Multiplicity of Equilibria in Environments with Limited Commitment*

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Abstract

In this paper, I study a class of decentralized contracting environments with multiple equilibria that can endogenously generate aggregate fluctuations. In these models agents sign long term contracts with risk-neutral intermediaries subject to voluntary participation constraints. Agents run production technologies by accumulating capital and hiring labor and are subject to idiosyncratic productivity shocks. They can at any time choose to default on their obligations to the intermediary and live in financial autarky forever. In autarky, agents can continue to run their technology but can no longer sign contracts with intermediaries. I consider two related environments; the first in which agents' actions are observable and the second in which these actions are hidden. In the second case, intermediaries cannot observe agents' choices of capital and labor. I show that in both cases, the decentralized environment has multiple equilibria. While this can be shown directly in the case with observable actions, in the case with hidden actions, the contracting problem is intractable. However, I prove that the set of equilibria is identical to a different environment in which agents trade Arrow securities subject to debt constraints that are determined endogenously in equilibrium. Equilibria can be characterized in the equivalent environment using standard techniques. In these models, multiple equilibria exist due to strategic complementarities in players' actions. For example, in the debt constrained setup, an autarkic equilibrium exists because if all agents expect that there will be no borrowing and lending in the future, no agent will be willing to lend today since the borrower will always choose to default. Similarly, in the contracting environment, if intermediaries believe that agents will not be able to borrow in the future, they will not be willing to lend today. This allows for the possibility of generating sunspot equilibria and hence endogenously generated aggregate fluctuations. To illustrate this, I compute a simple example in which there are sizable and persistent changes to aggregate output and investment due to changes in expectations about the tightness of debt constraints in the future.

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1 Introduction

In this paper, I study decentralized contracting environments with limited commitment in which agents sign long term contracts with risk-neutral intermediaries. The agents are risk-averse entrepreneurs who hire labor and accumulate capital to run a production technology subject to idiosyncratic productivity shocks. Agents sign $T-$ period contracts with intermediaries who can borrow and lend with each other at market determined prices. At the beginning of each period, agents can choose to default on their obligations to the intermediary and consequently live in financial autarky in all future periods. I consider two assumptions on the observability of the agents' actions. The first is that all actions are observable and contracts can be contingent on the level of capital and labor hired by the entrepreneurs. The second is that these actions are hidden. The paper has two main results. The first is that under both assumptions the economy has multiple equilibria and one can generate sunspot equilibria in which there are endogenous fluctuations in aggregate real variables. The second main result is that under the hidden action assumption, the set of equilibria is identical to the set in an environment in which agents trade arrow securities subject to state contingent debt constraints which are "Not-too-Tight" as defined in Alvarez and Jermann (2000).

The decentralized environment is similar to the ones studied in Prescott and Townsend (1984) and Golosov and Tsyvinski (2007). There are a large number of $T-$period lived intermediaries who can borrow and lend in a state contingent fashion with each other. In each period agents who haven't defaulted in the past can sign contracts with an intermediary. In the benchmark case where the agent's actions are observable, a contract consists of a sequence of state contingent transfers and production plans for the agents. Agents run Cobb-Douglass production technologies and are subject to idiosyncratic productivity shocks. They can hire labor and accumulate capital and can themselves work on projects run by other entrepreneurs. At the beginning of each period, any agent can choose to default on her contract and thereafter be disallowed from signing with any intermediary in the future. However, the agent can continue to run the production technology and use her capital stock to achieve some degree of smoothing. Any feasible contract must satisfy resource and voluntary participation constraints. The latter constraints state that at any date and state, the value of not defaulting on the contract must be at least weakly greater than the value of default. The intermediary chooses a feasible contract that maximizes the present discounted value of its profits. I show that this environment has multiple competitive equilibria. The result may seem surprising since in general optimal contracting problems of this nature are thought to be associated with unique equilibria especially in Prescott and Townsend (1984) whether the equilibrium corresponds to unique efficient allocation. However, this is not true in the environment I consider and in general even the equilibrium associated with the highest ex-ante welfare is not in general efficient.

Multiple equilbria exist due to strategic complimentarities in the actions of intermediaries. Intermediaries know that agents' incentives to not default on a contract depend on the types of contracts they will be able to sign with future intermediaries. For example if all intermediaries today believe that no intermediary in the future will be lend to agents, the only feasible contract they can offer is one with no insurance. To see why notice that in the last period of the contract T, any agent owing positive transfers to the intermediary will strictly prefer to default on them and therefore any feasible contract cannot have an agent making positive transfers in the last period. Next consider period $T-1$ and a transfer sequence that consists of a negative transfer to the agent in $T-1$ and a positive transfer in T. Since in period T, the intermediary cannot promise a negative transfer to any agent such a sequence must be financed through the intermediary borrowing and lending. The result in the section states that in equilibrium, the interest rates are such that such transfer sequences are not profitable for the intermediary and therefore the intermediary strictly prefers to offer the zero insurance contract. The next result in this section uses a continuity argument to show that another equilibrium exists in which intermediaries find it optimal to offer some insurance.

Next, I turn to the case in which the agent's actions are hidden. Here, the intermediary cannot observe the agent's actions (capital accumulation, labor etc) and hence a contract cannot be contingent on these as they could in the environment with observable actions. Hidden actions allow for the agents to engage in double deviations in which they deviate from the prescribed allocation in one period and default the following period. The optimal contract must satisfy incentive compatibility constraints that prevent such deviations. The nature of incentive compatibility constraints in this problem makes the characterization of the contract difficult. The first main result in this section proves an equivalence between the set of equilibria in this contracting environment and the set in a different environment where agents trade Arrow securities subject to state contingent debt constraints.

The model with debt constraints turns out to be a a direct extension of the Alvarez and Jermann (2000) state contingent,"not-too-tight" debt constraint setup to the case with idiosyncratic productivity heterogeneity and capital accumulation. Entrepreneurs run constant returns to scale production technologies that are subject to productivity shocks. They hire labor, accumulate capital and can supply their own labor. As is standard in the literature on financial frictions, agents know their next period productivity shock in the current period and so would like to accumulate capital accordingly. Markets are complete and agents can purchase arrow securities subject to state contingent debt constraints. At any date and state, an entrepreneur can choose to default on her obligations and live in financial autarky forever. In autarky, she can continue to run her production technology, accumulate capital and work but can no longer trade arrow securities. Debt constraints are determined endogenously in equilibrium and are chosen to be "not-too-tight" as in Alvarez and Jermann (2000). This means that an entrepreneur who has borrowed up to the limit is indifferent between paying back her (state contingent) debt and defaulting and living autarky forever. In an endowment economy, Alvarez and Jermann (2000) show that "not-too-tight" constraints (weakly) decentralize the efficient Kehoe and Levine (1993) allocation. However, this decentralization fails to hold in the environment I consider. To see why, notice that entrepreneur's default problem depends on equilibirum prices such as the wage rate, which implies that prices show up in the voluntary participation constraints a planner would have to respect. This results in a pecuniary externality due to the fact that agents do not internalize the effect of their choices on their default incentives through equilibrium prices.

The result is significant for two reasons. On one hand, it allows us to study properties of the optimal contract by studying a relatively easier problem. Moreover, the equivalence allows us to interpret a natural extension of not-too-tight debt constraints as being derived from a decentralized contracting problem. This is useful since these constraints can longer be thought of as an implementing the efficient allocation. Importantly, these debt constraints do not arbitrarily restrict the types of contracts borrowers and lenders can sign. The intuition for the equivalence results is as follows- first consider the model with debt constraints. Using the entrepreneurs' choice of asset holdings given constraints and prices it is straightforward to construct an insurance contract that is feasible for an intermediary who is subject to the same prices. I show that if any contract that provides more insurance to the agent and keeps the intermediary equally well off necessarily violates incentive compatibility in that agents will choose to default in some state. Next, consider an equilibrium of the contracting environment. I show that any equilibrium must satisfy a property of any competitive equilibrium with borrowing constraints, namely that the arrow prices must equal the maximum marginal rate of substitution of the agents. This implies that given appropriately constructed borrowing constraints, the consumption allocation from the contracting problem satisfies the inter-temporal optimality constraint in the debt constraint problem. The only involved aspect of the proof is the construction of asset holdings and collateral constraints. To do this I consider a T period truncated allocation in which agents are restricted to autarky after period T. I use results from Fudenberg and Levine (1983) and define the notion of an $\varepsilon-$perfect equilibrium where a deviating agent can gain at most ε by deviating from the truncated allocation. I show that the limit of this allocation as $T \to \infty$ converges to an equilibrium of an economy with not-too-tight debt constraints.

As in the case of observable actions, the hidden action environment also has multiple equilibria. I demonstrate this in a simple two type deterministic model where the entrepreneurs' productivity alternate between high and low. Agents can borrow and lend subject to debt constraints and so markets are complete. The simplicity of the model allows us to highlight the source of multiplicity to be these endogenous debt constraints and not for example market incompleteness where multiplicity is known to exist. It easy to see that autarky is an equilibrium since the value of default trivially equals the value of being able to continue to participate in financial markets. Next, I use continuity properties of entrepreneurs' value functions to show that if autarkic interest rates are low, there exists a positive level of borrowing and lending such that all agents strictly prefer it to autarky. Consequently it is straightforward to prove the existence of a stationary equilibrium with non-zero debt constraints that have the not-too-tight property. The intuition for the existence of multiple equilibria is straightforward- expectations of the future tightness of debt constraints affects the current availability of credit. For example, if all entrepreneurs believe that there will be no borrowing and lending in the future, there will be no borrowing and lending in the present as no agent will be willing to pay back her debt. Similarly, the expectation that constraints will be loose in the future, allows for there to be borrowing and lending in the present period. In addition to the two stationary equilibria, there is an indeterminacy of equilibria (as in Woodford (1986b) etc.). In particular, for a large space of parameters there are a continuum of non-stationary equilibria converging to the autarkic equilibrium. The existence of indeterminacy allows us to construct sunspot equilibria. I show in a numerical example that expectational shocks can affect investment and aggregate output and that the effects are persistent. As a result I show that these models have an endogenous source of aggregate fluctuations.

Literature: This paper is related to three strands of literature. First, it draws from and contributes to the theory of optimal contracting, examples of which include Green (1987), Thomas and Worrall (1990), Atkeson and Lucas (1992) and Albuquerque and Hopenhayn (2004). Albuquerque and Hopenhayn (2004) study a contracting problem where the agent has a limited liability problem and debt payment cannot be enforced. The show that in the optimal contract, borrowing must be constrained. However in contrast to models with exogenous collateral constraints, they find that the borrowing constraints have a forward looking component in which future profitability affects current access to credit. This paper is also related to a growing subset of the contracting literature that deals with principal-agent problems in which agents can engage in hidden actions. Papers that deal with the hidden action problem include Cole and Kocherlakota (2001), Golosov and Tsyvinski (2007) and Shimer and Werning (2008). Golosov and Tsyvinski (2007) study the problem of providing insurance to agents with hidden types who can borrow and lend in a hidden fashion. Unlike the classic results of Atkeson and Lucas (1992) and Prescott and Townsend (1984), they find that the competitive equilibrium is inefficient. The reason for inefficiency is that prices show up in the consumption sets of agents. Our contracting environment is similar to their decentralized environment but with a few important differences. First, this model only has hidden actions with no hidden types. The equivalence result will not hold with hidden types as agents will not be willing to trade state contingent securities since there is no mechanism enforcing these trades when types are private. In addition, while Golosov and Tsyvinski (2007) consider a single T period intermediary I consider the case in which agents sign consecutive T period contracts with different intermediaries. This introduces a dynamic complementarity between the actions of intermediaries which allows for the equivalence between with the contracting environment. The dynamic complementarity is the primary reason for equilibrium multiplicity in both environments.

The model with endogenous debt constraints is related to the literature on financial frictions originating with Kiyotaki and Moore (1997). More recent examples include Gertler and Kiyotaki (2010), Shourideh and Zetlin-Jones (2012) and Buera and Moll (2012). Shourideh and Zetlin-Jones (2012) Chari (2012) study the flow of funds data and show that if I consider a representative firm, on average, funds flow from firms to households. In particular, the representative firm has more than enough cash in hand to undertake their investment. Consequently they argue that any reasonable financial frictions model must have heterogeneous firms to account for this fact. I incorporate this into our model with collateral constraints by introducing heterogenous entrepreneurs. Buera and Moll (2012) study variants of standard financial frictions models and show how they can be mapped to back to different aggregate wedges. However they assume incomplete markets and model collateral constraints that are linear in the agent's capital stock. As mentioned earlier in the introduction, it is unclear where the restriction on the types of contracts borrowers and lender sign come from. Gertler and Kiyotaki (2010) on the other hand derive their constraint from Kehoe and Levine (1993) like voluntary participation constraints. Unlike the previous papers however they do not have heterogeneous producers.

I choose to model debt constraints differently than the papers mentioned above . I follow Alvarez and Jermann (2000)'s definition of not-too-tight and define credit constraints that are endogenously determined in equilibrium. In particular, the constraints are chosen so that an agent who has borrowed up to the limit the previous period is indifferent between paying back and defaulting and living in financial autarky forever. While Alvarez and Jermann (2000) consider an endowment economy, I consider one with production heterogeneity and capital accumulation in order to relate it to some of the more recent models with financial frictions like Shourideh and Zetlin-Jones (2012) and Buera and Moll (2012). Alvarez and Jermann (2000) show that the efficient allocation from Kehoe and Levine (1993) can be decentralized using complete markets and not-too-tight collateral constraints. The decentralization is weak as they show that autarky is always an equilibrium in their environment. Moreover in a recent paper Bloise, Reichlin, and Tirelli (2013) show that the equilibrium is indeterminate and that for any value of aggregate welfare between the efficient allocation and autarky once can construct an equilibrium with not-too-tight constraints that achieve it. I show via an example that the indeterminacy results carry through to a more complicated environment with production heterogeneity and capital accumulation.

Finally, this paper is related to the literature on multiple equilibria in general equilibrium models. Woodford (1986a) has a simple example of an economy with finance constraints and shows the existence of self-fulfilling fluctuations similar to overlapping generations economies. Moreover he shows that the persistence of these fluctuations is similar to those of business cycle fluctuations. In another important paper, Woodford (1986b) provides conditions in order for a steady state of non-linear model to be indeterminate and proves that indeterminacy is a necessary and sufficiency condition for the existence of sunspot equilibria. I use these results to argue the existence of sunspot equilibria in our example. A related to paper to ours is Azariadis and Kaas (2012). They study a simple financial frictions model with two types and linear constraints on debt of the form $d \leq \theta k$. However, unlike some of the other papers, the parameter θ is an equilibrium object and is determined using a not-too-tight constraint. I take this analysis a step further and do not assume apriori a form on the collateral constraint. Similar to them, I also find multiplicity in this environment and use this fact to generate sunspot equilibria. Another related paper is Gu, Mattesini, Monnet, and Wright (2013) who also prove the existence of multiple equilibria in a model with endogenous debt limits and show how one can generate endogenous fluctuations with sunspot dynamics.

The rest of the paper proceeds as follows: In section 2 and 3 I lay out the contracting model and the model with collateral constraints respectively. Then in section 4, I prove an equivalence between the two. Section 5 contains a simple example of the not-too-tight debt constraint economy and shows how one can use multiplicity to generate aggregate fluctuations Section 6 concludes.

