Efficiency and Policy in Models with Incomplete Markets and Borrowing Constraints*

Rishabh Kirpalani
Pennsylvania State University and NYU
rishabh.kirpalani@nyu.edu

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Abstract

I show that the equilibrium outcomes of long term contracting environments with certain informational and commitment frictions coincide with those in widely used models of exogenously incomplete markets. Under three frictions: private information, voluntary participation and hidden trading, equilibrium allocations and prices of the contracting environment are identical to one in which agents are restricted to trade a risk free bond subject to occasionally binding debt constraints. Despite this equivalence, policy implications in the two environments are very different. For example, equilibrium outcomes in models with exogenously incomplete markets are typically inefficient while the best equilibrium in my environment is efficient. This implies that imposing debt limits may be desirable when markets are exogenously incomplete, but not in my model. However, I show that this environment has multiple equilibria and that governments can play an important role as a lender of last resort in ensuring that the best equilibrium occurs.

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1 Introduction

A large and growing literature in macroeconomics and international economics uses models with incomplete markets and financial frictions for a variety of quantitative and policy exercises. Examples include the study of financial and sovereign debt crises, optimal taxation, and bankruptcy laws. The key assumption in these models is that markets are *exogenously* incomplete. In particular, strong assumptions are imposed on the types of contracts agents within the model can sign. The most commonly used assumption is that agents can trade an uncontingent risk-free bond subject to exogenous debt constraints. Examples of environments which make this assumption include Huggett (1993) and Aiyagari (1994) which are workhorse models in modern macroeconomics.[1] These models are used extensively for studying questions related to fiscal, monetary, and financial policy.

An alternate view, which I take in this paper, is to relax the assumptions of exogenous incompleteness and instead consider general contracting environments in which no restrictions are placed on the types of contracts agents can sign. I show that there exist informational and commitment frictions that endogenously generate the types of contracts assumed by much of the applied literature. In particular, under appropriate assumptions, the equilibrium outcomes of the contracting environment coincide with those in models in which agents are restricted to trade a risk-free bond subject to debt constraints. Next, I show that the best equilibrium in these environments is efficient. Finally, I show that models with endogenous incompleteness have substantially different implications for policy than those with exogenous incompleteness. To illustrate these differences, I study *macro-prudential* policies which have gained a lot of recent interest in the literature.

I study a dynamic environment with a large number of risk-averse households who receive stochastic endowments each period and seek to share risk with each other. I model trading among households by allowing them to sign contracts with competitive financial intermediaries. The contracting environment is subject to three key frictions: private information, voluntary participation and hidden trading. The first is that each household's endowment is private information and not observable to any other household. The second is that household participation in financial markets is voluntary in that in any period they can always choose autarky namely, to not participate in financial markets from then on and consume their endowments in every period. The third is that trades between households and intermediaries are hidden in that they are not observable by other households and other intermediaries. In particular, I allow households to sign contracts with multiple intermediaries in a hidden fashion.

[1] An alternative assumption is that agents can trade defaultable debt contracts. Such models are standard in the international macro and bankruptcy literature. I study the policy implications of endogenizing this assumption in a companion paper.

A well known feature of these environments is that risk-sharing is possible only if households that do not repay their debts suffer a cost. In my environment, I assume that if households do not repay their debts as specified in the contract, they are permanently excluded from financial markets and forced into autarky. With this assumption, I assume that financial intermediaries can only offer contracts that induce households to always repay their debts. I show that equilibrium outcomes in this environment are equivalent to those in a standard incomplete markets model in which households trade a risk-free bond subject to debt constraints. Moreover, these debt constraints are independent of households' histories and thus look exactly like those assumed in models with exogenous incompleteness.

The second main result of the paper is that the best equilibrium in this environment is constrained-efficient. By this I mean that a planner confronted with the same frictions as intermediaries cannot improve overall welfare. In particular, I show that in the presence of hidden markets, the amount of state contingency a planner can offer in a contract is severely limited. As a result, the planner cannot do better than offer short-term uncontingent contracts. In addition, transfers are further limited due to the interaction of hidden trading and voluntary participation. For example, unlike the case without hidden trading, it may be that overall welfare can be increased by not having voluntary participation constraints bind. The presence of hidden markets imply that such an allocation is not incentive feasible since borrowing constrained households will always borrow the maximal amount consistent with voluntary participation. However, while the best equilibrium is constrained efficient, the first welfare theorem does not hold since in general, the environment has multiple equilibria. This multiplicity is due to the presence of strategic complementarities in the actions of intermediaries.

The third set of results concern the lessons for policy. There are two important implications for policy. The first is that policies which might be considered desirable when markets are exogenously incomplete, may no longer improve welfare when markets are endogenous incomplete. In other words, the same frictions which restrict the set of incentive-feasible contracts also restrict the set of feasible policies. To illustrate this, I consider two types of inefficiencies that often arise in models with exogenous incompleteness which have been used to motivate policy. The first are *pecuniary* externalities which arise due to redistributionary effects that result from changing prices. See for example Lorenzoni (2008) and Davila and Korinek (2016). The second are *aggregate-demand* externalities which arise due to nominal rigidities and constraints on monetary policy. Aggregate-demand externalities have received a lot of recent attention in the study of liquidity traps and binding zero lower bound constraints. See for example Korinek and Simsek (2016) and Farhi and Werning (2016). Both types of externalities motivate the use of *macro-prudential* policies to limit the amount of debt in the economy. In contrast, equivalent models with endogenous incompleteness are constrained-efficient. In particular, such policies do not respect the underlying informational and commitment frictions. Intuitively, because debt limits arise from voluntary participation constraints and households can trade in hidden markets, they will use these markets to circumvent these limits. As a result, in the environment I study, imposing such limits will not increase overall welfare. The second important implication for policy concerns its role in uniquely implementing the best equilibrium. As mentioned earlier, this models features multiple Pareto-ranked equilibria. I show how simple lender of last resort policies can help uniquely implement the best one.

A key feature of this environment that allows for meaningful policy comparisons is that prices are *endogenously* determined to clear markets. While this complicates the notion of efficiency, it allows us to study the effects of policy on allocations through the general equilibrium channel. This is in sharp contract to much of the literature on hidden trading which treats prices are exogenous.

A final point worth noting is that all three frictions i.e. private information, limited commitment and hidden trading, are essential to the nature of the contract. Obviously, without private information, fully state-contingent contracts would be equilibrium outcomes. Without limited commitment, households will never be borrowing constrained. Without hidden trading, contracts will feature history contingency and equilibrium contracts will resemble those in Thomas and Worrall (1990) and Atkeson and Lucas (1992).

Literature: This paper is related to a large literature on dynamic contracts and its applications in macroeconomics. Green (1987), Thomas and Worrall (1990), Phelan and Townsend (1991) and Atkeson and Lucas (1992) are some of the important papers studying dynamic environments with private information. In general, efficient contracts in these environments feature lots of history contingency. As a result, these contracts are very different from the simple uncontingent borrowing and lending contracts assumed by the applied literature. In contrast, I show that when dynamic private information interacts with limited commitment and hidden trading, the equilibrium contracts are identical to those assumed in standard macroeconomic models.

Allen (1985), Cole and Kocherlakota (2001), and Ales and Maziero (2014) study dynamic private information environments with hidden trading and *exogenous* interest rates.[2] There are two significant differences between these papers and my work. The first is that no households/agents in these papers are borrowing constrained. In particular, Euler equations always hold with an equality. As a result the outcomes do not look like a Huggett model with occasionally binding constraints. Second, prices (interest rates) are *exogenous* in these models. As a result such models are not very useful for policy analysis.

[2] Bisin and Guaitoli (2004) and Bisin and Rampini (2006) study two period environments with moral hazard (hidden action) and hidden trading.

In particular, most of the interesting policy experiments in macroeconomics work through a general equilibrium channel by affecting prices. The key question in my paper is how the implications for policy differ if markets are endogenously incomplete versus the case in which they are exogenously incomplete. The existing literature on private information and hidden trading does not have much to say about such experiments since prices are exogenous. In contrast, prices are determined endogenously my model. This allows for a more meaningful study of policy experiments across models. Moreover, the key source of inefficiency in models with exogenous incompleteness are due to prices.[3] As a result, in order to compare the efficiency properties to a model with endogenously incomplete markets, one needs to also consider the effect on prices.

Golosov and Tsyvinski (2007) (henceforth GT07) study a dynamic Mirrleesian environment in which agents with hidden trading in which prices are endogenously determined. However, the efficient allocation looks very different from the equilibrium outcome of a Huggett model since it features more history contingency. In addition, they find that competitive equilibria are inefficient, which is in sharp contrast to this paper. The planning problem I study is related in that I also assume that households can trade in a hidden fashion. However, unlike their model, the best equilibrium in the environment I study is efficient even though the planner has control of the price in the hidden markets. In Section 4, I provide an explanation as to why the efficiency results are different.

In a recent important paper, Dovis (2014) studies an environment with both hidden types and limited commitment.[4] The efficient outcome in his model also features more history contingency than that associated with a standard incomplete markets environment with borrowing constraint. In particular, the assumption of hidden trading is necessary to obtain the equivalence result in this paper.

In seminal papers, Prescott and Townsend (1984) and Kehoe and Levine (1993) studied and defined constrained-efficiency for environments with moral hazard and limited commitment[5] respectively. The decentralized environment I study has both incentive compatibility and voluntary participation constraints as in these papers. However, in contrast to both papers, in my environment households can engage in hidden trading. As a result, their welfare theorems do not apply here.

Since the environment I study has multiple equilibria, I consider the role for policy to uniquely implement the best equilibrium. This paper uses techniques and language developed by Atkeson, Chari, and Kehoe (2010) and Bassetto (2002) which allows us to think about how policy can uniquely implement a desired competitive equilibrium.

[3]These are often referred to as pecuniary externalities.

[4]See Atkeson (1991), Atkeson and Lucas (1995) and Yared (2010) for other papers with both private information and limited commitment.

[5]See Kocherlakota (1996), Albuquerque and Hopenhayn (2004) and Kehoe and Perri (2002) for other papers studying models with limited commitment.

This paper is also related and contributes to the vast literature in macroeconomics that uses models with incomplete markets, two important examples of which are Huggett (1993) and Aiyagari (1994). These models have been used to study a variety of issues from optimal quantity of government debt by Aiyagari and McGrattan (1998) and more recently to studying the effects of exogenous shocks to the debt constraints as in Guerrieri and Lorenzoni (2011).

While the environment I consider is observationally equivalent to a large class of exogenously incomplete models, the approach to efficiency I take is substantially different. Usually, the approach taken is similar in spirit to Diamond (1967) who exogenously restricts the set of instruments available to the planner. Geanakoplos and Polemarchakis (1986) and more recently Dávila et al. (2012) study such planning problems and conclude that the equilibria with incomplete markets are constrained inefficient. However, I use an example to show that outcomes which would be considered constrained-inefficient when markets are exogenously incomplete are actually constrained-efficient when markets are endogenously incomplete.

The paper proceeds as follows. In Section 2 I describe the underling contracting environment and define an equilibrium. In Section 3, I the characterize the equilibrium of the contracting environment and prove the equivalence result. Next, Section 4 studies the efficiency properties of this environment. Finally, Section 5 discusses the role of various assumptions in generating the main results and Section 6 concludes. Most of the proofs are contained in Appendix A.

2 Environment

Consider an infinite horizon discrete time environment, $t = 1, 2, ...$ with a continuum of infinitely lived households $i \in I$ and a continuum of overlapping $\hat{T} < \infty$ period lived[6] risk-neutral intermediaries/firms born each period. Households are risk-averse with period utility functions $u(c_t)$ where $u : \mathbb{R}_+ \to \mathbb{R}$ is an increasing and strictly concave continuously differentiable function. I also assume that u satisfies Inada conditions, $\lim_{c\to 0} u'(c) = -\infty$ and $\lim_{c\to\infty} u'(c) = 0$. There is a single non-storable consumption good of which households receive a random endowment $\theta_t \in \Theta$, $\theta_t \in \mathbb{R}_{++}$ each period where Θ is a finite set. Denote the maximal and minimal element of Θ by $\bar{\theta}$ and $\underline{\theta}$ respectively. The endowment shock is independently and identically distributed over time and households with density function $\pi(\cdot)$. Intermediaries can borrow and lend with each other at a market determined interest rate $\frac{1}{q_t}$ each period.

[6] Intermediaries are assumed to be finitely lived so that their problem is always well defined. See section 5 for further discussion.

Households enter into long term contracts with intermediaries in order to smooth their consumption over time and can sign with multiple such intermediaries as described below. The timing of the game is as follows:

1. Endowments are realized and are private information to households. Households report endowments to all intermediaries they are currently signed to.

2. Households can voluntarily choose to not participate in financial markets and live in financial autarky forever.

3. Participating households receive transfers from or make transfers to incumbent intermediaries, namely those intermediaries with whom they have pre-existing contracts.

4. Intermediaries post contracts.

5. Households observe the offered contracts and can choose to sign with at most one new intermediary.

6. Consumption takes place

Elements 4 and 5 of the timing scheme refer to the hidden trading assumption of the game and deserve some comment. One interpretation of this that there are a large number of islands each with one intermediary. In every period, all intermediaries post contracts which are publicly observable. Households can go to any one of these islands and a sign a new contract. Transactions between an intermediary and a household on one island is unobservable to all other intermediaries. It is worth noting that the only outcomes that are publicly observable are posted contracts and whether the household has chosen to not participate in financial markets.

I begin the formal description of the game between intermediaries and households by first describing the information sets available to both types of players. Let $\hat{z}^{t-1} \in \hat{Z}^{t-1}$ denote the public history at the beginning of period t. A typical history $\hat{z}^{t-1} = \left(q^{t-1}, \mathcal{B}^{t-1}\right)$ consists of the history of prices q^{t-1} and posted contracts $\mathcal{B}^{t-1}$. Denote the public history when new contracts are offered by $z^t = \left(\hat{z}^{t-1}, q_t\right)$. Let $\omega^{t-1} \in \Omega^{t-1}$ denote the private history at the beginning of period t where $\omega^{t-1} = \left(\theta^{t-1}, B^{t-1}\right)$, where θ^{t-1} is the history of endowment realizations, and $B^{t-1} = (B_1, ..., B_{t-1})$ is the history of signed contracts. The private history for each household i in period t, after endowments have been realized, is denoted by $h^t \in H^t$ where $h^t = \left(\omega^{t-1}, \theta_t\right)$. If the household is not signed to any contract at the beginning of period t, I denote the contract history as $B^{t-1} = \varnothing$. Note that endowments and signed contracts are privately observed by the household. In each period, households report their endowment type θ_t to intermediaries who use the public history along with the history of reports and household strategies σ^{HH} to compute B^{t-1}. I consider symmetric equilibria in which all households with the same θ^t have identical contract histories B^{t-1}. Given the public history z^t, let $\tilde{\zeta}_t\left(h^t\right)$ denote the intermediaries' beliefs of private histories in period t. We can also define the true probability measure on the space of private histories: $\zeta_t\left(h^t\right)$ and $\wp\left(\omega^t\right)$ for histories h^t and ω^t respectively, which will be constructed after the formal definition of a contract.

A contract $B_t\left(z^t\right)$ offered in period t is defined as follows:

$$B_t\left(z^t\right) = \left({}_t\tau_{t+s}\left(z^{t+s}, m^s\right) : 0 \leq s \leq \hat{T} - 1\right)$$

where m^s denotes the history of type reports $(m_t, ..., m_{t+s})$ with $m_{t+j} \in \Theta$. Denote the space of all such contracts by $\mathbb{B}_t$. In general, given an element of a contract ${}_tx_s$, the left subscript denotes the period in which the contract is agreed to and the right subscript the current period. Here ${}_t\tau_{t+s}\left(z^{t+s}, m^s\right) \in \mathbb{R}$ denotes the transfers to the households from a contract signed in period t as a function of the history of reported types in period $t+s$.

Next, consider the problem of a household. A strategy for a household is σ_t^{HH} which maps the appropriate histories into $\{0,1\} \times \Sigma_t \times \mathcal{B}_t \times \mathbb{R}_+$ where Σ_t is the set of type reporting strategies and $\mathcal{B}_t$ denotes the set of posted contracts in period t. In each period, the household chooses whether to participate, what to report, whether to sign a new contract and how much to consume. A typical strategy, $\sigma_t^{HH} = \{\Delta_t, \sigma_t, B_t, c_t\}$ where each element depends on the appropriate histories. Let $\Delta_t \in \{0,1\}$ denote the participation strategy for the household which depends on h^t with $\Delta_t = 0$ implying that the household chooses to not participate in financial markets and consequently live in autarky forever. Let $\Sigma = (\Sigma_t)_{t\geq 1}$ with typical element $\sigma = \left(\{\sigma_t^s\}_{s\leq t}\right)_{t\geq 1}$ where σ_t^s is the household's type reporting strategy in period t, to the intermediary associated with contract B_s where $s \leq t$ which depends on h^t for $s < t$ and ω^t for $s = t$. In particular note that the household can potentially report different types to different intermediaries. I define the truth-telling strategy σ^*, to be one that satisfies $\sigma_t^{*s} = \theta_t$ for all $s \leq t$ where θ_t is the household's endowment. Given the structure of the game, participating households have the option to sign at most one new contract with another intermediary from the set of posted contracts which also depends on h^t. Note however that the consumption strategy depends on the new contract and hence on ω^t. Given a private history h^t and an associated vector of signed contracts B^{t-1}, it will be useful to define the following object

$$\tau_t^{old}\left(h^t \mid z^t\right) \equiv \sum_{s<t} {}_s\tau_t\left(z^t, \left(\sigma_s^s\left(z^s, h^s\right), ..., \sigma_t^s\left(z^t, h^t\right)\right)\right)$$

Here, $\tau_t^{old}\left(h^t \mid z^t\right)$ denotes the total transfers in period t from contracts signed prior to period t as a function of reports $\left(\sigma_s^s\left(z^s, h^s\right), ..., \sigma_t^s\left(z^t, h^t\right)\right)$. For ease of notation I will subsequently refer to this as $\tau_t^{old}\left(h^t\right)$. Given the definition of a contract we can now define how the true probabilities of private histories ω^t are constructed.[7] This is done recursively as follows[8]: $\wp\left(\omega^1\right) = \pi\left(\theta_1\right)$ and for all $t > 1$

$$\wp\left(\omega^t\right) = \wp\left(\omega^{t-1}\right)\pi\left(\theta_t\right)\mathbb{I}_{B_t \in \sigma_t^{HH}} \tag{1}$$

where $\omega^t = \left(\omega^{t-1}, \theta_t, B_t\right)$. In the first period, $\wp$ is the same as π. In subsequent periods, the first term $\wp\left(\omega^{t-1}\right)\pi\left(\theta_t\right)$ on the right hand side of (1) corresponds to the probability of h^{t-1} times the probability of the current realization of type θ_t multiplied by an indicator function which indicates whether the household strategy dictates that contract B_t be signed.

For any $t \geq 1$ and $h^t \in H^t$, the household of type h^t chooses a strategy σ_t^{HH} to maximize

$$\sum_{s=0}^{\infty}\beta^s \sum_{\omega^{t+s}\in\Omega^{t+s}} \wp\left(\omega^{t+s}\right) u\left(c_{t+s}\left(\omega^{t+s}\right)\right) \tag{2}$$

subject to a budget constraints: $\forall s \geq 0$, $h^{t+s} \in H^{t+s}$: for participating households,

$$c_{t+s}\left(\omega^{t+s}\right) \leq \theta_{t+s} + \tau_{t+s}^{old}\left(h^{t+s}\right) + {}_{t+s}\tau_{t+s}\left(\omega^{t+s}\right) \tag{3}$$

and $c_t\left(\omega^t\right) = \theta_t$ for non-participating households ($\Delta_t\left(h^t\right) = 0$). The term $\tau_t^{old}\left(h^t\right)$ denotes the transfers from contracts signed in periods prior to t, while ${}_t\tau_t\left(\omega^t\right) \equiv {}_t\tau_t\left(z^t, \sigma_t^t\left(\omega^t\right)\right)$ denotes the transfers from the contract B_t, signed in period t. B_t is chosen from the set of posted contracts $\mathcal{B}_t$. With slight abuse of notation I will sometimes denote the sum $\tau_t^{old}\left(h^t\right) + {}_t\tau_t\left(\omega^t\right)$ as $\tau_t\left(\omega^t\right)$. Note that if $B^{t-1} = \varnothing$, $\tau_t^{old}\left(h^t\right) = 0$. Denote the value of the above problem when the household is using reporting strategy σ by $V_t\left(h^t\right)(\sigma)$.

Finally, lets consider the problem of an intermediary. A strategy for an intermediary is $\sigma_t^{INT} : Z^t \rightarrow \mathbb{B}_t$ and a typical strategy $\sigma_t^{INT}\left(z^t\right) = B_t$. In each period, without loss of generality, we can consider intermediaries offering one contract for each type $h^t \in H^t$ and so $B_t = \left\{B_t^{h^t}\left(z^t\right) \in h^t \in H^t\right\}$. Here $B_t^{h^t}\left(z^t\right)$ is the contract *intended* for type h^t. Since households can choose any one of these contracts, each contract B_t must satisfy self-selection constraints which require that no type has an incentive to choose a contract intended for a different type. In any period t, after new contracts are posted, define $\hat{V}_t\left(h^t, B_t^{\hat{h}^t}\left(z^t\right)\right)$ to be the value for type h^t of choosing a contract intended for type $\hat{h}^t$. Contracts must satisfy the following self-selection constraints: for all t, $h^t \in H^t$,

$$\hat{V}_t\left(h^t, B_t^{h^t}\left(z^t\right)\right) \geq \hat{V}_t\left(h^t, B_t^{\hat{h}^t}\left(z^t\right)\right) \text{ for all } \hat{h}^t \tag{4}$$

[7]The probabilites for histories h^t are constructed similarly.

[8]Recall that households with identical type and have the same contract history B^{t-1}.

