Endogenously Incomplete Markets with Equilibrium Default*

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Abstract

An extensive literature in macroeconomics and international economics uses models with exogenously incomplete markets and financial frictions for a variety of quantitative and policy exercises. In this paper, I relax the assumptions of exogenous incompleteness and instead consider general contracting environments in which no restrictions are placed on the types of contracts agents can sign. I show that with three key frictions: private information, voluntary participation and hidden trading, equilibrium outcomes of the contracting environment coincide with those in models with exogenously incomplete markets. In particular, under appropriate assumptions, equilibrium outcomes are identical to either an environment with trades in a risk-free bond subject to occasionally binding debt constraints, or an environment with defaultable debt. The policy implications, however, are very different. For example, equilibrium outcomes in models with exogenously incomplete markets are typically inefficient while the best equilibrium in my environment is efficient. This implies that imposing borrowing limits may be desirable when markets are exogenously incomplete while such policies cannot improve welfare in my model. However, I show that this environment has multiple equilibria and that governments can play an important role as a lender of last resort in ensuring that the best equilibrium occurs.

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1 Introduction

A large and growing literature in macroeconomics and international economics uses models with incomplete markets and financial frictions for a variety of quantitative and policy exercises. Examples include the study of financial and sovereign debt crises, optimal taxation, and bankruptcy laws. The key assumption in these models is that markets are *exogenously* incomplete. In particular, strong assumptions are imposed on the types of contracts agents within the model can sign. Most of these models make one of two assumptions. The first type of assumption is that agents can trade an uncontingent risk-free bond subject to exogenous debt constraints. These include environments studied by Huggett (1993) and Aiyagari (1994). The second type of assumption is that agents can trade defaultable debt contracts. Such models are standard in the international macro and bankruptcy literature.

An alternate view, which I take in this paper, is to relax the assumptions of exogenous incompleteness and instead consider general contracting environments in which no restrictions are placed on the types of contracts agents can sign. I show that there exist informational and commitment assumptions that endogenously generate the types of contracts assumed by much of the applied literature. Next, I show that the best equilibrium in these environments is efficient. Finally, I show that models with endogenous incompleteness have substantially different implications for policy than those with exogenous incompleteness.

I study a dynamic environment with a large number of risk-averse households that receive stochastic endowments each period and seek to share risk with each other. I model trading among households by allowing them to sign contracts with competitive financial intermediaries. The contracting environment is subject to three key frictions: private information, voluntary participation and hidden trading. The first is that households' endowments are private information and not observable to any other household. The second is that household participation in financial markets is voluntary in that in any period they can always choose autarky namely, to not participate in financial markets from then on and consume their endowments in every period. The third is that trades between households and intermediaries are hidden in that they are not observable by other households and other intermediaries. In particular, I allow households to sign contracts with multiple intermediaries in a hidden fashion.

A well known feature of these environments is that risk-sharing is possible only if households that do not repay their debts suffer a cost. In my environment, I assume that if households do not repay their debts as specified in the contract, they are permanently banished from financial markets and forced into autarky. With this assumption I consider two environments. In the first, I assume that financial intermediaries can only offer contracts that induce households to always repay their debts. In the second, I introduce a technology that allows financial intermediaries to temporarily banish households from financial markets. Contracts can specify banishment and banished households consume their endowment and cannot trade with intermediaries. The contract also specifies a re-entry probability after which households are able to sign contracts again.

I show that equilibrium outcomes in the first environment are equivalent to those in a standard incomplete markets model in which households trade a risk-free bond subject to debt constraints. Moreover, these debt constraints are independent of households' histories and thus look exactly like those assumed in models with exogenous incompleteness.

In the second environment, I show that intermediaries will choose use the banishment technology in equilibrium. As a result, equilibrium contracts will feature temporary periods of financial autarky, much like in the sovereign default and bankruptcy literature. Under some sufficient conditions, equilibrium outcomes here are equivalent to those used widely in the sovereign default and bankruptcy literature, for example Eaton and Gersovitz (1981). In equilibrium, intermediaries use banishment as a way of introducing state and history contingency into contracts. In most contracting environments restricting to no-banishment contracts is without loss of generality. However in this environment, since banishment is publicly observable, it incentivizes truthful revelation of types. This might seem counterintuitive since the intermediary can always provide the value associated with autarky to the household without banishment. For example, the transfer scheme in which the intermediary makes zero transfers in all dates and states provides the autarkic value to the household. However, unlike banishment, the household still has the option of signing contracts with other intermediaries. As a result, with private information and hidden trading, such a transfer scheme is not in general incentive compatible. To understand this difference more starkly suppose that intermediaries have to pay an exogenous cost whenever they banish households. One can interpret this as a cost of monitoring households that are banished. In models with exclusive contracts and no hidden trading, equilibria with banishment are always Pareto-inferior to ones with no banishment. In contrast, with hidden trading, equilibria with banishment can Pareto-dominate any equilibrium without banishment. In this sense, hidden trading is necessary to get default in equilibrium.

The second main result of the paper is that the best equilibrium in these environments is efficient. By this I mean that a planner confronted with the same frictions as intermediaries cannot improve overall welfare. In particular, I show that in the presence of hidden markets, the amount of state contingency a planner can offer in a contract is severely limited. As a result, in the first environment with no banishment, for example, the planner cannot do better than offer short-term uncontingent contracts. However, the first welfare theorem does not hold since in general, the environment has multiple equilibria. This multiplicity is due to the presence of strategic complementarities in the actions of intermediaries.

The third set of results concern the lessons for policy. There are three important implications for policy. The first is that policies which might be considered desirable when markets are exogenously incomplete, may no longer improve welfare when markets are endogenous incomplete. For example, I illustrate how in models with exogenously incomplete markets, setting limits on how much households can borrow may increase overall welfare. However, I show that in models with hidden trading, households will use hidden markets to circumvent these limits. As a result, in the environment I study, imposing such limits will not increase overall welfare. Second, because of the multiplicity of equilibria there is an important role for policy to uniquely implement the best equilibrium. I show how simple lender of last resort policies can help achieve this. The third implication for policy is that with banishment, the efficient probability of re-entry is decreasing in the level of the debt defaulted on. This result has important implications for the bankruptcy policy. To understand these implications note that we can re-interpret this environment as one in which intermediaries decide whether or not to banish households and an outside authority enforces banishment and decides the probability of re-entry. Under this interpretation, a bankruptcy policy which allowed the probability of re-entry to depend on the level of defaulted debt can increase welfare.

Finally, I consider the positive implications of the model. As documented by Cruces and Trebesch (2013) in the case of sovereign defaults, larger haircuts are associated with a longer duration of banishment from capital markets. In my environment, even though default is associated with a 100 percent haircut, it is still true that the probability of re-entry is smaller if the level of defaulted debt is larger.

A final point worth noting is that all three frictions i.e. private information, limited commitment and hidden trading, are essential to the nature of the contract. Obviously, without private information, fully state-contingent contracts would be equilibrium outcomes. Without limited commitment, households will never be borrowing constrained. Without hidden trading, contracts will feature history contingency and equilibrium contracts will resemble those in Thomas and Worrall (1990) and Atkeson and Lucas (1992).