2 Observable Actions

Consider an environment with a representative $T-$period lived intermediary and I entrepreneurs (also referred to as agents). For ease of notation I assume that $T = 2$, but the argument generalizes for any finite T. Time $t = 0, 1, ..$ is discrete and let Z be the finite state space and S^t be set of histories till time t with typical element $s^t = (z_0, z_1, ..., z_{t+1})$ (the timing will be described below). The state is known to all agents and the intermediary. The transition probabilities are given by a matrix Π, with the unconditional probabilities of histories denoted as $\pi\left(s^t\right)$ and the conditional probabilities by $\pi\left(s^{t+1} \mid s^t\right)$. The symbol $\succeq$ is used the denote the partial order on histories. For example, $s^{t'} \succeq s^t$ for $t' \geq t$ denotes a possible continuation of history s^t. Given a random variable x, I use the notation $\{x\}_{t'}$ to denote the stochastic process $\left\{x_t\left(s^t\right); \forall\ t' \leq t \leq \infty,\ s^t \in S^t\right\}$. Given stochastic process for consumption $\left\{c^i\right\}_0$ and labor $\left\{n^i\right\}_0$, the utility for entrepreneur i is

$$\sum_{t=0}^{\infty} \sum_{s^t \in S^t} \beta^t \pi\left(s^t\right)\left[u\left(c_t^i\left(s^t\right) - v\left(n_t^i\left(s^t\right)\right)\right)\right]$$

I assume that u^i is strictly increasing, strictly concave and C^1 and that v is strictly increasing, strictly convex and C^1. Entrepreneurs run a production technology where k_t^i is the capital stock at the beginning of the period, $A_t^i\left(s^{t-1}\right)$ is the idiosyncratic productivity shock and $\hat{n}_t^i\left(s^t\right)$ is amount of labor hired.

$$\pi_t^i\left(s^t, k_t\right) = \max_{\hat{n}_t^i(s^t)} \left\{A_t^i\left(s^{t-1}\right)\left(k_t^i\right)^{\alpha}\left(\hat{n}_t^i\left(s^t\right)\right)^{1-\alpha} + (1-\delta)\, k_t^i - w_t\left(s^t\right)\hat{n}_t^i\left(s^t\right)\right\}$$

Given constant returns to scale, it is easy to see that

$$\begin{aligned}\hat{n}_t^i\left(s^t\right) &= \rho\left(w_t\left(s^t\right), A_t^i\left(s^{t-1}\right)\right) k_t^i \\ \pi_t^i\left(s^t, k_t\right) &= r\left(w_t\left(s^t\right), A_t^i\left(s^{t-1}\right)\right) k_t^i\end{aligned}$$

where

$$\rho\left(w_t\left(s^t\right), A_t^i\left(s^{t-1}\right)\right) = \left(\frac{(1-\alpha)\, A_t^i\left(s^{t-1}\right)}{w_t\left(s^t\right)}\right)^{\frac{1}{\alpha}} \tag{1}$$

$$r\left(w_t\left(s^t\right), A_t^i\left(s^{t-1}\right)\right) = A_t^i\left(s^{t-1}\right)\left(\frac{A_t^i\left(s^{t-1}\right)(1-\alpha)}{w_t\left(s^t\right)}\right)^{\frac{1-\alpha}{\alpha}} - w_t\left(s^t\right)\left(\frac{A_t^i\left(s^{t-1}\right)(1-\alpha)}{w_t\left(s^t\right)}\right)^{\frac{1}{\alpha}} + 1 - \delta \tag{2}$$

This simplifies the problem of the intermediary as I can just replace the intermediaries profit maximization problem with $r\left(w_t\left(s^t\right), A_t^i\left(s^{t-1}\right)\right) k_t^i$.

In this section I consider class of decentralized contracting problems except that competitive intermediaries can now observe and hence control the amount of capital accumulated by the agents/entrepreneurs. I show that there are multiple equilibria in this environment and thus demonstrate that the multiplicity is not special to the hidden trading setup. As mentioned earlier, an advantage of the hidden trading assumption is that I can directly deal with models with debt constraints which are in general easier to solve than contracting problems.

Intermediaries are risk-neutral and offer insurance contracts to entrepreneurs. Formally, given an intermediary born in period t and state s^t, a contract is a vector

$$C^{s^t} = \left(\begin{array}{c} c_t^i\left(s^t\right), k_{t+1}^i\left(s^t\right), n_t^i\left(s^t\right), m_t^{i,s^t}\left(s^t\right), \\ \left(\left\{c_{t'}^i\left(s^{t'}\right), k_{t'+1}^i\left(s^{t'}\right), n_{t'}^i\left(s^{t'}\right), m_{t'}^{i,s^t}\left(s^{t'}\right)\right\}_{s^{t'}\in S^{t'}}\right)_{t'\in\{t+1,\dots t+T-1\}} \\ , \left\{m_t^{i,s^t}\left(s^{t+T}\right)\right\}_{s^{t+T}\in S^{t+T}} \end{array} \right)$$

where $m_{t'}^{i,s^t}\left(s^{t'}\right)$, is the state contingent insurance offered by the intermediary s^t in state $s^{t'}$. Along with the insurance, the contract also specifies a set of allocations to each intermediary. Here the intermediaries can observe the aggregate state (the history of entrepreneurs productivity shocks), the actions of all other intermediaries and the actions of entrepreneurs.

The timing in the last period of a $T-$period contract is as follows[1]: At the beginning of period t, after z_{t+1} is known, entrepreneurs decide whether to default on last period's intermediary. Notice that each entrepreneurs knows her $t+1$ productivity shock in period t. This is a commonly used timing assumption in the literature and is assumed by Buera and Moll (2012) and Shourideh and Zetlin-Jones (2012). Next, production takes place following which agents can sign with new intermediaries if they haven't defaulted before while paying back existing ones.

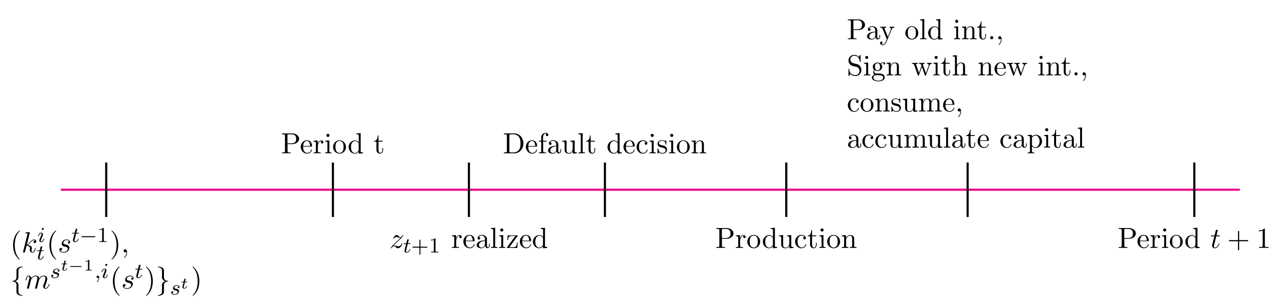

Figure 1: Timing

Taking as given the actions of other intermediaries, any feasible contract C^{s^t} that an intermediary born in s^t can offer must satisfy budget constraints in period t,

$$c_t^i\left(s^t\right) + k_{t+1}^i\left(s^t\right) = r\left(w_t\left(s^t\right), A_t^i\left(s^{t-1}\right)\right) k_t^i\left(s^{t-1}\right) + w_t\left(s^t\right) n_t^i\left(s^t\right) + m_t^{s^t,i}\left(s^t\right) + m_t^{s^{t-(T-1)},i}\left(s^t\right)$$

for all $t' \in \{t+1, ..., T-1\}$. Note that if an agent signs a new contract in period t, the previous

[1]The other periods are identical, except agents dont sign with new intermediaries the following period

contract was signed in $t-(T-1)$.

$$c_{t'}^{i}\left(s^{t'}\right)+k_{t'+1}^{i}\left(s^{t'}\right)=r\left(w_{t'}\left(s^{t'}\right),A_{t'}^{i}\left(s^{t'-1}\right)\right)k_{t'}^{i}\left(s^{t'-1}\right)+w_{t'}\left(s^{t'}\right)n_{t'}^{i}\left(s^{t'}\right)+m_{t'}^{s^{t},i}\left(s^{t}\right)$$

and in period $T-1$,

$$\begin{aligned}c_{T-1}^{i}\left(s^{T-1}\right)+k_{T}^{i}\left(s^{T-1}\right)&=r\left(w_{T-1}\left(s^{T-1}\right),A_{T-1}^{i}\left(s^{T-2}\right)\right)k_{T-1}^{i}\left(s^{T}\right)\\&\quad+w_{T-1}\left(s^{T-1}\right)n_{T-1}^{i}\left(s^{T-1}\right)+m_{T-1}^{i,s^{t}}\left(s^{T-1}\right)+m_{T-1}^{i,s^{T-1}}\left(s^{T-1}\right)\end{aligned}$$

a participation constraint

$$\sum_{t'=t}^{\infty}\sum_{s^{t'}\succeq s^{t}}\beta^{t'-t}\left[u\left(c_{t'}^{i}\left(s^{t'}\right)-v\left(n_{t'}^{i}\left(s^{t'}\right)\right)\right)\right]\geq\bar{V}_{t}$$

and a voluntary participation constraint,

$$\sum_{t'=t}^{\infty}\sum_{s^{t'}\succeq s^{t}}\beta^{t'-t}\pi\left(s^{t'}\mid s^{t}\right)\left[u\left(c_{t'}^{i}\left(s^{t'}\right)-v\left(n_{t'}^{i}\left(s^{t'}\right)\right)\right)\right]\geq V_{t}^{d}\left(s^{t},k_{t}^{i}\left(s^{t-1}\right);\mathbf{w}_{t}\right)\tag{3}$$

where $\mathbf{w}_t=\left\{w_{t'}\left(s^{t'}\right)\right\}_{s^{t'}\succeq s^{t},t'\geq t}$. $V_{t'}^{i,d}\left(s^{t'},k_{t'}\left(s^{t'}\right);\mathbf{w}_t\right)$ is the value of defaulting in t' and not being able to sign with an intermediary in the future. Given any t, s^t, $V_{t}^{i,d}\left(s^{t},k_{t}\left(s^{t}\right);\mathbf{w}_t\right)$ is defined as the solution to

$$V_{t}^{i,d}\left(s^{t},k^{i};\mathbf{w}_{t}\right)=\max\sum_{t'=t}^{\infty}\sum_{s^{t'}\succeq s^{t}}\beta^{t'-t}\left[u\left(c_{t'}^{i}\left(s^{t'}\right)-v\left(n_{t'}^{i}\left(s^{t'}\right)\right)\right)\right]\tag{4}$$

subject to

$$c_{t}^{i}\left(s^{t}\right)+k_{t+1}^{i}\left(s^{t}\right)\leq r\left(w_{t}\left(s^{t}\right),A_{t}^{i}\left(s^{t-1}\right)\right)k^{i}+w_{t}\left(s^{t}\right)n_{t}^{i}\left(s^{t}\right)$$

$$c_{t'}^{i}\left(s^{t'}\right)+k_{t'+1}^{i}\left(s^{t'}\right)\leq r\left(w_{t'}\left(s^{t'}\right),A_{t'}^{i}\left(s^{t'-1}\right)\right)k_{t'}^{i}\left(s^{t'-1}\right)+w_{t'}\left(s^{t'}\right)n_{t'}^{i}\left(s^{t'}\right)\quad\text{for all } t'\geq t$$

In particular, if the entrepreneur defaults she can continue to produce, accumulate capital, hire labor and work but is barred from financial markets in all future periods. This assumption on what happens after default is similar to Alvarez and Jermann (2000).

Intermediaries are risk neutral and maximize profits

$$\max_{\{c,k,n,m\}}-\sum_{i\in I}\sum_{t'=t}^{T}\sum_{s^{t'}\in S^{t'}}\left[\prod_{\hat{t}=t}^{t'}q\left(s^{\hat{t}}\mid s^{\hat{t}-1}\right)\right]m_{t'}^{s^{t},i}\left(s^{t'}\right)$$

Notice that I allow intermediaries to borrow and lend among each other at market determined prices $q\left(s^{t+1}\mid s^{t}\right)$.

Definition 1 *A competitive equilibrium in this contracting environment is prices* $\left\{q\left(s^{t+1}\mid s^{t}\right),w\left(s^{t}\right)\right\}_{t,s^{t}}$, *allocations for each intermediary* $\left(C^{s^t},\bar{V}_t\right)$ *such that*

- *Given prices and the actions of other intermediaries, the allocation for the* s^t *intermediary solves its problem,*

  $$\max_{\{c,k,n,m\}} -\sum_{i\in I}\sum_{t'=t}^{T}\sum_{s^{t'}\in S^{t'}}\left[\prod_{\hat{t}=t}^{t'} q\left(s^{\hat{t}}\mid s^{\hat{t}-1}\right)\right] m_{t'}^{s^t,i}\left(s^{t'}\right)$$

  subject to

  $$\begin{aligned}
  &c_t^i\left(s^t\right)+k_{t+1}^i\left(s^t\right)=r\left(w_t\left(s^t\right),A_t^i\left(s^{t-1}\right)\right)k_t^i\left(s^{t-1}\right)+w_t\left(s^t\right)n_t^i\left(s^t\right)+m_t^{s^t,i}\left(s^t\right)+m_t^{s^{t-(T-1)},i}\left(s^t\right),\\
  &c_{t'}^i\left(s^{t'}\right)+k_{t'+1}^i\left(s^{t'}\right)=r\left(w_{t'}\left(s^{t'}\right),A_{t'}^i\left(s^{t'-1}\right)\right)k_{t'}^i\left(s^{t'-1}\right)+w_{t'}\left(s^{t'}\right)n_{t'}^i\left(s^{t'}\right)+m_{t'}^{s^t,i}\left(s^t\right)\\
  &c_{T-1}^i\left(s^{T-1}\right)+k_T^i\left(s^{T-1}\right)=r\left(w_{T-1}\left(s^{T-1}\right),A_{T-1}^i\left(s^{T-2}\right)\right)k_{T-1}^i\left(s^T\right)\\
  &\qquad\qquad+w_{T-1}\left(s^{T-1}\right)n_{T-1}^i\left(s^{T-1}\right)+m_{T-1}^{i,s^t}\left(s^{T-1}\right)+m_{T-1}^{i,s^{T-1}}\left(s^{T-1}\right),\\
  &\sum_{t'=t}^{\infty}\sum_{s^{t'}\succeq s^t}\beta^{t'-t}\left[u\left(c_{t'}^i\left(s^{t'}\right)-v\left(n_{t'}^i\left(s^{t'}\right)\right)\right)\right]\geq\bar{V}_t\\
  &\sum_{t'=t}^{\infty}\sum_{s^{t'}\succeq s^t}\beta^{t'-t}\left[u\left(c_{t'}^i\left(s^{t'}\right)-v\left(n_{t'}^i\left(s^{t'}\right)\right)\right)\right]\geq V_t^d\left(s^t,k_t^i\left(s^{t-1}\right);\mathbf{w}_t\right)
  \end{aligned}$$

- *Intermediaries make zero profits*

- *Markets clear*

  $$\begin{aligned}
  \sum_{i\in I}\left[c_t^i\left(s^t\right)+k_{t+1}^i\left(s^t\right)\right]&=\sum_{i\in I}\left[A_t^i\left(s^t\right)\rho\left(w_t\left(s^t\right),A_t^i\left(s^t\right)\right)+1-\delta\right]k_t^i\left(s^{t-1}\right)\\
  \sum_{i\in I}\rho\left(w_t\left(s^t\right),A_t^i\left(s^{t-1}\right)\right)k_t^i&=\sum_{i\in I}n_t^i\left(s^t\right)
  \end{aligned}$$

An important implication of the fact that the intermediary has control of the agents' capital stock is that it can potentially lower it in order to relax future incentive compatibility constraints. It is this control that makes the existence of multiplicity rather surprising.