Second, each contract must satisfy incentive compatibility constraints at each date and history. A contract B_t is incentive compatible if for all t, and histories $h^t \in H^t$,

$$V_{t+s}\left(h^{t+s}\right)\left(\sigma^*\right) \geq V_{t+s}\left(h^{t+s}\right)\left(\tilde{\sigma}\right) \text{ for all } \tilde{\sigma} \in \Sigma \tag{5}$$

where $V_t\left(h^t\right)(\sigma)$ denotes the value to type h^t of following reporting strategy $\sigma \in \Sigma$ as defined in (2). The incentive compatibility constraints are the restrictions that private information places on the set of feasible contracts. In particular, all contracts must have the feature that no household has an incentive to misreport its type in any period. For ease of notation, I will sometimes denote the equilibrium value for a household following the truth telling strategy by $V_t\left(h^t\right)$.

Third, any contract B_t must satisfy voluntary participation constraints at each date t, and for each history h^t. At the beginning of each period, a household can choose to not repay their debts and thereafter live in autarky forever where it just consumes its endowment each period and cannot sign with new intermediaries. Formally, the voluntary participation constraint is

$$V_{t+s}\left(h^{t+s}\right)\left(\sigma^*\right) \geq V^d_{t+s}\left(h^{t+s}\right) \tag{6}$$

where $V^d_{t+s}\left(h^{t+s}\right)$ is the value of autarky which by assumption depends only on θ_t. This constraint captures the restrictions limited commitment places on the contract. I assume that if a household chooses to not participate, it lives in autarky in all future periods,[9] i.e.

$$V^d_t\left(h^t\right) = u\left(\theta_t\right) + \frac{\beta}{1-\beta}\mathbb{E}u\left(\theta'\right)$$

Intermediaries can borrow and lend at market determined rate $\frac{1}{q_t}$. Given public histories, σ^{HH}, the strategies of future intermediaries and reservation utilities $\left\{\tilde{V}_t\left(h^t\right)\right\}$, each intermediary chooses σ^{INT}_t to maximize

$$-\sum_{s=0}^{\hat{T}-1}\left(\prod_{j=0}^{s} q_{t+j}\right)\sum_{h^{t+s}\in H^{t+s}} \tilde{\zeta}_t\left(h^{t+s}\right) {}_t\tau_{t+s}\left(h^{t+s}\right) \tag{7}$$

subject to (4), (5), (6) and ex-ante participation constraints

$$\hat{V}_t\left(h^t, B^{h^t}_t\left(z^t\right)\right) \geq \underline{V}_t\left(h^t\right) \tag{8}$$

Clearly, to attract households, contracts must satisfy the above participation constraints.

[9] This assumption can be relaxed and we can introduce an exogenous probability of re-entry each period after default.

Of course in equilibrium, $\underline{V}_t\left(h^t\right)$ is such that intermediaries make zero profits. I now formally define a Perfect Bayesian Equilibrium of the game.

Definition 1. A Perfect Bayesian Equilibrium is a sequence of prices $\{q_t\}_{t\geq 1}$, reservation utilities $\{\underline{V}_t\left(h^t\right)\}_{t\geq 1}$, strategies $\{\sigma_t^{HH}, \sigma_t^{INT}\}_{t\geq 1}$, and beliefs $\{\tilde{\zeta}_t\}_{t\geq 1}$ such that

1. For all t, z^t, h^t, the strategy σ_t^{HH}solves the households problem (2)

2. For all t, z^t, given prices, reservation utilities, σ^{HH} and beliefs $\tilde{\zeta}_t$, the strategy $\sigma_t^{INT}\left(z^t\right)$ solves the intermediaries' problem (7)

3. Beliefs satisfy Bayes' rule wherever it applies

4. Markets clear: for all $t \geq 1$,

$$\sum_{h^t \in H^t} \wp\left(\omega^t\right) c_t\left(\omega^t\right) = \sum_{h^t \in H^t} \wp\left(\omega^t\right) \theta_t$$

Note that in any equilibrium $\tilde{\zeta}_t\left(h^t\right) = \zeta\left(h^t\right)$, the true probability of history h^t. Also, when characterizing the equilibrium contract, it is without loss of generality to restrict to equilibria in which households sign with only one intermediary at a time. As a final point about the setup, note that the space of contracts is very general. Intermediaries can decide to offer short or long term contracts depending on the actions of other intermediaries. A particularly useful contract which will play a central part in thinking about deviations is an uncontingent savings contract. Given any $\varepsilon \geq 0$ and $\delta \leq 1$, a contract $S_t^{\varepsilon,\delta}$ is called a $\varepsilon\delta-$savings contract if $S_t = \left({}_t\tau_t \ , {}_t\tau_{t+1}\right)$ where

$$\begin{aligned} {}_t\tau_t &= -q_t\varepsilon \\ {}_t\tau_{t+1} &= \delta\varepsilon. \end{aligned}$$

Note that the $S_t^{\varepsilon,\delta}$ contract is not contingent on report types. In particular if offered any household can choose to sign it. In the future, I will sometimes refer to the above as the environment with PI (private information), LC (Limited Commitment) and HT (Hidden Trading).[10]

3 Equilibrium Contracts

In this section, I characterize the equilibrium contracts when intermediaries can only offer contracts that induce households to always repay their debts. One interpretation of this

[10]By hidden trading I mean that households can sign a new contract each period in a hidden fashion.

assumption is that failure by households to not pay back their debts and consequently not participate imposes a large exogenous cost on intermediaries. The main result in this section says that under the above restrictions, the set of equilibria of the intermediary game is identical to the set in an incomplete markets model where households trade a risk-free bond subject to appropriately chosen debt constraints. I now describe this equivalent environment. For one direction of the equivalence result, namely that any equilibrium of the intermediary game is an equilibrium of the incomplete markets environment, we need only consider a standard model with exogenous debt constraints. For the other direction, we need a way of endogenizing debt constraints and this will require introducing a notion of default into the incomplete markets framework.

There are a continuum of infinitely lived households, $i \in I$, who each receive an i.i.d endowment shock each period $\theta_t \in \Theta$. All households begin the period with an existing stock of debt and after knowing their endowment shock, they can choose to default and live in autarky forever or not in which case they pay their debts and can continue to trade a risk-free bond subject to debt constraints. If a household chooses not to default, it chooses an allocation $\{l_{t+s}, c_{t+s}\}_{s\geq 0}$ to maximize

$$\mathbb{E}_t \sum_{s=0}^{\infty} \beta^s \sum_{\theta^{t+s}\in\Theta^{t+s}} \pi\left(\theta^{t+s}\right) u\left(c_{t+s}\left(\theta^{t+s}\right)\right)$$

subject to budget constraints in each period

$$c_{t+s} + l_{t+s+1} \leq \theta_{t+s} + R_{t+s} l_{t+s} \tag{9}$$

and debt constraints

$$l_{t+s+1} \geq -\phi_{t+s} \tag{10}$$

with $\phi_{t+s} \geq 0$. Note all agents face identical debt constraints ϕ_t which can only depend on calendar time and not a household's type. Denote the value of this problem at date t given existing debt level l_t by $W_t\left(\theta^t, l_t; \Phi_t\right)$ where $\Phi_t = \{\phi_{t+s}\}_{\geq 0}$. For ease of notation I denote the entire sequence $\left\{c^i_{t+s}\left(\theta^{t+s}\right)\right\}_{s\geq 0, \theta^{t+s}\in\Theta^{t+s}}$ by $\left\{c^i_t\right\}_t$. The household's problem at the beginning of date t if it hasn't defaulted in the past is to choose a default strategy $d_t \in \{0, 1\}$ to maximize

$$d_t W_t\left(\theta^t, l_t; \Phi_t\right) + [1 - d_t] V^d_t\left(\theta_t\right)$$

where as before, $V^d_t\left(\theta_t\right) = u\left(\theta_t\right) + \frac{\beta}{1-\beta}\mathbb{E}u\left(\theta'\right)$. Next, I define an equilibrium concept that endogenizes the sequence of debt constraints. I consider symmetric equilibria where allocations can only depend on the history of endowment shocks and no the household's identity i. As a result, I will drop i from the allocations. The equilibrium definition is similar to the concept introduced by Alvarez and Jermann (2000).

Definition 2. A Not-Too-Tight competitive equilibrium is a sequence of interest rates $\{R_t\}_{t\geq 0}$, debt constraints $\{\phi_t\}_{t\geq 0}$, allocations for households $\{d_t, c_t, l_t\}_{t\geq 0}$ such that

1. Given prices and debt constraints, the allocations solve each household's problem

2. Markets clear, $\forall t$
$$\sum_{\theta^t \in \Theta^t} \pi\left(\theta^t\right) l_{t+1}\left(\theta^t\right) = 0$$

3. The sequence $\{\phi_t\}_{t\geq 0}$ is chosen to be Not-Too-Tight, i.e. $\forall t$,
$$W_{t+1}\left(\theta^{t+1}, -\phi_t; \Phi_{t+1}\right) \geq V^d_{t+1}\left(\theta_{t+1}\right) \text{ for all } \theta^{t+1}$$
$$W_{t+1}\left(\hat{\theta}^{t+1}, -\phi_t; \Phi_{t+1}\right) = V^d_{t+1}\left(\hat{\theta}_{t+1}\right) \text{ for some } \hat{\theta}^{t+1}$$

Debt constraints are "Not-Too-Tight" if the following property is true; in equilibrium, at each date and given any history θ^t, if this household has borrowed up to the constraint the previous period, it weakly prefers to not default while there exists some type $\hat{\theta}^t$ who is exactly indifferent. The idea is to allow households to hold the maximum amount of debt consistent with no default. The primary difference between the above definition and the one in Alvarez and Jermann (2000) is that unlike their environment, here debt constraints are not state contingent. In particular, in their model, agents trade Arrow securities subject to state contingent debt constraints, while here since markets are incomplete, we have constraints that are independent of states. This equilibrium concept has also been studied by Zhang (1997).

It is worth noting that the usual incomplete markets environment with exogenous debt constraints can also be defined using the model described above.

Definition 3. Given a sequence of debt constraints $\Phi = \{\phi_t\}_{t\geq 0}$, a $\Phi-$competitive equilibrium is a sequence of interest rates $\{R_t\}_{t\geq 0}$, allocations for households $\{d_t, c_t, l_t\}_{t\geq 0}$ such that

1. Given prices and debt constraints, the strategies and allocations solve each household's problem

2. Markets clear, $\forall t$
$$\sum_{\theta^t \in \Theta^t} \pi\left(\theta^t\right) l_t\left(\theta^t\right) = 0 \tag{11}$$

The first main result of the paper proves an equivalence between the equilibria of the intermediary game (environment with PI, LC and HT) defined in the previous section and the model with a risk-free bond and endogenous debt constraints.

Theorem 1 (*Equivalence.*).

1. *An equilibrium outcome of the environment with PI, LC and HT, is an equilibrium outcome of the environment with incomplete markets and not-too-tight debt constraints.*
2. *An equilibrium outcome of the environment with incomplete markets and not-too-tight debt constraints is an equilibrium outcome of the environment with PI, LC and HT.*

A brief sketch of the proof is as follows. Consider the incomplete markets environment. A sequence of outcomes $\{q, \phi, c, l\}$ is an equilibrium of the incomplete markets environment where $q_t = \frac{1}{R_{t+1}}$, iff

1. $u'\left(c_t\left(\theta^t\right)\right) q_t \geq \beta \mathbb{E}_t u'\left(c_{t+1}\left(\theta^{t+1}\right)\right)$ for all t, θ^t.
2. $u'\left(c_t\left(\theta^t\right)\right) q_t > \beta \mathbb{E}_t u'\left(c_{t+1}\left(\theta^{t+1}\right)\right) \Rightarrow l_{t+1}\left(\theta^t\right) = -\phi$
3. Given $\{q, \phi\}$, $\{c, l\}$ satisfy the household's budget and debt constraints (9) and (10)
4. Market clearing conditions (11) hold
5. The debt constraints $\{\phi\}$ are chosen to be not-too-tight, i.e.

$$W_{t+1}\left(\theta^{t+1}, -\phi_t; \Phi_{t+1}\right) \geq V_{t+1}^d\left(\theta_{t+1}\right) \text{ for all } \theta^{t+1}$$
$$W_{t+1}\left(\hat{\theta}^{t+1}, -\phi_t; \Phi_{t+1}\right) = V_{t+1}^d\left(\hat{\theta}_{t+1}\right) \text{ for some } \hat{\theta}^{t+1}$$

The proof requires a series of preliminary results. The three main propositions that required to prove Theorem 1 are Proposition 1, Proposition 2 and Proposition 3. The first of these propositions shows that an equilibrium of the intermediary game satisfies conditions 1. and 5. above. The second proposition shows that these outcomes satisfy 3 and the final proposition shows 2. To prove these results, I consider limits of truncated T period economies. In such an environment, households sign contracts with intermediaries until period T and after that trade a risk free bond subject to debt constraints as long as they have not defaulted in the past. The infinite horizon environment is the limit of such a truncated environment.

The first main proposition required to prove Theorem 1 says that in any equilibrium of the intermediary game, households can only be borrowing constrained and never savings constrained. Further, if a household is borrowing constrained in a period then the voluntary participation constraint binds for some type in the following period.

Proposition 1. *In any non-autarkic equilibrium of the intermediary game*

1. *For all types ω^t,*
$$u'\left(c_t\left(\omega^t\right)\right) q_t \geq \beta \mathbb{E}_t u'\left(c_{t+1}\left(\omega^{t+1}\right)\right)$$
2. *In any period if for any ω^t,*
$$u'\left(c_t\left(\omega^t\right)\right) q_t > \beta \mathbb{E}_t u'\left(c_{t+1}\left(\omega^{t+1}\right)\right)$$

then there exists some $\tilde{h}^{t+1}$ such that

$$V_{t+1}\left(\tilde{h}^{t+1}\right) = V^{d}_{t+1}\left(\tilde{h}^{t+1}\right)$$

Proof. See Appendix A.1. □

If a household is savings constrained, at the second stage of the period a new intermediary can offer an $\varepsilon\delta$ savings contract[11] which would make both it and the household strictly better off. Since intermediaries write contracts that respect voluntary participation constraints, they are unwilling to lend too much since zero profits requires imposing negative transfers in subsequent periods on the household which would worsen its incentives to default. The second part of the proposition shows that if a household is Euler-constrained, then it must be that the voluntary participation constraints bind for some type the following period. The reason for this is clear: if not then an intermediary can increase transfers to the constrained household in the current period and reduce them in the following period in a way so as to make strictly positive profits since the shadow rate of interest of a constrained agent is higher than the market rate.

The second main proposition required for the equivalence result says that we can represent any $\hat{T}$ period contract as a sequence of 2 period contracts, each one of which makes zero profits.

Proposition 2. *Given an equilibrium of a truncated T-period environment with $\hat{T}$ period lived overlapping intermediaries, there exists an equilibrium with 2 period lived intermediaries with same allocations and prices.*

Proof. See Appendix A.1. □

This proposition establishes that in equilibrium, intermediaries can only offer short term contracts. Given a $\hat{T}-$period contract, one can construct contracts for two period lived intermediaries as follows: set the first period transfers to be the same and in subsequent periods, we split transfers from the original contract $\{\zeta_t\}$ into $\zeta_t = -\frac{{}_{t-1}\tau_t}{q_t} + {}_{t}\tau_t$ where ${}_{t-1}\tau_t$ is the period t transfer from a contract signed in period $t-1$ and ${}_{t}\tau_t$ is the transfer from an intermediary born in period t. Since the original $\hat{T}$ period lived intermediaries must make zero profits, to show that these 2 period contracts also make zero profits it is sufficient to show that the expected present discounted value of transfers from $\hat{T}-1$ onwards, is independent of the period $\hat{T}-1$ report of endowment. In particular, given a history $\omega^{\hat{T}-2}$, I show that the present discounted value of transfers in $\hat{T}-1$, is independent of $\theta_{\hat{T}-1}$. To see why, suppose we have two types $\left(\omega^{\hat{T}-2}, \theta\right)$ and $\left(\omega^{\hat{T}-2}, \theta'\right)$

[11] Recall that a contract $S_t^{\varepsilon,\delta}$ is called a $\varepsilon\delta-$savings contract if $S_t = \left({}_{t}\tau_t\ ,\ {}_{t}\tau_{t+1}\right)$ where ${}_{t}\tau_t = -q_t\varepsilon$ and ${}_{t}\tau_{t+1} = \delta\varepsilon$.

with $\theta > \theta'$, but type $\left(\omega^{\hat{T}-2}, \theta'\right)$ receives the higher present discounted value of transfers. There are two cases to consider. The first is that the difference in transfers is front-loaded and that period $\hat{T}-1$ transfers are higher for type $\left(\theta^{\hat{T}-2}, \theta'\right)$. In this case, type $\left(\theta^{\hat{T}-2}, \theta\right)$ will strictly prefer to lie and pretend to be $\left(\theta^{\hat{T}-2}, \theta'\right)$, and save with another intermediary. As mentioned earlier, intermediaries are always willing to over $\varepsilon\delta$ savings contracts and one can be constructed to make both the lying household and a new intermediary strictly better off. The second case in which the difference in transfers is back-loaded and both types are Euler-constrained, is a little more complicated . I show that if a lower type weakly prefers the backloaded transfer scheme (which should be true in equilibrium) type $\left(\omega^{\hat{T}-2}, \theta\right)$ will again strictly prefer to lie and pretend to be $\left(\omega^{\hat{T}-2}, \theta'\right)$. On the other hand, if $\left(\omega^{\hat{T}-2}, \theta\right)$ receives the higher present discounted value of transfers, then a perturbation which redistributes to types below θ increases ex-ante welfare since it increases the amount of insurance in $\hat{T}-1$. With participation constraints, such a perturbation might not be feasible. However, I show (see Lemma 3) that the voluntary participation constraints only bind for the lowest types. An important property in a 2-period lived intermediary environment is that for all t, and histories ω^{t-1}, the present discounted values of equilibrium transfers is independent of θ_t.

The results so far suggest that the equilibria in the intermediary environment are equivalent to one in which agents trade a risk-free bond subject to debt constraints. In particular, any equilibrium with incomplete markets and borrowing constraints must satisfy the constrained Euler equation and the above conditions on the transfers. The next few results will help us prove some properties about the corresponding debt constraints. The third key proposition required to prove Theorem 1 shows that in any period, all Euler-constrained households have identical debt constraints.

Proposition 3. *For any t, and h^t such that*

$$u'\left(\theta_t + {}_{t-1}\tau_t\left(h^t\right) + {}_t\tau_t\left(\omega^t\right)\right) q_t > \beta \mathbb{E}_t u'\left(\theta + {}_t\tau_{t+1}\left(h^{t+1}\right) + {}_{t+1}\tau_{t+1}\left(\omega^{t+1}\right)\right)$$

it must be that ${}_t\tau_t\left(\omega^t\right) = \varphi_t$ where φ_t is independent of the household's history.

Proof. See Appendix A.1. □

The proof follows from a preliminary result which states that in equilibrium, the value of not defaulting for any two types h^t and $\tilde{h}^t$ such that $\theta_t + {}_{t-1}\tau_t = \tilde{\theta}_t + {}_{t-1}\tilde{\tau}_t$ are identical. Notice that here ${}_{t-1}\tau_t$ corresponds to the transfer in period t from a contract signed in period $t-1$. I prove this using an induction argument. Given that we are working in a truncated economy, consider the last period T in which intermediaries are operational. Since from period T onwards households trade a risk-free bond, the household's value going forward depends on only its current endowment and transfer. Next, suppose the hypothesis is true from period $t+1$ onwards and so we want to establish that it is true in period t. For contradiction, suppose we have two histories such that $\theta_t + {}_{t-1}\tau_t = \tilde{\theta}_t + {}_{t-1}\tilde{\tau}_t$ but $V_t\left(h^t\right) > V_t\left(\tilde{h}^t\right)$. The idea of the proof is to show that a deviating intermediary can give household $\tilde{h}^t$ a contract similar to type h^t, which makes both the household and it strictly better off while still satisfying incentives. The key condition that needs to be checked is that such a contract does not violate voluntary participation the following period. Notice that household $\tilde{h}^t s$ incentives to not participate in period $t+1$ are exactly the same as household h^t if they receive the same transfers since the value of the two households going forward is identical by the induction assumption.

The result states that in the environment with 2 period lived intermediaries, the transfers received from new contracts signed in period t are identical for households that are Euler-constrained in period t. At the first glance, the result may seem surprising since in general the present discounted value of transfers need not be identical across all histories. Suppose we have two households with different histories who are Euler-constrained in period t. Given that each contract must make zero profits, contracts offered in period t are of the form $\left(\varphi, -\frac{\varphi}{q_t}\right)$. Competition among intermediaries will force φ to be as high as possible consistent with no default the following period for each Euler-constrained household. Then the previous proposition tells us that all households receiving $\left(\varphi, -\frac{\varphi}{q_t}\right)$ will have exactly the same incentives to participate independent of history. As a result, such a contract will always satisfy voluntary participation constraints.