Literature: This paper is related to a large literature on dynamic contracts and its applications in macroeconomics. Green (1987), Thomas and Worrall (1990), Phelan and Townsend (1991) and Atkeson and Lucas (1992) are some of the important papers studying dynamic environments with private information. In general, efficient contracts in these environments feature history contingency and no banishment/separation. As a result, these contracts are very different from the defaultable debt or the uncontingent borrowing and lending contracts assumed by the applied literature. In contrast, I show that when dynamic private information interacts with limited commitment and hidden trading, the equilibrium contracts are identical to those assumed in standard macroeconomic models.

Allen (1985), Cole and Kocherlakota (2001), Golosov and Tsyvinski (2007) and Ales and Maziero (2014) study dynamic private information environments with hidden trading.[1] While the first three papers assume a technology that allows agents to engage in hidden transactions, Ales and Maziero (2014) study an environment in which agents can sign non-exclusive contracts. The equilibrium contracts in these environments are also very different than those in the environment I study. While contracts in Golosov and Tsyvinski (2007) feature state contingency, the equilibrium contracts resulting from the other three papers are uncontingent. However, these contracts do not have separation on path and no agent is borrowing constrained. In the environment I study, if

[1] Bisin and Guaitoli (2004) and Bisin and Rampini (2006) study two period environments with moral hazard (hidden action) and hidden trading. In particular, Bisin and Rampini (2006) find that the ability to seize payoffs from secondary contracts is valuable and interpret this as bankruptcy. However, this requires that output is observable which is not true in my environment since endowments are private information.

intermediaries are allowed to banish households in equilibrium, contracts will feature separation on path. If they are not allowed to banish, the equilibria are equivalent to an incomplete markets environment with endogenous debt constraints. In particular, households will be borrowing constrained in equilibrium.

In a recent important paper, Dovis (2014) studies an environment with both hidden types and limited commitment.[2] He shows how one can decentralize the efficient allocation as an equilibrium of a sovereign debt game in which there is suspension of payments to lenders along the equilibrium path. There are two main differences between the environment in this paper and the one studied by Dovis (2014). First, in terms of the contracting problem my environment features hidden trading while in his, contracts are exclusive. This implies that if there was an exogenous cost of banishment, in his environment, separation would not be efficient. As mentioned earlier, even with a positive banishment cost, separation can be efficient in the environment I study. From an observational perspective, the key difference between our environments concerns the probability of re-entry after default. In Dovis (2014), the probability of re-entry is independent of the level of debt defaulted on. In contrast, in my environment the probability of re-entry is independent of the household's type but depends on the level of defaulted debt.

The efficient contracts studied by DeMarzo and Sannikov (2006) and Clementi and Hopenhayn (2006) feature inefficient terminations of the risk-sharing relationship on path. The reason for this is the presence of an exogenous outside option available to the lender. In particular, for certain regions of the contract space, the value of the outside technology is strictly greater than the value of firm. If the principal had access to same technology within the firm, separation would not be efficient. However, in the environment I consider, even though the intermediary can replicate the value of banishment or default on path, it is efficient to banish households in equilibrium.

This paper is also related to Hopenhayn and Werning (2008) who study a contracting environment in which the agents have a stochastic outside option that is unobservable to the principal. They show that the efficient contract features separation on path. In their model, separation also arises due the presence of an outside technology that is not available to the principal.[3] In particular, if in their model the principal had access to this technology within the firm and there was a small cost of separation, then default would not be efficient. However, in the environment I study, even though the intermediary can offer the outside option to the household without banishment, the presence of hidden trading implies that separation is necessary to achieve efficiency.

In seminal papers, Prescott and Townsend (1984) and Kehoe and Levine (1993) studied and defined constrained-efficiency for environments with moral hazard and limited commitment[4] respectively. The decentralized environment I study has both incentive compatibility and voluntary participation constraints as in these papers. However, in contrast to both papers, in my environ-

[2] See Atkeson (1991), Atkeson and Lucas (1995) and Yared (2010) for other papers with both private information and limited commitment.

[3] Note that in their model if the outside option is observable the efficient allocation will not feature separation as in Albuquerque and Hopenhayn (2004).

[4] See Kocherlakota (1996), Albuquerque and Hopenhayn (2004) and Kehoe and Perri (2002) for other papers studying models with limited commitment.

ment households can engage in hidden trading. As a result, their welfare theorems do not apply here.

Golosov and Tsyvinski (2007) study a dynamic Mirrleesian environment in which agents can trade a risk-free bond in a hidden market. They find that competitive equilibria are inefficient. The planning problem I study is related in that I also assume that households can trade in a hidden fashion. However, unlike their model, the best equilibrium in the environment I study is efficient even though the planner has control of the price in the hidden markets. This is because hidden trading in my environment implies that it is not incentive feasible to introduce any state contingency into contracts. As a result, the best the planner can do is to offer an uncontingent contract

Since the environment I study has multiple equilibria, I consider the role for policy to uniquely implement the best equilibrium. This paper uses techniques and language developed by Atkeson, Chari, and Kehoe (2010) and Bassetto (2002) which allows us to think about how policy can uniquely implement a desired competitive equilibrium.

The framework developed by Eaton and Gersovitz (1981) has been widely used in the sovereign debt literature[5]. Similar models have also been used to study the effects of changing bankruptcy laws as in Chatterjee et al. (2007) and Livshits et al. (2007). In particular, Chatterjee et al. (2007) use a model with exogenous incompleteness to understand the effects of changing bankruptcy laws. For example, they find substantial welfare gains to enacting a policy which prevents households with above median incomes from declaring bankruptcy. Since the best equilibrium is efficient in the environment I consider, such policies will not in general be welfare improving. However, I show that allowing the probability of re-entry after default to depend on the level of defaulted debt can improve welfare.

This paper is also related and contributes to the vast literature in macroeconomics that uses models with incomplete markets, two important examples of which are Huggett (1993) and Aiyagari (1994)[6].

While the environment I consider is observationally equivalent to a large class of exogenously incomplete models, the approach to efficiency I take is substantially different. Usually, the approach taken is similar in spirit to Diamond (1967) who exogenously restricts the set of instruments available to the planner. Geanakoplos and Polemarchakis (1986) and more recently Dávila et al. (2012) study such planning problems and conclude that the equilibria with incomplete markets are constrained inefficient. However, I use an example to show that outcomes which would be considered constrained-inefficient when markets are exogenously incomplete are actually constrained-efficient when markets are endogenously incomplete.

The paper proceeds as follows. In section 2 I describe the underling contracting environment and

[5]Quantitative versions of this model include Aguiar and Gopinath (2006) and Arellano (2008). See Aguiar and Amador (2014) for a short survey.

[6]These models have been used to study a variety of issues from optimal quantity of government debt by Aiyagari and McGrattan (1998) and more recently to studying the effects of exogenous shocks to the debt constraints as in Guerrieri and Lorenzoni (2011).

define an equilibrium. In section 3 and section 4, I study environments without and with banishment respectively and prove the equivalence results. Next, section 5 studies the efficiency properties of these environments while section 6 presents an application of this framework to bankruptcy policy. Finally section 7 discusses the role of various assumptions in generating the main results and section 8 concludes. Most of the proofs are contained in Appendix A.