I first show that there exist prices such that autarky is an equilibrium in the contracting environment and in particular, intermediaries find it optimal to not offer an insurance.

Proposition 2 *There exist prices* $\left\{q^a\left(s^t\mid s^{t-1}\right)\right\}_{s^t,s^{t-1}}$ *and allocations for the intermediary so that autarky is an equilibrium in the contracting environment. In particular, for all* i,s^t, $m_t^{s^t,i}\left(s^t\right)=0$

Proof. Appendix. ■

This proves that there is an equilibrium in which intermediaries offer no insurance. The proof use a construction argument. I construct prices such that given these prices, the profit maximizing contract is one in which no insurance is offered. To do this a consider a setup in which agents accumulate capital and hire labor (and work) but have no access to financial markets. This has a well defined equilibrium which exists and is unique. Using these equilibrium allocations, I construct appropriate Arrow security prices. Then given these prices, I show that the constructed allocations along with zero transfers solve the intermediaries' problem. I use Lagrangian techniques developed by Kehoe and Perri (2002) and Marcet and Marimon (2011) to show that these allocation satisfy the first order conditions of the intermediary's problem.

Next, I show that there is a *conditionally efficient*[2] allocation with borrowing and lending which can be decentralized an equilibrium of the contracting environment.

Definition 3 *Given wage rates $\{w_t(s^t)\}_{t,s^t}$, a conditonally efficient allocation is the solution to the problem*

$$\max_{c,n,k} \sum_{i\in I}\sum_{t=0}^{\infty}\sum_{s^t}\beta^t\pi\left(s^t\right)\left[u\left(c_t^i\left(s^t\right)-v\left(n_t^i\left(s^t\right)\right)\right)\right]$$

$s.t.$

$$\left(\lambda^c\left(s^t\right)\right):\sum_{i\in I}\left[c_t^i\left(s^t\right)+k_{t+1}^i\left(s^t\right)\right]\leq\sum_i\left[r\left(w_t\left(s^t\right),A_t^i\left(s^{t-1}\right)\right)k_t^i\left(s^{t-1}\right)+w_t\left(s^t\right)n_t^i\left(s^t\right)\right] \tag{5}$$

$$\left(\beta^t\pi\left(s^t\right)\mu^{c,i}\left(s^t\right)\right):\sum_{t'=t}^{\infty}\sum_{s^{t'}\succeq s^t}\beta^{t'-t}\pi\left(s^{t'}\mid s^t\right)\left[u\left(c_{t'}^i\left(s^{t'}\right)-v\left(n_{t'}^i\left(s^{t'}\right)\right)\right)\right]\geq V_t^d\left(s^t,k_t^i\left(s^{t-1}\right)\right)$$

Lemma 4 *In any conditionally efficient allocation, for any t and history s^t, only the technology with the highest productivity is operated, i.e. for i^* such that $A_t^{i^*}\left(s^{t-1}\right)=\max_{i\in I}A_t^i\left(s^{t-1}\right)$, $k_t^{i^*}\left(s^{t-1}\right),n_t^{i^*}\left(s^t\right)>0$ while $k_t^i\left(s^{t-1}\right),n_t^i\left(s^t\right)=0$ for all $i\neq i^*$.*

Proof. The result follows from the fact that the return on investing a unit of capital in any technology is given by $r\left(w_t\left(s^t\right),A_t^i\left(s^{t-1}\right)\right)$ which is increasing in productivity. As a result, only the most efficient technology is used. ■

Lemma 5 *A conditionally efficient allocation given wage rates $\{w_t(s^t)\}_{t,s^t}$ can be decentralized as an equilibrium of the contracting problem if and only if for each t,s^t*

$$w_t\left(s^t\right)=\left(\frac{\sum_{i\in I}\left(\left(1-\alpha\right)A_t^i\left(s^t\right)\right)^{\frac{1}{\alpha}}k_t^i\left(s^{t-1}\right)}{\sum_i n_t^i\left(s^t\right)}\right)^{\frac{1}{\alpha}} \tag{6}$$

Proof. Appendix. ■

The proposition gives a necessary and sufficient condition for any conditionally efficient allocation to constitute a competitive equilibrium. The condition just follows from labor market clearing $\sum_{i\in I}\rho\left(w_t\left(s^t\right),A_t^i\left(s^t\right)\right)k_t^i\left(s^{t-1}\right)=\sum_i n_t^i\left(s^t\right)$. The first step in the proof is to construct an appropriate sequence of Arrow security prices and show that given these, the contract corresponding to

[2] As in Kehoe and Levine (1993)

the conditionally efficient allocation solves the intermediary's problem. Finally, the above condition ensures that markets clear.

Proposition 6 *There exists a non-autarkic competitive equilibrium*

Proof. It suffices to prove the existence of a fixed point w^* of the following map: Define the operator $\Gamma : l^\infty \to l^\infty$ such that

$$\Gamma(w)\left(s^t\right) = \left(\frac{\sum_{i\in I}\left((1-\alpha) A_t^i\left(s^t\right)\right)^{\frac{1}{\alpha}} k_t^i\left(s^{t-1}\right)}{\sum_i n_t^i\left(s^t\right)}\right)^{\frac{1}{\alpha}}$$

where $k_t^i\left(s^{t-1}\right), n_t^i\left(s^t\right)$ are solution to the conditionally efficient planning problem

In general, this map is not a contraction. And so one cannot apply the standard contraction mapping theorem in order to prove the existence of a fixed point. We need to rely on more general arguements.

For some W large but finite, consider the set of infinite sequences where the index set spans t, s^t, such that for each t, s^t, $w_t\left(s^t\right) \in [0, W]$. Let $\mathcal{W}$ the subset of l^∞ containing such sequences. Then $\mathcal{W}$ is compact in the weak topology by Tychonoff's theorem. Next, I show that the map Γ is continuous. Given sequence of wage rates w, let correspondence $G(w)$ denote the constraint set of the conditionally efficient planning problem. By the maximum theorem, is straightforward to show that $V_t^d\left(s^t, k_t^i\left(s^{t-1}\right)\right)$ is continuous in $w_t\left(s^t\right)$. As result, $G(w)$ is a continuous correspondence since the utility function is continuous. Then again, by the maximum theorem, the policy functions are continuous in $w_t\left(s^t\right)$ (assuming uniqueness). As a result $\Gamma(w)\left(s^t\right)$ is continuous. Then from Schauder's Fixed Point theorem, I know there exists a fixed point of Γ in $\mathcal{W}$. This establishes the existence of an equilibrium different from autarky. ■

3 Hidden Actions

In this section I present a similar environment to the one in the previous setions except that the agents' actions are hidden. As in the previous section given an intermediary born in period t and state s^t, a contract is a vector

$$C^{s^t} = \left(\begin{array}{c} c_t^i\left(s^t\right), k_{t+1}^i\left(s^t\right), n_t^i\left(s^t\right), m_t^{i,s^t}\left(s^t\right), \\ \left(\left\{c_{t'}^i\left(s^{t'}\right), k_{t'+1}^i\left(s^{t'}\right), n_{t'}^i\left(s^{t'}\right), m_{t'}^{i,s^t}\left(s^{t'}\right)\right\}_{s^{t'}\in S^{t'}}\right)_{t'\in\{t+1,\ldots t+T-1\}} \\ , \left\{m_t^{i,s^t}\left(s^{t+T}\right)\right\}_{s^{t+T}\in S^{t+T}} \end{array}\right)$$

where $m_{t'}^{i,s^t}\left(s^{t'}\right)$, is the state contingent insurance offered by the intermediary s^t in state $s^{t'}$. Along with the insurance, the intermediary recommends a set of allocations to each intermediary. However unlike the previous section, while intermediaries can observe the aggregate state (the history of entrepreneurs productivity shocks) and the actions of all other intermediaries, an important friction in the contracting environment is that the entrepreneurs' actions are unobservable. In particular the intermediary cannot see the amount of capital being accumulated by an agent.

As in the previous section a feasible contract must satisfy resource feasiblity and an ex-ante participation constraint. However now, the contract must satisfy an incentive compatibility constraint,

$$\sum_{t'=t}^{\infty}\sum_{s^{t'}\succeq s^t}\beta^{t'-t}\left[u\left(c_{t'}^i\left(s^{t'}\right)-v\left(n_{t'}^i\left(s^{t'}\right)\right)\right)\right]\geq\hat{V}_t^i\left(s^t,m^{s^t,i},k_t^i\left(s^{t-1}\right),\mathbf{w}_t\right)\tag{7}$$

where $\mathbf{w}_t=\left\{w_{t'}\left(s^{t'}\right)\right\}_{s^{t'}\succeq s^t,t'\geq t}$. $\hat{V}_t^i\left(s^t,m,k_t^i,\mathbf{w}_t\right)$ represents the best deviation an agent can undertake and is the solution to

$$\begin{aligned}\hat{V}_t^i\left(s^t,m,k_t^i,\mathbf{w}_t\right)=\max_{\left\{\tilde{c}_t,\tilde{k}_{t+1},\Delta\right\}}\left(1-\Delta^i\left(s^t\right)\right)&\left(\sum_{t'=t}^{T-1}\beta^{t'-1}\prod_{\hat{t}=t+1}^{t'-1}\left(1-\Delta^i\left(s^{\hat{t}}\right)\right)\left[\begin{array}{c}\left(1-\Delta^i\left(s^{t'}\right)\right)u\left(\tilde{c}_{t'}^i\left(s^{t'}\right)\right)\\+\Delta^i\left(s^{t'}\right)V_{t'}^{i,d}\left(s^{t'},k_{t'}^i,\mathbf{w}_{t'}\right)\end{array}\right]\right.\\&\left.+\prod_{\hat{t}=t+1}^{T-1}\left(1-\Delta^i\left(s^{\hat{t}}\right)\right)\left[\begin{array}{c}\left(1-\Delta^i\left(s^T\right)\right)\hat{V}_T^i\left(s^T,m,k_T^i\right)\\+\Delta^i\left(s^T\right)V_T^{i,d}\left(s^T,k_T^i,\mathbf{w}_T\right)\end{array}\right]\right)\\&+\Delta^i\left(s^t\right)V_t^{i,d}\left(s^t,k_t^i,\mathbf{w}_t\right)\end{aligned}\tag{8}$$

subject to

$$\tilde{c}_t^i\left(s^t\right)+\tilde{k}_{t+1}^i\left(s^t\right)=r\left(w_t\left(s^t\right),A_t^i\left(s^{t-1}\right)\right)k_t^i+w_t\left(s^t\right)\tilde{n}_t^i\left(s^t\right)+\prod_{j=0}^{t}\left[1-\Delta\left(s^j\right)\right]\left[\begin{array}{c}m_t^{s^{t-1},i}\left(s^t\right)\\+m_t^{s^t,i}\left(s^t\right)\end{array}\right]$$

$$\tilde{c}_{t'}^i\left(s^t\right)+\tilde{k}_{t'+1}^i\left(s^{t'}\right)=\left[\begin{array}{c}r\left(w_{t'}\left(s^{t'}\right),A_{t'}^i\left(s^{t'-1}\right)\right)k_{t'}^i+w_{t'}\left(s^{t'}\right)\tilde{n}_{t'}^i\left(s^{t'}\right)\\+\prod_{j=0}^{t'}\left[1-\Delta\left(s^j\right)\right]m_{t'}^{s^t,i}\left(s^{t'}\right)\end{array}\right],\ t'=t+1,...,T-1$$

$$\tilde{c}_T^i\left(s^t\right)+\tilde{k}_{T+1}^i\left(s^T\right)=\left[\begin{array}{c}r\left(w_T\left(s^T\right),A_T^i\left(s^{T-1}\right)\right)\tilde{k}_T^i\left(s^{T-1}\right)+w_T\left(s^T\right)\tilde{n}_T^i\left(s^T\right)\\+\prod_{j=0}^{T}\left[1-\Delta\left(s^j\right)\right]\left[m_T^{s^t,i}\left(s^T\right)+m_T^{s^T,i}\left(s^T\right)\right]\end{array}\right]$$

Here $\Delta^i\left(s^{t'}\right)\in\{0,1\}$ are the agent's default decision in t', $r\left(w_{t'}\left(s^{t'}\right),A_{t'}^i\left(s^{t'-1}\right)\right)$ is the return on capital accumulated the previous period (profits from production) as defined above, $\tilde{n}_{t'}^i\left(s^{t'}\right)$ is the amount the entrepreneur chooses to work herself in any other production technology, $\hat{V}_T^i\left(s^T,m,k_T^i\right)$ is her continuation value of not defaulting in T given her chosen capital stock (potentially different from intermediaries' recommendations) and $V_{t'}^{i,d}\left(s^{t'},k_{t'}\left(s^{t'}\right),\mathbf{w}_t\right)$ was defined in (4).

In each period, if the agent chooses to not default, she can sign with a new intermediary who offers her an insurance contract along with recommended allocations. Since actions are unobservable, the agent can choose different levels of consumption and capital accumulation. Again, in period $t+1$, she can choose to default on her obligations, take her current capital stock and live in autarky forever. The term $\prod_{j=0}^{t}\left[1-\Delta\left(s^{j}\right)\right]\begin{bmatrix} m_t^{s^{t-1},i}\left(s^t\right) \\ +m_t^{s^t,i}\left(s^t\right)\end{bmatrix}$ captures whether the agent has defaulted in the past or not. Notice that this formulation allows for a rich set of deviations an agent can undertake. For example, she can engage in a "double" deviation where she chooses to accumulate a different amount of capital in s^t, and default the following period. The profit maximizing contract must prevent such deviations.

Definition 7 *A competitive equilibrium in this contracting environment is prices* $\left\{q\left(s^{t+1}\mid s^t\right), w\left(s^t\right)\right\}_{t,s^t}$, *allocations for each intermediary* $\left(C^{s^t}, \bar{V}_t\right)$ *such that*

- *Given prices and the actions of other intermediaries, the allocation for the* s^t *intermediary solves its problem,*

$$\max_{\{c,k,n,m\}} -\sum_{i\in I}\sum_{t'=t}^{T}\sum_{s^{t'}\in S^{t'}}\left[\prod_{\hat{t}=t}^{t'} q\left(s^{\hat{t}}\mid s^{\hat{t}-1}\right)\right] m_{t'}^{s^t,i}\left(s^{t'}\right)$$

subject to

$$c_t^i\left(s^t\right)+k_{t+1}^i\left(s^t\right)=r\left(w_t\left(s^t\right),A_t^i\left(s^{t-1}\right)\right)k_t^i\left(s^{t-1}\right)+w_t\left(s^t\right)n_t^i\left(s^t\right)+m_t^{s^t,i}\left(s^t\right)+m_t^{s^{t-(T-1)},i}\left(s^t\right),$$

$$c_{t'}^i\left(s^{t'}\right)+k_{t'+1}^i\left(s^{t'}\right)=r\left(w_{t'}\left(s^{t'}\right),A_{t'}^i\left(s^{t'-1}\right)\right)k_{t'}^i\left(s^{t'-1}\right)+w_{t'}\left(s^{t'}\right)n_{t'}^i\left(s^{t'}\right)+m_{t'}^{s^t,i}\left(s^t\right)$$

$$\begin{aligned} c_{T-1}^i\left(s^{T-1}\right)+k_T^i\left(s^{T-1}\right)=&\; r\left(w_{T-1}\left(s^{T-1}\right),A_{T-1}^i\left(s^{T-2}\right)\right)k_{T-1}^i\left(s^T\right) \\ &+w_{T-1}\left(s^{T-1}\right)n_{T-1}^i\left(s^{T-1}\right)+m_{T-1}^{i,s^t}\left(s^{T-1}\right)+m_{T-1}^{i,s^{T-1}}\left(s^{T-1}\right), \end{aligned}$$

$$\sum_{t'=t}^{\infty}\sum_{s^{t'}\succeq s^t}\beta^{t'-t}\left[u\left(c_{t'}^i\left(s^{t'}\right)-v\left(n_{t'}^i\left(s^{t'}\right)\right)\right)\right]\geq\bar{V}_t$$

$$\sum_{t'=t}^{\infty}\sum_{s^{t'}\succeq s^t}\beta^{t'-t}\left[u\left(c_{t'}^i\left(s^{t'}\right)-v\left(n_{t'}^i\left(s^{t'}\right)\right)\right)\right]\geq\hat{V}_t^i\left(s^t,m^{s^t,i},k_t^i\left(s^{t-1}\right),\mathbf{w}_t\right)$$

- *Intermediaries make zero profits*

- *Markets clear*

$$\sum_{i\in I}\left[c_t^i\left(s^t\right)+k_{t+1}^i\left(s^t\right)\right]=\sum_{i\in I}\left[A_t^i\left(s^t\right)\rho\left(w_t\left(s^t\right),A_t^i\left(s^t\right)\right)+1-\delta\right]k_t^i\left(s^{t-1}\right)$$

$$\sum_{i\in I}\rho\left(w_t\left(s^t\right),A_t^i\left(s^{t-1}\right)\right)k_t^i=\sum_{i\in I}n_t^i\left(s^t\right)$$

4 Model with Debt Constraints

Consider a model with $i\in I$ entrepreneurs. Time $t=0,1,..$ is discrete and let Z be the finite state space and S^t be set of histories till time t with typical element $s^t=(z_0,z_1,...,z_{t+1})$.

