. The metric

$$d(x,z) = \sup_t \left[\frac{1}{t}\min\left\{|x_t - z_t|, 1\right\}\right]$$

induces the product topology on X. We can see that $x^{i,T} \to x^{i,*}$ in this metric.

Definition 9 *A allocation $\left(\left\{c^{T,i}, m^{T,i}, l^{T,i}, a^{T,i}, \Delta^{T,i}\right\}_0\right)_{i\in I} \in \Gamma(T)$ is an $\varepsilon-$perfect equilibrium ($\varepsilon -$ perfect) if for each t, s^t, any agent i's best deviation in $\Gamma^i(T)$ from the prescribed allocation yields her a welfare gain of at most ε.*

Given this definition and the previous discussion we have that x^T is ε^T perfect in $\Gamma(T)$ and that $x^T \to x^*$. Notice that since the contracting equilibrium allocations satisfy incentive compatibility (11), $\varepsilon^T \to 0$ since the agent does not want to deviate. Next, I adapt an argument from Theorem 3.3. in Fudenberg and Levine (1983) and prove that

Theorem 3 *A sufficient condition that x^* be perfect in $\Gamma(\infty)$ is that there be a sequences ε^T, x^T such that x^T is ε^T-perfect in $\Gamma(T)$ and as $T \to \infty$, $\varepsilon^T \to 0$ and $x^T \to x^*$.*

Similar to ε^T, I define w^T to be the greatest variation in any agent's payoff due to events after $T-1:$ for any t, s^t let

$$\tilde{V}_t^i\left(s^t, a_{s_t}^i\left(s^{t-1}\right); \Phi^i\left(s^t\right)\right) = \max\left\{V_t^{i,d}\left(s^t\right) V_t^{i,c}\left(s^t, a_{s_t}^i\left(s^{t-1}\right); \Phi^i\left(s^t\right)\right)\right\}$$

and define

$$w^t = \max_{i\in I}\left(\max_{\substack{x_1,x_2\in X \\ x_1,x_2 \text{ feas} \\ x_1(T-1)=x_2(T-1)}} \tilde{V}_0^i\left(s_0, a_0^i; \Phi^i\left(s_0\right)\right)\right)$$

The restriction $x_1(T-1) = x_2(T-1)$ just means that the allocation is identical for all dates and states prior to period T.The following two lemmas whose proofs can be found in Fudenberg and Levine (1983) will be useful,

Lemma 7 *If x is $\varepsilon-$perfect in $\Gamma(T)$ then x is $\left(\varepsilon + w^T\right)$ perfect in $\Gamma(\infty)$*

Lemma 8 *Let x be $\varepsilon-$perfect in $\Gamma(\infty)$ and $x \to x^*$. Then x^* is also ε-perfect.*

Proof of Theorem 3. From lemma 7 we know that x^T is $\varepsilon^T + w^T$-perfect in $\Gamma(\infty)$. Since $\varepsilon^T + w^T \to 0$ (because $x^T \to x^*$) for each $\delta > 0$ there is some T^*such that $\varepsilon^{T'} + w^{T'} < \delta$ for all $T' > T^*$. From lemma 8, x^* is $\delta-$perfect in $\Gamma(\infty)$. Since this is true for all $\delta > 0$, x^* is perfect in $\Gamma(\infty)$. ■

The theorem says that the truncated allocations converge to an equilibrium of the model with debt constraints. The last thing which needs to be checked is that the not-too-tight property is satisfied, but this follows from the construction of the debt limits. In particular if for any agent i

$$q_{s_{t+1}}\left(s^t\right) > \frac{\beta\pi\left(s^{t+1} \mid s^t\right) u'\left(c^i_{t+1}\left(s^{t+1}\right)\right)}{u'\left(c^i_t\left(s^t\right)\right)}$$

this agent sets $a^i_{s_{t+1}}\left(s^t\right) = \phi^i_{s_{t+1}}\left(s^t\right) < 0$. But then then from (26) I see that

$$\hat{V}^i_t\left(s^t, \boldsymbol{\zeta}^{i,j}\left(s^t\right); \mathbf{p}_t, \delta\left(s^{t+1}\right) = 0\right) = \hat{V}^i_t\left(s^t, \boldsymbol{\zeta}^{i,j}\left(s^t\right); \mathbf{p}_t, \delta\left(s^{t+1}\right) = 1\right)$$

which proves that the debt constraints are not-too-tight.