2 Environment

Consider an infinite horizon discrete time environment, $t = 1, 2, ...$ with a continuum of infinitely lived households $i \in I$ and a continuum of overlapping $\hat{T} < \infty$ period lived[7] risk-neutral intermediaries/firms born each period. Households are risk-averse with period utility functions $u(c_t)$ where $u : \mathbb{R}_+ \to \mathbb{R}$ is an increasing and strictly concave continuously differentiable function. I also assume that u satisfies Inada conditions, $\lim_{c\to 0} u'(c) = -\infty$ and $\lim_{c\to\infty} u'(c) = 0$. There is a single non-storable consumption good of which households receive a random endowment $\theta_t \in \Theta$, $\theta_t \in \mathbb{R}_{++}$ each period where Θ is a finite set. Denote the maximal and minimal element of Θ by $\bar{\theta}$ and $\underline{\theta}$ respectively. The endowment shock is independently and identically distributed over time and households with density function $\pi(\cdot)$. Intermediaries can borrow and lend with each other at a market determined interest rate $\frac{1}{q_t}$ each period.

Households enter into long term contracts with intermediaries in order to smooth their consumption over time and can sign with multiple such intermediaries as described below. An important feature of the contracting environment is that I will endow intermediaries with a banishment technology. Banishment is publicly observable and a banished household consumes its endowment and cannot sign with other intermediaries. Once banished, the intermediary can also choose a probability of re-entry in subsequent periods. Households can also choose to not repay their debts to intermediaries and subsequently live in financial autarky in all future periods.

1. With some probability previously banished households are allowed to contract with intermediaries[8]

2. Endowments are realized and are private information to households. Households report endowments to all intermediaries they are currently signed to.

3. Households that are not banished receive transfers from or make transfers to incumbent intermediaries, namely those intermediaries with whom they have pre-existing contracts. At this time households can voluntarily choose to not participate in financial markets and live in financial autarky forever.

4. Intermediaries post contracts.

[7]Intermediaries are assumed to be finitely lived so that their problem is always well defined. See section 7 for further discussion.

[8]Note that the actual signing of a new contract takes place later in the period.

5. Households observe the offered contracts and can choose to sign with at most one new intermediary.

6. Consumption takes place

An important assumption I make throughout the paper is that any contract signed between a household and intermediary is not observable to any other intermediary. The only outcomes that are publicly observable are posted contracts, banishment histories and whether the household has chosen to not participate in financial markets.

I begin the formal description of the game between intermediaries and households by first describing the information sets available to both types of players. Let $\hat{z}^t \in \hat{Z}^t$ denote the public history at the beginning of period t. A typical history $\hat{z}^t = \left(q^{t-1}, \mathcal{B}^{t-1}, \left(\gamma^{i,t-1}\right)_{i \in I}\right)$ consists of the history of prices, posted contracts $\mathcal{B}^{t-1}$ and banishment histories for all households $\left(\gamma^{i,t-1}\right)_{i \in I}$. Denote the public history when new contracts are offered by $z^t = \left(\hat{z}^t, \left(\gamma_t^i\right)_{i \in I}, q_t\right)$. Let $\omega^{t-1} \in \Omega^{t-1}$ denote the private history at the beginning of period t where $\omega^{t-1} = \left(\theta^{t-1}, \gamma^{t-1}, B^{t-1}\right)$, θ^{t-1} is the history of endowment realizations, γ^{t-1} is the banishment history, where $\gamma_{t-1} \in \{0,1\}$ and $\gamma_t = 1$ means that the household is banished from the contracting environment, and $B^{t-1} = (B_1, ..., B_{t-1})$ is the history of signed contracts. The private history for each household i at the in period t, after endowments have been realized, is denoted by $h^t \in H^t$ where $h^t = \left(\omega^{t-1}, \gamma_t, \theta_t\right)$. If the household is not signed to any contract at the beginning of period t, I denote the contract history as $B^{t-1} = \varnothing$. Note that endowments and signed contracts are privately observed by the household while γ^t is publicly observed. In each period, households report their endowment type θ_t to intermediaries who use the public history along with the history of reports and σ^{HH} to compute B^{t-1}. It is without loss of generality to assume that all households with the same $\left(\gamma^t, \theta^t\right)$ have identical contract histories B^{t-1}. Given the public history z^t, let $\tilde{\zeta}_t\left(h^t\right)$ denote the intermediaries' beliefs of personal histories in period t. We can also define the true probability measure on the space of personal histories $\zeta_t\left(h^t\right)$ and $\wp\left(\omega^t\right)$ for histories h^t and ω^t respectively, which will be constructed after the formal definition of a contract.

A contract $B_t\left(z^t\right)$ offered in period t is defined as follows:

$$B_t\left(z^t\right) = \left({}_t\tau_{t+s}\left(z^{t+s}, m^s\right), {}_t\delta_{t+s}\left(z^{t+s}, m^s\right), {}_t\mu_{t+s}\left(z^{t+s}, m^s\right) : 0 \le s \le \hat{T} - 1\right)$$

where m^s denotes the history of type reports $(m_t, ..., m_{t+s})$. Denote the space of all such contracts by $\mathbb{B}_t$. In general, given an element of a contract ${}_tx_s$, the left subscript denotes the period in which the contract is agreed to and the right subscript the current period. Here ${}_t\tau_{t+s}\left(z^{t+s}, m^s\right) \in \mathbb{R}$ denotes the transfers to the households as a function of the history of reported types in period $t+s$, ${}_t\delta_{t+s}\left(z^{t+s}, m^s\right) \in \{0,1\}$ denotes the banishment decision in period $t+s$ with ${}_t\delta_{t+s}\left(z^{t+s}, m^s\right) = 1$ meaning that the household is banished, and ${}_t\mu_{t+s}\left(z^{t+s}, m^s\right) \in [0,1]$ is the subsequent probability of re-entry if the household is banished.

Next, I consider the problem of a household. A strategy for a household is σ_t^{HH} which maps the appropriate histories into $\{0,1\} \times \Sigma_t \times \mathcal{B}_t \times \mathbb{R}_+$ where Σ_t is the set of type reporting strategies and

$\mathcal{B}_t$ denotes the set of posted contracts in period t. In each period, the household chooses whether to participate, what to report, whether to sign a new contract and how much to consume. A typical strategy, $\sigma_t^{HH} = \{\Delta_t, \sigma_t, B_t, c_t\}$ where each element depends on the appropriate histories. Let $\Delta_t \in \{0,1\}$ denote the participation strategy for the household which depends on h^t with $\Delta_t = 0$ implying that the household chooses to not participate in financial markets and consequently live in autarky forever. Let $\Sigma = (\Sigma_t)_{t\geq 1}$ with typical element $\sigma = \left(\{\sigma_t^s\}_{s\leq t}\right)_{t\geq 1}$ where σ_t^s is the household's type reporting strategy in period t, to the intermediary associated with contract B_s where $s \leq t$ which depends on h^t for $s < t$ and ω^t for $s = t$. In particular note that the household can potentially report different types to different intermediaries. I define the truth-telling strategy σ^*, to be one that satisfies $\sigma_t^{*s} = \theta_t$ for all s and t where θ_t is the household's endowment. Given the structure of the game, if a household is not banished at the initial stage it has the option to sign at most one new contract with another intermediary from the set of posted contracts which also depends on h^t. Note however that the consumption strategy depends on the new contract and hence on ω^t. Given a private history h^t and an associated vector of signed contracts B^{t-1}, it will be useful to define the following objects

$$\tau_t^{old}\left(h^t \mid z^t\right) \equiv \sum_{s<t} {}_s\tau_t\left(z^t, \left(\sigma_s^s\left(z^s, h^s\right), ..., \sigma_t^s\left(z^t, h^t\right)\right)\right)$$

$$\delta_t\left(h^t \mid z^t\right) \equiv \min\left(\sum_{s\leq t-1} {}_s\delta_{t-1}\left(z^t, \left(\sigma_s^s\left(z^s, h^s\right), ..., \sigma_t^s\left(z^t, h^t\right)\right)\right), 1\right)$$

$$\mu_t\left(h^t \mid z^t\right) \equiv \prod_{s<t} {}_s\mu_t\left(z^t, \left(\sigma_s^s\left(z^s, h^s\right), ..., \sigma_t^s\left(z^t, h^t\right)\right)\right)$$