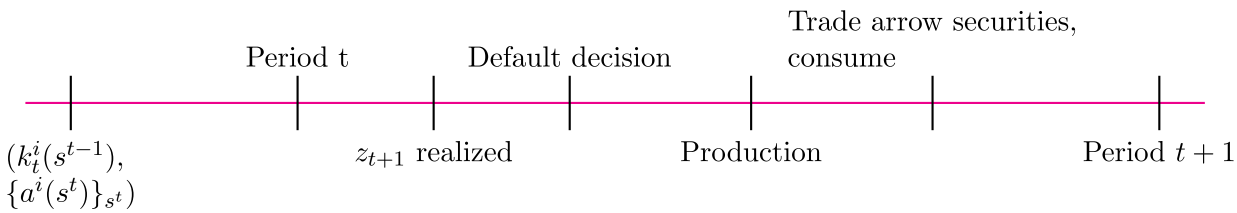

Figure 2: Timing

The timing is as follows: each entrepreneur enters the period with capital stock and a vector of arrow securities
$\left(k_t^i\left(s^{t-1}\right), \left\{a_t^i\left(s^{t-1}, z_{t+1}\right)\right\}_{z_{t+1}\in Z}\right)$. At the start of period t, the state next period z_{t+1} is realized and as a consequence entrepreneurs know their next period productivity shock $A_{t+1}^i\left(s^t\right)$. Next, agents make default decisions on their (state contingent) debt, following which production takes place. Finally, agents make consumption decisions and if they haven't defaulted in the past, purchase arrow securities for the next period.

As described earlier, entrepreneurs are subject to idiosyncratic productivity shocks $A_t^i\left(s^{t-1}\right)$ which are known the previous period. Formally the entrepreneur's ex-ante problem is to maximize her preferences

$$\sum_{t=0}^{\infty}\sum_{s^t\in S^t}\beta^t\pi\left(s^t\right)\left[u\left(c_t^i\left(s^t\right)-v\left(n_t^i\left(s^t\right)\right)\right)\right]$$

subject to the budget constraint at each t, s^t

$$c_t^i\left(s^t\right)+\sum_{z_{t+2}\in Z}q\left(s^{t+1}\mid s^t\right)a^i\left(s^t, z_{t+2}\right)+k_{t+1}^i\left(s^t\right)=\pi_t^i\left(s^t, k_t\right)+w_t\left(s^t\right)n_t^i\left(s^t\right)+a^i\left(s^t\right) \tag{9}$$

where $a^i\left(s^t\right)$ is the current holdings of arrow securities, $\left\{a^i\left(s^t, z_{t+2}\right)\right\}_{z_{t+2}\in Z}$ is the portfolio of securities chosen in period t, $k_{t+1}^i\left(s^t\right)$ is the capital stock chosen, $n_t^i\left(s^t\right)$ is the amount the entrepreneur chooses to work (for some other entrepreneur) and $\pi_t^i\left(s^t, k_t\right)$ is the profits from running her production technology,

$$\pi_t^i\left(s^t, k_t\right)=\max_{\hat{n}_t^i(s^t)}\left\{A_t^i\left(s^{t-1}\right)\left(k_t^i\right)^{\alpha}\left(\hat{n}_t^i\left(s^t\right)\right)^{1-\alpha}+(1-\delta)k_t^i-w_t\left(s^t\right)\hat{n}_t^i\left(s^t\right)\right\}$$

Note that $\hat{n}_t^i\left(s^t\right)$ is the amount of labor the entrepreneur chooses to hire for her own production technology. As in Alvarez and Jermann (2000), an entrepreneur's purchases of arrow securities are constrained by state contingent debt constraints

$$a^i\left(s^t, z_{t+2}\right)\geq\phi^i\left(s^t, z_{t+2}\right) \tag{10}$$

Notice that the form of these constraints is not prespecified and in general these can be fairly complicated equilibrium objects. They can depend on the entrepreneur's past history of types and choices and her future profitability. This is in sharp contrast to the literature that assumes a linear from on debt/collateral constraints. Using equation (2) from the previous section I can simplify the entrepreneurs budget constraint to

$$c_t^i\left(s^t\right)+\sum_{z_{t+2}\in Z} q\left(s^{t+1}\mid s^t\right)a^i\left(s^t,z_{t+2}\right)+k_{t+1}^i\left(s^t\right)=r\left(w_t\left(s^t\right),A_t^i\left(s^{t-1}\right)\right)k_t^i\left(s^{t-1}\right)+w_t\left(s^t\right)n_t^i\left(s^t\right)+a^i\left(s^t\right)$$

Let $V_t^{i,c}\left(s^t,k^i,a\left(s^t\right);\Phi^i\left(s^t\right)\right)$ denote the value for an entrepreneur in s^t with capital stock k^i and holdings of arrow securities $a^i\left(s^t\right)$ of being allowed to participate in financial markets at all dates and states in the future given a sequence of borrowing constraints $\Phi^i\left(s^t\right)=\left\{\left(\phi^i\left(s^{t'}\right)\right)_{s^{t'}\in S^{t'}}\right\}_{t'=t}^{\infty}$. At any date and state, the entrepreneur can choose to default on her obligations, take her existing capital stock and live in autarky forever. In autarky, she can continue to produce and work in each period but is barred from financial markets in all future periods. $V^{i,d}\left(s^t,k^i,\mathbf{w}_t\right)$ denote the value of autarky in state s^t which is defined identically to (4).

One can now define a competitive equilibrium with debt constraints.

Definition 8 *A Competitive equilibrium with not-too-tight debt constraints consists of prices* $\left\{q\left(s^{t+1}\mid s^t\right),w_t\left(s^t\right)\right\}_{t,s^t}$ *,debt constraints* $\left\{\left(\phi^i\right)_{i\in I}\right\}_0$ *and allocations* $\left\{\left(c_t^i\left(s^t\right),n_t^i\left(s^t\right),k_{t+1}^i\left(s^t\right),\left\{a^i\left(s^t,z_{t+2}\right)\right\}_{s_{t+1}\in S}\right)_{i\in I}\right\}_{t,s^t}$ *such that*

- *Given prices and debt constraints, the allocations for entrepreneur i solve her problem,*

$$V_t^{i,c}\left(s^t,k^i,a\left(s^t\right);\Phi^i\left(s^t\right)\right)=\max_{\{c^i,n^i,k^i,a^i\}_0}\sum_{t=0}^{\infty}\sum_{s^t\in S^t}\beta^t\pi\left(s^t\right)\left[u\left(c_t^i\left(s^t\right)-v\left(n_t^i\left(s^t\right)\right)\right)\right] \tag{11}$$

subject to

$$c_t^i\left(s^t\right)+\sum_{z_{t+2}\in Z} q\left(s^{t+1}\mid s^t\right)a^i\left(s^t,z_{t+2}\right)+k_{t+1}^i\left(s^t\right) \tag{12}$$

$$=r\left(w_t\left(s^t\right),A_t^i\left(s^t\right)\right)k_t^i\left(s^{t-1}\right)+w_t\left(s^t\right)n_t^i\left(s^t\right)+a^i\left(s^t\right) \tag{13}$$

$$a^i\left(s^t,z_{t+2}\right)\geq\phi^i\left(s^t,s_{t+1}\right)$$

- *Markets clear: for each t,s^t*

$$\sum_{i\in I}\left[c_t^i\left(s^t\right)+k_{t+1}^i\left(s^t\right)\right]=\sum_{i\in I}\left[A_t^i\left(s^t\right)\rho\left(w_t\left(s^t\right),A_t^i\left(s^t\right)\right)+1-\delta\right]k_t^i\left(s^{t-1}\right)$$

$$\sum_{i\in I}\rho\left(w_t\left(s^t\right),A_t^i\left(s^t\right)\right)k_t^i\left(s^{t-1}\right)=\sum_i n_t^i\left(s^t\right)$$

$$\sum_{i\in I}a^i\left(s^t,z_{t+2}\right)=0 \text{ for all } z_{t+2}$$

- *Debt constraints are chosen to be not-too-tight: for each i,t,s^t*

$$V_t^{i,c}\left(s^t,k_t^i\left(s^{t-1}\right),\phi^i\left(s^t\right);\Phi^i\left(s^t\right)\right)=V_t^{i,d}\left(s^t,k_t^i\left(s^{t-1}\right),\mathbf{w}_t\right) \tag{14}$$

Equation (14) is the not-too-tight constraint and is defined in a similar fashion to Alvarez and Jermann (2000). The constraints are chosen so that in each t and state s^t, an agent who has borrowed up to the limit $\phi^i(s^t)$ the previous period is indifferent between paying back her debt and defaulting (and living in autarky for all future periods). As mentioned earlier, $\phi^i(s^t)$ depends on the history of entrepreneur's productivity shocks and choices as well the future profitability of the firm which is captured in $V_t^{i,c}\left(s^t, k_t^i\left(s^{t-1}\right), \phi^i\left(s^t\right); \Phi^i\left(s^t\right)\right)$. As a result the nature of the debt constraints is similar to that of Albuquerque and Hopenhayn (2004).

Next, I prove that any equilibrium in this environment is inefficient. As stated in the Introduction, this is because the entrepreneur does not internalize the effects of her capital choice on her future incentives to default. In particular, some agents accumulate too much capital relative to the efficient level.

The efficient allocation is the solution to the following Planning Problem

$$\max_{\{c^i,n^i,k^i,a^i\}} \sum_{t=0}^{\infty}\sum_{s^t\in S^t}\sum_{i\in I}\beta^t\pi\left(s^t\right)\left[u\left(c_t^i\left(s^t\right)\right)-v\left(n_t^i\left(s^t\right)\right)\right]$$

subject to

$$\sum_{i\in I}\left[c_t^i\left(s^t\right)+k_{t+1}^i\left(s^t\right)\right]\leq\sum_{i\in I}\left[A_t^i\left(s^t\right)\rho\left(w_t\left(s^t\right),A_t^i\left(s^t\right)\right)+1-\delta\right]k_t^i\left(s^{t-1}\right),\ \text{for reach } t, s^t$$

$$\sum_{t'=t}^{\infty}\sum_{s^{t'}\succeq s^t}\beta^{t'-t}\pi\left(s^{t'}\mid s^t\right)\left[u\left(c_{t'}^i\left(s^{t'}\right)\right)-v\left(n_{t'}^i\left(s^{t'}\right)\right)\right]\geq V_t^{i,d}\left(s^t,k\left(s^{t-1}\right),\mathbf{w}_t\right),\ \text{for reach } i,\ t, s^t$$

$$w_t\left(s^t\right)=\left(\frac{\sum_{i\in I}\left(\left(1-\alpha\right)A_t^i\left(s^{t-1}\right)\right)^{\frac{1}{\alpha}}k_t^i\left(s^{t-1}\right)}{\sum_i n_t^i\left(s^t\right)}\right)^{\alpha},\ \text{for reach } t, s^t \tag{15}$$

Proposition 9 *Any competitive equilibrium is constrained-inefficient.*

Proof. Let $\lambda\left(s^t\right)$ be the multiplier on the resource constraint for the planner and $\mu^i\left(s^{t+1}\right)$, the multiplier for the voluntary participation constraints. The first order condition for capital $k_{t+1}^i\left(s^t\right)$ is given by

$$0=\lambda\left(s^t\right)+\sum_i\mu^i\left(s^{t+1}\right)\frac{\partial}{\partial k_{t+1}^i\left(s^t\right)}V_{t+1}^{i,d}\left(s^{t+1},k_{t+1}^i\left(s^t\right),\mathbf{w}_{t+1}\right)$$
$$-\sum_{s^{t+1}}\lambda\left(s^{t+1}\right)\left[A_t^i\left(s^t\right)\rho\left(w_t\left(s^t\right),A_t^i\left(s^t\right)\right)+1-\delta\right]$$

where

$$\frac{\partial}{\partial k_{t+1}^i\left(s^t\right)}V_{t+1}^{i,d}\left(s^{t+1},k_{t+1}^i\left(s^t\right),\mathbf{w}_{t+1}\right)$$
$$=\frac{\partial}{\partial k_{t+1}^i\left(s^t\right)}V_{t+1}^{i,d}\left(s^{t+1},k_{t+1}^i\left(s^t\right),\left(\frac{\sum_{i\in I}\left(\left(1-\alpha\right)A_{t+1}^i\left(s^t\right)\right)^{\frac{1}{\alpha}}k_{t+1}^i\left(s^t\right)}{\sum_i n_{t+1}^i\left(s^{t+1}\right)}\right)^{\alpha},\mathbf{w}_{t+2}\right)$$

since the planner internalizes the effect of capital accumulation on wages using (15). Compare this with the problem faced by an entreprenuer in a competitive equilibrium. The entreprenuer is this case does not internalize the effect her choice of capital has on future debt constraints. Therefore if for some i and s^t, $\mu^i\left(s^t\right) > 0$ (which is the case we are interested in), this agent will over-acummulate capital relative to the efficient amount. This implies that equilibrium wages are higher than the efficient level. ■

In the light of this result, the next theorem offers an interpretation of these constraints in environments where they no longer decentralize the efficient allocation.

5 Equivalence Result

One main result of this paper proves an equivalence between the two sets of equilibria defined in the previous two sections.