Using these characterization results, we can proceed to proof of the equivalence theorem (see Appendix A.1). The proof of the first part of the theorem is a direct consequence of the properties proved in the previous section. The necessary and sufficient conditions for an allocation-price pair to constitute a $\Phi-$competitive equilibrium are, for all t, θ^t

$$u'\left(c_t\left(\theta^t\right)\right) \geq \beta R_{t+1} \mathbb{E}_t u'\left(c_{t+1}\left(\theta^{t+1}\right)\right)$$

and

$$u'\left(c_t\left(\theta^t\right)\right) > \beta R_{t+1} \mathbb{E}_t u'\left(c_{t+1}\left(\theta^{t+1}\right)\right)$$
$$\Rightarrow l_{t+1} = -\phi_t$$

Finally, the budget constraint must hold at each date and state. The equilibrium interest rates are chosen so that $R_{t+1} = \frac{1}{q_t}$ where $\{q_t\}$ are the equilibrium prices from the intermediary game. The first two properties are satisfied in equilibrium of the intermediary game as described earlier. The second follows from the fact that in any equilibrium of the intermediary game, the equilibrium expected present discounted value of transfers,

$A_1(h_1) = 0$ and for all t, and histories ω^{t-1}, $A_t(\omega^{t-1}, \theta) = A_t(\omega^{t-1}, \theta')$ for all $\theta, \theta' \in \Theta$ where

$$A_t\left(\omega^{t-1}, \theta\right) \equiv \tau_t\left(\omega^{t-1}, \theta_t\right) + q_t \sum_{\theta' \in \Theta} \pi(\theta') A_{t+1}(\omega^t, \theta')$$

For the converse, we need to show that if all intermediaries are offering $\Phi-$contracts[12], no existing or new intermediary has an incentive to deviate and offer contracts that make positive profits. First consider the case of a new intermediary. The only type of deviating contract we need to consider is one in which an Euler-constrained household at some date receives an increased transfer. Incentive compatibility requires that the contract make a negative transfer in the following period. However, any Not-too-tight competitive equilibrium has the property that some type's voluntary participation constraint binds the following period and therefore this negative transfer cannot be uncontingent. I show that such a deviating contract is never incentive compatible in any period since households are not constrained to reporting the same type to different intermediaries. In particular, they can always report the type that results in the highest transfer to the new intermediary while reporting their true type to the original intermediary. Finally we need to consider the incentives for an existing intermediary to modify its contract. As in the case with the new intermediary, the relevant deviations involve increasing transfers to Euler-constrained households at some t, and reducing transfers the following period. Since some type's voluntary participation constraint binds in $t+1$, the negative transfer must be state-contingent. Consider imposing the negative transfer on those households that are Euler constrained in $t+1$. Since the lowest type falls into this category, clearly this is not possible since its voluntary participation constraint is binding. On the other hand, if the negative transfers are imposed on those households that are not Euler-constrained, these agents will strictly prefer to lie and pretend to be a lower type. Therefore such contracts are not incentive compatible.

It is worth noting that all three frictions, i.e., private information, limited commitment and hidden trading are *necessary* to obtain the above characterization. Environments with only private information, for example Atkeson and Lucas (1992) or private information and limited commitment as in Dovis (2014) cannot be decentralized with only a short term uncontingent bond. In particular, it is not true in such environments that the present discounted value of transfers is independent of current type. Environments with private information and hidden trading imply contracts that resemble trade in a risk-free bond as was shown by Allen (1985). Cole and Kocherlakota (2001) also prove a similar result in an environment with hidden savings. However in both these environments, no agent is Euler-constrained in equilibrium and as a result the efficient allocation cannot be

[12] Simple borrowing and lending contract subject to debt constraints

decentralized as an environment with a risk-free bond and binding (endogenous) debt constraints. In particular, the efficient allocation in models with private information and hidden savings will not in general satisfy voluntary participation constraints introduced in the previous sections. Moreover, the hidden savings rate in these papers is *exogenous* in contrast to this environment where it is determined in equilibrium. This will be key when thinking about efficiency and policy in the later sections.

3.1 Equilibrium Existence and Multiplicity

Next, I study the existence of equilibria in the intermediary game. Given the equivalence result, it suffices to prove the existence of a Not-too-tight competitive equilibrium. To show existence, I focus on stationary recursive competitive equilibria and show that these are well defined and exist. The main theorem in this subsection is that are multiple competitive equilibria.

We can write the problem of a household recursively as follows:

$$W(\theta, l, \phi; \Phi) = \max_{c, l'} u(c) + \beta \mathbb{E} W(l', \phi'; \Phi')$$

$$\text{subject to}$$

$$c + l' \leq \theta + Rl$$

$$l' \geq -\phi$$

where θ is the household's current endowment, l its assets and ϕ, the current debt constraint which is determined by the rule $\phi' = \Phi(\phi)$ where Φ is known to all households. In this case the value of default is given by

$$V^d(\theta) = u(\theta) + \mathbb{E} V^d(\theta')$$

As earlier we can define the notion of a $\Phi-$Recursive competitive equilibrium and finally a Not-Too-Tight RCE. Let $\mathbb{A}$ be the bounded space of assets and $\mathcal{P}(\mathbb{A})$ the set of probability measures on $\mathbb{A}$.

Definition 4. A $\Phi-$Recursive Competitive Equilibrium is price function $R(\phi)$, a law of motion $\phi' = \Phi(\phi)$, a measurable map $G : R_+ \times \mathcal{P}(\mathbb{A})$,value functions $W(\theta, b, \phi; \Phi)$, policy functions $l'(\theta, l, \phi)$ such that

1. Given R and Φ, the value functions and policy functions solve the households' problems and

2. The sequence of distributions generated by G is such that markets clear

$$\int_{\mathbb{A}\times\Theta} l'(\theta, l, \phi)\, d\lambda(l, \Theta) = 0$$

where

$$\lambda' = G(\phi, \lambda)$$

Definition 5. A NTT−Recursive Competitive Equilibrium is price function $R(\phi)$, a law of motion $\phi' = \Phi(\phi)$, a measurable map $G : R_+ \times \mathcal{P}(\mathbb{A})$,value functions $W(\theta, l, \phi; \Phi)$, policy functions $l'(\theta, l, \phi)$ such that

1. Given R and Φ, the value functions and policy functions solve the household's problem

2. The sequence of distributions generated by G is such that markets clear

$$\int_{\mathbb{A}\times\Theta} l'(\theta, l, \phi)\, d\lambda(l, \Theta) = 0$$

where

$$\lambda' = G(\phi, \lambda)$$

3. If $\phi' \in \Phi(\phi)$ then

$$\begin{aligned} W\left(\theta, -\phi', \phi'; \Phi\right) &\geq V^d(\theta) \text{ for all } \theta \in \Theta \\ W\left(\theta^*, -\phi', \phi'; \Phi\right) &= V^d(\theta^*) \text{ for some } \theta^* \in \Theta \end{aligned}$$

Define $\eta \equiv \sum_{\theta\in\Theta} \pi(\theta)\, u'(\theta)$ and let

$$\kappa = \min_{\theta} \frac{u'(\theta) + \beta\eta}{u'(\theta) + \beta\eta + \beta^2\eta}$$

Theorem 2 (Existence.)**.** *Under the following sufficient condition*

$$\frac{u'(\bar{\theta})}{\beta\eta} < \kappa$$

there exist multiple NTT−Recursive Competitive Equilibria.

Proof. See Appendix A.1.1. □

The first step in the proof is to show that given a measurable map Φ, a $\Phi-$RCE always exists. Next, it always true that a $\Phi-$RCE with Φ being the zero map is NTT-RCE.

The reason for this is clear. If debt constraints are zero each period, then in equilibrium, households consume their endowment which trivially implies that the voluntary participation constraint binds for each period and each type. The final and key proposition that completes the proof of Theorem 2 is to show that there exists a NTT-RCE with $\Phi \neq 0$. The idea is to show that for each θ, there exists Φ^{θ}, such that debt constraints are ϕ^{θ} each period and

$$W\left(\theta, -\phi^{\theta}, \phi^{\theta}; \Phi^{\theta}\right) = V^{d}\left(\theta\right)$$

Then setting $\phi = \min_{\theta} \phi^{\theta}$ gives us a $\Phi-$RCE with debt constraints that are not too tight.

The above result along with Theorem 1 shows that the intermediary game has multiple equilibria. There exists an equilibrium of the decentralized contracting environment in which all intermediaries offer null contracts (no insurance) to households. A simple way of understanding this result is to notice a *strategic complementarity* in the actions of intermediaries. In particular, if an intermediary believes that no future intermediary is willing to lend to households, it will be unwilling to lend since the household will choose to default in subsequent periods.

On the surface this might seem a surprising result since one would expect a intermediary to always be able to construct a deviating contract that offers some insurance and hence make positive profits. To see why this is not possible, consider a $\hat{T}$ lived intermediary born at date $t+1$. In the last period of the contract, $\hat{T}$ it must be that ${}_{t+1}\tau_{\hat{T}}\left(h^{\hat{T}}\right) \geq 0$ since no intermediary in the future is offering any insurance. If ${}_{t+1}\tau_{\hat{T}}\left(h^{\hat{T}}\right) < 0$ for any $h^{\hat{T}}$ that household will strictly prefer to default. Now consider $\hat{T}-1$. For any $h^{\hat{T}-1}$ it must be that ${}_{t+1}\tau_{\hat{T}-1}\left(h^{\hat{T}-1}\right) \leq 0$ since if it is strictly positive then in order to preserve incentive compatibility and make positive profits the intermediary will have to set transfers negative for some type in $\hat{T}$. Therefore the only feasible perturbation in $\hat{T}-1$ must be ${}_{t+1}\tau_{\hat{T}-1}\left(h^{\hat{T}-1}\right) < 0$ and ${}_{t+1}\tau_{\hat{T}}\left(h^{\hat{T}-1}\right) > 0$. Note again that if ${}_{t+1}\tau_{\hat{T}}\left(h^{\hat{T}-1}\right)$ depended on $\theta_{\hat{T}}$ incentive compatibility would be violated. The perturbation resembles a savings contract. However if the interest rates are such that $R_{\hat{T}} \leq \frac{u'\left(\bar{\theta}\right)}{\beta \mathbb{E} u'(\theta)}$ such a contract would have to offer a return on savings $> R_{\hat{T}}$ which would mean that the intermediary makes negative profits. For any $R \leq R_{\hat{T}}$ the household prefers the transfer schedule ${}_{t+1}\tau_{\hat{T}-1}\left(h^{\hat{T}-1}\right) = 0$, ${}_{t+1}\tau_{\hat{T}}\left(h^{\hat{T}-1}\right) = 0$ to the one offered by the deviating contract. Therefore in $\hat{T}-1$ it must be that ${}_{t+1}\tau_{\hat{T}-1}\left(h^{\hat{T}-1}\right) \geq 0$. A similar argument works in $\hat{T}-2$ and hence for previous periods.

4 Efficiency

The first step in asking whether the equilibria characterized in the previous sections are efficient is to define the right notion of constrained-efficiency. In environments with private information and limited commitment this is well understood and has been studied by Prescott and Townsend (1984) and Kehoe and Levine (1993). However, the definition of constrained-efficiency is less clear in environments with non-exclusive contracts in which interest rates are endogenously determined.

To begin, I consider a setup with a fictitious social planner and continuum of infinitely lived households who receive an unobservable perishable endowment each period. An allocation for the planner consists of a sequence $\left\{c_t\left(\theta^t\right), \tau_t\left(\theta^t\right)\right\}_{t\geq 0, \theta^t\in\Theta^t}$ which correspond to the consumption and transfer sequences to households in the mechanism. An allocation is *incentive-feasible* if it satisfies the following conditions. First, it must be resource feasible; for each t,

$$\sum_{\theta^t\in\Theta^t}\pi\left(\theta^t\right)c_t\left(\theta^t\right)=\sum_{\theta^t\in\Theta^t}\pi\left(\theta^t\right)\theta_t \tag{12}$$

Next, the contract must satisfy voluntary participation constraints: for all t and $\theta\in\Theta^t$,

$$V_t\left(\theta^t\right)\geq V_t^d\left(\theta^t\right) \tag{13}$$

As in the decentralized environment, I assume that at the beginning of each date, each household can voluntarily default on the planner and consequently live in autarky forever. In autarky, the household consumes its endowment each period. Next, the allocation must be incentive compatible

$$V_t\left(\theta^t\right)\left(\sigma^*\right)\geq \hat{V}_t\left(\theta^t,\{\tau\},\{q\}\right)(\sigma), \tag{14}$$

Here $V_t\left(\theta^t\right)\left(\sigma^*\right)$ denotes the value of the contract to type θ^t of following truth-telling strategy σ^*. $\hat{V}_t\left(\theta^t,\{\tau\},\{q\}\right)(\sigma)$ denotes the value to the household of using reporting strategy σ and trading in a *hidden market*. I consider a hidden market in which households can sign hidden contracts with T period lived intermediaries. The equilibrium of the hidden market is identical to that characterized in the previous section. As a result we can restrict ourselves to a hidden market in which households can trade a risk free bond subject to *endogenous* debt constraints. Therefore,

$$\hat{V}_t\left(\theta^t;\{\tau\},\{q\}\right)(\sigma)=\max\sum_{s=0}^{\infty}\beta^s\sum_{\theta^{t+s}\in\Theta^{t+s}}\pi\left(\theta^t\right)u\left(x_{t+s}\left(\theta^{t+s}\right)\right) \tag{15}$$

$$\begin{aligned}&\text{subject to for all } s\geq 0,\ h^{t+s}\\ &x_{t+s}\left(\theta^{t+s}\right)+q_{t+s}l_{t+s+1}\left(\theta^{t+s}\right)\geq\theta_{t+s}+\tau_{t+s}\left(\sigma_{t+s}\left(\theta^{t+s}\right)\right)+l_{t+s}\left(\theta^{t+s-1}\right)\\ &l_{t+s+1}\left(\theta^{t+s}\right)\geq-\phi_{t+s}\end{aligned}$$

Here x_t and l_t denote final consumption and bond holdings in the hidden market, $\tau_{t+s}\left(\sigma_{t+s}\left(\theta^{t+s}\right)\right)$ denotes the transfer from the planner when type θ^{t+s} reports $\sigma_{t+s}\left(\theta^{t+s}\right)$ and ϕ_{t+s}, the (endogenous) debt constraints. We can rewrite $\hat{V}_t\left(\theta^t;\{\tau\},\{q\}\right)$ as

$$\begin{aligned}J_t\left(\theta^t,l_t;\{\tau\},\{q\},\Phi_t\right)=\max u\left(x_t\right)+\beta\mathbb{E}_tJ_t\left(\theta^{t+1},l_{t+1};\{\tau\},\{q\},\Phi_t\right)\\ \text{subject to}\\ x_t+q_tl_{t+1}\geq\theta_t+\tau_t\left(\theta^t\right)+l_t\\ l_{t+1}\geq-\phi_t\end{aligned}$$

where Φ_t denotes the sequence of current and future debt constraints which each household takes as given.

Definition 6. An equilibrium in the hidden market given a transfer sequence $\{\tau\}$ consists of prices $\{q_t\}$, allocations $\{x_t, s_t\}$ and debt constraints $\{\phi_t\}$ such that

1. Households solve their problem defined above,
2. Markets clear: for all t,

$$\sum_{h^t\in H^t}\pi\left(\theta^t\right)x_t\left(\theta^t\right)=\sum_{h^t\in H^t}\pi\left(\theta^t\right)\left[\theta_t+\tau_t\left(\theta^t\right)\right]$$

3. Debt constraints are chosen to be Not-Too-Tight, i.e.

$$\begin{aligned}J_t\left(\theta^t,-\phi_t;\{\tau\},\{q\},\Phi\right)\geq V_t^d\left(\theta^t\right)\ \text{for all }\theta^t\\ J_t\left(\hat{\theta}^t,-\phi_t;\{\tau\},\{q\},\Phi\right)=V_t^d\left(\hat{\theta}^t\right)\ \text{for some }\hat{\theta}^t\end{aligned}$$

The definition of the hidden market is similar in spirit to Golosov and Tsyvinski (2007)(GT07). In their model, agents trade a risk free bond with the interest rate determined in equilibrium. Here, households trade these bonds subject to debt constraints which along with the interest rates are also determined in equilibrium. I assume that households can also default on their hidden debt obligations. As in the intermediary game, default in the hidden markets is publicly observable and consequently households live in autarky in all future periods. Debt constraints are chosen in equilibrium so that all households weakly prefer not to default on their debt if they have borrowed up to the debt limit the previous period while some household is indifferent between the two options. It is clear that in any constrained-efficient allocation, there will be no trade in these markets. In particular, the efficient allocation will have the property that for any Euler-constrained household, borrowing more in the hidden market will incentivize default the next period. Moreover the price q_t will be such that no household will wish to save in these markets and as a result we a have a well defined equilibrium with no hidden trades.

The main result in this subsection is that the efficient allocation can be decentralized as an equilibrium of the intermediary game.

Theorem 3 (*Efficiency.*)**.** *The constrained efficient allocation can be implemented as an equilibrium of the intermediary game.*

To prove this is result, I first prove properties that any efficient allocation must satisfy. In particular, I show that the planner cannot do better than simple borrowing and lending contracts. Then, I show that if all intermediaries are offering the efficient contract, no incumbent or new intermediary has any incentive to offer a deviating contract. As in the intermediary game I consider limits of $T-$period truncated environments in which from period 1 to T, the planner provides transfers and after T those households that have not defaulted can trade a risk-free bond subject to exogenous debt constraints.

Proposition 4. *Any $T-$period truncated incentive feasible allocation must satisfy*

$$q_t \geq \beta \frac{\mathbb{E}_{t+1} u'\left(c_{t+1}\left(\theta^{t+1}\right)\right)}{u'\left(c_t\left(\theta^t\right)\right)} \text{ for all } t, \theta^t \in \Theta^t$$

and

$$\sum_{t=1}^{T}\left(\prod_{s=1}^{t} q_s\right) \tau_t\left(\theta^T\right) = 0 \text{ for all } \theta^T \in \Theta^T \tag{16}$$

Proof. See Appendix A.2.1. □

Notice that the proposition says that the efficient contract is also a simple borrowing and lending contract subject to debt constraints. In particular, the presence of the hidden markets prevents the planner from introducing state-contingency in contracts. The intuition for this is exactly the same as in the intermediary game. If lower types receive a larger present discounted value of transfers, then higher types will lie and use the hidden markets to save. On the other hand if higher types receive a larger present discounted value of transfers then redistribution is welfare increasing. Given that voluntary participation constraints induce some agents to be Euler-constrained in the efficient allocation, the planner will allow agents to borrow the largest amount consistent with no default in the subsequent period. As a result the voluntary participation constraints will be binding for some type in the following period. These two conditions imply that as in the intermediary game, the efficient contract are simple uncontingent borrowing and lending contracts. The next result provides necessary and sufficient conditions for an allocation to induce an equilibrium of the hidden market with no trades.

Lemma 1. *An allocation induces no trades in the hidden market if and only if for all $t, \theta^t \in \Theta^t$*

$$q_t \geq \beta \frac{\mathbb{E}_{t+1} u'\left(c_{t+1}\left(\theta^{t+1}\right)\right)}{u'\left(c_t\left(\theta^t\right)\right)} \text{ for all } t, \theta^t \in \Theta^t \tag{17}$$

and

$$\left[q_t - \frac{\beta \mathbb{E}_t u'\left(c_{t+1}\left(\theta^{t+1}\right)\right)}{u'\left(c_t\left(\theta^t\right)\right)}\right] \min_{\tilde{\theta}^{t+1} \in \tilde{\Theta}^{t+1}} \left[V_{t+1}\left(\tilde{\theta}^{t+1}\right) - V^d_{t+1}\left(\tilde{\theta}^{t+1}\right)\right] = 0 \tag{18}$$

Proof. See Appendix A.2.1. □

The second condition says that if a household is Euler-constrained in period t, then it must be that in the following period, the voluntary participation for some type binds. The reason for this is that if not, then debt constraints in the hidden market will not satisfy the Not-too-tight property. In other words intermediaries will be willing to lend more to households without fearing default in the subsequent periods. Next, as in Golosov and Tsyvinski (2007) we can re-write the planner's problem as one in which the planner also chooses the prices in the hidden markets subject to additional conditions. An allocation in this case is a sequence of transfers $\left\{\tau_t\left(\theta^t\right)\right\}_{t\geq 0, h^t \in H^t}$ and prices $\{q_t\}_{t\geq 0}$.

Lemma 2. *The constrained efficient allocation $\left\{c_t\left(h^t\right), \tau_t\left(h^t\right)\right\}_{t\geq 0, h^t \in H^t}$ and prices $\{q_t\}_{t\geq 0}$ is a solution to the following programming problem*

$$\max_{\{c,\tau,q\}} \sum_{t=1}^{T} \beta^{t-1} \sum_{\theta^t \in \Theta^t} \pi\left(\theta^t\right) u\left(c_t\left(\theta^t\right)\right)$$

subject to (12), (16), (13), (17), and (18).

To prove Theorem 3, I show that if all intermediaries are offering the efficient contract, no individual intermediary has an incentive to deviate and offer a different contract. In particular, it will not be able to offer some Euler-constrained individuals the option to borrow more since they will default the following period. This establishes that the efficient allocation can be decentralized as an equilibrium of the intermediary game. Note that even though as in Golosov and Tsyvinski (2007), the planner controls the price in the hidden market, he is unable to achieve outcomes better than the best competitive equilibrium. The planner chooses q_t consistent with best competitive equilibrium from the set of equilibria which we know is not a singleton. The reason for this is that incentive compatibility dictates that in any incentive feasible allocation no state contingency is possible. As a result, the best the planner can do is to choose the allocation that corresponds to loosest borrowing constraints which in turn corresponds to the best competitive equilibrium. Unlike GT07, in this model output is not publicly observable. Therefore, the planner cannot use incentives to work to provide state-contingency in contracts as in their paper.

It is widely believed that the presence of prices in constraints such as incentive compatibility leads to the failure of the Welfare theorems in these models. While this is certainly true in GT07, it is not true in the environment I consider. A natural question that arises is what exactly leads to the failure of the First Welfare theorem in GT07. While studying the environment with PI, LC and HT in a Mirrleesian environment with capital is outside the scope of this paper, I provide some intuition within the context of this environment. The key difference between the two papers, is how the hidden market is modeled. In GT, there are restrictions on the types of trades that can be undertaken in the hidden market while this is not true in the environment I study. In particular, agents in GT07 are restricted to trading a risk-free bond in the hidden market while the planner can in general provide a lot more insurance than that associated with a risk-free bond. In contrast, households in the hidden market studied in this paper can sign *any* contract that respects the underlying frictions, namely PI, LC and HT. To illustrate this point I consider a minor modification to the above planning problem in which in the hidden market, households are only allowed to borrow and lend subject to an exogenously specified borrowing constraint. The prices are still endogenously determined to clear markets. Note that it is now no longer true that the equilibrium of this hidden market coincides with that of the intermediary game. The incentive compatibility constraint is still (14) where $\hat{V}_t\left(\theta^t;\{\tau\},\{q\}\right)$ is defined as in (15) except that $\phi_{t+s}=\phi$ is exogenously specified.