Here, $\tau_t^{old}\left(h^t \mid z^t\right)$ denotes the total transfers in period t from contracts signed prior to period t as a function of reports $\left(\sigma_s^s\left(z^s, h^s\right), ..., \sigma_t^s\left(z^t, h^t\right)\right)$, $\delta_t\left(h^t \mid z^t\right)$ denotes the banishment indices of contracts signed prior to t and similarly $\mu_t\left(h^t \mid z^t\right)$ is re-entry probability prescribed by these contracts. In particular, a household is banished if at least one of the contracts it is signed to prescribes banishment. For ease of notation I will subsequently refer to these objects as $\tau_t^{old}\left(h^t\right)$, $\delta_t\left(h^t\right)$ and $\mu_t\left(h^t\right)$. It is worth noting the difference between $\gamma_t\left(h^t\right)$ and $\delta_t\left(h^t\right)$. $\gamma_t\left(h^t\right)$ denotes the state of banishment at the beginning of the period, after previously banished households are stochastically allowed to sign contracts again. For example, if $\delta_{t-1}\left(h^{t-1}\right) = 1$, and the household was not allowed to contract with intermediaries at the beginning of period t, then $\gamma_t\left(h^t\right) = 1$. If it was allowed to sign contract with intermediaries, then $\gamma_t\left(h^t\right) = 0$. When a household is allowed to sign contracts with intermediaries after being banished, it starts afresh, i.e. $B^{t-1} = \varnothing$ and it can sign a new contract with any intermediary. Given the definition of a contract we can now define how the true probabilities of personal histories h^t are constructed.[9] This is done recursively as follows[10]:

[9] The probabilites for histories h^t are constructed similarly.

[10] Recall that households with identical type and exclusion histories have the same contract history B^{t-1}.

$\wp\left(\omega^1\right)=\pi\left(\theta_1\right)$ and for all $t>1$

$$\begin{aligned}\wp\left(\omega^t\right)=&\wp\left(\omega^{t-1}\right)\pi\left(\theta_t\right)\left(\left[1-\delta_{t-1}\left(h^{t-1}\right)\right]\mathbf{1}_{B_t}\right.\\&\left.+\delta_{t-1}\left(h^{t-1}\right)\left[\gamma_t\left(h^t\right)\left[1-\mu_t\left(h^{t-1}\right)\right]+\left(1-\gamma_t\left(h^t\right)\right)\mu_t\left(h^{t-1}\right)\mathbf{1}_{B_t}\right]\right)\end{aligned}\tag{2.1}$$

In the first period, $\wp$ is the same as π. In subsequent periods, the first term $\wp\left(\omega^{t-1}\right)\pi\left(\theta_t\right)$ on the right hand side of (2.1) corresponds to the probability of h^{t-1} times the probability of the current realization of type θ_t multiplied by an indicator function which indicates that contract B_t has been signed. Next, if the household was not banished last period and $\delta_{t-1}\left(h^{t-1}\right)=0$, the probability is just these two terms. However, if the household was banished in $t-1$ then with probability $\mu_t\left(h^{t-1}\right)$ it is allowed to sign contracts again and with probability $1-\mu_t\left(h^{t-1}\right)$ it is still banished.

For any $t\geq 1$ and $h^t\in H^t$, the household of type h^t chooses a strategy σ_t^{HH} to maximize

$$\sum_{s=0}^{\infty}\beta^s\sum_{\omega^{t+s}\in\Omega^{t+s}}\wp\left(\omega^{t+s}\right)u\left(c_{t+s}\left(\omega^{t+s}\right)\right)\tag{2.2}$$

subject to a budget constraints: $\forall s\geq 0,\ h^{t+s}\in H^{t+s}$ such that $\gamma_{t+s}=0$ and $\delta_{t+s}\left(h^{t+s}\right)=0$ i.e., the household is not banished,

$$c_{t+s}\left(h^{t+s}\right)\leq\Delta_{t+s}\left(h^{t+s}\right)\left[\theta_{t+s}+\tau_{t+s}^{old}\left(h^{t+s}\right)+{}_{t+s}\tau_{t+s}\left(h^{t+s}\right)\right]+\left(1-\Delta_{t+s}\left(h^{t+s}\right)\right)\theta_{t+s}\tag{2.3}$$

and $c_t\left(\omega^t\right)=\theta_t$ if $\delta_t\left(h^t\right)=1$ or $\gamma_t=1$. The term $\tau_t^{old}\left(h^t\right)$ denotes the transfers from contracts signed in periods prior to t, while ${}_t\tau_t\left(h^t\right)\equiv{}_t\tau_t\left(z^t,\sigma_t^t\left(h^t\right)\right)$ denotes the transfers from the contract B_t, signed in period t. B_t is chosen from the set of posted contracts $\mathcal{B}_t$. With slight abuse of notation I will sometimes denote the sum $\tau_t^{old}\left(h^t\right)+{}_t\tau_t\left(h^t\right)$ as $\tau_t\left(h^t\right)$. Note that if $B^{t-1}=\varnothing$, $\tau_t^{old}\left(h^t\right)=0$. The second term on the right hand side of the budget constraint says that if the household voluntarily chooses to not participate, it consumes its endowment that period. Denote the value of the above problem when the household is using reporting strategy σ by $V_t\left(h^t\right)(\sigma)$.

Note that if $\delta_t\left(h^t\right)=1$, the value of a banished household is given by

$$V_t\left(h^t\right)=u\left(\theta_t\right)+\beta\mathbb{E}_t\left[\mu_{t+1}\left(h^t\right)V_{t+1}\left(h^t,\left(\varnothing,0,\theta_{t+1}\right)\right)+\left[1-\mu_{t+1}\left(h^t\right)\right]V_{t+1}\left(h^t,\left(\varnothing,1,\theta_{t+1}\right)\right)\right]$$