Theorem 10 *1. Given an equilibrium of the not-too-tight debt constraint economy,*
$\left(\left\{q\left(s^{t+1} \mid s^t\right), w_t\left(s^t\right)\right\}_{t,s^t}, \left\{\left(\phi^i\right)_{i\in I}\right\}_0, \left\{\left(c_t^i\left(s^t\right), n_t^i\left(s^t\right), k_{t+1}^i\left(s^t\right), \left\{a^i\left(s^t, z_{t+2}\right)\right\}_{s_{t+1}\in S}\right)_{i\in I}\right\}_{t,s^t}\right)$
there exist, $\left\{\left(m_t^{s^t,i}\left(s^t\right), \bar{V}_t^i\left(s^t\right)\right)_{i\in I}\right\}_{t,s^t}$ *such that*
$\left(\left\{q\left(s^{t+1} \mid s^t\right), w\left(s^t\right)\right\}_{t,s^t}, \left\{\left(c_t^i\left(s^t\right), k_t^i\left(s^t\right), n_t^i\left(s^t\right), m_t^{s^t,i}\left(s^t\right), \bar{V}_t^i\left(s^t\right)\right)_{i\in I}\right\}_{t,s^t}\right)$ *constitute an equilibrium in the contracting environment.*

2. Given an equilibrium of the contracting environment
$\left(\left\{q\left(s^{t+1} \mid s^t\right), w\left(s^t\right)\right\}_{t,s^t}, \left\{\left(c_t^i\left(s^t\right), k_t^i\left(s^t\right), n_t^i\left(s^t\right), m_t^{s^t,i}\left(s^t\right), \bar{V}_t^i\left(s^t\right)\right)_{i\in I}\right\}_{t,s^t}\right)$, *there exist debt constraints* $\left\{\left(\phi^i\right)_{i\in I}\right\}_0$, *such that*
$\left(\left\{q\left(s^{t+1} \mid s^t\right), w_t\left(s^t\right)\right\}_{t,s^t}, \left\{\left(\phi^i\right)_{i\in I}\right\}_0, \left\{\left(c_t^i\left(s^t\right), n_t^i\left(s^t\right), k_{t+1}^i\left(s^t\right), \left\{a^i\left(s^t, z_{t+2}\right)\right\}_{s_{t+1}\in S}\right)_{i\in I}\right\}_{t,s^t}\right)$
constitute an equilibrium with not-too-tight debt constraints.

I prove this is several steps starting with part 1.

Proof of part 1. Consider a competitive equilibrium of the debt constraint problem. Define

$$m_t^{s^{t-1},i}\left(s_t\right) = a^i\left(s_t\right)$$
$$m_t^{s^t,i}\left(s_t\right) = -\sum_{z_{t+2}} q\left(s^{t+1} \mid s^t\right) a^i\left(s^t, z_{t+2}\right)$$

Now given our proposed m^i along with the allocation $\left\{\left(c_t^i\left(s^t\right), n_t^i\left(s^t\right), k_{t+1}^i\left(s^t\right)\right)_{i\in I}\right\}_{t,s^t}$ from the debt constrained competitive equilibrium it must be that

$$\hat{V}_t^i\left(s^t, m^{s^t,i}; k_t^i\left(s^{t-1}\right)\right) = V_t^{i,c}\left(s^t, k_t^i\left(s^{t-1}\right), a^i\left(s^t\right); \Phi^i\left(s^t\right)\right)$$

where the term on the left hand side is defined using (8) and the term on the right hand side using (11) . This is because given the construction of m^i and the fact that the allocation satisfies the entrepreneur's optimality conditions in the debt constraint environment, it must be that the best the agent can do given the insurance contract is what she does in the debt constraint environment. Since the debt limits are chosen to be not-too-tight the allocation $\left\{\left(c_t^i\left(s^t\right), n_t^i\left(s^t\right), k_{t+1}^i\left(s^t\right)\right)_{i\in I}\right\}_{t,s^t}$ along with m constructed above satisfies incentive compatibility and so is feasible for the intermediary given the prices.

Suppose that there exists an allocation that is feasible and gives the intermediary strictly higher profit. Note that perfect competition implies that such an allocation must increases both the ex-ante welfare of the firm and the agent. The only way an intermediary can do is to increase the amount of insurance provided to the agent. Consider any such contract with greater insurance that satisfies incentive compatibility.

Then it must be that for some i, s^t, s^{t+1},

$$q\left(s^{t+1} \mid s^t\right) > \frac{\beta\pi\left(s^{t+1} \mid s^t\right) u'\left(c_{t+1}^i\left(s^{t+1}\right)\right)}{u'\left(c_t^i\left(s^t\right)\right)} \tag{16}$$

and

$$V_t^{i,c}\left(s^t, k_t^i\left(s^{t-1}\right), a^i\left(s^t\right); \Phi^i\left(s^t\right)\right) > V_t^{i,d}\left(s^t, k_t^i\left(s^{t-1}\right)\right) \tag{17}$$

To see why notice that if (16) held with an equality, I would have full insurance among agents and as a result an intermediary cannot offer a contract with greater ex-ante insurance. On the other hand if (17) held with an equality for all agents in all dates and states in which insurance is being increased, the constraint will be violated for some agent as a result of increased insurance which in turn violates incentive compatibility. However equations (16) and (17) contradict the not-too-tightness requirement of collateral constraints. ■

The intuition behind this result is fairly straightforward. If the stochastic process for m is chosen to be the entrepreneur's choices of arrow security holdings in all dates and states, then the agent's best deviation given this contract coincides with her value of not defaulting in the competitive equilibrium with collateral constraints. As a result, given this contract, the agent will not want to default nor deviate from the recommended actions. The only thing that needs to be checked is that there is no alternative contract an intermediary can offer which makes him better off while leaving the agents equally as well off. As shown in the proof, the not-too-tight feature of the competitive equilibrium rules such a contract out.

Next, I prove the converse. For the proof, I assume that each intermediary lives for two periods. I do this for presentational purposes but the argument holds for any finite $T < \infty$. The following lemmas will be useful

Lemma 11 *Consider an equilibrium of the contracting environment. Then*

$$q\left(s^{t+1} \mid s^t\right) \geq \max_{i\in I}\left\{\frac{\beta\pi\left(s^{t+1} \mid s^t\right) u'\left(c_{t+1}^i\left(s^{t+1}\right)\right)}{u'\left(c_t^i\left(s^t\right)\right)}\right\} \tag{18}$$

with equality if there is some insurance being offered by the intermediary.

Proof. See Appendix A. ■

This lemma gives us a sense of why the second part of the theorem is true. Any equilibrium of the debt constrained environment must satisfy (18) and with equality is some agent is constrained. The proof illustrates the fact that if it did not hold an intermediary could offer a different contract that would the agent strictly better off while leaving him equally well off. As an example consider the case in which $q\left(s^{t+1} \mid s^t\right) > \max_{i \in I}\left\{\frac{\beta\pi\left(s^{t+1}|s^t\right)u'\left(c_{t+1}^i\left(s^{t+1}\right)\right)}{u'\left(c_t^i\left(s^t\right)\right)}\right\}$ and some insurance is being offered. Since insurance is being offered there must exist some agent (receiving a positive transfer) who in s^{t+1} strictly prefers to stay in the contract rather than default. Then the intermediary can offer this agent a little more insurance $q\left(s^{t+1} \mid s^t\right)\varepsilon$ today and the cost of reducing the transfer to her in s^{t+1} by ε. I can approximate her change in utility using a Taylor expansion

$$\Delta u = \left[q\left(s^{t+1} \mid s^t\right)u'\left(c_t^i\left(s^t\right)\right) - \beta\pi\left(s^{t+1} \mid s^t\right)u'\left(c_{t+1}^i\left(s^{t+1}\right)\right)\right]\varepsilon$$

which is greater than zero since $q\left(s^{t+1} \mid s^t\right)$ is greater than the entrepreneur's marginal rate of substitution. Morever, given that intermediaries can borrow and lend at prices $q\left(s^{t+1} \mid s^t\right)$, such a contract is payoff neutral for the intermediary. This violates the zero profit condition and so is a contradiction. A similar argument applies for the reverse inequality.

Lemma 12 *In the competitive equilibrium, for any $\left(s^t, s_{t+1}\right)$, if full insurance is not being provided by the intermediary, then for some i*

$$\hat{V}_t^i\left(s^t, \tilde{m}^{s^t,j}; k_t^j\left(s^{t-1}\right), \delta\left(s^{t+1}\right) = 0\right) = \hat{V}_t^i\left(s^t, \tilde{m}^{s^t,j}; k_t^j\left(s^{t-1}\right), \delta\left(s^{t+1}\right) = 1\right)$$

Proof. See Appendix A. ■

Lemma 12 states that if the intermediary is not offering full insurance, then there exists an agent who is indifferent between defaulting and staying the contract where $\hat{V}_t^i\left(s^t, \tilde{m}^{s^t,j}; k_t^j\left(s^{t-1}\right), \delta\left(s^{t+1}\right) = 0\right)$ is the value conditional on the agent not defaulting. The idea is that if this were not the case, the intermediary could increase the amount of insurance being offered while continuing to respect the incentive compatibility constraints. This will be useful in our construction of not-too-tight debt limits.

The proof of the part 2 of the theorem relies on a limiting argument due to Fudenberg and Levine (1983). The idea is to construct truncated allocations of the debt

constraint environment, the limit of which converges to an equilibrium with not-too-tight debt constraints. I turn to this construction next (recall that I are assuming that each intermediary lives for 2 periods for ease of notation).

Let $\left\{\left(c_t^i\left(s^t\right), k_t^i\left(s^t\right), n_t^i\left(s^t\right), m_t^{s^t,i}\left(s^t\right), \bar{V}_t^i\left(s^t\right)\right)_{i \in I}\right\}_{t,s^t}$ be the allocation associated with the competitive equilibrium in the contracting environment. I first construct a truncated $T-$period allocation for the economy with debt constraints as follows: Define $\{a^{T,i}(s^t)\}$ using the equations

$$a^{T,i}(s^T) = m_T^{s^{T-1},i}(s^T) \tag{19}$$

$$a^{T,i}(s^t) - \sum_{s^{t+1}} q(s^{t+1} \mid s^t)\, a^{T,i}(s^t, s_{t+1}) = m_t^{s^{t-1},i}(s^t) + m_t^{s^t,i}(s^t) \text{ for all } t < T \tag{20}$$

For all $t < T$, let

$$\phi^{T.i}(s^{t+1}) = a^i(s^{t+1})$$

for i, s^{t+1} such that

$$\hat{V}_t^i\left(s^t, \tilde{m}^{s^t,j}; k_t^j(s^{t-1}), \delta(s^{t+1}) = 0\right) = \hat{V}_t^i\left(s^t, \tilde{m}^{s^t,j}; k_t^j(s^{t-1}), \delta(s^{t+1}) = 1\right) \tag{21}$$

where I know that there exists i for an s^t, s^{t+1} such that the above holds true from lemma 12 . For $t > T$ define

$$a^{T,i}(s^t) = 0,$$
$$\phi^{T,i}(s^{t+1}) = 0$$

Next, for all $t \leq T$, $c_t^{T,i}(s^t) = c_t^i(s^t)$, $n_t^{T,i}(s^t) = n_t^i(s^t)$, $k_t^{T,i}(s^{t-1}) = k_t^i(s^{t-1})$ and for $t > T$, $c_t^{T,i}(s^t)$, $n_t^{T,i}(s^t)$, $k_t^{T,i}(s^{t-1})$ be the best autarkic allocation that can be chosen by agent i given prices. Clearly the above allocation does not constitute a competitive equilibrium with not-too-tight constraints since in period T, given that there is no borrowing and lending the future, no agent will be willing to honor her debt. Let $\Delta^i(s^t) \in \{0,1\}$ denote the agent's default decision in any period and state. Define the sequence $\{\Delta^{T,i}\}_0$ where

$$\Delta^{T,i}(s^t) = 0 \text{ for } t \leq T$$
$$\Delta^{T,i}(s^t) = 1 \text{ for } t > T$$

For each entrepreneur/agent i, define

$$\Gamma^i(T) = \left\{ \begin{array}{c} \{c^i, k^i, n^i, a^i, \Delta\}_0 : \text{For all } t, s^t, \ \left(c_t^i(s^t), k_{t+1}^i(s^t), n_t^i(s^t), \{a^i(s^t, z_{t+2})\}_{z_{t+2}}, \Delta^i(s^t)\right) \text{ satisfies} \\ \text{equations (9), (10) and for } t' > T \text{ correspond to the best autarkic allocation given prices} \end{array} \right\}$$

and $\Gamma(T) = \prod_{i \in I} \Gamma^i(T)$. $\Gamma^i(T)$ consists of choices that are budget feasible for the agent and involve living in autarky after period T. $\Gamma(\infty)$ is the untruncated choice sets for the agent. Clearly, the truncated allocation constructed above in an element of this set for each i given the debt constraints.

Define $x^{i,T} = \{c^{T,i}, k^{T,i}, n^{T,i}, a^{i,T}, \Delta^{i,T}\}_0$ and $x^{i,*} = \{c^i, k^i, n^i, a^i, \Delta^i\}_0$ where $\Delta^i(s^t) = 0$ for all t and $\{c^i, k^i, n^i\}_0$ correspond to the competitive equilibrium allocation in the contracting problem and a^i is the limit as $T \to \infty$ of asset holdings constructed using (19) and (20).

For any i consider the best deviation in $\Gamma^i(T)$ from the truncated allocation constructed above given the debt constraints $\{\phi^{T.i}\}$. From lemma 11 I know that the in any equilibrium of the contracting environment $q(s^{t+1} \mid s^t) \geq \max_{i\in I}\left\{\frac{\beta\pi(s^{t+1}|s^t)u'(c^i_{t+1}(s^{t+1}))}{u'(c^i_t(s^t))}\right\}$. As a result for all $t < T$, conditional on not defaulting the truncated allocation is optimal for each agent given the debt constraints constructed above. Therefore, the best possible deviation (if one exists) involves default by some agent i in some period $t \leq T$. Define $\varepsilon^{i,T} \geq 0$ to be the value of the best possible deviation for agent i in $\Gamma^i(T)$ for any $t \leq T$,

$$V_t^{i,d}\left(s^t, k_t^i\left(s^{t-1}\right)\right) - V_t^{i,c}\left(s^t, k_t^i\left(s^{t-1}\right), a^i\left(s^t\right); \Phi^i\left(s^t\right)\right) = \varepsilon^{i,T}$$

and let $\varepsilon^T = \max_i \varepsilon^{i,T}$. In particular, ε^T corresponds to the best possible deviation from the truncated allocation that can be achieved by any player who chooses from choice set $\Gamma^i(T)$. Let

$$X_t = R_+ \times R_+ \times [0,1] \times R_+ \times \{0,1\}$$

$X_{t'} = \prod_{t=0}^{t'} X_t$ and $X = \prod_{t=0}^{\infty} X_t$. I have $x^{i,T} \in X$ and $x^{i,*} \in X$. The metric

$$d(x,z) = \sup_t \left[\frac{1}{t}\min\{|x_t - z_t|, 1\}\right]$$

induces the product topology on X. I can see that $x^{i,T} \to x^{i,*}$ in this metric.

Definition 13 *A allocation $\left(\{c^{T,i}, k^{T,i}, n^{T,i}, a^{T,i}, \Delta^{T,i}\}_0\right)_{i\in I} \in \Gamma(T)$ is an $\varepsilon-$perfect equilibrium ($\varepsilon -$ perfect) if for each t, s^t, any agent i's best deviation in $\Gamma^i(T)$ from the prescribed allocation yields her a welfare gain of at most ε.*

Given this definition and the previous discussion I have that x^T is ε^T perfect in $\Gamma(T)$ and that $x^T \to x^*$. Notice that since the contracting equilibrium allocations satisfy incentive compatibility (7), $\varepsilon^T \to 0$ since the agent does not want to deviate. Next, I adapt an argument from Theorem 3.3. in Fudenberg and Levine (1983) which proves that

Theorem 14 *A sufficient condition that x^* be perfect in $\Gamma(\infty)$ is that there be a sequences ε^T, x^T such that x^T is ε^T-perfect in $\Gamma(T)$ and as $T \to \infty$, $\varepsilon^T \to 0$ and $x^T \to x^*$.*

Proof. See Appendix ■

The theorem says that the truncated allocations converge to an equilibrium of the model with debt constraints. The last thing which needs to be checked is that the not-too-tight property is satisfied, but this follows from the construction of the debt limits. In particular if for any agent i

$$q\left(s^{t+1} \mid s^t\right) > \frac{\beta\pi\left(s^{t+1} \mid s^t\right)u'\left(c_{t+1}^i\left(s^{t+1}\right)\right)}{u'\left(c_t^i\left(s^t\right)\right)}$$

this agent sets $a^i\left(s^t, s_{t+1}\right) = \phi^i\left(s^t, s_{t+1}\right) < 0$. But then then from (21) I see that

$$\hat{V}_t^i\left(s^t, \tilde{m}^{s^t,j}; k_t^j\left(s^{t-1}\right), \delta\left(s^{t+1}\right) = 0\right) = \hat{V}_t^i\left(s^t, \tilde{m}^{s^t,j}; k_t^j\left(s^{t-1}\right), \delta\left(s^{t+1}\right) = 1\right)$$

which proves that the debt constraints are not-too-tight.