Recall that the proof assumed that μ^{*i} was a dirac measure. If in equilibrium, multiple contracts are offered, the proof proceeds in an identical fashion except that the debt constraints associated with each contract will be associated with a corresponding measure of agents in the debt-constrained environment.

Proof of Proposition 8. Suppose not. There are two possibilities to consider. The first is one in which in equilibrium all agents are eligible for this savings scheme i.e $\chi(C) = 1$ and $\mu^{i*}(C) = 1$ for all $i \in I$. The only relevant case we need to consider is if for any t, s^t $R^g_t\left(s^t\right) > \frac{p_t\left(s^t\right)}{p_{t-1}\left(s^t\right)}$ and $\tau_t\left(s^t\right) > 0$. It is straightforward to notice that this savings scheme is equivalent to a subsidy on labor. To see this notice since all productive agents strictly prefer to hold their output with government, the optimal contract must satisfy

$$l^i\left(s^t\right) : \beta^t\pi\left(s^t\right) = R^g \sum_{s^{t+1}} \lambda^i\left(s^{t+1}\right)$$

$$\beta^t\pi\left(s^t\right) u'\left(c^i_t\left(s^t\right)\right) = \frac{1}{\beta}\sum_{s^{t+1}} \lambda^i\left(s^{t+1}\right)$$

Here I have assumed that $R^g_t\left(s^t\right)$ can always be chosen so that the CIA constraint is slack (??). Therefore

$$\frac{1}{u'\left(c^i_t\left(s^{t-1}, i\right)\right)} = \beta R^g$$

and so one can reinterpret this as the agents believing that they now have a production technology that transforms one unit of labor into $\beta R^g > 1$ units of output. If this scheme is offered on the equilibrium path, all productive agents will choose it and as a result while the total amount of consumption goods produced in t, s^t is $l^i_t\left(s^t\right)$, the amount of consumption goods owed by the government owes is

$$\sum_{i \in I} \int_{\xi \in \mathcal{C}} \chi_{s^t}(\xi) R^g_t\left(s^t\right) l^{\xi,i}_{t-1}\left(s^{t-1}\right) d\mu^{i*}_{s^t}(\xi) = \sum_{i \in I} R^g_t\left(s^t\right) l^{C,i}_{t-1}\left(s^{t-1}\right)$$

which is strictly greater than the total amount of consumption goods in the economy $\sum_{i \in I} \frac{p_t\left(s^t\right)}{p_{t-1}\left(s^{t-1}\right)} l^{C,i}_{t-1}\left(s^{t-1}\right)$.

As a result if all contracts offered in equilibrium are eligible, such a scheme is never feasible.

Now consider a second possibility in which not all contracts offered in equilibrium are eligible for the scheme. As a result policy may now satisfy feasibility. For any contract $\hat{C}$ such that $\chi\left(\hat{C}\right) = 1$, it must be that if $\chi\left(\hat{C}\right) = 0$, the contract is no longer feasible. Else we could just set $\chi\left(\hat{C}\right) = 0$ for all such contracts and so vacuously $\tau_t\left(s^t\right) = 0$. As a result we can restrict to the case in which all contracts eligible for the scheme offer strictly greater insurance than those that are not. Therefore all agents will strictly prefer signing with firms offering eligible contracts. Since intermediaries can only distinguish types but not within types, as in the previous case all agents will be eligible for the scheme which is infeasible. ■

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