Finally, lets consider the problem of an intermediary. A strategy for an intermediary is $\sigma_t^{INT}:Z^t\rightarrow\mathbb{B}_t$ and a typical strategy $\sigma_t^{INT}\left(z^t\right)=B_t$. In each period, without loss of generality, we can consider intermediaries offering one contract for each type $h^t\in H^t$ and so $B_t=\left\{B_t^{h^t}\left(z^t\right)\in h^t\in H^t\right\}$. Here $B_t^{h^t}\left(z^t\right)$ is the contract *intended* for type h^t. Since households can choose any one of these contracts, each contract B_t must satisfy self-selection constraints which require that no type has an incentive to choose a contract intended for a different type. In any period t, after new contracts are posted, define $\hat{V}_t\left(h^t,B_t^{\hat{h}^t}\left(z^t\right)\right)$ to be the value for type h^t of choosing a contract intended for type $\hat{h}^t$. Clearly, a type h^t can only choose contracts associated with histories consistent with the publicly observable component of its history, γ^t. Given a history h^t, define $H^c\left(h^t\right)$ to be the set of histories with same banishment histories as h^t. Contracts must satisfy the following self-selection constraints: for all $t,\ h^t \in H^t$,

$$\hat{V}_t\left(h^t, B_t^{h^t}\left(z^t\right)\right) \geq \hat{V}_t\left(h^t, B_t^{\hat{h}^t}\left(z^t\right)\right) \text{ for all } \hat{h}^t \in H^c\left(h^t\right) \tag{2.4}$$

Second, each contract must satisfy incentive compatibility constraints at each date and history. A contract B_t is incentive compatible if for all t, and histories $h^t \in H^t$,

$$V_{t+s}\left(h^{t+s}\right)\left(\sigma^*\right) \geq V_{t+s}\left(h^{t+s}\right)\left(\tilde{\sigma}\right) \text{ for all } \tilde{\sigma} \in \Sigma \tag{2.5}$$

where $V_t\left(h^t\right)(\sigma)$ denotes the value to type h^t of following reporting strategy $\sigma \in \Sigma$ as defined in (2.2). The incentive compatibility constraints are the restrictions that private information places on the set of feasible contracts. In particular, all contracts must have the feature that no household has an incentive to misreport its type in any period. For ease of notation, I will sometimes denote the equilibrium value for a household following the truth telling strategy by $V_t\left(h^t\right)$.

Third, any contract B_t must satisfy voluntary participation constraints at each date t, and for each history h^t. At the beginning of each period, a household can choose to not repay their debts and thereafter live in autarky forever where it just consumes its endowment each period and cannot sign with new intermediaries. Formally, the voluntary participation constraint is

$$\left[1 - {}_t\delta_{t+s}\left(h^{t+s}\right)\right] V_{t+s}\left(h^{t+s}\right)\left(\sigma^*\right) \geq \left[1 - {}_t\delta_{t+s}\left(h^{t+s}\right)\right] V_{t+s}^d\left(h^{t+s}\right) \tag{2.6}$$

where ${}_t\delta_{t+s}\left(h^{t+s}\right) \in \{0,1\}$ is the banishment index prescribed by contract B_t in period $t+s$ and $V_{t+s}^d\left(h^{t+s}\right)$ is the value of autarky which by assumption depends only on θ_t.This constraint captures the restrictions limited commitment places on the contract. It says that all households not being banished must want to participate in financial markets. I assume that if a household chooses to not participate, it lives in autarky in all future periods,[11] i.e.

$$V_t^d\left(h^t\right) = u\left(\theta_t\right) + \frac{\beta}{1-\beta^2}\mathbb{E}u\left(\theta'\right)$$

Intermediaries can borrow and lend at market determined rate $\frac{1}{q_t}$. Given public histories, σ^{HH}, the strategies of future intermediaries and reservation utilities $\left\{\tilde{V}_t\left(h^t\right)\right\}$, each intermediary chooses σ_t^{INT} to maximize

$$-\sum_{s=0}^{\hat{T}-1}\left(\prod_{j=0}^{s} q_{t+j}\right)\sum_{h^{t+s}\in H^{t+s}} \tilde{\zeta}_t\left(h^{t+s}\right)\left(\left[1 - {}_t\delta_{t+s}\left(h^{t+s}\right)\right] {}_tb_{t+s}\left(h^{t+s}\right)\right) \tag{2.7}$$

[11]This assumption can be relaxed and we can introduce an exogenous probability of re-entry each period after default.

subject to (2.4), (2.5), (2.6) and ex-ante participation constraints

$$\hat{V}_t\left(h^t, B_t^{h^t}\left(z^t\right)\right) \geq \underline{V}_t\left(h^t\right) \tag{2.8}$$

Clearly, to attract households, contracts must satisfy the above participation constraints. Of course in equilibrium, $\underline{V}_t\left(h^t\right)$ is such that intermediaries make zero profits.

I now formally define a Perfect Bayesian Equilibrium of the game.

Definition 1 *A Perfect Bayesian Equilibrium is a sequence of prices* $\{q_t\}_{t\geq 1}$, *reservation utilities* $\left\{\underline{V}_t\left(h^t\right)\right\}_{t\geq 1}$, *strategies* $\left\{\sigma_t^{HH}, \sigma_t^{INT}\right\}_{t\geq 1}$, *and beliefs* $\left\{\tilde{\zeta}_t\right\}_{t\geq 1}$ *such that*

1. *For all* t, z^t, h^t, *the strategy* σ_t^{HH} *solves the households problem (2.2)*

2. *For all* t, z^t, *given prices, reservation utilities,* σ^{HH} *and beliefs* $\tilde{\zeta}_t$, *the strategy* $\sigma_t^{INT}\left(z^t\right)$ *solves the intermediaries' problem (2.7)*

3. *Beliefs satisfy Bayes' rule wherever it applies*

4. *Markets clear: for all* $t \geq 1$,

$$\sum_{h^t \in H^t} \wp\left(\omega^t\right) c_t\left(\omega^t\right) = \sum_{h^t \in H^t} \wp\left(\omega^t\right) \theta_t$$
$$\delta_t\left(h^t\right) c_t\left(\omega^t\right) = \delta_t\left(h^t\right) \theta_t$$

Note that in any equilibrium $\tilde{\zeta}_t\left(h^t\right) = \zeta\left(h^t\right)$, the true probability of history h^t. It is also worth noting that in any equilibrium, for all dates $t \leq s \leq \hat{T} - 1$, and each history $h^t \in H^t$, contracts must satisfy budget feasibility,

$$c_{t+s}\left(h^{t+s}\right) \leq \theta_{t+s} + \tau_{t+s}^{old}\left(h^{t+s}\right) + {}_{t+s}\tau_{t+s}\left(h^{t+s}\right) \tag{2.9}$$

where as before $\tau_{t+s}^{old}\left(h^{t+s}\right)$ denotes the total transfers received from contracts signed prior to period $t+s$, including those associated with B_t. ${}_{t+s}\tau_{t+s}\left(h^{t+s}\right)$ denotes the transfers associated with a potential new (hidden) contract that households can sign in period $t+s$. Both the consumption strategy $c_{t+s}\left(h^{t+s}\right)$ and ${}_{t+s}\tau_{t+s}\left(h^{t+s}\right)$ can be computed using the household's strategy σ^{HH}. Note that when characterizing the equilibrium contract, it is without loss of generality to restrict to equilibria in which households sign with only one intermediary at a time. If the equilibrium strategy doesn't satisfy (2.9), either households are not maximizing or markets cannot clear.