6 An Example

In this section I present a simple example of an economy with not-too-tight debt constraints to illustrate some equilibrium properties and show how one can generate endogenous fluctuations in these models. In addition, one might also be interested in the contracting problem in order to understand the nature of optimal contracts with hidden actions and while this is a fairly intractable problem, I show that the model with debt constraints is easier to solve. An important feature of these models is that there are multiple equilibria. The existence of multiple equilibria provides a natural source for aggregate fluctuations in these models. The model I present is stripped down to highlight the role of endogenous debt constraints in generating multiple equilibria.

Consider a deterministic two-type version of the model presented in section 3. Entrepreneurs' productivity alternate between high and low, $A_t\left(s^t\right) \in \left\{A^h, A^l\right\}$. In addition, I assume that there is another class of hand-to-mouth workers who supply labor inelastically. This is just to make the computation simpler and the multiplicity result is unaffected having the entrepreneur's work themselves on other projects. The entrepreneur's objective function and budget constraint is

$$\sum_{t=0}^{\infty} \beta^t \log\left(c_t^i\right)$$

$$c_t^i + q_t a_{t+1}^i + k_{t+1}^i = r_t\left(A, w_t\right) k_t^i + a_t^i$$

Entrepreneurs are subject to borrowing constraints

$$a_{t+1}^i \geq \phi_{t+1}^i$$

that are determined endogenously. The equilibrium is defined analogously to section 3. The first observation is that autarky is always an equilibrium with not-too-tight collateral constraints. To see why consider the equilibrium requirement that

$$V_t^{i,c}\left(s^t, k_t^i\left(s^{t-1}\right), \phi^i\left(s^t\right); \Phi^i\left(s^t\right)\right) = V_t^{i,d}\left(s^t, k_t^i\left(s^{t-1}\right)\right)$$

In the case of autarky,

$$V_t^{i,c}\left(s^t, k_t^i\left(s^{t-1}\right), \phi^i\left(s^t\right); \Phi^i\left(s^t\right)\right) = V_t^{i,c}\left(s^t, k_t^i\left(s^{t-1}\right), 0; \mathbf{0}\right)$$

which is just the value of default since there is no borrowing and lending in equilibrium. In this equilibrium each entrepreneur solves

$$\sum_{t=0}^{\infty} \beta^t \log\left(c_t^i\right)$$

subject to

$$c_t^i + q_t a_{t+1}^i + k_{t+1}^i = r_t\left(A, w_t\right) k_t^i + a_t^i$$

$$a_{t+1}^i \geq 0$$

with the equilibrium defined in a standard fashion.

Given the difference in productivity, the autarkic allocation is inefficient. The efficient allocation would involve only the high productivity technology to be used and a transfer from the entrepreneur owning this technology to the other type. Let the transfer associated with the efficient allocation be ξ^*. As is common in most of the literature with limited enforcement (Kehoe and Levine (1993) etc.), I assume that the efficient allocation violates the participation constraint of the high entrepreneur who would rather live in autarky forever than provide the transfer.

Next, I show that there exists a stationary equilibrium with borrowing and lending

Proposition 15 *If the autarkic return on the low productivity technology* $r\left(A^l, w^{aut}\right) < 1$, *there exists a stationary equilibrium with borrowing and lending, i.e.* $\phi^i < 0$ *for some* i.

Proof. See Appendix. ■

The proposition establishes that there are multiple equilibria in the model. The condition $r\left(A^l, w^{aut}\right) < 1$ ensures that if prices were fixed at the autarkic level, some insurance between the high and low types would be beneficial to the high type. A similar condition is needed in the analogous environment with endowments as in Alvarez and Jermann (2000). As they show in proposition 4.8, if the implied interest rates at autarky are "high" then autarky is the only equilibrium allocation in the model. By high interest rates they mean that the present discounted value of consumption is finite which is ensures if the interest rates at autarky are larger than 1. In our environment with heterogenous productivity, the relevant interest rate is $r\left(A^l, w^{aut}\right)$ since in a stationary autarkic equilibrium this is the interest rate on capital faced by agents who currently have high productivity (and have low productivity the next period). Moreover, in any equilibrium with borrowing and lending, the equilibrium interest rate R_{t+1} must be greater or equal to $r\left(A^l, w_{t+1}\right)$ in order for the current high type to lend to the low type. The stationary equilibrium with borrowing and lending I consider has $R = r\left(A^l, w\right)$.

The intuition for why multiple equilibria exist in this model is the dynamic complementarity present in entrepreneurs' decisions to borrow and lend. For example, if entrepreneurs believe that there will be no borrowing and lending in the future, there will be no borrowing and lending in the present as they will strictly prefer to default on their debts than pay back. Similarly, the expectation that debt constraints will be loose in the future allows for current borrowing and lending. A similar intuition holds in the contracting environment. Here the dynamic complementarity is present in the actions of the intermediaries. If an intermediary today believes that all future intermediaries will not lend to entrepreneurs, it will not be willing to offer any insurance today.

Next I will show that for a large class of parameter values, there is equilibrium indeterminacy in that there are a continuum of non-stationary equilibria converging to the autarkic one. I will use this to construct a sunspot equilibrium and look at the impulse response of current aggregate variables to an expectational shock of the state of collateral constraints the following period. Consider the equilibrium conditions of the model

$$\frac{c_t^h}{\beta c_{t+1}^l} = r\left(A^l, w_{t+1}\right) \tag{22}$$

$$\frac{c_t^l}{\beta c_{t+1}^h} = r\left(A^h, w_{t+1}\right) \tag{23}$$

$$c_t^h + k_{t+1}^l = r\left(A^h, w_t\right) k_t^h - R_t\phi_t - \phi_{t+1} \tag{24}$$

$$c_t^l + k_{t+1}^h = r\left(A^l, w_t\right) k_t^l + R_t\phi_t + \phi_{t+1} \tag{25}$$

$$\rho\left(w_t, A^h\right) k_t^h + \rho\left(w_t, A^l\right) k_t^l = 2 \tag{26}$$

$$c_t^h + c_t^l + k_{t+1}^h + k_{t+1}^l = r\left(A^l, w_t\right) k_t^l + r\left(A^h, w_t\right) k_t^h \tag{27}$$

$$\log c_t^h + \beta \log c_{t+1}^l = a \log r\left(A^h, w_{t+1}\right) k^h + b_t^h - \beta^2\left[a \log r\left(A^h, w_{t+2}\right) k_{t+2}^h + b_{t+2}^h\right] \tag{28}$$

$$b_t^h = \log\gamma + \beta\left[a \log\left(r\left(A^l, w_{t+1}\right)(1-\gamma)\right) + b_{t+1}^l\right] \tag{29}$$

$$b_t^l = \log\gamma + \beta\left[a \log\left(r\left(A^h, w_{t+1}\right)(1-\gamma)\right) + b_{t+1}^h\right] \tag{30}$$

where

$$a = \frac{1}{1-\beta}$$
$$\gamma = 1-\beta$$

(22) and (23) are the first order conditions for the two types with respect to capital accumulation, (24) and (25) are the entrepreneurs' budget constraints assuming that the debt constraint is binding, (26) is the labor market clearing conditions where $\rho\left(w_t, A^h\right)$ is defined as in (1) and (28) is the difference of the not-too-tight conditions of the model in t and $t+2$. Note the timing in the model- k_{t+1}^h is the capital invested by the current low productivity agent for use in $t+1$ when she has high productivity. Similarly k_{t+1}^l is the capital invest by the current high productivity agent. The entrepreneur's problem on default has a very simple structure given the log utility assumption and hence takes the form in the equation. In particular, after default, the entrepreneur saves and consumes a constant amount of her wealth in all future periods. Her value of default when she has high productivity and capital stock k_t^h is of the form

$$a \log r\left(A^h, w_{t+1}\right) k_t^h + b_t^h$$

where b_t^h is defined recursively using (29) and (30). I can linearize this system of equations around the autarkic steady state. Let

$x = \left(\tilde{c}_t^h, \tilde{k}_t^h, \tilde{k}_t^l, \tilde{k}_{t+1}^h, \tilde{k}_{t+1}^l, \tilde{b}_t^h, \tilde{b}_t^l, \tilde{b}_{t+1}^h, \tilde{b}_{t+1}^l\right)$ where the tilde variables are deviations from the steady state. The linear system is of the form $x_{t+1} = Gx_t$. As in standard (see for example Stokey, Lucas, and Prescott (1989)) the Jordan decomposition allows us the write $G = P\Lambda P^{-1}$ where Λ is a diagonal matrix consisting of the eigenvalues of G. I know from Woodford (1986b) that if the number of eigenvalues with absolute value greater than 1 is less than the number of forward looking variables, the equilibrium is indeterminate. Furthermore also from Woodford (1986b) I know that indeterminacy of equilibria is a sufficient condition for the existence of sunspot equilibria. Next, I construct an example of a sunspot equilibrium and examine the impulse response from the autarkic steady state to a one time shock to agents expectation of the tightness of collateral constraints the following period. To compute the model I use a technique developed by Farmer and Khramov (2013) which modifies the tools developed by Sims (2002) to deal with indeterminate equilibria[3]. The parameter values are

β	.98
α	.33
A^l	1
A^h	2
δ	.05

Starting at the autarkic steady I consider a one time expectational shock to future collateral constraints of 0.5 and consider the impulse responses of the collateral constraint ϕ_t, aggregate consumption $c_t = c_t^h + c_t^l$ and the capital accumulated by the current low type (high tomorrow) k_t^h.

The impulse responses highlight the persistent effects of a one time expectational shock to future collateral constraints. In the first period of the shock, due to the anticipation of loose constraints in the future, the agents with high technology lend to the agents with low current technology. This reduces the high productivity entrepreneur's investment in capital (recall the timing) and increases the low productivity agent's investment in capital. Additionally aggregate consumption rises since the entrepreneurs with low productivity are consuming more. Note the persistence of the shocks- it takes over a 100 periods for aggregate consumption to return to its original autarkic steady state level.

[3]See the note on computation at the end

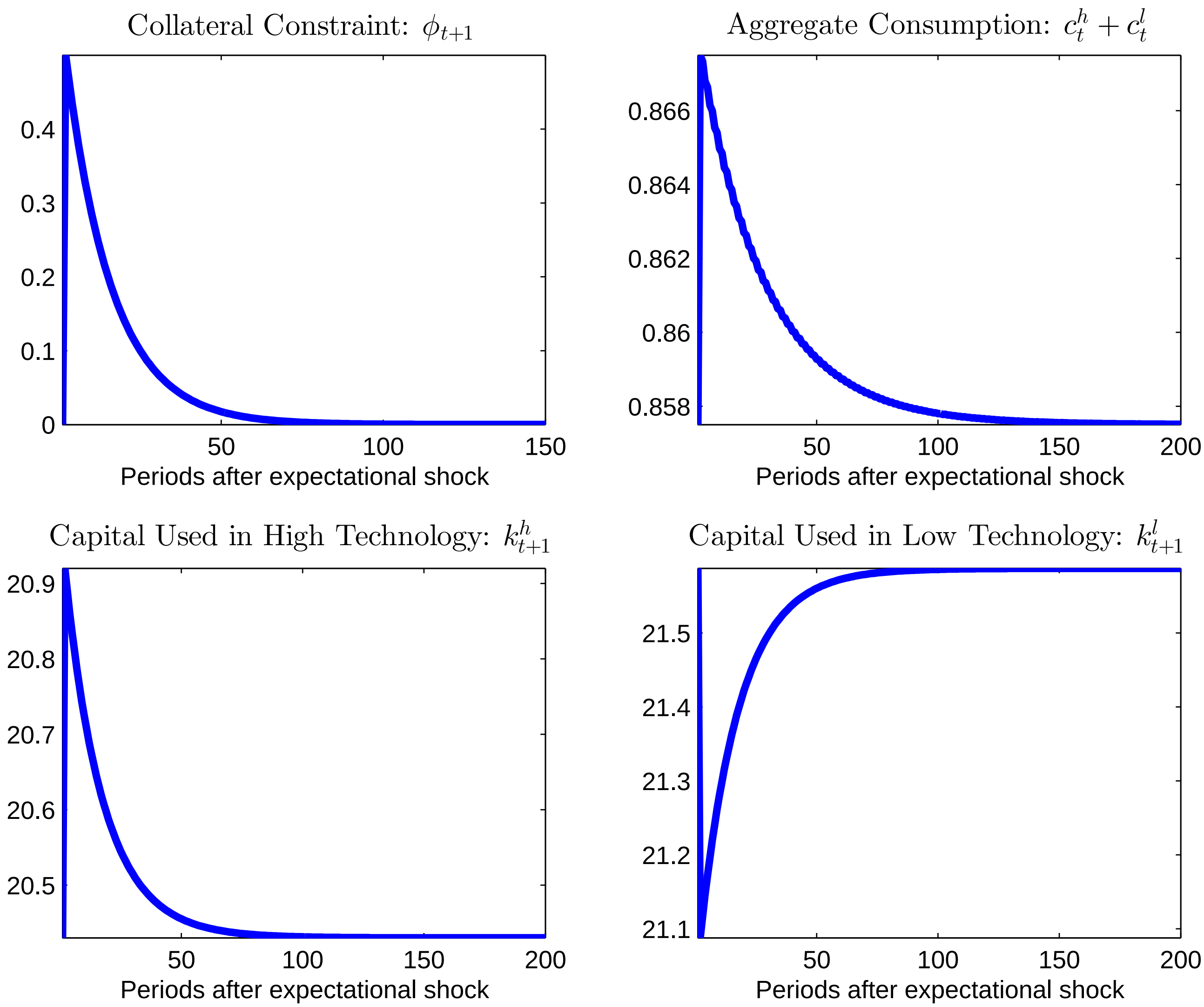

Figure 3: Impulse Responses

7 Conclusion

In this paper, I consider a class of models with debt constraints that can endogenously generate aggregate fluctuations. A crucial difference between the setup I consider and standard models is that I do not arbitrarily restrict the set of contracts borrowers and lenders can sign. In particular, I prove an equivalence between a class of decentralized contracting problems and models with debt constraints that are not-too-tight. I consider an environment in which intermediaries offer insurance contracts to entrepreneurs with heterogeneous productivity where the aggregate state is common knowledge but the actions of the intermediary are unobservable. The debt constraint setup is an extension of that by Alvarez and Jermann (2000) to an environment with heterogenous entrepreneurs and capital accumulation. As in their paper, debt constraints are equilibrium objects.