Suppose that $\Theta=\{\theta_H,\theta_L\}$ and consider the truncated economy with $T=2$. Since T is finite, I assume that there is an exogenous cost of defaulting.

Proposition 5. *Suppose that ϕ is small. Then if the coefficient of absolute risk aversion is sufficiently large, and θ_H and θ_L are sufficiently far apart, the efficient allocation cannot be implemented as an equilibrium of the intermediary game.*

Proof. We know that in any equilibrium of the intermediary game in which type θ_L is Euler-constrained (i.e. Euler equation does not hold with equality) in period 1, it must be that the voluntary participation constraint is binding in period 2 for some type (θ_L, θ_2). We will show that for ϕ small enough, given an allocation from the intermediary game, the planner can improve upon it by introducing slack in the participation constraint of type (θ_L, θ_2). Consider the transfer sequence τ associated with an equilibrium allocation of the intermediary game. We can define (for some given θ_2)

$$\begin{aligned}\mathbb{T}_1(\theta_H, \theta_2) =& \tau_1(\theta_H) + q_1 \tau_2(\theta_H, \theta_2) \\ \mathbb{T}_1(\theta_L, \theta_2) =& \tau_1(\theta_L) + q_1 \tau_2(\theta_H, \theta_2)\end{aligned}$$

An incentive feasible allocation must satisfy $\mathbb{T}_1(\theta_H, \theta_2) = \mathbb{T}_1(\theta_L, \theta_2) = 0$. Consider a perturbation, $\mathbb{T}_1^{\varepsilon}(\varepsilon, \theta_H, \theta_2)$ and $\mathbb{T}_1^{\varepsilon}(\varepsilon, \theta_L, \theta_2)$ where

$$\begin{aligned}\mathbb{T}_1^{\varepsilon}(\varepsilon, \theta_H, \theta_2) &= \tau_1(\theta_H) + q_1 \varepsilon + q_1(\varepsilon)[\tau_2(\theta_H, \theta_2) - \varepsilon] \\ \mathbb{T}_1^{\varepsilon}(\varepsilon, \theta_L, \theta_2) &= \tau_1(\theta_L) - q_1 \varepsilon + q_1(\varepsilon)[\tau_2(\theta_L, \theta_2) + \varepsilon]\end{aligned}$$

where the planner internalizes the effect of the perturbation on price in the hidden market. In particular, recall that

$$q_1 = \frac{\beta \mathbb{E}_1 u'(c_2(\theta_H, \theta'))}{u'(c_1(\theta_H))}$$

Notice that at the original price q_1, such a perturbation will leave $\mathbb{T}_1(\theta_H, \theta_2)$ unchanged. We can use the above equation to compute how the price changes for a small perturbation:

$$q_1'(\varepsilon) u'(c_1) + q_1 u''(c_1) \left[q_1'(\varepsilon) \tau_1 + q_1'(\varepsilon) \varepsilon + q_1(\varepsilon) \right] = -\beta \mathbb{E}_1 u''(c_2) \tag{19}$$

which implies that for $\varepsilon = 0$

$$q_1'(0) = \frac{-\beta \mathbb{E} u_1''(c_2) - q_1^2 u''(c_1)}{[u'(c_1) + q_1 u''(c_1) \tau_1]}$$

Since $\tau_1(\theta_H) < 0$, $q_1'(0) > 0$. Notice that

$$\frac{\partial \mathbb{T}_1^{\varepsilon}(\varepsilon, \theta_H, \theta_2)}{\partial \varepsilon} = q_1'(\varepsilon) \tau_2(\theta_H, \theta_2)$$

Given that $\theta_H > \theta_L$, we know that $\tau_1(\theta_H) < 0 < \tau_1(\theta_L)$. Therefore, $\frac{\partial \mathbb{T}_1^{\varepsilon}(\varepsilon, \theta_H, \theta_2)}{\partial \varepsilon} > 0$ and $\frac{\partial \mathbb{T}_1^{\varepsilon}(\varepsilon, \theta_L, \theta_2)}{\partial \varepsilon} < 0$. As a result, this perturbation introduces a slack in the incentive constraint. And so the planner can increase the transfer to θ_L by $2q_1(\varepsilon) \tau_2(\theta_H, \theta_2)$ since $\tau_2(\theta_H, \theta_2) = -\tau_2(\theta_L, \theta_2)$ from market clearing. In particular the planner increases both the period 1 and 2 transfer by $q_1(\varepsilon) \tau_2(\theta_H, \theta_2)$. Since $\varepsilon > 0$, the transfer in period 2 to the period 1 low type is strictly larger. To show that type θ_L is made strictly better off by this perturbation it suffices to show that

$$-\varepsilon + q_1(\varepsilon)\,\tau_2(\theta_H, \theta_2) > 0$$

or for εsmall, $q_1'(0)\,\tau_2(\theta_H, \theta_2) > 1$. From (19) we have,

$$q_1'(0)\,\tau_1 = \frac{-q_1^2 u''(c_1) - \beta \mathbb{E}_1 u''(c_2) - q_1'(0)\,u'(c_1)}{q_1 u''(c_1)}$$

Using the fact that $\tau_1(\theta_H) = -q_1 \tau_2(\theta_H, \theta_2)$ we obtain,

$$\begin{aligned} q_1'(0)\,\tau_2 &= \frac{-q_1^2 u''(c_1) - \beta \mathbb{E}_1 u''(c_2) - q_1'(0)\,u'(c_1)}{-q_1^2 u''(c_1)} \\ &= \frac{-q_1^2 u''(c_1) - \beta \mathbb{E}_1 u''(c_2)}{-q_1^2 u''(c_1)} - \frac{q'(0)\,u'(c_1)}{-q_1^2 u''(c_1)} \\ &= 1 + \frac{\beta \mathbb{E}_1 u''(c_2)}{q_1^2 u''(c_1)} - \frac{q'(0)\,u'(c_1)}{-q_1^2 u''(c_1)} \end{aligned}$$

Consider the third term on the RHS of the above equation

$$\frac{q'(0)\,u'(c_1)}{-q_1^2 u''(c_1)} = \frac{\frac{\beta}{q_1^2}\frac{\mathbb{E}_1 u''(c_2)}{u''(c_1)} + 1}{\left[1 + q_1 \frac{u''(c_1)}{u'(c_1)} \tau_1\right]}$$

Therefore,

$$\frac{\beta \mathbb{E}_1 u''(c_2)}{q_1^2 u''(c_1)} - \frac{q'(0)\,u'(c_1)}{-q_1^2 u''(c_1)} = \frac{\beta \mathbb{E}_1 u''(c_2)}{q_1^2 u''(c_1)} - \frac{\frac{\beta}{q_1^2}\frac{\mathbb{E}_1 u''(c_2)}{u''(c_1)} + 1}{\left[1 + q_1 \frac{u''(c_1)}{u'(c_1)} \tau_1\right]}$$

Notice that if the coefficient of absolute risk-aversion $-\frac{u''(c_1)}{u'(c_1)}$ is large and θ_H and θ_L are sufficiently far apart (which implies that τ_1 is large), then the the above is ≥ 0. Therefore, $q_1'(0)\,\tau_2(\theta_H, \theta_2) > 1$. Finally notice that this perturbation makes the voluntary participation constraints in period 1 slack for type (θ_L, θ_2). However, if ϕ_2 is small enough, the low type cannot use the hidden market to borrow any more. Finally note that while the high type is made worse off, the planner is made strictly better off by this perturbation since the low type has strictly larger marginal utility. □

The intuition behind this result is that efficient allocation may be feature non-binding voluntary participation constraints even though households are borrowing constrained.

Introducing slack in the participation constraint allows the planner to transfer resources from type (θ_L, θ_2) to (θ_H, θ_2) which in turn allows it to construct a transfer scheme which lowers the hidden market price q_1. Since the present discounted value of transfers across the two types must be identical, lowering q_1 introduces a wedge between the original transfer schemes. Under some sufficient conditions, this allows the planner to transfer more to the low type in period 1. Since the low type is borrowing constrained, overall utility is higher. Notice that what allows such a perturbation to be feasible with exogenous ϕ is that even though the voluntary participation constraint is slack, the low type cannot use the hidden market to borrow if ϕ is small enough. As a result the efficient allocation cannot be implemented as an equilibrium of the intermediary game or the Huggett economy with Not-too-tight debt constraints because they require voluntary participation constraints to be binding. In the next section, I provide an example of this to illustrate the differences between efficiency with exogenous and endogenously incomplete markets.

It is also illustrative to consider the environment with commitment on the part of households. This is similar to the Golosov and Tsyvinski (2007) environment without capital and private information on endowments rather than productivity. Here, households can always commit to repay their debts. Moreover, in the hidden market, households can trade a risk-free bond with non-binding borrowing constraints. The price is determined to clear the hidden market. Similar to the environment studied in this paper, there are no exogenous restrictions placed on trades in the hidden market. In particular the equilibrium outcome of the intermediary game with commitment is equivalent to an environment in which households can trade a risk-free uncontingent security with non-binding borrowing constraints. It is easy to see that in this case that the equilibrium of the intermediary game is constrained-efficient. Consider the transfer sequence associated with equilibrium allocation. For our two type, two period example studied above, the transfer sequence associated with an equilibrium allocation must again satisfy $\mathbb{T}_1(\theta_H, \theta_2) = \mathbb{T}_1(\theta_L, \theta_2) = 0$ for all θ_2. However, unlike the previous case, any attempt by the planner to introduce a wedge between $\mathbb{T}_1(\theta_H, \theta_2)$ and $\mathbb{T}_1(\theta_L, \theta_2)$ will incentive trades in the hidden market. Moreover, since households have full commitment, lenders will be willing to lend up until the natural borrowing constraint which is non-binding. As a result, the transfer sequence is efficient.

4.1 Efficiency with Exogenous Incompleteness

A general result when markets are exogenously incomplete is that equilibrium outcomes are constrained inefficient. This literature considers a planner who is restricted from making state-contingent transfers to agents but internalizes the effect of its allocations on prices. Geanakoplos and Polemarchakis (1986) find that equilibrium outcomes are generically inefficient in an exchange economy with multiple goods. In particular, they find that aggregate welfare can be increased if households are induced to save different amounts. More recently, Dávila et al. (2012) find that the equilibria in the model studied by Aiyagari (1994) are also constrained inefficient. Consumers do not internalize the effects of their choices on factor prices which in a model with uninsurable risk implies that there can be over-saving or under-saving relative to the constrained efficient equilibrium.

While the environment I consider is observationally equivalent to a large class of exogenously incomplete models, the approach to efficiency I take is substantially different. Rather than exogenously restrict the set of instruments available to the planner, I derive the incompleteness as a consequence of informational and commitment frictions. In particular, the planner can offer any allocation subject to these underlying frictions. In this section, I explore whether for two observationally equivalent models, the two notions of efficiency have different implications for whether the competitive equilibria are efficient. I consider two types of externalities that arise in models with exogenous incompletess which have been studied in the literature: *pecuniary* externalities and *aggregate demand* externalities. As I show using simple examples, it is possible that outcomes that are considered inefficient when markets are exogenously incomplete are no longer so when they are endogenously incomplete. Both types of inefficiency results have been used to motivate the use of macro-prudential policies in limiting the amount of debt in the economy.

4.1.1 Pecuniary Externalities

Consider a simple two period environment with $t = 1, 2$ and a continuum of households. In period 1, households can receive endowment shocks $\theta_i \in \Theta = (\theta_h, \theta_l)$ with probability π_i, $i \in \{h, l\}$. In period 2, households receive endowment shocks $x_i \in \mathbb{X} = (x_h, x_l)$ with probability κ_j, $j \in \{h, l\}$. The shocks are i.i.d over time and across households. As in previous sections, there are a large number of intermediaries who sign 2 period contracts with households. The timing of the game is follows:

1. Households can sign a contract with a single intermediary before period 1 types are realized
2. In period 1, after types are realized, households receive transfers from original the intermediary
3. Next, households can sign a contract with another intermediary. This contract is unobservable to the original intermediary and vice-versa.
4. At the beginning period 2, households can default on their obligations to the intermediary and receive utility

$$u\left(x_j\right) - \psi$$

Note here that since the horizon is finite I need to assume an exogenous cost of default. If $\psi = 0$, no household would ever have an incentive to pay back in period 2. A contract for the date 0 intermediary is $B = \{\tau_1(i), \tau_2(i)\}$. While the equilibrium contract is derived in a similar fashion to the general case, it suffices to notice from Theorem 1 that the equilibrium is equivalent to one in which households trade a risk free bond subject to debt constraints ϕ. In particular households choose $s \geq -\phi$ to maximize

$$u(\theta_i - qs_i) + \beta \mathbb{E} u\left(x_j + s_i\right)$$

where q and ϕ are chosen to clear markets and satisfy not-too-tight restrictions respectively. Moreover from Theorem 3 we know that given ψ, the equilibrium outcome is efficient. Under the following parametrization, $\beta = .9$; $\pi_i = 1/2, \kappa_h = .8$, $\theta_l = .3$, $\theta_h = 2$, $x_l = .5$, $x_h = 1.4$, in Fig. 1, I plot the change in the ex-ante welfare and debt levels for $\psi \in [0, 2]$.

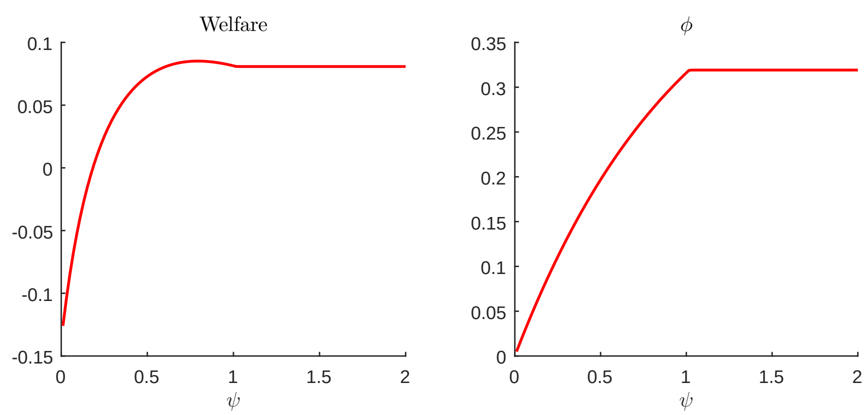

Figure 1: Welfare and Debt Levels

As one would expect, initially, as ψ increases, welfare increases and for ψ large enough, the change in welfare is zero after the low type ceases to be Euler-constrained. In addition, the endogenous debt levels ϕ increase and eventually flatten out. The key portion of Fig. 1 to notice is the downward sloping part of the welfare plot. In a region around $\psi = 1$, welfare *decreases* as ψ increases. The reason for this is a price effect which redistributes wealth from the period 1 low to the high type. This can be seen easily in the example by computing how the ex-ante welfare

$$W(\psi) = \pi_h \left[u(\theta_h - qs) + \beta \mathbb{E} u\left(x_j + s\right)\right] + \pi_l \left[u(\theta_l + qs) + \beta \mathbb{E} u\left(x_j - s\right)\right]$$

changes with ψ. One can show using simple algebra that

$$W'(\psi) = q'(\psi) s \left[-\pi_h u'(\theta_h - qs) + \pi_l u'(\theta_l + qs)\right] + \nu(\psi)$$

where $q'(\psi)$ is the change in price as a function of ψ and $\nu(\psi)$ is the multiplier on the borrowing constraint for the low type. Since risk sharing is imperfect, in general, $u'(\theta_h - qs) \leq u'(\theta_l + qs)$. Further, $q'(\psi) \leq 0$ since interest rates need to rise to clear markets as ψ increases. Given that the multiplier $\nu(\psi) \geq 0$, the change in welfare as ψ is increases is ambiguous. For ψ small enough, s will be small and so the multiplier effect will dominate and hence $W'(\psi) > 0$. However, as we can see from the picture as ψ get larger, s gets larger and $\nu(\psi)$ smaller, which causes the redistribution effect to dominate and $W'(\psi) < 0$.

Suppose we were to take as given the exogenously incomplete market structure and ask if the debt-constrained economy is efficient by considering a planning problem similar to Diamond (1967). For ϕ corresponding to the downward sloping portion of the welfare plot, we would conclude that outcomes are inefficient. In this case, imposing additional borrowing limits will implement the desired allocation. As we have seen, when markets are endogenously incomplete, the outcome is efficient. The same frictions, namely PI, LC and HT which restrict the set of feasible allocations also restrict the set of feasible policies. Intuitively, this is because of hidden trading and in particular the fact that if the planner tried to transfer an amount smaller than ϕ to the low type in period 1, the household's voluntary participation constraints in period 2 would be slack. Therefore, households would use the hidden markets to borrow which would make these additional limits ineffective. In other words, the allocation would no longer satisfy the no-hidden-trades condition in Lemma 1.

The key difference between these two environments is presence of hidden markets. If in the exogenously incomplete world, the assumption is that contracts are observable and exclusivity can be enforced, then the planner should be able do much better than offer uncontingent transfers. However, if we think that market incompleteness and borrowing constraints are a result of deeper frictions then those same frictions also put constraints on policy which in turn imply that the outcomes are efficient.

4.1.2 Aggregate Demand Externalities

I now consider an extension of the environment to study the effect of policy in mitigating *aggregate demand* externalities. A large recent literature has studied such externalities in environments with nominal rigidities. In this section I consider a very similar setup to Korinek and Simsek (2016) and highlight the differences in implications for policy when markets are exogenously vs endogenously incomplete.

Consider a three period environment with $t = 0, 1, 2$. Households can be one of two types $\theta \in \{\theta_h, \theta_l\}$ where $\theta_h < \theta_l$ will represent the disutility from working. Consider the problem of a household after period 0 types have been realized and subsequently a household can be of type θ_h (θ_l) with probability π $(1-\pi)$. Households have GHH preferences $u\left(\tilde{c_t}-\theta_t v\left(n_t\right)\right)$ and define $c_t=\tilde{c_t}-\theta_t v\left(n_t\right)$. Households solve

$$\max \sum_{t=0}^{2} \beta^{t-1} \mathbb{E}_0\left[u\left(c_t\right)\right]$$

subject to

$$c_t+\frac{1}{1+r_{t+1}} s_{t+1} \leq e_t\left(\theta_t\right)-s_t$$

and

$$s_{t+1} \geq -\phi_t$$

where

$$e_t\left(\theta_t\right)=w_t n_t+\Pi_t-\theta_t v\left(n_t\right)$$

denotes net income, w_t is the wage rate, Π_t are profits from firms. The economy is also subject to a lower bound (ZLB) on nominal interest rates $i_{t+1} \geq 0$. As in Korinek and Simsek (2016), I assume that prices are perfectly sticky so that this constraint translates into a lower bound on the real interest rate $r_{t+1} \geq 0$ as well as implications for real allocations.

A standard New-Keynesian production setup is assumed with a competitive final goods sector as well as a monopolistically competitive intermediate goods sector with a continuum of varieties. The linear production technology for the monopolistic firms implies that $w_t=1$. The details of the production side are omitted as they are standard and not crucial to understanding the results With perfectly sticky prices, output will be determined by aggregate demand. Monetary policy follows a Taylor rule, i.e. $i_{t+1}=r_{t+1}=\max\left(0, r_{t+1}^*\right)$ where r_{t+1}^* denotes the real rate in the frictionless benchmark. Given the functional forms, without the the ZLB constraint, the optimal labor supply of each household type is given by $n^*\left(\theta\right)=\arg\max_n n-\theta v\left(n\right)$. Define $e^*\left(\theta\right)=n^*\left(\theta\right)-\theta v\left(n^*\left(\theta\right)\right)$.

The equilibrium can be characterized by backward induction. In period 2, households work and consume and do not issue new debt. So $c_2\left(\theta^2\right)=e_2^*\left(\theta_2\right)-s_2\left(\theta^1\right)$. Next consider period 1. As is standard in models with occasionally binding debt constraints, the interest rate will be determined by the savers i.e.

$$1+r_2=\min \frac{u'\left(c_1\left(\theta^1\right)\right)}{\beta \mathbb{E}_1 u'\left(c_2\left(\theta^2\right)\right)}$$

Assume that the history corresponding to the type is (θ_h, θ_h) while all other types are borrowing constrained.[13] As in Korinek and Simsek (2016), the lower bound on the interest

[13]This will true for θ_h small enough and θ_l large enough.

rate generates an upper bound on the consumption for this type $\bar{c}_1$

$$u'(\bar{c}_1) = \beta \mathbb{E}_1 u'\left(e_2^*(\theta_2) + \bar{s}\right)$$

where $\bar{s} = \phi \frac{\sum_{\theta^1 \neq (\theta_h, \theta_h)} \pi(\theta^1)}{\pi(\theta_h, \theta_h)}$. Next, assume that $\phi_0 > \phi_1$ to highlight the effect of deleveraging in period 1. Then the equilibrium in period 1 depends on whether $e^*(\theta_h) + \phi_0 - \phi_1 < \bar{c}_1$ or not. In the first case $e_1(\theta_h) = e^*(\theta_h)$ while in the second the interest rate cannot fall enough and so $e_1(\theta_h) < e^*(\theta_h)$ and $c_1(\theta_h, \theta_h) = \bar{c}_1$. This is often called a demand-driven recession and is standard in such environments. Let

$$W_1\left(\theta^1, s, S\right) = u\left(e_1\left(\theta^1, S\right) + s(\theta_0) - \frac{s'(\theta_1, s)}{1 + r_2(S)}\right)$$

denote the welfare for a household of type θ^1 in period 1 with debt s and aggregate debt S. As in Korinek and Simsek (2016) we can show that

$$\frac{\partial}{\partial S} \sum_{\theta^1 \in \Theta^1} \pi\left(\theta^1\right) W_1\left(\theta^1, s_1(\theta_0), S_1\right) < 0$$

which corresponds to the aggregate demand externalities due to binding zero-lower bound constraints. The reason for this is that households do not internalize the effect of their period 0 debt choices on the zlb constraint which in turn forces the high types to work less and lowers output. It is worth noting that if $e^*(\theta_h) - \phi_0 + \phi_1 < \bar{c}_1$ then all other types work at the efficient level e^*. KS use this result to argue that macro-prudential policy which limits period 0 debt can increase welfare. Notice that this environment maps into the environment with endowment risk studied in this paper and thus all the main theorems hold. In particular, with endogenously incomplete markets, the best equilibrium is constrained-efficient and thus such policies are welfare neutral. This is exactly as in the case with pecuniary externalities studied in the previous section. The same frictions which generate the incompleteness also limit the policies that can be undertaken. Informally, such macro-prudential policies will incentivize trades in the hidden markets which will undo the effects of the policy.