As a final point about the setup, note that the space of contracts is very general. Intermediaries can decide to offer short or long term contracts depending on the actions of other intermediaries. A particularly useful contract which will play a central part in thinking about deviations is an uncontingent savings contract. Given any $\varepsilon \geq 0$ and $\delta \leq 1$, a contract $S_t^{\varepsilon,\delta}$ is called a $\varepsilon\delta-$savings contract if $S_t = ({}_t\tau_t \ , {}_t\tau_{t+1})$ where

$$\begin{aligned} {}_t\tau_t &= -q_t \varepsilon \\ {}_t\tau_{t+1} &= \delta \varepsilon. \end{aligned}$$

Note that the $S_t^{\varepsilon,\delta}$ contract is not contingent on report types. In particular if offered any household can choose to sign it. In the future, I will sometimes refer to the above as the environment with PI (private information), LC (Limited Commitment) and HT (Hidden Trading).[12]

3 Contracts without Banishment

In this section, I consider an environment in which intermediaries do not have access to the banishment technology and that they can only offer contracts that induce households to always repay their debts. One interpretation of this assumption is that failure by households to not pay back their debts and consequently not participate imposes a large exogenous cost on intermediaries. Under this assumption we can work with a simpler private history space $H^t = \Theta^t \times \mathbb{B}^{t-1}$ and the corresponding probability measure π. Since by assumption all households with identical endowment histories θ^t have the same B^{t-1}, for ease of notation I will denote a history simply by θ^t and probability $\pi\left(\theta^t\right)$. The definition of equilibrium without banishment is identical to the one section 2 except that $\delta_t\left(\theta^t\right) = 0$ for all $t, \ \theta^t \in \Theta^t$.

The main result in this section says that under the above restrictions, the set of equilibria of the intermediary game is identical to the set in an incomplete markets model where households trade a risk-free bond subject to appropriately chosen debt constraints. I now describe this equivalent environment. For one direction of the equivalence result, namely that any equilibrium of the intermediary game is an equilibrium of the incomplete markets environment, we need only consider a standard model with exogenous debt constraints. For the other direction, we need a way of endogenizing debt constraints and this will require introducing a notion of default into the incomplete markets framework.

There are a continuum of infinitely lived households, $i \in I$, who each receive an i.i.d endowment shock each period $\theta_t \in \Theta$. All households begin the period with an existing stock of debt and after knowing their endowment shock, they can choose to default and live in autarky forever or not in which case they pay their debts and can continue to trade a risk-free bond subject to debt constraints. If a household chooses not to default, it chooses an allocation $\{l_{t+s}, c_{t+s}\}_{s\geq 0}$ to maximize

$$\mathbb{E}_t \sum_{s=0}^{\infty} \beta^s \sum_{\theta^{t+s} \in \Theta^{t+s}} \pi\left(\theta^{t+s}\right) u\left(c_{t+s}\left(\theta^{t+s}\right)\right)$$

[12]By hidden trading I mean that households can sign a new contract each period in a hidden fashion.

subject to budget constraints in each period

$$c_{t+s} + l_{t+s+1} \leq \theta_{t+s} + R_{t+s} l_{t+s} \tag{3.1}$$

and debt constraints

$$l_{t+s+1} \geq -\phi_{t+s} \tag{3.2}$$

with $\phi_{t+s} \geq 0$. Note all agents face identical debt constraints ϕ_t which can only depend on calendar time and not a household's type. Denote the value of this problem at date t given existing debt level l_t by $W_t\left(\theta^t, l_t; \Phi_t\right)$ where $\Phi_t = \{\phi_{t+s}\}_{\geq 0}$.

For ease of notation I denote the entire sequence $\left\{c_t^i\left(\theta^t\right)\right\}_{\theta^t \in \Theta^t}$ by $\left\{c_t^i\right\}$. The household's problem at the beginning of date t if it hasn't defaulted in the past is to choose a default strategy $d_t \in \{0, 1\}$ to maximize

$$d_t W_t\left(\theta^t, l_t; \Phi_t\right) + \left[1 - d_t\right] V_t^d\left(\theta_t\right)$$

where as before, $V_t^d\left(\theta_t\right) = u\left(\theta_t\right) + \frac{\beta}{1-\beta^2}\mathbb{E}u\left(\theta'\right)$.

Next, I define an equilibrium concept that endogenizes the sequence of debt constraints. This is similar to the concept introduced by Alvarez and Jermann (2000).

Definition 2 *A Not-Too-Tight competitive equilibrium is a sequence of interest rates* $\{R_t\}_{t\geq 0}$, *debt constraints* $\{\phi_t\}_{t\geq 0}$, *allocations for households* $\{d_t, c_t, l_t\}_{t\geq 0}$ *such that*

1. *Given prices, the allocations solve each household's problem*

2. *Markets clear,* $\forall t$

$$\sum_{\theta^t \in \Theta^t} \pi\left(\theta^t\right) l_{t+1}\left(\theta^t\right) = 0$$

3. *The sequence* $\{\phi_t\}_{t\geq 0}$ *is chosen to be Not-Too-Tight, i.e.* $\forall t$,

$$W_{t+1}\left(\theta^{t+1}, -\phi_t; \Phi_{t+1}\right) \geq V_{t+1}^d\left(\theta_{t+1}\right) \text{ for all } \theta^{t+1}$$
$$W_{t+1}\left(\hat{\theta}^{t+1}, -\phi_t; \Phi_{t+1}\right) = V_{t+1}^d\left(\hat{\theta}_{t+1}\right) \text{ for some } \hat{\theta}^{t+1}$$

Debt constraints are "Not-Too-Tight" if the following property is true; in equilibrium, at each date and given any history θ^t, if this household has borrowed up to this constraint the previous period, it weakly prefers to not default while there exists some type $\hat{\theta}^t$ who is exactly indifferent. The idea is to allow households to hold the maximum amount of debt consistent with no default. The primary difference between the above definition and the one in Alvarez and Jermann (2000) is that unlike their environment, here debt constraints are not state contingent. In particular, in their model, agents trade Arrow securities subject to state contingent debt constraints, while here since markets are incomplete, we have constraints that are independent of states. This equilibrium concept has also been studied by Zhang (1997).

It is worth noting that the usual incomplete markets environment with exogenous debt constraints can also be defined using the model described above.

Definition 3 *A* $\Phi-$*competitive equilibrium is a sequence of interest rates* $\{R_t\}_{t\geq 0}$ *, debt constraints* $\Phi = \{\phi_t\}_{t\geq 0}$ *, allocations for households* $\{d_t, c_t, l_t\}_{t\geq 0}$ *such that*

1. *Given prices, the strategies and allocations solve each household's problem*
2. *Markets clear,* $\forall t$

$$\sum_{\theta^t \in \Theta^t} \pi\left(\theta^t\right) l_t\left(\theta^t\right) = 0 \tag{3.3}$$

The first main equivalence result of the paper proves an equivalence between the banishment-free equilibria of the intermediary game defined in the previous section and the model with a risk-free bond and endogenous debt constraints.