I view the equivalence result as important for several reasons. First, it offers a new interpretation of not-too-tight collateral constraints in commonly used financial frictions environment. This is in sharp contrast to much of the literature that imposes exogenous constraints on credit. A serious understanding of the origins of collateral constraints is essential for any policy analysis. For example, in our model if agents expect the government to intervene in credit markets in some future date, it will have an effect on the current level of borrowing and lending. Next, it allows us to understand the decentralized contracting environment better which on its own is intractable. As I show in an example, I can compute the equilibria of the debt constrained problem relatively easily. The most interesting feature of both these environments is the equilibrium multiplicity. Entrepreneurs' expectations of the tightness of future credit constraints affect the current state of debt constraints. I construct a simple example to show the effect of such an expectational shock. Our model has no extrinsic uncertainty and highlights the role of this dynamic complementarity present in both models. I see that the shock causes aggregate consumption and the capital used in the high technology to increase while the capital used in the low technology decreases.

As mentioned earlier I view this model as in interesting benchmark in which to contrast the effect of standard policy prescriptions on models with endogeous constraints versus one in which they are exogenous and linear. One such policy recommendation that economists have been proposing is "Macro-prudential" regulation which involves restraining the amount of credit offered by institutions in certain times. While such policy is shown the welfare enhancing in models with exogenous constraints, the mechanism in this paper suggests that agents' expectations of these policies might result in perversely limiting credit in "good" times as well. If I consider models with exogenous debt constraints, this channel is absent and so is ignored when formulating policy. As a result, models with reduced-form constraints should be used for policy exercises with extreme caution.

Appendix A

Proof of Lemma 11. I prove this by contradiction. Suppose that in the competitive equilibrium, for some i, s^t, s^{t+1}

$$q\left(s^{t+1} \mid s^t\right) < \frac{\beta\pi\left(s^{t+1} \mid s^t\right) u'\left(c^i_{t+1}\left(s^{t+1}\right)\right)}{u'\left(c^i_t\left(s^t\right)\right)} \tag{31}$$

Consider the following contract intermediary s^t can offer to this agent

$$\tilde{m}^{s^t,i}_t\left(s^t\right) = m^{s^t,i}_t\left(s^t\right) - q\left(s^{t+1} \mid s^t\right)\varepsilon$$
$$\tilde{m}^{s^t,i}_{t+1}\left(s^{t+1}\right) = m^{s^t,i}_t\left(s^{t+1}\right) + \varepsilon$$

where $\varepsilon > 0$ and with the rest of the contract being unchanged. For ε small but positive the change in welfare to agent i is

$$\left[-q\left(s^{t+1} \mid s^t\right) u'\left(c^i_t\left(s^{t+1}\right)\right) + \beta\pi\left(s^{t+1} \mid s^t\right) u'\left(c^i_{t+1}\left(s^{t+1}\right)\right)\right]\varepsilon > 0$$

because of (31), while the change in intermediary welfare is

$$q\left(s^{t+1} \mid s^{t}\right)\varepsilon - q\left(s^{t+1} \mid s^{t}\right)\varepsilon = 0$$

As a result an intermediary can offer a deviating contract and make strictly positive profit which contradicts the definition of a competitive equilibrium.

Therefore,

$$q\left(s^{t+1} \mid s^{t}\right) \geq \max_{i\in I}\left\{\frac{\beta\pi\left(s^{t+1} \mid s^{t}\right)u'\left(c_{t+1}^{i}\left(s^{t+1}\right)\right)}{u'\left(c_{t}^{i}\left(s^{t}\right)\right)}\right\}$$

Now consider an equilibrium in which

$$q\left(s^{t+1} \mid s^{t}\right) > \max_{i\in I}\left\{\frac{\beta\pi\left(s^{t+1} \mid s^{t}\right)u'\left(c_{t+1}^{i}\left(s^{t+1}\right)\right)}{u'\left(c_{t}^{i}\left(s^{t}\right)\right)}\right\}$$

and some insurance is being offered between periods t and $t+1$. Since insurance is being offered there must exist some agent j such that

$$\hat{V}_{t}^{j}\left(s^{t}, m^{s^{t},j}; k_{t}^{j}\left(s^{t-1}\right), \delta\left(s^{t+1}\right)=0\right) > \hat{V}_{t}^{j}\left(s^{t}, m^{s^{t},j}; k_{t}^{j}\left(s^{t-1}\right), \delta\left(s^{t+1}\right)=1\right) \tag{32}$$

where $\hat{V}_{t}^{j}\left(s^{t}, m^{s^{t},j}; k_{t}^{j}\left(s^{t-1}\right), \delta\left(s^{t+1}\right)=0\right)$ is the agent's best deviation conditional on not defaulting while the term on the right hand side is the best deviation conditional on defaulting. Inequality (32) says that the value for some agent j (who receives positive transfers) staying in the contract is strictly greater than defaulting. Consider the following contract an intermediary could offer agent j

$$\tilde{m}_{t}^{s^{t},j}\left(s^{t}\right) = m_{t}^{s^{t},j}\left(s^{t}\right) + q\left(s^{t+1} \mid s^{t}\right)\varepsilon$$
$$\tilde{m}_{t+1}^{s^{t},j}\left(s^{t+1}\right) = m_{t+1}^{s^{t},j}\left(s^{t+1}\right) - \varepsilon$$

For ε small the change in welfare is

$$\left[q\left(s^{t+1} \mid s^{t}\right)u'\left(c_{t}^{i}\left(s^{t+1}\right)\right) - \beta\pi\left(s^{t+1} \mid s^{t}\right)u'\left(c_{t+1}^{i}\left(s^{t+1}\right)\right)\right]\varepsilon > 0$$

while

$$\hat{V}_{t}^{j}\left(s^{t}, \tilde{m}^{s^{t},j}; k_{t}^{j}\left(s^{t-1}\right), \delta\left(s^{t+1}\right)=0\right) \geq \hat{V}_{t}^{j}\left(s^{t}, \tilde{m}^{s^{t},j}; k_{t}^{j}\left(s^{t-1}\right), \delta\left(s^{t+1}\right)=1\right)$$

And so in period t and state s^t, the agent is strictly better off under this contract. By a similar argument to above, since such a pertubation is welfare neutral to the intermediary, it can offer a contract and make strictly positive profits contradiction the competitive equilibrium assumption. This proves the claim. ■

Proof of Lemma 12. Suppose not and that full insurance is not being provided. Then there is some agent i, states s^t, s^{t+1} such that

$$q\left(s^{t+1}\mid s^t\right)>\frac{\beta\pi\left(s^{t+1}\mid s^t\right)u'\left(c^i_{t+1}\left(s^{t+1}\right)\right)}{u'\left(c^i_t\left(s^t\right)\right)}$$

and by assumption

$$\hat{V}^i_t\left(s^t,\tilde{m}^{s^t,j};k^j_t\left(s^{t-1}\right),\delta\left(s^{t+1}\right)=0\right)>\hat{V}^i_t\left(s^t,\tilde{m}^{s^t,j};k^j_t\left(s^{t-1}\right),\delta\left(s^{t+1}\right)=1\right)$$

Then an intermediary at s^t can offer the following contract to agent i

$$\begin{aligned}\tilde{m}^{s^t,i}_t\left(s^t\right)&=m^{s^t,i}_t\left(s^t\right)+q\left(s^{t+1}\mid s^t\right)\varepsilon\\ \tilde{m}^{s^t,i}_{t+1}\left(s^{t+1}\right)&=m^{s^t,i}_{t+1}\left(s^{t+1}\right)-\varepsilon\end{aligned}$$

with the rest of the contract unchanged. For ε small enough this contract gives the agent greater utility to the agent while leaving the intermediary equally well off. To see that all the incentive compatibility constraints are still satisfied notice that since $\frac{\partial\hat{V}^i_{t+1}\left(s^{t+1},k_{t+1},m^i_t\right)}{\partial m^i\left(s^{t+1}\right)}>0$, and $\hat{V}^i_{t+1}\left(s^{t+1},k_{t+1},m^i_t\right)$ is continuous for ε small enough the voluntary participation constraint in s^{t+1} still holds while in s^t, the value of not defaulting increases. As a result there exists a contract with gives the intermediary positive profit which contradicts the allocation/price being a competitive equilibrium ■

Proof of Theorem 14

Similar to ε^T, I define w^T to be the greatest variation in any agent's payoff due to events after $T-1$: for any t,s^t let

$$\tilde{V}^i_t\left(s^t,k^i_t\left(s^{t-1}\right),a^i\left(s^t\right);\Phi^i\left(s^t\right)\right)=\max\left\{V^{i,d}_t\left(s^t,k^i_t\left(s^{t-1}\right)\right)V^{i,c}_t\left(s^t,k^i_t\left(s^{t-1}\right),a^i\left(s^t\right);\Phi^i\left(s^t\right)\right)\right\}$$

and define

$$w^t=\max_{i\in I}\left(\max_{\substack{x_1,x_2\in X\\ x_1,x_2\text{ feas}\\ x_1(T-1)=x_2(T-1)}}\tilde{V}^i_0\left(s_0,k^i_0,a^i_0;\Phi^i\left(s_0\right)\right)\right)$$

The restriction $x_1\left(T-1\right)=x_2\left(T-1\right)$ just means that the allocation is identical for all dates and states prior to period T.The following two lemmas whose proofs can be found in Fudenberg and Levine (1983) will be useful,

Lemma 16 *If x is $\varepsilon-$perfect in $\Gamma\left(T\right)$ then x is $\left(\varepsilon+w^T\right)$ perfect in $\Gamma\left(\infty\right)$*

Lemma 17 *Let x be $\varepsilon-$perfect in $\Gamma\left(\infty\right)$ and $x\rightarrow x^*$. Then x^* is also ε-perfect.*

Proof of Theorem 14. From lemma 16 we know that x^T is $\varepsilon^T + w^T$-perfect in $\Gamma(\infty)$. Since $\varepsilon^T + w^T \to 0$ (because $x^T \to x^*$) for each $\delta > 0$ there is some T^*such that $\varepsilon^{T'} + w^{T'} < \delta$ for all $T' > T^*$. From lemma 17, x^* is $\delta-$perfect in $\Gamma(\infty)$. Since this is true for all $\delta > 0$, x^* is perfect in $\Gamma(\infty)$. ■

Proof of Proposition 15. Consider the following problem for the entrepreneur in which she is not allowed to borrow and lend, but is subject to a state contingent transfer $\xi > 0$[4].

$$\begin{aligned} &\max_{\{c,k\}} \sum_{t=0}^{\infty} \beta^t u(c_t) \\ &s.t. \\ &c_t + k_{t+1} = r(A_t, w_t) k_t - \xi 1_{A_t = A^h} + \xi 1_{A_t = A^l} \end{aligned}$$

An equilibrium is defined in a straightforward manner with the hand to mouth workers. It is easy to show that this economy has a stationary equilibrium $\left(c^h(\xi), c^l(\xi), k^h(\xi), k^l(\xi)\right)$ where$\left(c^l(\xi), k^h(\xi)\right)$ is chosen by the agent with high current productivity and $\left(c^l(\xi), k^h(\xi)\right)$ by the entrepreneur with low current productivity. Let $\left(c^h(0), c^l(0), k^h(0), k^l(0)\right)$ be the autarkic stationary allocation.

Define the value for the high agent in the stationary equilibrium as $V\left(h, k^h(\xi), \xi\right)$.

Next, I consider an alternate world without these exogenous state contingent transfers and consider the relative payoffs for the agent with high productivity in both environments. Let the value of autarky for an agent starting off with capital stock $k^h(\xi)$ with her current productivity is high be $V\left(h, k^h(\xi), 0\right)$. Define the map

$$\Lambda(\xi) = V\left(h, k^h(\xi), \xi\right) - V\left(h, k^h(\xi), 0\right)$$

We are interested in how this difference changes ξ changes. Consider the effect of increasing ξ around the autarkic allocation. Totally differentiating Λ,

$$\begin{aligned} &V_k\left(h, k^h(0), 0\right) k^{h\prime}(0) + V_\xi\left(h, k^h(0), 0\right) - V_k\left(h, k^h(0), 0\right) \\ &= V_\xi\left(h, k^h(0), 0\right) \end{aligned}$$

Notice that$V_\xi\left(h, k^h(0), 0\right)$ measures the change in the continuation value for the agent when ξ is increased slightly if the her capital holdings was fixed at the autarkic level. This can be written as

$$\begin{aligned} V_\xi\left(h, k^h(0), 0\right) &= \frac{d}{d\xi}\left(u\left(c^h(0) - \xi\right) + \beta u\left(c^l(0) + \xi\right)\right)\Big|_{\xi=0} \\ &= -u'\left(c^h(0)\right) + \beta u'\left(c^l(0)\right) \end{aligned}$$

[4]One can even think of this as an endowment shock.

Since $\frac{u'\left(c^h(0)\right)}{\beta u'\left(c^l(0)\right)} = r\left(A^l, w^{aut}\right) < 1,\ V_\xi\left(h, k^h(0), 0\right) > 0$ for ξ small but positive and so for this $\xi,\ \Lambda(\xi) > 0$. Also for ξ large enough $\Lambda(\xi) < 0$. Since Λ is continuous in ξ, by the intermediate value theorem there exists some $\bar{\xi}$ such that $\Lambda\left(\bar{\xi}\right) = 0$.

One can easily reinterpret this setup as one with not-too-tight collateral constraints. Set

$$R = r\left(A^l, w\right)$$
$$q = \frac{1}{R}$$
$$a = \phi = \frac{\bar{\xi}}{1-q}$$

And so the entrepreneur with high productivity lends ϕ to the agent with low productivity. Given our construction and the fact that $a - qa = \bar{\xi}$, I know that ϕ is not-too-tight in that the high productivity agent who owes ϕ, is indifferent between defaulting and paying back. ■

Appendix B

Proof of Proposition 2. To prove this result, we will construct prices and allocations so that these along with the no insurance contract constitute a competitive equilibrium.