As a final point, note that the deleveraging can be a result of shocks to the value of default in the endogenously incomplete world. In particular, recall that for a finite period setup we need exogenous costs of default ψ_t in order to sustain debts. If $\psi_1 > \psi_0$, then $\phi_0 > \phi_1$ which will imply the deleveraging needed to deliver the result.

4.2 Unique Implementation

The results in this section so far have two important implications for policy in the context of models with incomplete markets. The first is that interventions which may be desirable when markets are exogenously incomplete, might be ineffective when markets are endogenously incomplete. The second important message is that there is a role for policy to uniquely implement the best equilibrium. This motivates the use of credible off-equilibrium policies which will ensure that the best outcome will occur on path. To this end, I consider the effect of simple lender of last resort policies. In particular, I introduce a third strategic player into the game, namely a government.

Consider the intermediary game. Recall that the public history at the beginning of each period was denoted by $\hat{z}^{t-1} = \left(q^{t-1}, \mathcal{B}^{t-1}\right)$. Note that I am assuming that signed contracts between private agents are still unobservable to any outside authority. A *lender of last resort* policy is vector $G_t = \left\{q_t^G, \phi_t^G\right\}$ which consists of an interest rate $\frac{1}{q_t^G}$ and debt constraint ϕ_t^G for all $t \geq 1$.[14] In particular, under such a policy

1. Households can borrow and lend with the government at prices q_t^G subject to debt constraints ϕ_t^G
2. Intermediaries can borrow and lend with the government at prices q_t^G in an unconstrained fashion.

Given government policy G_t, we can define a competitive equilibrium given $\{G_t\}_{t\geq 1}$ in an analogous fashion to Section 2.

Notice that a lender of last resort policy does not in general depend on the public history $\hat{z}^t$. Using the language of Atkeson, Chari, and Kehoe (2010) we can define a *sophisticated lender of last resort* policy to be a vector $\mathcal{G}_t\left(\hat{z}^t\right) = \left(q_t^G\left(\hat{z}^t\right), \phi_t^G\left(\hat{z}^t\right)\right)$ that depends on the public history $\hat{z}^t$. Given that we are including a third player into the game, the government, we need to modify the structure of the game. The timing within a period is identical to Section 2, except that after private transactions have taken place, the government implements a policy $\mathcal{G}_t\left(\hat{z}^t\right)$ and finally private agents transact with the government. I now define the strategies of the players in this game. Given any history we can define a *continuation competitive equilibrium* as one that requires optimality by intermediaries and households. An equilibrium outcome is a collection $a_t = \{B_t, q_t, \mathcal{G}_t\}$ of contracts offered by intermediaries, prices q_t and government policy $\mathcal{G}_t$. Denote the government's strategy by σ^G. After any history, these strategies induce continuation outcomes in a standard fashion. Given this setup, we can define a *sophisticated equilibrium* as in Atkeson, Chari, and Kehoe (2010).

[14] The government uses lump-sum taxes to balance its budget.

Definition 7. A sophisticated equilibrium is a collection of strategies $\left(\sigma^{HH}, \sigma^{INT}, \sigma^{G}\right)$ such that after all histories, the continuation outcomes induced by $\left(\sigma^{HH}, \sigma^{INT}, \sigma^{G}\right)$ constitute a continuation competitive equilibrium.

We can define a sophisticated outcome to be the equilibrium outcome associated with a sophisticated equilibrium. A policy σ_G^* *uniquely implements* a desired competitive equilibrium $a_t^* = \{B_t^*, q_t^*, \mathcal{G}_t^*\}$ if the sophisticated outcome associated with any sophisticated equilibrium of the form $\left(\sigma^{HH}, \sigma^{INT}, \sigma^{*G}\right)$ coincides with the desired competitive equilibrium. The main result in this section is that there exists a sophisticated lender of last resort policy that uniquely implements the best equilibrium.

Proposition 6. *Given a desired competitive equilibrium a^*, there exists a sophisticated policy that uniquely implements it.*

Proof. We know from Theorem 1 that the equilibrium contract B_t^* is a simple borrowing and lending contract with debt constraints ϕ_t^*. Consider a history $\hat{z}^t$ with $\tilde{q}_t \neq q_t^*$ where q_t^* is the interest rate associated with ϕ_t^*. In this case $\tilde{B}_t \neq B_t^*$. $\tilde{B}_t$ is also an uncontingent contract and is characterized by debt constraints $\tilde{\phi}_t$. As a result to each price $\tilde{q}_t$ we can associate a private debt constraint $\phi_t^{\tilde{q}_t}$. Consider the following lender of last resort policy: for all $t \geq 0$

$$\mathcal{G}_t^*\left(\hat{z}^{t-1}, \hat{z}_t\right) = (0,0) \text{ if } q_t = q_t^*$$

$$\mathcal{G}_t^*\left(\hat{z}^{t-1}, \hat{z}_t\right) = \left(q_t^*, \max\left(\phi_t^* - \phi_t^{\tilde{q}_t}, 0\right)\right) \text{ if } q_t \neq q_t^* \text{ and } q_{t-j} = q_{t-j}^* \text{ for all } j \geq 1$$

$$\mathcal{G}_{t+s}^*\left(\hat{z}^{t-1}, \hat{z}_t, \cdot\right) = \left(q_{t+s}^*, \phi_{t+s}^*\right) \text{ for all } s \geq 1 \text{ if } q_t \neq q_t^* \text{ and } q_{t-j} = q_{t-j}^* \text{ for all } j \geq 1$$

where (q_t^*, ϕ_t^*) correspond to the price and debt constraint associated with the desired equilibrium. Given strategy σ_G^* and associated policy, $\{\mathcal{G}_t^*\}_{t\geq 0}$, it is easy to see that a^* is an equilibrium outcome of the game. We want to show that it is the unique outcome. Given a period t, consider whether outcome $\left\{\tilde{B}_t, \tilde{q}_t, \mathcal{G}_t^*\right\}$ with $\tilde{q}_t \neq q_t$ can ever occur on the equilibrium path. It is easy to see that if $\tilde{q}_t \neq q_t$ then arbitrage opportunities exist and so in any equilibrium, it must be that $\tilde{q}_t = q_t^*$. As a result, since the only equilibrium contract consistent with q^* is B^*, it must be that $\tilde{B}_t = \tilde{B}_t^*$. Finally, we need to show that the continuation outcomes after any history constitute continuation competitive equilibria. In this case, after an undesirable history, Euler-constrained households will borrow from the government. In following periods, given $\mathcal{G}_{t+s}^*\left(\hat{z}^{t-1}, \hat{z}_t, \cdot\right)$, market prices will be q_{t+s}^* and private intermediaries will only offer uncontingent savings contracts, and households will only transact with the government. Consider the incentives for any household to default in $t+1$ given this policy. Since the equilibrium outcome $\left(q_{t+s}^*, \phi_{t+s}^*\right)$ is consistent with no default, all households will weakly prefer to pay the government back in all future periods. □

The policies that uniquely implement the desired equilibrium are simple. After any undesired history, the government announces a sophisticated lender of last resort policy that allows private agents to borrow and lend with it at prices $\left\{q^*_{t+s}\right\}_{s\geq 0}$. In period t, households can borrow up to an amount so that the total debt is at most ϕ^*_t while in all future periods, they can borrow the full amount ϕ^*_t from the government. After any undesired history, in the continuation equilibrium, households will only transact with the government while intermediaries will offer uncontingent savings contracts. As a result the policy is well defined. It is then easy to see that the only equilibrium consistent with this policy is the desired one since no-arbitrage will ensure that $\tilde{q}_t = q^*_t$.

5 Discussion of Assumptions

In this section, I discuss the role of some of the assumptions in the model.

1. *Finitely lived intermediaries*: The reason for this is an existence problem. In the model, tighter debt constraints imply lower interest rates or higher q_t. In particular, it may be that the value of default is large enough so that the equilibrium debt constraints imply an interest rate that is less than 1. In this case, the present discounted value of transfers to the household is ∞ and as a result we cannot have infinitely lived intermediaries in the model. One example of such an environment is Hellwig and Lorenzoni (2009).

2. *i.i.d endowments*: I have assumed that the endowment shocks are independently and identically distributed across time and households. The reason for this is tractability. Introducing persistent endowment complicates the environment further but would be an interesting extension of the model.

3. *Restriction to signing with only one new intermediary at a time*: While the environment allows households to sign multiple contracts in a hidden fashion, I only allow them to sign at most one new contract each period. The reason for this is that if all intermediaries posted identical contracts and households could sign multiple hidden contracts, households could in theory borrow an infinite large amount and default the next period.

6 Conclusion

Models with exogenously incomplete markets have been widely used to study a variety of quantitative questions in macroeconomics and international economics. The purpose of this paper is to complement this literature by providing a framework to think about policy questions in the context of these models. The main advantage of my approach is that unlike the majority of the contracting literature, the resulting contracts are identical to the ones assumed by the applied literature. In particular, I show that uncontingent contracts with debt constraints endogenously arise under appropriate assumptions from a contracting environment with private information, limited commitment and hidden trading. I show that the best equilibrium outcome in this case is efficient but that there are multiple equilibria. This result has two important implications for policy. The first is that outcomes that might appear inefficient with exogenously incomplete markets may not be so when we explicitly model the underlying frictions. The second is that there is an important role for policy to implement the best equilibrium.

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A Appendix: Proofs From the Main Text

This appendix contains proofs from the main text.

A.1 Proofs from Section 3

Proof of Proposition 1. Suppose that in some equilibrium, for some t and history θ^t,

$$u'\left(c_t\left(\omega^t\right)\right)q_t < \beta\mathbb{E}_t u'\left(c_{t+1}\left(\omega^{t+1}\right)\right)$$

Since we are considering symmetric equilibria in which ex-ante identical intermediaries offer the same contract in equilibrium, we consider the incentives for a deviating intermediary to offer a different contract and make strictly positive profits. Consider an intermediary offering a $\varepsilon\delta-$savings contract $S_t^{\varepsilon,\delta}$ for some $\varepsilon > 0$ and $\delta < 1$. Notice that the intermediary makes positive profits whenever this contract is accepted. Since $u'\left(c_t\left(\omega^t\right)\right)q_t < \beta\mathbb{E}_t u'\left(c_{t+1}\left(\omega^{t+1}\right)\right)$, there exists $\varepsilon > 0, \delta < 1$ such that type θ^t will strictly prefer to sign an $\varepsilon\delta$ savings contract if offered. These contracts are by construction incentive compatible and satisfy voluntary participation constraints. As a result an intermediary offering such a contract will make positive profits which is a contradiction. The proof of part 2 is also straightforward. Suppose for contradiction we have a household who is Euler-constrained in period t, and in period $t+1$, for all $\tilde{h}^{t+1}$,

$$V_{t+1}\left(\tilde{h}^{t+1}\right) > V_{t+1}^d\left(\tilde{h}^{t+1}\right)$$

Consider the following deviating contract

$$\begin{aligned} {}_t\tilde{\tau}_t &= \delta\varepsilon \\ {}_t\tilde{\tau}_{t+1} &= -\frac{\varepsilon}{q_t} \end{aligned}$$

where $\varepsilon > 0, \delta < 1$ and the contract is not contingent on reported type. Clearly we can find an ε, δ such a borrowing constrained household accepting is made strictly better off. Moreover for ε, δ small, incentives are preserved for all households since the voluntary participation constraints are assumed to be slack. Since intermediaries make a strictly positive contract by offering such a contract, we have a contradiction. □

Proof of Proposition 2: The proof requires a series of preliminary results.

The first intermediate lemma tells us that we need only consider a relaxed problem and drop all voluntary participation constraints besides those for the lowest type.

Lemma 3. *In any equilibrium, for at any date t and history ω^{t-1}, if the voluntary participation constraint for type $\left(\omega^{t-1}, \underline{\theta}\right)$ is satisfied, then it is satisfied for all types $\left(\omega^{t-1}, \theta\right), \theta \in \Theta$.*

Proof of Lemma 3. Let $W_{\omega^{t-1}}\left(\theta, \hat{\theta}\right) = u\left(\theta + \tau_t\left(\omega^{t-1}, \hat{\theta}\right)\right) + \beta\mathbb{E}_t V_{t+1}\left(\hat{h}^{t+1}\right)$ be the equilibrium value for type $\left(\omega^{t-1}, \theta\right)$ pretending to be $\left(\omega^{t-1}, \hat{\theta}\right)$ but not signing a hidden contract.[15] Suppose first that the VP constraint for type $\left(\omega^{t-1}, \underline{\theta}\right)$ is satisfied and that $\tau_t\left(\omega^{t-1}, \underline{\theta}\right) \leq 0$. Then

$$\begin{aligned}
W_{\omega^{t-1}}(\theta, \theta) &= u\left(c_t\left(\omega^{t-1}, \theta\right)\right) + \beta \mathbb{E}_t V_{t+1}\left(h^{t+1}\right) \\
&\geq W_{\omega^{t-1}}(\theta, \underline{\theta}) \\
&= W_{\omega^{t-1}}(\underline{\theta}, \underline{\theta}) + u\left(\theta + \tau_t\left(\omega^{t-1}, \underline{\theta}\right)\right) - u\left(\underline{\theta} + \tau_t\left(\omega^{t-1}, \underline{\theta}\right)\right) \\
&= V^d(\underline{\theta}) + u\left(\theta + \tau_t\left(\omega^{t-1}, \underline{\theta}\right)\right) - u\left(\underline{\theta} + \tau_t\left(\omega^{t-1}, \underline{\theta}\right)\right) \\
&= V^d(\theta) + u(\underline{\theta}) - u(\theta) + u\left(\theta + \tau_t\left(\omega^{t-1}, \underline{\theta}\right)\right) - u\left(\underline{\theta} + \tau_t\left(\omega^{t-1}, \underline{\theta}\right)\right)
\end{aligned}$$

Since $\tau_t\left(\theta^{t-1}, \underline{\theta}\right) \leq 0$,

$$\begin{aligned}
& u(\underline{\theta}) - u(\theta) + u\left(\theta + \tau_t\left(\omega^{t-1}, \underline{\theta}\right)\right) - u\left(\underline{\theta} + \tau_t\left(\omega^{t-1}, \underline{\theta}\right)\right) \\
&= -u'(x)\left[\theta - \underline{\theta}\right] + u'(y)\left[\theta - \underline{\theta}\right] \\
&= \left[\theta - \underline{\theta}\right]\left(u'(y) - u'(x)\right) \\
&\geq 0
\end{aligned}$$

where $x \in [\underline{\theta}, \theta]$ and $y \in \left[\underline{\theta} + \tau_t\left(\omega^{t-1}, \underline{\theta}\right), \theta + \tau_t\left(\omega^{t-1}, \underline{\theta}\right)\right]$. Next, suppose that $\tau_t\left(\omega^{t-1}, \underline{\theta}\right) > 0$. Then the VP constraint for type $\left(\theta^{t-1}, \underline{\theta}\right)$ is slack. Suppose that the the VP constraint binds for some other type $\left(\omega^{t-1}, \theta\right)$. Then

$$\begin{aligned}
W_{\omega^{t-1}}(\underline{\theta}, \underline{\theta}) &\leq W_{\omega^{t-1}}(\theta, \theta) + u\left(\underline{\theta} + \tau_t\left(\omega^{t-1}, \underline{\theta}\right)\right) - u\left(\theta + \tau_t\left(\omega^{t-1}, \underline{\theta}\right)\right) \\
&= V^d(\theta) + u\left(\underline{\theta} + \tau_t\left(\omega^{t-1}, \underline{\theta}\right)\right) - u\left(\theta + \tau_t\left(\omega^{t-1}, \underline{\theta}\right)\right) \\
&= u\left(\underline{\theta} + \tau_t\left(\omega^{t-1}, \underline{\theta}\right)\right) + \mathbb{E}V^d(\theta') + u(\theta) - u\left(\theta + \tau_t\left(\omega^{t-1}, \underline{\theta}\right)\right) \\
&< u\left(\underline{\theta} + \tau_t\left(\omega^{t-1}, \underline{\theta}\right)\right) + \mathbb{E}V^d(\theta') \\
&\leq W_{\omega^{t-1}}(\underline{\theta}, \underline{\theta})
\end{aligned}$$

which is a contradiction. In particular, if $\tau_t\left(\omega^{t-1}, \underline{\theta}\right) > 0$ the the VP constraints for all types $\left(\omega^{t-1}, \theta\right)$ are slack. □

Recall that $\tau_t\left(\omega^t\right) = \tau_t^{old}\left(h^t\right) + {}_t\tau_t\left(\omega^t\right)$. The result states that in general, the voluntary participation constraints will bind for the lowest type $\underline{\theta}$. This is true generally in models with private information and limited commitment, for example in Dovis (2014). Note the binding pattern of these constraints is the opposite of models with only limited commitment such as Kehoe and Levine (1993) and Alvarez and Jermann (2000). For any

[15] As mentioned earlier, we consider equilibria in which the household signs with at most one intermediary at at time.

household define $A_t\left(\omega^{t-1},\theta\right)$ to be the equilibrium expected present discounted value of future transfers for type $\left(\theta^{t-1},\theta\right)$

$$A_t\left(\omega^{t-1},\theta\right) \equiv \tau_t\left(\omega^{t-1},\theta_t\right) + q_t \sum_{\theta'\in\Theta} \pi\left(\theta'\right) A_{t+1}\left(\omega^{t-1},\theta,\theta'\right)$$

Similarly, given a contract B_t, let

$${}_t\mathcal{P}_s\left(z^{t+s},\theta^{t+s}\right) \equiv {}_t\tau_s\left(z^{t+s},\theta^{t+s}\right) + q_t \sum_{\theta'\in\Theta} \pi\left(\theta^{t+s},\theta'\right) {}_t\mathcal{P}_{s+1}\left(\omega^{t+s+1},\theta^{t+s+1}\right)$$

denote the expected present discounted value of transfers associated with contract B_t from period s onward. The next set of results will be used to prove that in any equilibrium, the expected present discounted value of transfers to households with the same history ω^{t-1}, is independent of their period t reports.

Lemma 4. *In any equilibrium, for any t and any contract offered by an intermediary born at date t, ${}_t\mathcal{P}_t\left(z^t,\theta^t\right) = 0$ for all θ^t.*

Proof of Lemma 4. Suppose not. Clearly, ${}_t\mathcal{P}_t\left(z^t,\theta^t\right) > 0$ for all θ^t is not possible since the intermediary would making negative profits. On the other hand if ${}_t\mathcal{P}_t\left(z^t,\theta^t\right) \leq 0$ for all θ^t with strict inequality for some, then a deviating intermediary can offer a contract which transfers a little more to some types and still continue to make positive profits. As a result these types will strictly prefer to sign with the deviating intermediary. Finally, suppose that there exists $\theta^t, \tilde{\theta}^t$ such that ${}_t\mathcal{P}_t\left(z^t,\theta^t\right) > 0$ and ${}_t\mathcal{P}_t\left(z^t,\tilde{\theta}^t\right) < 0$. Then at the beginning of period t, consider a deviating intermediary offering the following contract,

$$\begin{aligned}
{}_t\tilde{\mathcal{P}}_t\left(z^t,\tilde{\theta}^t\right) &= {}_t\mathcal{P}_t\left(z^t,\tilde{\theta}^t\right) + \varepsilon \\
{}_t\tilde{\tau}_{t+s}\left(z^{t+s},\hat{\theta}^{t+s}\right) &= 0 \text{ for all } s \geq 0, \text{ for } \hat{\theta}^t \neq \tilde{\theta}^t
\end{aligned}$$

where $\varepsilon > 0$ and small. Notice that types θ^t strictly prefer the original contract while types $\tilde{\theta}^t$ strictly prefer ${}_t\tilde{\mathcal{P}}_t$ to ${}_t\mathcal{P}_t$. As a result, these households will strictly prefer to sign with the deviating intermediary who makes a positive profit. □

The lemma shows that all contracts offered by intermediaries must make zero profits and as a result there is no cross subsidization between contracts. The result is a direct consequence of perfect competition among intermediaries. If there is cross subsidization between initial types, a deviating intermediary can offer only the contract that yields positive profits and make strictly positive profits in equilibrium. In an environment with

2 period lived intermediaries for any $t \geq 0$,

$$
\begin{aligned}
{}_t\mathcal{P}_t\left(z^t,\theta^t\right) &= {}_t\tau_t\left(z^t,\theta^t\right) + q_t \sum_{\theta' \in \Theta} \pi\left(\theta^t,\theta'\right) {}_t\tau_{t+1}\left(z^t,\theta^t,\theta'\right) \\
{}_t\mathcal{P}_{t+1}\left(z^{t+1},\theta^{t+1}\right) &= {}_t\tau_{t+1}\left(z^{t+1},\theta^{t+1}\right)
\end{aligned}
$$

The final result required for the proof of Proposition 2 shows that higher types will always strictly prefer transfer sequences with a larger present discounted value even if they are Euler-constrained. Given an equilibrium transfer sequence A, define

$$
\begin{aligned}
A_{\varepsilon_+}\left(\omega^{t-1},\theta\right) &\equiv \tau_t\left(\omega^{t-1},\theta\right) + \varepsilon + q_t \sum_{\theta'} \pi\left(\theta'\right)\left[A_{t+1}\left(\omega^t,\theta'\right) - a\varepsilon\right] \\
A_{\varepsilon_-}\left(\omega^{t-1},\theta\right) &\equiv \tau_t\left(\omega^{t-1},\theta\right) - \varepsilon + q_t \sum_{\theta'} \pi\left(\theta'\right)\left[A_{t+1}\left(\omega^t,\theta'\right) + a\varepsilon\right]
\end{aligned}
$$

Notice that if $A_{\varepsilon_+}\left(\omega^{t-1},\theta_t\right) > A_t\left(\omega^{t-1},\theta_t\right)$ then it must be that $a < R_{t+1} = \frac{1}{q_t}$ and if $A_{\varepsilon_-}\left(\omega^{t-1},\theta_t\right) > A_t\left(\omega^{t-1},\theta_t\right)$ then $a > R_{t+1}$.Given a transfer schedule A and the associated transfer sequence, define

$$
\begin{aligned}
Z_t\left(\theta^t,s_t,\tau_t;A\right) = \max_{s_{t+1}} & \, u\left(c_t\right) + \beta\mathbb{E}_t Z_{t+1}\left(\theta^{t+1},s_{t+1},\tau_{t+1};A\right) \\
& s.t. \\
& c_{t'} + s_{t'+1} \leq \theta_{t'} + \tau_{t'}\left(h^{t'}\right) + R_{t'}s_{t'},\ \forall t' \geq t \\
& s_{t'+1} \geq 0,\ \forall t' \geq t
\end{aligned}
$$

where $R_{t'} = \frac{1}{q_{t'}}$. Here $Z_t\left(\theta^t,s_t,\tau;A\right)$ denotes the continuation value for a household of type θ^t who receives transfers according to A and can save at rate R_{t+s}, $s \geq 0$. The reason this will be useful is that in general, deviating intermediaries are always willing to provide savings contracts since they have no fear of default the following period. Therefore, if it is true that a household can do strictly better by lying and and saving, there exists a deviating contract that makes both the intermediary and household strictly better off. This will be particularly useful in the proof of Proposition 2.