Theorem 1 (Equivalence: No banishment)

1. *A no-banishment equilibrium outcome of the environment with PI, LC and HT, is an equilibrium outcome of the environment with incomplete markets and not-too-tight debt constraints.*
2. *An equilibrium outcome of the environment with incomplete markets and not-too-tight debt constraints is a no-banishment equilibrium outcome of the environment with PI, LC and HT.*

A brief sketch of the proof is as follows. Consider the incomplete markets environment. A sequence of outcomes $\{q, \phi, c, s\}$ is an equilibrium of the incomplete markets environment iff

1. $u'\left(c_t\left(\theta^t\right)\right) q_t \geq \beta \mathbb{E}_t u'\left(c_{t+1}\left(\theta^{t+1}\right)\right)$ for all t, θ^t.
2. $u'\left(c_t\left(\theta^t\right)\right) q_t > \beta \mathbb{E}_t u'\left(c_{t+1}\left(\theta^{t+1}\right)\right) \Rightarrow l_{t+1}\left(\theta^t\right) = -\phi$
3. Given $\{q, \phi\}$, $\{c, l\}$ satisfy the household's budget and debt constraints (3.1) and (3.2)
4. Market clearing conditions (3.3) hold
5. The debt constraints $\{\phi\}$ are chosen to be not-too-tight, i.e.

$$W_{t+1}\left(\theta^{t+1}, -\phi_t; \Phi_{t+1}\right) \geq V^d_{t+1}\left(\theta_{t+1}\right) \text{ for all } \theta^{t+1}$$
$$W_{t+1}\left(\hat{\theta}^{t+1}, -\phi_t; \Phi_{t+1}\right) = V^d_{t+1}\left(\hat{\theta}_{t+1}\right) \text{ for some } \hat{\theta}^{t+1}$$

The proof requires a series of preliminary results. The three main propositions that required to prove Theorem 1 are Proposition 1, Proposition 2 and Proposition 3. The first of these propositions shows that an equilibrium of the intermediary game satisfies conditions 1. and 5. above. The second proposition shows that these outcomes satisfy 3 and the final proposition shows 2.

The first main proposition required to prove Theorem 1 says that in any equilibrium of the intermediary game, households can only be borrowing constrained and never savings constrained. Further, if a household is borrowing constrained in a period then the voluntary participation constraint binds for some type in the following period.

Proposition 1 *In any non-autarkic equilibrium of the intermediary game*

1. *For all types* θ^t,
$$u'\left(c_t\left(\theta^t\right)\right) q_t \geq \beta \mathbb{E}_t u'\left(c_{t+1}\left(\theta^{t+1}\right)\right)$$

2. *In any period if for any* θ^t,
$$u'\left(c_t\left(\theta^t\right)\right) q_t > \beta \mathbb{E}_t u'\left(c_{t+1}\left(\theta^{t+1}\right)\right)$$
then there exists some $\tilde{\theta}^{t+1}$ *such that*
$$V_{t+1}\left(\tilde{\theta}^{t+1}\right) = V_{t+1}^{d}\left(\tilde{\theta}^{t+1}\right)$$

Proof. See Appendix A. ■

If a household was savings constrained, at the second stage of the period a new intermediary can offer an $\varepsilon\delta$ savings contract[13] which would make both it and the household strictly better off. In the case in which intermediaries write default-free contracts, they will be unwilling to lend too much since zero profits requires imposing negative transfers in subsequent periods on the household which would worsen its incentives to default. The second part of the proposition shows that if a household is Euler-constrained, then it must be that the voluntary participation constraints bind for some type the following period. The reason for this is clear, if not then an intermediary can increase transfers to the constrained household in the current period and reduce them in the following period in a way so as to make strictly positive profits since the shadow rate of interest of a constrained agent is higher than the market rate.

The second main proposition required for the equivalence result says that we can represent any $\hat{T}$ period contract as a sequence of 2 period contracts, each one of which makes zero profits.

Proposition 2 *Given an equilibrium of a truncated* T*-period environment with* $\hat{T}$ *period lived overlapping intermediaries, there exists an equilibrium with 2 period lived intermediaries with same allocations and prices.*

Proof. See Appendix A. ■

This proposition establishes that in equilibrium, intermediaries can only offer short term contracts. Given a $\hat{T}-$period contract we set the first period transfers to be the same and in subsequent periods, we split transfers from the original contract into $\zeta_t = -\frac{{}_{t-1}\tau_t}{q_t} + {}_t\tau_t$ where ${}_{t-1}\tau_t$ is the period t transfer from a contract signed in period $t-1$ and ${}_t\tau_t$ is the transfer from an intermediary born in period t. Since the original $\hat{T}$ period lived intermediaries must make zero profits, to show that these 2 period contracts also make zero profits it is sufficient to show that the expected present discounted value of transfers from $\hat{T}-1$ onwards, is independent of the period $\hat{T}-1$ report of endowment. In particular, given a history $\theta^{\hat{T}-2}$, I show that the present discounted value of transfers

[13] Recall that a contract $S_t^{\varepsilon,\delta}$ is called a $\varepsilon\delta-$savings contract if $S_t = \left({}_t\tau_t\ ,\ {}_t\tau_{t+1}\right)$ where ${}_t\tau_t = -q_t\varepsilon$ and ${}_t\tau_{t+1} = \delta\varepsilon$.

in $\hat{T}-1$, is independent of $\theta_{\hat{T}-1}$. To see why, suppose we have two types $\left(\theta^{\hat{T}-2},\theta\right)$ and $\left(\theta^{\hat{T}-2},\theta'\right)$ with $\theta>\theta'$, but type $\left(\theta^{\hat{T}-2},\theta'\right)$ receives the higher present discounted value of transfers. There are two cases to consider. The first is that the difference in transfers is front-loaded and that period $\hat{T}-1$ transfers are higher for type $\left(\theta^{\hat{T}-2},\theta'\right)$. In this case, type $\left(\theta^{\hat{T}-2},\theta\right)$ will strictly prefer to lie and pretend to be $\left(\theta^{\hat{T}-2},\theta'\right)$, and save with another intermediary. As mentioned earlier, intermediaries are always willing to over $\varepsilon\delta$ savings contracts and one can be constructed to make both the lying agent and a new intermediary strictly better off. The second case is a little more complicated in the case in which the difference in transfers is back-loaded and both types are Euler-constrained. However, I show that if a lower type weakly prefers the backloaded transfer scheme (which should be true in equilibrium) type $\left(\theta^{\hat{T}-2},\theta\right)$ will again strictly prefer to lie and pretend to be $\left(\theta^{\hat{T}-2},\theta'\right)$. On the other hand, if $\left(\theta^{\hat{T}-2},\theta\right)$ receives the higher present discounted value of transfers, then a perturbation which redistributes to types below θ increases ex-ante welfare since it increases the amount of insurance in $\hat{T}-1$. Such a perturbation always satisfies voluntary participation constraints since one can show (see Lemma 7) that these constraints only bind for the lowest types. An important property in a 2-period lived intermediary environment is that for all t, and histories θ^{t-1}, the present discounted values of equilibrium transfers is independent of θ_t.

The results so far suggest that the equilibria in the intermediary environment are equivalent to one in which agents trade a risk-free bond subject to debt constraints. In particular, any equilibrium with incomplete markets and borrowing constraints must satisfy the constrained Euler equation and the above conditions on the transfers. The next few results will help us prove some properties about the corresponding debt constraints. The third key proposition required to prove Theorem 1 shows that in any period, all Euler-constrained households have identical debt constraints.

Proposition 3 *For any t, and θ^t such that*

$$u'\left(\theta_t+{}_{t-1}\tau_t\left(\theta^t\right)+{}_t\tau_t\left(\theta^t\right)\right)q_t>\beta\mathbb{E}_t u'\left(\theta+{}_t\tau_{t+1}\left(\theta^{t+1}\right)+{}_{t+1}\tau_{t+1}\left(\theta^{t+1}\right)\right)$$

it must be that ${}_t\tau_t\left(\theta^t\right)=\varphi_t$ where φ_t is independent of the household's history.