First consider an environment in which agents accumulate capital and hire labor but do not have access to financial markets. Given a sequence of wage rates $\left\{w_t\left(s^t\right)\right\}$, each agent solves

$$\max_{\{c^i, n^i, a^i\}_0} \sum_{t=0}^{\infty} \sum_{s^t \in S^t} \beta^t \pi\left(s^t\right)\left[u\left(c_t^i\left(s^t\right) - v\left(n_t^i\left(s^t\right)\right)\right)\right] \tag{33}$$

subject to

$$c_t^i\left(s^t\right) + k_{t+1}^i\left(s^t\right) \leq r\left(w_t\left(s^t\right), A_t^i\left(s^t\right)\right) k_t^i\left(s^{t-1}\right) + w_t\left(s^t\right) n_t^i\left(s^t\right) \tag{34}$$

An equilibrium for this setup is defined in the standard fashion and exists. Denote the solution to it with superscript a. Define a sequence of Arrow security prices as follows

$$q^a\left(s^{t+1} \mid s^t\right) = \max_{i \in I}\left\{\frac{\beta u'\left(c_{t+1}^{a,i}\left(s^{t+1}\right)\right)}{u'\left(c_t^{a,i}\left(s^t\right)\right)}\right\}$$

Now consider the contracting environment in which intermediaries can borrow and lend with each other at prices$\left\{q^a\left(s^{t+1} \mid s^t\right)\right\}_{s^{t+1}, s^t}$ and the wage rates are $\left\{w_t^a\left(s^t\right)\right\}_{t, s^t}$. We will show that the allocation constructed above form an equilibrium in this environment and in particular intermediaries offer no insurance to agents. Consider the period 0 intermediary's problem (the multipliers on the constraints are in parentheses)

$$\max_{\{c,k,n,m\}} \sum_i \sum_{t=0}^{T} \sum_{s^t} \prod_{t'=1}^{t} q^a\left(s^{t'} \mid s^{t'-1}\right) m^i_{t'}\left(s^{t'}\right)$$

s.t.

$$\left(\lambda^i\left(s^t\right)\right) : c^i_t\left(s^t\right) + k^i_{t+1}\left(s^t\right) \leq r\left(w_t\left(s^t\right), A^i_t\left(s^{t-1}\right)\right) k^i_t\left(s^{t-1}\right) + w_t\left(s^t\right) n^i_t\left(s^t\right) - m^i_t\left(s^t\right)$$

$$\left(\beta^t \pi\left(s^t\right) \mu^i\left(s^t\right)\right) : \sum_{t'=t}^{\infty} \sum_{s^{t'} \succeq s^t} \beta^{t'-t} \pi\left(s^{t'} \mid s^t\right)\left[u\left(c^i_{t'}\left(s^{t'}\right) - v\left(n^i_{t'}\left(s^{t'}\right)\right)\right)\right] \geq V^d_t\left(s^t, k^i_t\left(s^{t-1}\right)\right)$$

$$\left(\nu^i\right) : \sum_{t'=0}^{\infty} \sum_{s^{t'}} \beta^{t'-t} \pi\left(s^{t'}\right)\left[u\left(c^i_{t'}\left(s^{t'}\right) - v\left(n^i_{t'}\left(s^{t'}\right)\right)\right)\right] \geq \bar{V}$$

$$(\eta) : \sum_i \sum_{t=0}^{T} \sum_{s^t} \prod_{t'=1}^{t} q^a\left(s^{t'} \mid s^{t'-1}\right) m^i_{t'}\left(s^{t'}\right) \geq 0$$

As in Kehoe and Perri (2002), we can define $M^i\left(s^t\right) = M^i\left(s^{t-1}\right) + \mu^i\left(s^t\right)$ and $M^i\left(s_0\right) = \nu^i + \mu^i\left(s_0\right)$. Then we can re-write the problem using a "partial" Lagrangian,

$$\max_{\{c,k,n,m\}} \sum_i \sum_{t=0}^{T} \sum_{s^t} \prod_{t'=1}^{t} q^a\left(s^{t'} \mid s^{t'-1}\right) m^i_{t'}\left(s^{t'}\right) + \sum_i \sum_{t=0}^{T} \sum_{s^t} \beta^t \pi\left(s^t\right) M^i\left(s^t\right)\left[u\left(c^i_t\left(s^t\right) - v\left(n^i_t\left(s^t\right)\right)\right)\right]$$

$$- \sum_i \sum_{t=0}^{T} \sum_{s^t} \beta^t \pi\left(s^t\right) \mu^i\left(s^t\right) V^d_t\left(s^t, k^i_t\left(s^{t-1}\right)\right) - \sum_i \nu^i \bar{V}$$

s.t.

$$\left(\lambda^i\left(s^t\right)\right) : c^i_t\left(s^t\right) + k^i_{t+1}\left(s^t\right) \leq r\left(w_t\left(s^t\right), A^i_t\left(s^{t-1}\right)\right) k^i_t\left(s^{t-1}\right) + w_t\left(s^t\right) n^i_t\left(s^t\right) - m^i_t\left(s^t\right)$$

$$(\eta) : \sum_i \sum_{t=0}^{T} \sum_{s^t} \prod_{t'=1}^{t} q^a\left(s^{t'} \mid s^{t'-1}\right) m^i_{t'}\left(s^{t'}\right) \geq 0$$

The necessary first order conditions of the problem are

$$0 = \lambda^i\left(s^t\right) - \beta^t \pi\left(s^t\right) M^i\left(s^t\right) u'\left(c^i_t\left(s^t\right)\right) \tag{35}$$

$$0 = \lambda^i\left(s^{t+1}\right) - \beta^{t+1} \pi\left(s^{t+1}\right) M^i\left(s^{t+1}\right) u'\left(c^i_{t+1}\left(s^{t+1}\right)\right) \tag{36}$$

$$0 = -\beta^t \pi\left(s^t\right) M^i\left(s^t\right) v'\left(n^i_t\left(s^t\right)\right) - \lambda^i\left(s^t\right) w_t\left(s^t\right) \tag{37}$$

$$0 = \lambda^i\left(s^t\right) - \sum_{s^{t+1}} \left[\begin{array}{c} \lambda^i\left(s^{t+1}\right) r\left(w_{t+1}\left(s^{t+1}\right), A^i_{t+1}\left(s^t\right)\right) \\ -\beta^{t+1} \pi\left(s^{t+1}\right) \mu^i\left(s^{t+1}\right) \frac{\partial V^d_{t+1}\left(s^{t+1}, k^i_{t+1}\left(s^t\right)\right)}{\partial k^i_{t+1}\left(s^t\right)} \end{array}\right] \tag{38}$$

$$\prod_{t'=0}^{t} q\left(s^{t'} \mid s^{t'-1}\right) = \lambda^i\left(s^t\right) - \eta \prod_{t'=0}^{t} q\left(s^t \mid s^{t-1}\right) \tag{39}$$

$$\prod_{t'=0}^{t+1} q\left(s^{t'} \mid s^{t'-1}\right) = \lambda^i\left(s^{t+1}\right) - \eta \prod_{t'=0}^{t+1} q\left(s^{t'} \mid s^{t'-1}\right) \tag{40}$$

Combining (35), (36), (39) and (40) we obtain

$$\frac{M^i\left(s^t\right)u'\left(c_t^i\left(s^t\right)\right)}{M^i\left(s^{t+1}\right)\beta\pi\left(s^{t+1}\mid s^t\right)u'\left(c_{t+1}^i\left(s^{t+1}\right)\right)}=\frac{1}{q\left(s^{t+1}\mid s^t\right)} \tag{41}$$

Next using (38) we get

$$0=u'\left(c_t^i\left(s^t\right)\right)-\sum_{s^{t+1}}\left[\begin{array}{c}\frac{M^i\left(s^{t+1}\right)}{M^i\left(s^t\right)}\pi\left(s^{t+1}\mid s^t\right)\beta u'\left(c_{t+1}^i\left(s^{t+1}\right)\right)r\left(w_{t+1}\left(s^{t+1}\right),A_{t+1}^i\left(s^t\right)\right)\\ -\frac{\beta^{t+1}\pi\left(s^{t+1}\right)\mu^i\left(s^{t+1}\right)}{\beta^t\pi\left(s^t\right)M^i\left(s^t\right)}\frac{\partial V_t^d\left(s^{t+1},k_{t+1}^i\left(s^t\right)\right)}{\partial k_{t+1}^i\left(s^t\right)}\end{array}\right] \tag{42}$$

Finally, the intra-temporal equation is

$$\frac{u'\left(c_{t+1}^i\left(s^{t+1}\right)\right)}{-v'\left(n_t^i\left(s^t\right)\right)}=\frac{1}{w_t\left(s^t\right)} \tag{43}$$

In summary we want to show that autarkic allocation satisfies equations (41), (42) and (43). (43) follows straight away from the intra-temporal first order condition from the model with debt constraints. Next, consider (42) for any set of non-negative multipliers

$$u'\left(c_t^{a,i}\left(s^t\right)\right)-\sum_{s^{t+1}}\left[\begin{array}{c}\frac{M^i\left(s^{t+1}\right)}{M^i\left(s^t\right)}\pi\left(s^{t+1}\mid s^t\right)\beta u'\left(c_{t+1}^{a,i}\left(s^{t+1}\right)\right)r\left(w_{t+1}^a\left(s^{t+1}\right),A_{t+1}^i\left(s^t\right)\right)\\ -\frac{\beta^{t+1}\pi\left(s^{t+1}\right)\mu^i\left(s^{t+1}\right)}{\beta^t\pi\left(s^t\right)M^i\left(s^t\right)}\frac{\partial V_t^d\left(s^{t+1},k_{t+1}^{a,i}\left(s^t\right)\right)}{\partial k_{t+1}^{a,i}\left(s^t\right)}\end{array}\right] \tag{44}$$

$$\Rightarrow u'\left(c_t^{a,i}\left(s^t\right)\right)-\sum_{s^{t+1}}\left[\begin{array}{c}\frac{M^i\left(s^{t+1}\right)}{M^i\left(s^t\right)}\pi\left(s^{t+1}\mid s^t\right)\beta u'\left(c_{t+1}^{a,i}\left(s^{t+1}\right)\right)r\left(w_{t+1}^a\left(s^{t+1}\right),A_{t+1}^i\left(s^t\right)\right)\\ -\frac{\beta\pi\left(s^{t+1}\right)\mu^i\left(s^{t+1}\right)}{\pi\left(s^t\right)M^i\left(s^t\right)}u'\left(c_{t+1}^{a,i}\left(s^{t+1}\right)\right)r\left(w_{t+1}^a\left(s^{t+1}\right),A_{t+1}^i\left(s^t\right)\right)\end{array}\right] \tag{45}$$

where the second line follows from the Envelope Theorem. Then we can rewrite the above expression as

$$u'\left(c_t^{a,i}\left(s^t\right)\right)-\sum_{s^{t+1}}\left[\pi\left(s^{t+1}\mid s^t\right)\beta u'\left(c_{t+1}^{a,i}\left(s^{t+1}\right)\right)r\left(w_{t+1}^a\left(s^{t+1}\right),A_{t+1}^i\left(s^t\right)\right)\right]$$

which equals 0 since the autarkic allocation satisfies the Euler equation of the model with debt constraints. I construct the multipliers as follows: For any s^t, s^{t+1} for any i, such that

$$q^a\left(s^{t+1}\mid s^t\right)=\frac{\beta\pi\left(s^{t+1}\mid s^t\right)u'\left(c_{t+1}^{a,i}\left(s^{t+1}\right)\right)}{u'\left(c_t^{a,i}\left(s^t\right)\right)}$$

set $\mu^i\left(s^{t+1}\right)=0$. Otherwise define μ recursively so that

$$\frac{M^i\left(s^t\right)u'\left(c_t^{a,i}\left(s^t\right)\right)}{M^i\left(s^{t+1}\right)\beta\pi\left(s^{t+1}\mid s^t\right)u'\left(c_{t+1}^{a,i}\left(s^{t+1}\right)\right)}=\frac{1}{q^a\left(s^{t+1}\mid s^t\right)}$$

then define $\lambda^i\left(s^t\right)$ from

$$0=\lambda^i\left(s^t\right)-\beta^t\pi\left(s^t\right)M^i\left(s^t\right)u'\left(c_t^i\left(s^t\right)\right)$$

Then by construction, the autarkic allocation satisfies (41). Therefore given our constructed prices, the profit maximizing contract is one that offers no insurance to agents. ■

Proof of Lemma 5. For each i define the transfers as follows

$$m_t^i\left(s^t\right)=r\left(w_t\left(s^t\right),A_t^i\left(s^{t-1}\right)\right)k_t^i\left(s^{t-1}\right)+w_t\left(s^t\right)n_t^i\left(s^t\right)-c_t^i\left(s^t\right)-k_{t+1}^i\left(s^t\right)$$

Define the arrow security prices as

$$q\left(s^{t+1}\mid s^t\right)=\max_{i\in I}\left\{\frac{\beta u'\left(c_{t+1}^i\left(s^{t+1}\right)\right)}{c_t^i\left(s^t\right)}\right\}$$

Define the multipliers on the incentive constraint as those corresponding the multipliers on the voluntary participation constraints from the planning problem. Next define the multipliers on the budget constraint to be $\lambda^i\left(s^t\right)=\lambda^c\left(s^t\right)\prod_{t'=0}^{t}q\left(s^{t'}\mid s^{t'-1}\right)$.

To see that the first order conditions of the intermediary's problem is satisfied notice that (41) and (42) follow from the above construction and the intra-temporal condition from the conditionally efficient planning problem. From Lemma 4, I know that only the most efficient technology is operated. The Euler equation from the planning problem (given that only the most efficient technology is operated) is

$$u'\left(c_t^i\left(s^t\right)\right)-\sum_{s^{t+1}}\left[\begin{array}{c}\frac{M^{c,i}\left(s^{t+1}\right)}{M^{c,i}\left(s^t\right)}\pi\left(s^{t+1}\mid s^t\right)\beta u'\left(c_{t+1}^i\left(s^{t+1}\right)\right)r\left(w_{t+1}\left(s^{t+1}\right),A_{t+1}^i\left(s^t\right)\right)\\-\frac{\beta^{t+1}\pi\left(s^{t+1}\right)\mu^{c,i}\left(s^{t+1}\right)}{\beta^t\pi\left(s^t\right)M^{c,i}\left(s^t\right)}\frac{\partial V_t^d\left(s^{t+1},k_{t+1}^i\left(s^t\right)\right)}{\partial k_{t+1}^i\left(s^t\right)}\end{array}\right]$$

But since the multipliers are chosen to be identical, the conditionally efficient allocation satisfies the corresponding first order condition from the intermediary's problem. Finally I need to check that the market clearing conditions are satisfied. The security market clearing follows directly from the resource constraint. To see that the labor market clears, notice that the labor market clearing condition is

$$\sum_{i\in I}\rho\left(w_t\left(s^t\right),A_t^i\left(s^t\right)\right)k_t^i\left(s^{t-1}\right)=\sum_i n_t^i\left(s^t\right)$$

where $\rho\left(w_t\left(s^t\right), A_t^i\left(s^t\right)\right)$ is defined as in (1). However this follows from (6) since

$$\begin{aligned}
&\sum_{i\in I}\rho\left(w_t\left(s^t\right), A_t^i\left(s^t\right)\right)k_t^i\left(s^{t-1}\right)=\sum_i n_t^i\left(s^t\right)\\
&\Rightarrow\sum_{i\in I}\left(\frac{(1-\alpha)A_t^i\left(s^t\right)}{w_t\left(s^t\right)}\right)^{\frac{1}{\alpha}}k_t^i\left(s^{t-1}\right)=\sum_i n_t^i\left(s^t\right)\\
&\Rightarrow w_t\left(s^t\right)=\left(\frac{\sum_{i\in I}\left((1-\alpha)A_t^i\left(s^t\right)\right)^{\frac{1}{\alpha}}k_t^i\left(s^{t-1}\right)}{\sum_i n_t^i\left(s^t\right)}\right)^{\frac{1}{\alpha}}
\end{aligned}$$

■

A Note on Computation

One commonly used method to solve DSGE models and look at impulse responses around a deterministic steady state is to linearize the system and use techniques developed by Sims (2002) and other authors. However when the number of non-predetermined (forward looking) variables exceeds the number of explosive eigenvalues of the system, the technique is no longer applicable. Farmer and Khramov (2013) develop a way to append the standard approach to solve the model. The idea is simple- take some of the expectational variables and treat them as exogenous processes. As an example consider a linearized system in which there is one less explosive eigenvalues than forward looking variables (as is true in our model). As shown in Sims (2002), the way to incorporate expectational variables for example $E_t x_{t+1}$ is to define a new variable $y_t = E_t x_{t+1}$ and include a new equation in system $x_{t+1} = y_t + \eta_{t+1}$ where η_t is an endogenous shock process. Farmer and Khramov (2013) deal with the indeterminacy issue by redefining η_t as an *exogenous* mean zero process and specifying a variance for it. This reduces the dimension of the forward looking variables by one thus allowing us to apply standard techniques. Note that this is only one such solution- for example specifying a different variance will result in different dynamics. In the context of our simple model, the variable I choose to redefine is the expectation of the following periods collateral constraint. In the linearized system associated with (22)- (30) this would involve defining redefining the $E_t\tilde{\phi}_{t+2}$ term where the expectations is purely in terms of a "sunspot" variable. Our results confirm the intuition laid out in the previous sections that such sunspot equilibria can exist.

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