Lemma 5.

Lemma 6. *If $A_{\varepsilon_+}\left(\omega^{t-1},\theta_t\right) > A\left(\omega^{t-1},\theta_t\right)$ then for ε small, $Z\left(\left(\theta^{t-1},\theta'\right),0,\tau_t+\varepsilon;A_{\varepsilon_+}\right) > Z\left(\left(\theta^{t-1},\theta'\right),0,\tau_t;A\right)$ for all $\theta' > \theta$*

If $A_{\varepsilon_-}\left(\omega^{t-1},\theta_t\right) > A\left(\omega^{t-1},\theta_t\right)$ and $Z\left(\left(\theta^{t-1},\theta'\right),0,\tau_t-\varepsilon;A_{\varepsilon_-}\right) \geq Z\left(\left(\theta^{t-1},\theta'\right),0,\tau_t;A\right)$ then for εsmall, $Z\left(\left(\theta^{t-1},\theta'\right),0,\tau_t-\varepsilon;A_{\varepsilon_-}\right) > Z\left(\left(\theta^{t-1},\theta'\right),0,\tau_t;A\right)$ for $\theta' > \theta$

Proof. Part 1. To prove this I show that $\frac{\partial}{\partial\varepsilon} Z\left(\left(\theta^{t-1},\theta'\right),0,\tau_t+\varepsilon;A_{\varepsilon_+}\right)\Big|_{\varepsilon=0} > 0$.

We have that

$$
\begin{aligned}
&\frac{\partial}{\partial \varepsilon} Z_t\left(\left(\theta^{t-1}, \theta'\right), 0, \tau_t+\varepsilon; A_{\varepsilon_+}\right) = u'\left(\theta' - s_{t+1} + \tau_t + \varepsilon\right)\left[-\frac{\partial}{\partial \varepsilon} s_{t+1} + 1\right] \\
&+ \beta \frac{\partial}{\partial \varepsilon} \mathbb{E}_t Z_{t+1}\left(\left(\theta^{t-1}, \theta', \theta_{t+1}\right), s_{t+1}, \tau_{t+1} - a\varepsilon\right) \\
&= u'\left(\theta' - s_{t+1} + \tau_t\left(\theta^t\right) + \varepsilon\right)\left[-\frac{\partial}{\partial \varepsilon} s_{t+1} + 1\right] + \beta\left[\mathbb{E}_t Z_{2,t+1} \frac{\partial}{\partial \varepsilon} s_{t+1} - a \mathbb{E}_t Z_{3,t+1}\right] \\
&= u'\left(c_t\left(\theta^{t-1}, \theta'\right)\right)\left[-\frac{\partial}{\partial \varepsilon} s_{t+1} + 1\right] + \beta\left[\begin{array}{c} R_{t+1} \mathbb{E}_t u'\left(c_{t+1}\left(\theta^{t-1}, \theta', \theta_{t+1}\right)\right) \frac{\partial}{\partial \varepsilon} s_{t+1} \\ -a \mathbb{E}_t u'\left(c_{t+1}\left(\theta^{t-1}, \theta', \theta_{t+1}\right)\right) \end{array}\right] \\
&= \frac{\partial}{\partial \varepsilon} s_{t+1}\left[-u'\left(c_t\left(\theta^{t-1}, \theta'\right)\right) + \beta R_{t+1} \mathbb{E}_t u'\left(c_{t+1}\left(\theta^{t-1}, \theta', \theta_{t+1}\right)\right)\right] \\
&+ u'\left(c_t\left(\theta^{t-1}, \theta'\right)\right) - \beta a \mathbb{E}_t u'\left(c_{t+1}\left(\theta^{t-1}, \theta', \theta_{t+1}\right)\right) \\
&= -\frac{\partial}{\partial \varepsilon} s_{t+1} \mu_t\left(\theta^{t-1}, \theta'\right) + u'\left(c_t\left(\theta^{t-1}, \theta'\right)\right) - \beta a \mathbb{E}_t u'\left(c_{t+1}\left(\theta^{t-1}, \theta', \theta_{t+1}\right)\right)
\end{aligned}
$$

$$
\begin{aligned}
&> -\frac{\partial}{\partial \varepsilon} s_{t+1} \mu_t\left(\theta^{t-1}, \theta'\right) + u'\left(c_t\left(\theta^{t-1}, \theta'\right)\right) - \beta R_{t+1} \mathbb{E}_t u'\left(c_{t+1}\left(\theta^{t-1}, \theta', \theta_{t+1}\right)\right) \qquad (20) \\
&= -\frac{\partial}{\partial \varepsilon} s_{t+1} \mu_t\left(\theta^{t-1}, \theta'\right) + \mu_t\left(\theta^{t-1}, \theta'\right) \\
&\geq 0
\end{aligned}
$$

where $\mu_t\left(\theta^{t-1}, \theta'\right)$ is the multiplier on the non-negative savings constraint. The strict inequality in (20) follows since $a < R$ and

$$
\begin{aligned}
& c_t + s_{t+1} = \theta_t + \tau_t + \varepsilon \\
& \Rightarrow \frac{\partial}{\partial \varepsilon} c_t + \frac{\partial}{\partial \varepsilon} s_{t+1} = 1 \\
& \Rightarrow \frac{\partial}{\partial \varepsilon} s_{t+1} < 1
\end{aligned}
$$

Part 2. Notice that

$$
\begin{aligned}
&\frac{\partial}{\partial \varepsilon} Z\left(\left(\theta^{t-1}, \theta'\right), 0, \tau_t - \varepsilon; A_{\varepsilon_-}\right) = u'\left(\theta' - s_{t+1} + \tau_t\left(\theta^t\right) - \varepsilon\right)\left[-\frac{\partial}{\partial \varepsilon} s_{t+1} - 1\right] \\
&+ \beta \frac{\partial}{\partial \varepsilon} \mathbb{E} Z_{t+1}\left(\left(\theta^{t-1}, \theta', \theta_{t+1}\right), s_{t+1}, \tau_{t+1} + a\varepsilon\right) \\
&= u'\left(c_t\left(\theta^{t-1}, \theta'\right)\right)\left[-\frac{\partial}{\partial \varepsilon} s_{t+1} - 1\right] + \beta\left[\mathbb{E} Z_{2,t+1} \frac{\partial}{\partial \varepsilon} s_{t+1} + a \mathbb{E} Z_{3,t+1}\right] \\
&= u'\left(c_t\left(\theta^{t-1}, \theta'\right)\right)\left[-\frac{\partial}{\partial \varepsilon} s_{t+1} - 1\right] + \beta\left[\begin{array}{c} R \mathbb{E} u'\left(c_{t+1}\left(\theta^{t-1}, \theta', \theta_{t+1}\right)\right) \frac{\partial}{\partial \varepsilon} s_{t+1} \\ + a \mathbb{E} u'\left(c_{t+1}\left(\theta^{t-1}, \theta', \theta_{t+1}\right)\right) \end{array}\right] \\
&= \frac{\partial}{\partial \varepsilon} s_{t+1}\left[-u'\left(c_t\left(\theta^{t-1}, \theta'\right)\right) + \beta R \mathbb{E} u'\left(c_{t+1}\left(\theta^{t-1}, \theta', \theta_{t+1}\right)\right)\right] \\
&- u'\left(c_t\left(\theta^t\right)\right) + \beta a \mathbb{E} u'\left(c_{t+1}\left(\theta^{t-1}, \theta', \theta_{t+1}\right)\right) \\
&= -\frac{\partial}{\partial \varepsilon} s_{t+1} \mu_t\left(\theta^{t-1}, \theta'\right) - u'\left(c_t\left(\theta^{t-1}, \theta'\right)\right) + \beta a \mathbb{E} u'\left(c_{t+1}\left(\theta^{t-1}, \theta', \theta_{t+1}\right)\right)
\end{aligned}
$$

If type $\left(\theta^{t-1}, \theta'\right)$ is Euler-unconstrained then clearly

$$
\begin{aligned}
&-\frac{\partial}{\partial \varepsilon} s_{t+1} \mu_t\left(\theta^{t-1}, \theta'\right) - u'\left(c_t\left(\theta^{t-1}, \theta'\right)\right) + \beta a \mathbb{E}_t u'\left(c_{t+1}\left(\theta^{t-1}, \theta', \theta_{t+1}\right)\right) \\
&= -u'\left(c_t\left(\theta^{t-1}, \theta'\right)\right) + \beta a \mathbb{E}_t u'\left(c_{t+1}\left(\theta^{t-1}, \theta', \theta_{t+1}\right)\right) > 0
\end{aligned}
$$

since $a > R$. Suppose however that the type $\left(\theta^{t-1}, \theta'\right)$ is Euler-constrained at $\varepsilon = 0$. Then $s_{t+1} = 0$ and so $\beta a E u'\left(c_{t+1}\left(\theta^{t-1}, \theta', \theta_{t+1}\right)\right) = \beta a E u'\left(c_{t+1}\left(\theta^{t+1}\right)\right)$ since type $\left(\theta^{t-1}, \theta\right)$ will also be Euler-constrained. Moreover since $\theta' > \theta$, we must have that $\mu_t\left(\theta^{t-1}, \theta'\right) < \mu_t\left(\theta^{t-1}, \theta\right)$. Therefore

$$
\begin{aligned}
&= -\frac{\partial}{\partial \varepsilon} s_{t+1} \mu_t\left(\theta^{t-1}, \theta'\right) - u'\left(c_t\left(\theta^{t-1}, \theta'\right)\right) + \beta a \mathbb{E}_t u'\left(c_{t+1}\left(\theta^{t-1}, \theta', \theta_{t+1}\right)\right) \\
&> -\mu_t\left(\theta^{t-1}, \theta'\right) - u'\left(c_t\left(\theta^{t-1}, \theta'\right)\right) + \beta a \mathbb{E}_t u'\left(c_{t+1}\left(\theta^{t-1}, \theta', \theta_{t+1}\right)\right) \\
&\geq -\mu_t\left(\theta^t\right) - u'\left(c_t\left(\theta^t\right)\right) + \beta a \mathbb{E}_t u'\left(c_{t+1}\left(\theta^{t-1}, \theta', \theta_{t+1}\right)\right) \\
&= -\mu_t\left(\theta^t\right) - u'\left(c_t\left(\theta^t\right)\right) + \beta a \mathbb{E}_t u'\left(c_{t+1}\left(\theta^{t+1}\right)\right) \\
&\geq 0
\end{aligned}
$$

since by assumption $\frac{\partial}{\partial \varepsilon} Z\left(\left(\theta^{t-1}, \theta\right), 0, \tau_t - \varepsilon; A_{\varepsilon-}\right)\big|_{\varepsilon=0} \geq 0.$ □

Lemma 7. *Given an equilibrium transfer sequence A, if for any date and history θ^{t-1},*

$$
Z_t\left(\left(\theta^{t-1}, \theta\right), s_t, \tau_t\left(\omega^{t-1}, \tilde{\theta}\right); A\right) > Z_t\left(\left(\theta^{t-1}, \theta\right), s_t, \tau_t\left(\omega^{t-1}, \theta\right); A\right)
$$

then there exists a deviating contract that makes both the intermediary and type (θ^{t-1},θ) strictly better off.

Proof. It is clear from the definition of Z that such a contract will be savings contract. In particular the deviating intermediary can offer an $\varepsilon\delta$ savings contract that make both it and the household strictly better off. Such a contract will always be incentive compatible and satisfy voluntary participation constraints. □

Proof of Proposition 2. Without loss of generality, we can just consider the truncated $T-$ period economy with a T lived intermediaries. Suppose we have an equilibrium in this environment. Let the equilibrium transfer sequence for the households be denoted by $\{\zeta_t(\omega^t)\}_{t,\omega^t}$ where in each period $\zeta_t(\omega^t) = \zeta_t^{old}(h^t) + {}_t\zeta_t(\omega^t)$ for all $\omega^t \in \Omega^t$. Let $R_t = \frac{1}{q_t}$ and construct a sequence of contracts for 2 period intermediaries $({}_t\tau_t(\omega^t), {}_t\tau_{t+1}(h^{t+1}))$ as follows

$$\begin{aligned}
{}_1\tau_1(\omega_1) &= \zeta_1(\omega_1) \\
&\vdots \\
{}_{t-1}\tau_t(h^t) &= -R_t\ {}_{t-1}\tau_{t-1}(\omega^t) \\
{}_t\tau_t(\omega^t) &= \zeta_t(\omega^t) - {}_{t-1}\tau_t(h^t) \\
&\vdots \\
{}_{T-1}\tau_{T-1}\left(\omega^{T-1}\right) &= \zeta_{T-1}\left(\omega^{T-1}\right) - {}_{T-2}\tau_{T-1}\left(h^{T-1}\right) \\
{}_{T-1}\tau_T\left(h^T\right) &= \zeta_T\left(h^T\right)
\end{aligned}$$

We know from Lemma 4 that the expected present discounted value of transfers associated with the sequence $\{\zeta_t(\omega^t)\}_{t,\theta^t}$, $A_1(\omega^1) = 0$. By construction[16],

$$\begin{aligned}
A_1(\omega_1) &= \zeta_1(\omega_1) + q_1\sum_{\theta_2}\pi(\theta_2)A_2(\omega_1,\theta_2) \\
&= \zeta_1(\theta_1) + q_1\sum_{\theta_2}\pi(\theta_2)\left[\zeta_2(\omega_1,\theta_2) + \ldots\left[\ldots + \sum_{\theta_{T-1}}\pi(\theta_{T-1})\left[\zeta_{T-1}\left(\omega^{T-1}\right) + q_{T-1}\sum_{\theta_T}\pi(\theta_T)\zeta_T\left(\omega^T\right)\right]\right]\right] \\
&= {}_1\tau_1(\omega_1) + q_1\sum_{\theta_2}\pi(\theta_2)\left[{}_1\tau_2 + {}_1\tau_2 + \ldots\left[\ldots + \sum_{\theta_{T-1}}\pi(\theta_{T-1})\left[\ldots + {}_{T-1}\tau_{T-1} + q_{T-1}\sum_{\theta_T}\pi(\theta_T)\ {}_{T-1}\tau_T\right]\right]\right] \\
&= q_1\sum_{\theta_2}\pi(\theta_2)\left[\left[\ldots + \ldots\left[\ldots + \sum_{\theta_{T-1}}\pi(\theta_{T-1})\left[{}_{T-2}\tau_{T-1} + {}_{T-1}\tau_{T-1} + q_{T-1}\sum_{\theta_T}\pi(\theta_T)\ {}_{T-1}\tau_T\right]\right]\right]\right] \\
&= \prod_{s=1}^{T-2} q_s\sum_{\theta^{T-2}}\pi\left(\theta^{T-2}\right)\sum_{\theta_{T-1}}\pi(\theta_{T-1})\left[{}_{T-1}\tau_{T-1}\left(\omega^{T-2},\theta_{T-1}\right) + q_{T-1}\sum_{\theta_T}\pi(\theta_T)\ {}_{T-1}\tau_T\left(\omega^{T-1},\theta_T\right)\right]
\end{aligned}$$

[16] Note that I have dropped some of the history dependence, wherever clear, for ease of notation.

Since $A_1^1(\omega_1) = 0$,

$$\sum_{\theta^{T-2}} \pi\left(\theta^{T-2}\right) \sum_{\theta_{T-1}} \pi\left(\theta_{T-1}\right) \left[{}_{T-1}\tau_{T-1}\left(\omega^{T-2}, \theta_{T-1}\right) + q_{T-1} \sum_{\theta_T} \pi\left(\theta_T\right) {}_{T-1}\tau_T\left(\omega^{T-1}, \theta_T\right) \right] = 0$$

We want to show that for all θ^{T-2}, $\left[{}_{T-1}\tau_{T-1}\left(\omega^{T-2}, \theta_{T-1}\right) + q_{T-1} \sum_{\theta_T} \pi\left(\theta_T\right) {}_{T-1}\tau_T\left(\omega^{T-2}, \theta_{T-1}, \theta_T\right) \right]$ is independent of θ_{T-1}. By construction, this is equivalent to showing that

$$A_{T-1}\left(\omega^{T-2}, \theta_{T-1}\right) = \zeta_{T-1}\left(\omega^{T-2}, \theta_{T-1}\right) + q_{T-1} \sum_{\theta_T} \pi\left(\theta_T\right) \zeta_T\left(\omega^{T-2}, \theta_{T-1}, \theta_T\right)$$

is independent of θ_{T-1}. It is easy to see that $\zeta_T\left(\omega^{T-2}, \theta_{T-1}, \theta_T\right)$ must be independent of θ_T else households would always announce the type consistent with the largest transfer. Therefore, ${}_{T-1}\tau_{T-1}\left(\omega^{T-2}, \theta_{T-1}, \theta_T\right)$ must also be independent of θ_T. Suppose for some history ω^{T-2} and $\theta, \theta' \in \Theta$,

$$A_{T-1}\left(\omega^{T-2}, \theta\right) > A_{T-1}\left(\omega^{T-2}, \theta'\right)$$

First suppose that $\theta < \theta'$. There exists some $\delta > 0$

$$A_{T-1}\left(\omega^{T-2}, \theta\right) = A_{T-1}\left(\omega^{T-2}, \theta'\right) + \delta$$

Since this excess transfer can either be front or back-loaded, we need to consider two cases. If the transfer is front loaded then

$$A_{T-1}\left(\omega^{T-2}, \theta\right) = \zeta_{T-1}\left(\omega^{T-2}, \theta'\right) + \varepsilon + q_{T-1} \sum_{\theta' \in \Theta} \pi\left(\theta'\right) \left[\zeta_T\left(\omega^{T-2}, \theta'\right) - a\varepsilon\right]$$

where $\varepsilon > 0, a < R_T$ and $\varepsilon - q_{T-1} a \varepsilon = \delta$. Similarly if the transfers are back-loaded then

$$A_{T-1}\left(\omega^{T-2}, \theta\right) = \zeta_{T-1}\left(\omega^{T-2}, \theta'\right) - \varepsilon + q_1 \sum_{\theta' \in \Theta} \pi\left(\theta'\right) \left[\zeta_T\left(\omega^{T-2}, \theta'\right) + a\varepsilon\right]$$

where $\varepsilon > 0, a > R_T$ and $\varepsilon - q_{T-1} a \varepsilon = \delta$. In the first case, the first part of Lemma 5 along with Lemma 7 tells us that type $\left(\omega^{T-2}, \theta'\right)$ would strictly prefer to lie and pretend to be type $\left(\omega^{T-2}, \theta\right)$ and save with another intermediary and so incentive compatibility constraints are violated. In the second case notice that in any equilibrium type $\left(\omega^{T-2}, \theta\right)$ must weakly prefer to tell the truth than announce $\left(\omega^{T-2}, \theta'\right)$. As a result this type must weakly prefer transfer scheme $A_{T-1}\left(\omega^{T-2}, \theta\right)$ to $A_{T-1}\left(\omega^{T-2}, \theta'\right)$. Then part 2 of Lemma 5 along with Lemma 7 implies that the incentive compatibility constraint for $\left(\omega^{t-1}, \theta'\right)$ is violated again. Next, suppose $\theta > \theta'$. We know from Lemma 3 that if the voluntary participation constraint binds, it does so for the lowest type and hence

$$V_{T-1}\left(\omega^{T-2},\theta\right) > V^{d}_{T-1}\left(\theta\right)$$

so that the household with a larger present discounted value of transfers strictly prefers the existing contract to defaulting. In this case consider an intermediary modifying the original contract as follows; for some $\delta > 0$, small

$$\tilde{\zeta}_{T-1}\left(\omega^{T-2},\theta\right) = \zeta_{T-1}\left(\omega^{T-2},\theta\right) - \delta$$
$$\tilde{\zeta}_{1}\left(\omega^{T-2},\hat{\theta}\right) = \zeta_{T-1}\left(\omega^{T-2},\hat{\theta}\right) + \frac{\delta}{\sum_{\theta'<\theta}\pi\left(\theta^{T-2},\theta'\right)} \text{ for all } \hat{\theta} < \theta$$

Since this provides more insurance in period $T-1$, it increases the expected welfare of household ω^{T-2}. The perturbation continues to satisfy incentive compatibility and also the participation constraints for δ small enough. Clearly, the sequence of constructed transfers is budget feasible and satisfies incentive compatibility and participation constraints. Moreover as demonstrated above, each two-period contract makes 0 profits and hence there is no cross-subsidization. We only need to check that a particular two period intermediary cannot do strictly better. But this is clear since if it could then a T intermediary could just modify its contract and also make positive profits. □

Proof of Proposition 3: The proof requires the following result

Proposition 7. *In any equilibrium with two period lived intermediaries, for any t and $h^t, \hat{h}^t$ such that $\theta_t + {}_{t-1}\tau_t\left(h^t\right) = \hat{\theta}_t + {}_{t-1}\tau_t\left(\hat{h}^t\right)$,*

$$V_t\left(h^t\right) = V_t\left(\hat{h}^t\right)$$

Proof of Proposition 7. Because of the assumption that after period T, households can only trade a risk free bond subject to exogenous debt constraints, it is easy to see that the statement holds in period T, since all that matters for the households' choices is the sum $\theta_T + {}_{T-1}\tau_T\left(h^T\right)$. In period $T-1$ suppose h^{T-1} and $\hat{h}^{T-1}$ such that $\theta_{T-1} + {}_{T-2}\tau_{T-1}\left(h^{T-1}\right) = \hat{\theta}_{T-1} + {}_{T-2}\tau_{T-1}\left(\hat{h}^{T-1}\right)$ and

$$V_{T-1}\left(h^{T-1}\right) > V_{T-1}\left(\hat{h}^{T-1}\right)$$

For ease of notation denote the corresponding transfers by ${}_{T-2}\tau_{T-1}$ and ${}_{T-2}\hat{\tau}_{T-1}$ We need to consider a few cases. Suppose first that for both $\theta^{T-1}, \hat{\theta}^{T-1}$

$$u'\left(\theta_{T-1} + {}_{T-2}\tau_{T-1} + {}_{T-1}\tau_{T-1}\right) = \beta R_T \mathbb{E}_{T-1} u'\left(\theta_T + {}_{T-1}\tau_T + \psi_{T+1}\left(\theta_T + {}_{T-1}\tau_T\right)\right) \quad (21)$$
$$u'\left(\hat{\theta}_{T-1} + {}_{T-2}\hat{\tau}_{T-1} + {}_{T-1}\hat{\tau}_{T-1}\right) = \beta R_T \mathbb{E}_{T-1} u'\left(\theta_T + {}_{T-1}\hat{\tau}_T + \psi_{T+1}\left(\theta_T + {}_{T-1}\hat{\tau}_T\right)\right) \quad (22)$$

where $\psi_{T+1}\left(\theta_T + {}_{T-1}\tau_T\right)$ is the savings choice for the household (given that it is subject to debt constraint ϕ^e_{T+1}). Since ${}_{T-1}\tau_{T-1} + \frac{{}_{T-1}\tau\tau}{q_{T-1}} = {}_{T-1}\hat{\tau}_{T-1} + \frac{{}_{T-1}\hat{\tau}_T}{q_{T-1}} = 0$ and the savings choice ψ_{T+1} depends only on the sum $\theta_T + {}_{T-1}\tau_T$, it must be that ${}_{T-1}\tau_{T-1} = {}_{T-1}\hat{\tau}_{T-1}$ and $V\left(\theta^t\right) = V\left(\hat{\theta}^{T-1}\right)$ and so we have a contradiction. Suppose on the other hand that (21) holds with equality and (22) with strictly inequality. Again, since ${}_{T-1}\tau_{T-1} + \frac{{}_{T-1}\tau_T}{q_{T-1}} = {}_{T-1}\hat{\tau}_{T-1} + \frac{{}_{T-1}\hat{\tau}_T}{q_{T-1}} = 0$, it must be that ${}_{T-1}\tau_{T-1} > {}_{T-1}\hat{\tau}_{T-1}$. Since the household is Euler-constrained, assume that ${}_{T-1}\hat{\tau}_{T-1} > 0$. It is easy to see that that giving type $\hat{\theta}^{T-1}$ the contract associated with θ^{T-1} makes it strictly better off. Consider modifying the original contract so that

$$\begin{aligned} {}_{T-1}\tilde{\tau}_{T-1} &= {}_{T-1}\hat{\tau}_{T-1} + \varepsilon \\ {}_{T-1}\tilde{\tau}_T &= {}_{T-1}\hat{\tau}_T - \delta\varepsilon \end{aligned}$$

where ε chosen so that ${}_{T-1}\hat{\tau}_T - \delta\varepsilon \geq {}_{T-1}\tau_T$ and

$$\frac{u'\left(\hat{\theta}_{T-1} + {}_{T-2}\hat{\tau}_{T-1} + {}_{T-1}\hat{\tau}_{T-1}\right)}{\beta \mathbb{E}_{T-1} u'\left(\theta_T + {}_{T-1}\hat{\tau}_T + \psi_{T+1}\left(\theta_T + {}_{T-1}\hat{\tau}_T\right)\right)} > \delta > R_T$$

This perturbation makes type $\hat{\theta}^{T-1}$ strictly better off. To see that voluntary participation constraints continue to hold for type $\hat{\theta}^{T-1}$ in period t, notice that this household's value in period T is exactly the same as θ^{T-1}. Since the original transfer scheme was incentive compatible and satisfied voluntary participation constraints in period T, it must be that for all $\theta \in \Theta$

$$u\left(\theta + {}_{T-1}\tilde{\tau}_T + \psi_{T+1}\left(\theta + {}_{T-1}\tilde{\tau}_T\right)\right) + \beta \mathbb{E}_T V_{T+1}\left(\theta, \psi_{T+1}\right) \geq V^d_T\left(\theta\right)$$

As a result, these constraints continue to hold under this deviation. Finally since $\delta > R_T$, the deviating intermediary makes strictly positive profits. Therefore it must be that ${}_{T-1}\tau_{T-1} = {}_{T-1}\hat{\tau}_{T-1}$. Note that a similar argument holds if both (21) and (22) hold with inequality and $V_{T-1}\left(\theta^T\right) > V_{T-1}\left(\hat{\theta}^{T-1}\right)$ .Given that the property holds for $\hat{T}-1$, assume that this property holds for some $t+1 < \hat{T}-1$. Our goal is to show that the property holds in t. Suppose for contradiction we have some $\theta^t, \hat{\theta}^t$ such that $\theta_t + {}_{t-2}\tau_{t-1}\left(\hat{h}^{t-1}\right) =$

$\hat{\theta}_t + {}_{t-2}\tau_{t-1}\left(\hat{h}^{t-1}\right)$ and

$$V_t\left(\hat{h}^t\right) > V_t\left(\hat{h}^t\right)$$

Again, denote the transfers by ${}_{t-2}\tau_{t-1}$ and ${}_{t-2}\hat{\tau}_{t-1}$. As before, first consider the case in which both type's Euler equations hold with equality. Suppose ${}_t\tau_t < {}_t\hat{\tau}_t$. Then it is easy to see that an intermediary can offer an $\varepsilon\delta$ savings contract which will be accepted by this agent making both intermediary strictly better off. To see why notice that since ${}_t\tau_t + \frac{{}_t\tau_{t+1}}{q_t} = {}_t\hat{\tau}_t + \frac{{}_t\hat{\tau}_{t+1}}{q_t} = 0$ and $V\left(\theta^t\right) > V\left(\hat{\theta}^t\right)$ it must be that there exists some $\varepsilon > 0$ such that the transfer scheme ${}_t\hat{\tau}_t - \varepsilon + \frac{{}_t\hat{\tau}_{t+1} + \varepsilon}{q_t}$ makes this type strictly better off. Next suppose that

$$\begin{aligned} u'\left(\theta_t + {}_{t-1}\tau_t + {}_t\tau_t\right) &= \beta R_{t+1}\mathbb{E}_t u'\left(\theta_{t+1} + {}_t\tau_{t+1} + {}_{t+1}\tau_{t+1}\right) \\ u'\left(\hat{\theta}_t + {}_{t-1}\hat{\tau}_t + {}_t\hat{\tau}_t\right) &> \beta R_{t+1}\mathbb{E}_t u'\left(\theta_{t+1} + {}_t\hat{\tau}_{t+1} + {}_{t+1}\hat{\tau}_{t+1}\right) \end{aligned}$$

As in the period $T-1$ case, consider modifying the original contract

$$\begin{aligned} {}_t\tilde{\tau}_t &= {}_t\hat{\tau}_t + \varepsilon \\ {}_t\tilde{\tau}_{t+1} &= {}_t\hat{\tau}_{t+1} - \delta\varepsilon \end{aligned}$$

where

$$\frac{u'\left(\hat{\theta}_t + {}_{t-1}\hat{\tau}_t + {}_t\hat{\tau}_t\right)}{\beta\mathbb{E}_t u'\left(\theta_{t+1} + {}_t\hat{\tau}_{t+1} + {}_{t+1}\hat{\tau}_{t+1}\right)} > \delta > R$$

independently of reported type. To see that no agent would choose to default on this intermediary notice that for any type that signs this contract will have the same value in $t+1$ by the induction assumption. Therefore since type θ^{t+1} preferred not to default under the original contract, type $\hat{\theta}^t$ will not want to default under the deviating contract. If the original contract was incentive compatible, the deviating one will be as well. Finally, since there exists a type, $\hat{\theta}^t$ who is made strictly better off for some $\delta < 1$, the deviating intermediary makes strictly positive profits. Therefore by induction the claim must hold in period t and by induction for all previous periods as well. □

Proof of Proposition 3. Note that the proposition is written in terms of the equivalent 2 period contracts. We know from Proposition 7 that for all θ^t, $V_t\left(h^t\right)$ only depends on $\theta_t + {}_{t-1}\tau_t\left(h^t\right)$. Given the nature of these two period contracts, we consider transfers of the form $\left({}_t\tau_t\left(\omega^t\right), {}_t\tau_{t+1}\left(h^t\right)\right) = \left(\varphi\left(\omega^t\right), \frac{-\varphi\left(\omega^t\right)}{q_t}\right)$. Let φ^* be largest such $\varphi\left(\omega^t\right)$ given to all households that are Euler-constrained and denote the corresponding history by ω^{*t}

. Given some $\varphi\left(\omega^t\right)$ define

$$R^{\varphi\left(\theta^t\right)} = \frac{u'\left(\theta_t + {}_{t-1}\tau_t\left(h^t\right) + \varphi\left(\omega^t\right)\right)}{\beta\mathbb{E}_t u'\left(\theta + \frac{-\varphi\left(\omega^t\right)}{q_t} + {}_{t+1}\tau_{t+1}\left(\omega^{t+1}\right)\right)}$$

Since this household is Euler-constrained, $R^{\varphi\left(\omega^t\right)} > R_{t+1}$. Suppose there exists an Euler-constrained household $\tilde{\omega}^t$ such that $\varphi\left(\tilde{\omega}^t\right) < \varphi^*$. Consider modifying the original contract as follows

$$\begin{aligned} {}_t\tilde{\tau}_t\left(\tilde{\omega}^t\right) &= \varphi\left(\tilde{\omega}^t\right) + \varepsilon \\ {}_t\tilde{\tau}_{t+1}\left(\tilde{\omega}^t\right) &= -\frac{\varphi\left(\tilde{\omega}^t\right)}{q_t} - \frac{\varepsilon}{\hat{q}_t} \end{aligned}$$

where

$$R_{t+1} = \frac{1}{q_t} < \frac{1}{\hat{q}_t} < R^{\varphi\left(\tilde{\omega}^t\right)}$$

Notice that for ε small, type $\tilde{\theta}^t$ will be made strictly better off by signing such a contract since $R^{\varphi\left(\tilde{\omega}^t\right)} > \frac{1}{\hat{q}_t}$ and the household is Euler-constrained. For ε small enough, ${}_t\tilde{\tau}_{t+1}\left(\tilde{\omega}^t\right) \geq \frac{-\varphi^*}{q_t}$. Since we have shown earlier that equilibrium continuation value for any agent going forward only depends on the sum $\theta + {}_t\tau_{t+1}\left(h^t\right)$, if

$$u\left(\theta^* + \frac{-\varphi^*}{q_t} + {}_{t+1}\tau_{t+1}\left(\omega^{*t+1}\right)\right) + \beta\mathbb{E}_{t+1}V_{t+2}\left(h^{*t+2}\right) \geq V^d_{t+1}\left(h^{*t+1}\right)$$

then all households accepting the deviating contract will also prefer not to default. To check incentive compatibility, notice that if the original contract was incentive compatible and all other types preferred their transfers to $\left(\varphi^*, \frac{-\varphi^*}{q_t}\right)$, clearly the modified transfer sequence will be incentive compatible as well. Finally, since $\frac{1}{q_t} < \frac{1}{\hat{q}_t}$, the deviating intermediary is also made strictly better off. □

Using these results, we can proceed to the proof of the equivalence theorem.

Proof of Theorem 1. Given an equilibrium of the decentralized contracting problem with equilibrium transfer schedules $\zeta_t\left(\omega^t\right)$, construct the equivalent 2 period contracts (which we proved exists earlier). As a result we have a sequence of transfers $\left\{{}_t\tau_t\left(\omega^t\right), {}_t\tau_{t+1}\left(h^{t+1}\right)\right\}_{\theta^t,t}$. Construct bond holdings after each history for the agent as follows (assume that agents start off with 0 initial wealth)

$$l_2(\theta_1) = -\, {}_1\tau_1(\omega_1)$$
$$\vdots$$
$$l_{t+1}(\theta^t) = -\, {}_t\zeta_t(\omega^t)$$
$$\vdots$$

Let the interest rates $\{R_t\}$ be defined such that $R_{t+1} = \frac{1}{q_t}$. Given that the sequence of transfers satisfies the zero profit condition we know that $-R_t l_t(\theta^{t-1}) = {}_{t-1}\zeta_t(h^{t-1})$ and therefore the constructed bond holdings satisfy the household's budget constraints. To construct the sequence of debt constraints recall that we showed that in contracting environment, that for any t, and θ^t such that

$$u'\left(\theta_t + {}_{t-1}\zeta_t\left(h^{t-1}\right) + {}_t\zeta_t\left(\omega^t\right)\right) > \beta R_{t+1}\mathbb{E}_t u'\left(\theta_{t+1} + {}_t\zeta_{t+1}\left(h^t\right) + {}_{t+1}\zeta_{t+1}\left(\omega^{t+1}\right)\right)$$

it must be that ${}_t\zeta_t(\omega^t) = \varphi_t$ where φ_t is independent of the agent's history. Let

$$\phi_{t+1} = \varphi_t$$

for all t. The necessary and sufficient conditions for agent optimality in the bond trading economy are

$$u'\left(c_t\left(\theta^t\right)\right) \geq \beta R_{t+1}\mathbb{E}_t u'\left(c_{t+1}\left(\theta^{t+1}\right)\right)$$

with strict inequality if

$$l_{t+1}\left(\theta^t\right) = -\phi_{t+1}$$

along with budget feasibility (which we have already established). We know from earlier results that any allocation from the decentralized contracting environment satisfies exactly these conditions which shows that the constructed allocation is optimal for all agents. It only remains to show that these debt constraints are not-too-tight which follows from Proposition 1 and Proposition 3.

For part 2, consider an equilibrium of the debt constrained environment. Construct transfer schedules for $\hat{T}$ period lived intermediaries as follows

$$
\begin{aligned}
{}_1\tau_1\left(\omega_1\right) &= -l_2\left(\theta_1\right) \\
{}_1\tau_2\left(h^1\right) &= Rl_2\left(\theta_1\right) - l_3\left(\theta^2\right) \\
&\vdots \\
{}_1\tau_{\hat{T}}\left(h^{\hat{T}}\right) &= R_{\hat{T}} l_{\hat{T}}\left(\theta^{\hat{T}-1}\right) \\
{}_{\hat{T}}\tau_{\hat{T}}\left(\omega^{\hat{T}}\right) &= -l_{\hat{T}+1}\left(\theta^{\hat{T}}\right) \\
&\vdots
\end{aligned}
$$

And let $q_t = \frac{1}{R_{t+1}}$. Note that we are constructing an equilibrium in which each intermediary born at date $1, \hat{T}, 2\hat{T}-1, \ldots$ offers a single contract with transfers as constructed. All intermediaries born at other dates offer simple uncontingent savings contracts. While these will never be signed in equilibrium, a deviating contract that offers some state-contingency will never be profitable since households can always lie and use these savings contracts to smooth any excess transfers. This similar to the "latent contracts" used by Ales and Maziero (2014) to sustain their equilibrium. Suppose these contracts and prices did not constitute an equilibrium. There are two cases to consider:

1. Given prices and the contract offered by this intermediary, no new intermediary has an incentive to offer a contract and make strictly positive profits.

2. The existing intermediary has no incentive to modify its contract and make strictly positive profits.

Consider the first case. Suppose that this was a $\tilde{T}$ period contract that spanned dates $t \to \tilde{T}-1$. Notice that the only way in which households will strictly prefer to sign with such a deviating contract and the intermediary make a positive profit is if it increases increases insurance in some period. First consider the last period $\tilde{T}-1$. It is easy to see that in this period, the transfers from the intermediary to the household cannot depend on $\theta_{\tilde{T}-1}$ else the household would always announce the type consistent with the highest transfer. Next consider period $\tilde{T}-2$. Suppose the contract made a positive transfer to some type who is Euler constrained in period $\tilde{T}-1$. Incentive compatibility requires that this type must receive a negative uncontingent transfer in period $\tilde{T}$ otherwise households would lie to get this increased transfer. Since the household is Euler constrained and debt constraints are chosen to be Not-too-tight, we know that some type's voluntary participation constraint holds with equality in $\tilde{T}-1$ and so such a perturbation is not possible. If the household is unconstrained in this state, a perturbation that makes both the intermediary and the agent strictly better off is not possible. On the other hand, suppose the contract made a negative transfer to some type. Again incentive compatibility dictates that a positive uncontingent transfer be made to this type in period $\tilde{T}-1$. However, this is exactly a pure savings contract and since the households are not savings constrained, this will never be profitable. Now consider period $\tilde{T}-3$. First, consider a state contingent positive transfer to some type who is constrained. This must be compensated for by a negative transfer in period $\tilde{T}-2$. This transfer cannot be independent of state since some household's voluntary participation constraint binds. It also cannot be state contingent by the previous argument. As before, a negative transfer followed by an uncontingent transfer at date $\tilde{T}-2$ can never make both the intermediary and agent strictly better off. A similar argument holds for all previous periods by induction. Finally, consider a positive transfer to some type in $\tilde{T}-3$ who is not constrained. Incentive compatibility requires that a negative uncontingent transfer be made in period $\tilde{T}-2$. However such a perturbation can never be welfare enhancing if the present discounted value of transfers is less than zero and so the intermediary can never make a positive profit on this particular deviation.

Next, we need to check that the *existing* intermediary has no incentive to modify its contract given prices. As above, the only such modifications will involve providing some type in some period a more insurance. Consider a period t, and a type ω^t who is Euler-constrained under the original contract. Given our equilibrium definition, we know that there exists some θ^c such that the voluntary participation constraint for type $\left(\omega^t,\theta^c\right)$ holds with equality in $t+1$. We consider a deviation in which the intermediary increases the transfer to this household by some $\varepsilon > 0$. It is easy to see that incentive compatibility requires that the intermediary make a negative transfer at some future date, say $t+1$. So there exists some $\left(\omega^t,\theta^*\right)$ who receives a negative transfer δ in $t+1$. Note that the negative transfer cannot be uncontingent since for some type in $t+1$, the voluntary participation constraint holds with equality. Therefore, the transfer δ must be state contingent. We can group states into two classes; the first $Con_{t+1}\left(\omega^t\right)$ are those that are Euler-constrained at $t+1$, i.e.

$$u'\left(c_{t+1}\left(\omega^t,\theta\right)\right) > \beta R_{t+2}\mathbb{E}_{t+1}u'\left(c_{t+2}\left(\omega^t,\theta,\theta'\right)\right)$$

and the second $Uncon_{t+1}\left(\omega^t\right)$, those that are not, i.e.

$$u'\left(c_{t+1}\left(\omega^t,\theta\right)\right) = \beta R_{t+2}\mathbb{E}_{t+1}u'\left(c_{t+2}\left(\omega^t,\theta,\theta'\right)\right)$$

We know that $\theta^c \in Con_{t+1}\left(\omega^t\right)$. Also, we know that the equilibrium contract satisfies $A_{t+1}\left(\omega^t,\theta\right) = A_{t+1}\left(\omega^t,\theta'\right)$ for any $\theta,\theta' \in Uncon_{t+1}\left(\omega^t\right)$. Therefore a negative transfer δ' will have to be imposed on all such types. However, since under the original contract, $A_{t+1}\left(\omega^t,\theta\right)$ is independent of θ, the perturbation implies that $A_{t+1}\left(\omega^t,\theta\right) >$

$A_{t+1}\left(\omega^t, \hat{\theta}\right)$ for any $\theta, \hat{\theta}$ in $Con_{t+1}\left(\omega^t\right)$ and $Uncon_{t+1}\left(\omega^t\right)$ respectively. Therefore, all types in $Uncon_{t+1}\left(\omega^t\right)$ will strictly prefer to lie and announce some type in $Con_{t+1}\left(\omega^t\right)$ and save with some other intermediary. □

A.1.1 Proofs from Section 3.1

Proof of Theorem 2: The first step in the proof is to show that given a measurable map Φ, a $\Phi-$RCE always exists.

Proposition 8. *For any finite measurable map $\Phi : \mathbb{R}_+ \to \mathbb{R}_+$, a $\Phi-$RCE exists.*

Proof. The first step of the proof is to show that given continuous pricing functions $R\left(\phi\right)$, there exists a unique list of value functions W and policy functions $l'\left(\theta, l, \phi\right)$ that solve the individual household's problems. This part of the proof uses arguments developed in Miao (2006). Let $\mathbb{A} \subset \mathbb{R}$ be the compact feasible asset space, $\mathbb{D} \subset \mathbb{R}_+$ the compact space of debt constraints and the $\mathbb{V}$ denote the set of uniformly bounded and continuous real valued functions on $\Theta \times \mathbb{A} \times \mathbb{D}$. Define operator $\mathbb{T}$ as follows: Given some $w \in \mathbb{V}$,

$$\left(\mathbb{T}w\right)\left(\theta, l, \phi\right) = \max_{l' \in \Gamma(l,\phi)} u\left(\theta - Rl - l'\right) + \beta \mathbb{E} w\left(\theta', l', \phi'; \Phi'\right)$$

where $\Gamma\left(l, \phi\right) = \left[-\phi, \theta + Rl\right]$. In order to apply the contraction mapping theorem I first show $\mathbb{T}w \in \mathbb{V}$. Boundedness follows. To show continuity, consider a sequence $\left(\theta, l, \phi\right)^n \to \left(\theta, l, \phi\right)$. Given our restriction to continuous pricing functions, $R\left(\phi^n\right) \to R\left(\phi\right)$. As a result correspondence Γ is continuous. Then first term on the right hand side of the above dynamic program is continuous since u is continuous. Consider second term. We want to show that

$$\left|\mathbb{E}w\left(\theta^n, l^n, \phi^n\right) - \mathbb{E}w\left(\theta, l, \phi\right)\right| \to 0$$

Since $\mathbb{A} \times \mathbb{D} \times \Theta$ is compact by Tychonoff's theorem, w is uniformly continuous and as a result $w\left(\theta^n, l^n, \phi^n\right) \to w\left(\theta, l, \phi\right)$ uniformly. As a result we can interchange the limit and integrals. Therefore by Maximum theorem, $\mathbb{T}w$ is also continuous and hence $\mathbb{T}w \in \mathbb{V}$. It is easy to see that the operator satisfies Blackwell's sufficiency conditions. As a result operator $\mathbb{T}$ is a contraction and so by the Contraction Mapping Theorem we have unique sequence of functions w^* and corresponding policy functions l'^*. Next, we can use the individual policy function to compute the aggregate distribution

$$\lambda\left(A \times B\right) = \mu\left(i \in I : \left(l'\left(i\right), \theta\left(i\right)\right) \in A \times B,\ A \times B = \mathcal{B}\left(A\right) \times \mathcal{B}\left(\Theta\right)\right)$$

Consequently

$$\lambda'(A \times B) = \int \mu\left(i \in I, \theta'(i) \in A, l'(\theta, l, \phi) \in B\right) d\lambda(\theta, l)$$

which defines the measurable mapping G. Next, it is straightforward to note that the policy functions $l'(\theta, l, \phi)$ are strictly increasing in R for all $b' > -\phi$ and that $l'(\theta, l, \phi) = -\phi$ for R small enough. As a result given ϕ, for $R(\phi)$ large enough

$$\int_{\mathbb{A}\times\Theta} l'(\theta, l, \phi)\, d\lambda(l, \Theta) > 0$$

and for $R(\phi)$ small enough

$$\int_{\mathbb{A}\times\Theta} l'(\theta, l, \phi)\, d\lambda(l, \Theta) = -\phi < 0$$

As a result continuity implies that there exists $R(\phi)$ such that

$$\int_{\mathbb{A}\times\Theta} l'(\theta, l, \phi)\, d\lambda(l, \Theta) = 0$$

□

Next, it always true that a $\Phi-$RCE with Φ being the zero map is NTT-RCE

Lemma 8. *There exists an NTT-RCE in which $\Phi = 0$.*

Proof. Consider the $\Phi-$RCE in which Φ is the zero map i.e. $\phi = 0$ and $\Phi(\phi) = 0$. We know that such an equilibrium exists from the previous lemma. To show that these also constitutes a NTT-RCE we also need to show that

$$W\left(\theta, 0, 0; \Phi^0\right) = V^d(\theta)$$

which is straightforward since

$$W\left(\theta, 0, 0; \Phi^0\right) = u(\theta) + \mathbb{E}u\left(\theta'\right) = V^d(\theta)$$

□

The reason for this is clear. If debt constraints are zero each period, then in equilibrium agents consume their endowment which trivially implies that the voluntary participation constraint binds for each period and each type. The final and main proposition that completes the proof of Theorem 2 is to show that there exists a NTT-RCE with $\Phi \neq 0$.

Proposition 9. *If*

$$\frac{u'\left(\bar{\theta}\right)}{\beta\eta} < \kappa$$

then there exists a NTT-RCE in which $\Phi > 0$.

Proof. Define $\phi_\varepsilon = \phi + \varepsilon$ and Φ_ε such that $\Phi_\varepsilon\left(\phi + \varepsilon\right) = \phi' + \varepsilon$. The first step in the proof is to compute the sign of the following object

$$\lim_{\substack{\phi\to 0\\ \mathbf{\Phi}\to\mathbf{0}}} \frac{\partial}{\partial\varepsilon} W\left(\theta, -\phi_\varepsilon, \phi_\varepsilon; \Phi_\varepsilon\right)\Big|_{\varepsilon=0}$$

In words, this measures the change in equilibrium welfare of the $\Phi-$RCE as we change Φ from zero the something positive. In equilibrium we must have from the agent's problem

$$W\left(\theta, -\phi_\varepsilon, \phi_\varepsilon; \Phi_\varepsilon\right) = u\left(z - R\phi - R\varepsilon - l'\left(\theta, -\phi_\varepsilon, \phi_\varepsilon\right)\right) + \beta\mathbb{E}W\left(\theta', l'\left(\theta, -\phi_\varepsilon, \phi_\varepsilon\right), \phi'; \Phi'_\varepsilon\right)$$

where $l'\left(\theta, -\phi_\varepsilon, \phi_\varepsilon, R\right)$ denote the policy function for bond holdings. We can then compute the following derivative (which is well defined)

$$\begin{aligned}\frac{\partial}{\partial\varepsilon} W\left(\theta, -\phi_\varepsilon, \phi_\varepsilon; \Phi_\varepsilon\right) &= u'\left(\theta - R\phi - R\varepsilon - l'\right)\left[-R - R_\varepsilon\varepsilon - l'_\varepsilon\right] \\ &+ \beta E W_1\left(\theta', l', \phi'_\varepsilon; \Phi'_\varepsilon\right) l'_\varepsilon + \beta\mathbb{E}W_2\left(\theta', l'\left(\theta, -\phi_\varepsilon, \phi_\varepsilon\right), \phi'_\varepsilon; \Phi'_\varepsilon\right)\end{aligned}$$

This implies that

$$\begin{aligned}\frac{\partial}{\partial\varepsilon} W\left(\theta, -\phi_\varepsilon, \phi_\varepsilon; \Phi_\varepsilon\right)\Big|_{\varepsilon=0} &= u'\left(\theta - R\phi - l'\left(\theta, -\phi, \phi\right)\right)\left[-R - l'_\varepsilon\left(\theta, -\phi, \phi\right)\right] \\ &+ \beta\mathbb{E}W_1\left(\theta', l'\left(\theta, -\phi, \phi\right), \phi'; \Phi'\right) l'_\varepsilon\left(\theta, -\phi, \phi\right) + \beta\mathbb{E}W_2\left(l'\left(\theta, -\phi, \phi\right), \phi'; \Phi'\right)\end{aligned}$$

Given the continuity of the policy and price functions

$$\begin{aligned}\lim_{\substack{\phi\to 0\\ \mathbf{\Phi}\to\mathbf{0}}} \frac{\partial}{\partial\varepsilon} W\left(\theta, -\phi_\varepsilon, \phi_\varepsilon; \Phi_\varepsilon\right)\Big| &= u'\left(\theta\right)\left[-R - l'_\varepsilon\right] + \beta\mathbb{E}W_1\left(\theta', 0, 0; 0\right) l'_\varepsilon + \beta\mathbb{E}W_2\left(\theta', 0, 0; 0\right) \\ &= -Ru'\left(\theta\right) - u'\left(\theta\right) l'_\varepsilon + \beta\mathbb{E}W_1\left(\theta', 0, 0; 0\right) l'_\varepsilon + \beta\mathbb{E}W_2\left(\theta', 0, 0; 0\right)\end{aligned}$$

From the first order conditions of the above problem where $\mu\left(\theta, l, \phi\right)$ is the multiplier on the debt constraint, we have

$$\beta\mathbb{E}W_1\left(\theta', l'\left(\theta, l, \phi\right), \phi'; \Phi\right) = u'\left(c\left(\theta, l, \phi\right)\right) - \mu\left(\theta, l, \phi\right)$$

we see that

$$\lim_{\substack{\phi \to 0 \\ \Phi \to \mathbf{0}}} \frac{\partial}{\partial \varepsilon} W\left(\theta, -\phi_\varepsilon, \phi_\varepsilon; \Phi_\varepsilon\right)\Big|_{\varepsilon=0} = -Ru'\left(\theta\right) - \mu\left(\theta, 0, 0\right) l'_\varepsilon + \beta \mathbb{E} W_2\left(\theta', l'\left(\theta, 0, 0\right), 0; 0\right)$$

From the complementary slackness condition we have that

$$\begin{aligned}
&\mu\left(z, 0, 0\right)\left[l'\left(\theta, -\phi, \phi\right) + \phi\right] = 0 \\
&\Rightarrow \mu_\varepsilon\left(\theta, 0, 0\right)\left[l'\left(\theta, -\phi, \phi\right) + \phi\right] + \mu\left(\theta, 0, 0\right)\left[l'_\varepsilon\left(\theta, -\phi, \phi\right) + \phi_\varepsilon\right] = 0 \\
&\Rightarrow \mu_\varepsilon\left(\theta, 0, 0\right)\left[l'\left(\theta, -\phi, \phi\right) + \phi\right] + \mu\left(\theta, 0, 0\right)\left[l'_\varepsilon\left(\theta, -\phi, \phi\right) + 1\right] = 0
\end{aligned}$$

As $\phi, \Phi \to 0$ we have

$$\begin{aligned}
&\mu\left(\theta, 0, 0\right)\left[l'_\varepsilon\left(\theta, -\phi, \phi\right) + 1\right] = 0 \\
&\Rightarrow \mu\left(\theta, 0, 0\right) l'_\varepsilon\left(\theta, 0, 0\right) = -\mu\left(\theta, 0, 0\right)
\end{aligned}$$

Therefore

$$\lim_{\substack{\phi \to 0 \\ \Phi \to \mathbf{0}}} \frac{\partial}{\partial \varepsilon} W\left(\theta, -\phi_\varepsilon, \phi_\varepsilon; \Phi_\varepsilon\right)\Big|_{\varepsilon=0} = -Ru'\left(\theta\right) + \mu\left(\theta, 0, 0\right) + \beta \mathbb{E} W_2\left(\theta', l'\left(\theta, 0, 0\right), 0; 0\right)$$

From the first order conditions we have

$$\begin{aligned}
\mu\left(\theta, 0, 0\right) &= u'\left(\theta\right) - \beta \mathbb{E} W_1\left(\theta', 0, 0; 0\right) \\
&= u'\left(\theta\right) - \beta R \sum_{\theta' \in \Theta} \pi\left(\theta'\right) u'\left(\theta'\right)
\end{aligned}$$

Define $\eta = \sum_{\theta' \in \Theta} \pi\left(\theta'\right) u'\left(\theta'\right)$. Therefore

$$\mu\left(\theta, 0, 0\right) = u'\left(\theta\right) - \beta R \eta$$

and

$$\begin{aligned}
\mathbb{E} W_2\left(\theta', 0, 0; 0\right) &= \sum_{\theta' \in \Theta} \pi\left(\theta'\right) \mu\left(\theta', 0, 0\right) \\
&= \sum_{\theta' \in \Theta} \pi\left(\theta'\right)\left[u'\left(\theta'\right) - \beta R \eta\right] \\
&= \eta - \beta R \eta
\end{aligned}$$

As a result

$$\lim_{\substack{\phi\to 0\\ \Phi\to\mathbf{0}}} \frac{\partial}{\partial\varepsilon} W\left(\theta, -\phi_\varepsilon, \phi_\varepsilon; \Phi_\varepsilon\right)\Big|_{\varepsilon=0} = -Ru'\left(\theta\right) + u'\left(\theta\right) - \beta R\eta + \beta\left[\eta - \beta R\eta\right]$$
$$= -R\left[u'\left(\theta\right) + \beta\eta + \beta^2\eta\right] + u'\left(\theta\right) + \beta\eta \quad (23)$$

Notice that if

$$R < \frac{u'\left(\theta\right) + \beta\eta}{\left[u'\left(\theta\right) + \beta\eta + \beta^2\eta\right]}$$

then (23)> 0 since $\eta > 0$. In any $\Phi-$RCE, when $\phi = 0$ the interest rate must satisfy

$$u'\left(\bar{\theta}\right) \geq \beta R\eta$$
$$\Rightarrow R \leq \frac{u'\left(\bar{\theta}\right)}{\beta\eta}$$

Therefore if

$$\frac{u'\left(\bar{\theta}\right)}{\beta\eta} < \frac{u'\left(\theta\right) + \beta\eta}{\left[u'\left(\theta\right) + \beta\eta + \beta^2\eta\right]}$$

then we know that

$$\lim_{\substack{\phi\to 0\\ \Phi\to\mathbf{0}}} \frac{\partial}{\partial\varepsilon} W\left(\theta, -\phi_\varepsilon, \phi_\varepsilon; \Phi_\varepsilon\right)\Big|_{\varepsilon=0} > 0$$

But since $\frac{u'(\theta)+\beta\eta}{[u'(\theta)+\beta\eta+\beta^2\eta]} \geq \kappa$ by assumption the property is true. Next notice because of Inada conditions that

$$\lim_{\substack{\phi\to\infty\\ \Phi\to\infty}} W\left(\theta, -\phi, \phi; \Phi\right) \to -\infty$$

since eventually, the debt constraints cease to bind for all agents. And so continuity implies that there exists ϕ^θ such that

$$W\left(\theta, -\phi^\theta, \phi^\theta; \Phi^\theta\right) = V^d\left(\theta\right)$$

with $\phi > 0$ and $\Phi\left(\phi^\theta\right) = \phi^\theta$. If there are many such we pick the one closest to 0. In this equilibrium all agents are subject to debt constraints ϕ^θ in each period. However it might be that for some $\tilde{\theta}$

$$W\left(\tilde{\theta}, -\phi^\theta, \phi^\theta; \Phi^\theta\right) < V^d\left(\tilde{\theta}\right)$$

and as a result this would cease to be a NTT-RCE. Using a similar procedure, we can construct debt constraints $\phi^{\tilde{\theta}}$ for any $\tilde{\theta}$ such that the above constraint holds with equality.

Consider $\phi = \min_\theta \phi^\theta$. By continuity it must be that

$$\frac{\partial}{\partial \varepsilon} W(\theta, -\phi - \varepsilon, \phi + \varepsilon)\Big|_{\varepsilon=0} \leq 0$$

Therefore for all $\theta \in \Theta$,

$$W(\theta, -\phi, \phi; \Phi) \geq W\left(\theta, -\phi^\theta, \phi^\theta\right) = V^d(\theta)$$

which proves the claim. □

A.2 Proofs from Section 5

This section contains proofs from section 5 of the main text.

A.2.1 Proofs from Section 5.1

Proof of Proposition 4. The first part is as in Proposition 1. Next, given a date t and history θ^{t-1}, if $\theta > \theta'$ we can use an identical argument as in the intermediary game to show that it must be that $A_t\left(\theta^{t-1}, \theta\right) \geq A_t\left(\theta^{t-1}, \theta'\right)$ where $A_t\left(\theta^{t-1}, \theta\right) = \tau_t\left(\theta^{t-1}, \theta\right) + q_t \sum_{\theta_{t+1}} \pi\left(\theta^{t-1}, \theta, \theta_{t+1}\right) A_{t+1}\left(\theta^{t-1}, \theta, \theta_{t+1}\right)$ is the expected present discounted value of transfers to type $\left(\theta^{t-1}, \theta\right)$. In particular, if this did not hold, type θ will strictly prefer to lie and pretend to be type θ' and use the hidden markets to save. Suppose that $A_t\left(\theta^{t-1}, \theta\right) > A_t\left(\theta^{t-1}, \theta'\right)$. There are two cases to consider. First suppose that $q_t > \frac{\beta \mathbb{E}_t u'\left(c_{t+1}\left(\theta^{t-1}, \theta, \theta_{t+1}\right)\right)}{u'\left(c_t\left(\theta^{t-1}, \theta\right)\right)}$. Then as in the intermediary game we can find a perturbation which involves a small transfer of wealth between type $\left(\theta^{t-1}, \theta\right)$ and the types below that increases ex-ante welfare. The second case to consider is one in which $q_t = \frac{\beta \mathbb{E}_t u'\left(c_{t+1}\left(\theta^{t-1}, \theta, \theta_{t+1}\right)\right)}{u'\left(c_t\left(\theta^{t-1}, \theta\right)\right)}$. We want to consider a wealth transfer from type $\left(\theta^{t-1}, \theta\right)$ that leaves this equation unchanged. Choose $(\varepsilon, a\varepsilon)$ where

$$u'\left(c_t\left(\theta^{t-1}, \theta\right) - \varepsilon\right) q - \beta \mathbb{E}_t u'\left(c_{t+1}\left(\theta^{t-1}, \theta, \theta_{t+1}\right) - a\varepsilon\right) = 0$$

Modify the transfer sequence as follows: $\tilde{\tau}_t\left(\theta^{t-1}, \theta\right) = \tau_t\left(\theta^{t-1}, \theta\right) - \varepsilon$ and $\tilde{\tau}_t\left(\theta^{t-1}, \theta, \theta_{t+1}\right) = \tau_t\left(\theta^{t-1}, \theta, \theta_{t+1}\right) - a\varepsilon$ for all θ_{t+1}. This constitutes a wealth transfer from type $\left(\theta^{t-1}, \theta\right)$ which can be redistributed to lower types. For ε small, the voluntary participation constraints are still satisfied and the pricing equation is unchanged since given the choice of a. As a result any solution to the constrained-efficient problem must satisfy $A_t\left(\theta^{t-1}, \theta\right) = A_t\left(\theta^{t-1}, \theta'\right)$. Moreover, it must be that $A_1(\theta_1) = 0$ for all θ_1. These two conditions imply that $\sum_{t=1}^{T} \left(\prod_{s=1}^{t} q_s\right) \tau_t\left(\theta^T\right) = 0$ for all $\theta^T \in \Theta^T$. □

Proof of Lemma 1. We have already established the first part in an earlier proposition. Next, suppose that

$$q_t > \frac{\beta \mathbb{E}_t u'\left(c_{t+1}\left(\theta^{t+1}\right)\right)}{u'\left(c_t\left(\theta^t\right)\right)}$$

for some type θ^t and

$$V_{t+1}\left(\tilde{\theta}^{t+1}\right) - V_{t+1}^d\left(\tilde{\theta}^{t+1}\right) > 0 \text{ for all } \tilde{\theta}^{t+1}$$

In this case, zero debt constraints would no longer be be not-too-tight in the hidden market. More generally, intermediaries can find a deviating contract that makes both it and the household strictly better off. □

Proof of Theorem 3. Given the previous result we know that constrained-efficient allocation looks like uncontingent borrowing and lending subject to debt constraints. In particular we can decompose the sequence of efficient transfers $\left\{\tau_t\left(\theta^t\right)\right\}$ into $\tau_t\left(\theta^t\right) = \tilde{\tau}_t\left(\theta^{t-1}\right) + \tilde{\tau}_t\left(\theta^t\right)$. We can construct contracts for T period lived intermediaries as follows:

$$\begin{aligned}
&{}_1\zeta_1\left(\theta_1\right) = \tau_1\left(\theta_1\right) \\
&{}_1\zeta_t\left(\theta^t\right) = \tau_t\left(\theta^t\right),\ t < T \\
&{}_1\zeta_T\left(\theta^T\right) = \tilde{\tau}_t\left(\theta^{t-1}\right) \\
&{}_T\zeta_T\left(\theta^T\right) = \tilde{\tau}_T\left(\theta^T\right) \\
&\vdots
\end{aligned}$$

while all other intermediaries (for example those born in period $2, T+1, ..$) offer simple uncontingent savings contracts. Given the prices from the planning problem consider the incentives of any particular intermediary to deviate when all other intermediaries are offering the uncontingent contracts constructed above. Given that intermediaries are offering savings contracts, a deviating intermediary cannot offer a contract with state-contingency. Therefore, the best this intermediary can do is to offer an agent who is Euler constrained the opportunity to borrow more at date t. Consider some t and history $\theta^t \in \Theta^t$ such that $u'\left(c_t\left(\theta^t\right)\right) q_t > \beta E_{t+1} u'\left(c_{t+1}\left(\theta^{t+1}\right)\right)$. We know from (18) and Lemma 2 that in period $t+1$, $V_{t+1}\left(\theta^t, \theta\right) = V_{t+1}^d\left(\theta^t, \theta\right)$ for some $\theta \in \Theta$. Therefore, the deviating contract will violate voluntary participation constraints for the agent in some state at date $t+1$. Notice that offering a savings contract can never lead to positive profits for any deviating intermediary since it would have to offer a return $\tilde{R}_{t+1} < \frac{1}{q_t}$ no household will ever accept such a contract. □