Proof. See Appendix A. ■

The proof follows from a preliminary result which states that in equilibrium, the value of not defaulting for any two types θ^t and $\tilde{\theta}^t$ such that $\theta_t+{}_{t-1}\tau_t=\tilde{\theta}_t+{}_{t-1}\tilde{\tau}_t$ are identical. Notice that here ${}_{t-1}\tau_t$ corresponds to the transfer in period t from a contract signed in period $t-1$. I prove this using an induction argument. Given that we are working in a truncated economy, consider the last period T in which intermediaries are operational. Since from period T onwards households trade a risk-free bond, the household's value going forward depends on only its current endowment and transfer. Next, suppose the hypothesis is true from period $t+1$ onwards and so we want to establish that it is true in period t. For contradiction, suppose we have two histories such that $\theta_t+{}_{t-1}\tau_t=\tilde{\theta}_t+{}_{t-1}\tilde{\tau}_t$ but $V_t\left(\theta^t\right)>V_t\left(\tilde{\theta}^t\right)$. The idea of the proof is to show that a deviating intermediary can give agent $\tilde{\theta}^t$ a contract similar to type θ^t, which makes both the household and it strictly better off while still satisfying incentives. The key condition that needs to be checked is that such a contract does not incentivize default the following period. Notice that household $\tilde{\theta}^t s$ incentives to default in period $t+1$ are exactly the same as household θ^t if they receive the same transfers since the value of the two households going forward is identical by the induction assumption.

The result states that in the environment with 2 period lived intermediaries, the transfers received from new contracts signed in period t are identical for households that are Euler-constrained in period t. At the first glance, the result may seem surprising since in general the present discounted value of transfers is not identical across all histories. Suppose we have two households with different histories who are Euler-constrained in period t. Given that each contract must make zero profits, contracts offered in period t are of the form $\left(\varphi, -\frac{\varphi}{q_t}\right)$. Competition among intermediaries will force φ to be as high as possible consistent with no default the following period for each Euler-constrained household. Then the previous proposition tells us that all agents receiving $\left(\varphi, -\frac{\varphi}{q_t}\right)$ will have exactly the same incentives to default independent of history. As a result, such a contract will always satisfy voluntary participation constraints.

Using these characterization results, we can proceed to proof of the equivalence theorem (see Appendix A). The proof of the first part of the theorem is a direct consequence of the properties proved in the previous section. The necessary and sufficient conditions for an allocation-price pair to constitute a $\Phi-$competitive equilibrium are, for all t, θ^t

$$u'\left(c_t\left(\theta^t\right)\right) \geq \beta R_{t+1} \mathbb{E}_t u'\left(c_{t+1}\left(\theta^{t+1}\right)\right)$$

and

$$u'\left(c_t\left(\theta^t\right)\right) > \beta R_{t+1} \mathbb{E}_t u'\left(c_{t+1}\left(\theta^{t+1}\right)\right)$$
$$\Rightarrow b_{t+1} = -\phi_t$$

Finally, the budget constraint must hold at each date and state. The first two properties are satisfied in equilibrium of the intermediary game as described earlier. The second follows from the

fact that in any equilibrium of the intermediary game, the equilibrium expected present discounted value of transfers, $A_1\left(\theta_1\right) = 0$ and for all t, and histories θ^{t-1}, $A_t\left(\theta^{t-1}, \theta\right) = A_t\left(\theta^{t-1}, \theta'\right)$ for all $\theta, \theta' \in \Theta$ where

$$A_t\left(\theta^{t-1}, \theta\right) \equiv b_t\left(\theta^{t-1}, \theta_t\right) + q_t \sum_{\theta' \in \Theta} \pi\left(\theta'\right) A_{t+1}\left(\theta^{t-1}, \theta, \theta'\right)$$

For the converse, we need to show that if all intermediaries are offering $\Phi-$contracts[14], no existing or new intermediary has an incentive to deviate and offer contracts that make positive profits. First consider the case of a new intermediary. The only type of deviating contract we need to consider is one in which an Euler-constrained household at some date receives an increased transfer. Incentive

[14]Simple borrowing and lending contract subject to debt constraints

compatibility requires that the contract make a negative transfer in the following period. However, any Not-too-tight competitive equilibrium has the property that some type's voluntary participation constraint binds the following period and therefore this negative transfer cannot be uncontingent. I show that such a deviating contract is never incentive compatible in any period since households are not constrained to reporting the same type to different intermediaries. In particular, they can always report the type that results in the highest transfer to the new intermediary while reporting their true type to the original intermediary. Finally we need to consider the incentives for an existing intermediary to modify its contract. As in the case with the new intermediary, the relevant deviations involve increasing transfers to Euler-constrained households at some t, and reducing transfers the following period. Since some type's voluntary participation constraint binds in $t + 1$, the negative transfer must be state-contingent. Consider imposing the negative transfer on those households that are Euler constrained in $t + 1$. Since the lowest type falls into this category, clearly this is not possible since his voluntary participation constraint is binding. On the other hand, if the negative transfers are imposed on those households that are not Euler-constrained, these agents will strictly prefer to lie and pretend to be a lower type. Therefore such contracts are not incentive compatible.

It is worth noting that all three frictions, i.e., private information, limited commitment and hidden trading are necessary to obtain the above characterization. Environments with only private information, for example Atkeson and Lucas (1992) or private information and limited commitment as in Dovis (2014) cannot be decentralized with only a short term uncontingent bond. In particular, it is not true in such environments that the present discounted value of transfers is independent of current type. Environments with private information and hidden trading imply contracts that resemble trade in a risk-free bond as was shown by Allen (1985). Cole and Kocherlakota (2001) Also prove a similar result in an environment with hidden savings. However in both these environments, no agent is Euler-constrained in equilibrium and as a result the efficient allocation cannot be decentralized as an environment with a risk-free bond and binding (endogenous) debt constraints. In particular, the efficient allocation in models with private information and hidden savings will not in general satisfy voluntary participation constraints introduced in the previous sections.

3.1 Equilibrium Existence and Multiplicity

Next, I consider whether equilibria of the intermediary game without banishment exist. Given the equivalence result, it suffices to prove the existence of a Not-too-tight competitive equilibrium. To show existence, I focus on stationary recursive competitive equilibria and show that these are well defined and exist. The main theorem in this subsection is that are multiple competitive equilibria.

We can write the problem of a household recursively as follows:

$$W(\theta, l, \phi; \Phi) = \max_{c,b} u(c) + \beta \mathbb{E} W(l', \phi'; \Phi')$$

$$\text{subject to}$$

$$c + l' \leq \theta + Rl$$

$$l' \geq -\phi$$

where θ is the household's current endowment, l its assets and ϕ, the current debt constraint which is determined by the rule $\phi' = \Phi(\phi)$ where Φ is known to all households.

In this case the value of default is given by

$$V^d(\theta) = u(\theta) + \mathbb{E} V^d(\theta')$$

As earlier we can define the notion of a $\Phi-$Recursive competitive equilibrium and finally a Not-Too-Tight RCE. Let $\mathbb{A}$ be the bounded space of assets and $\mathcal{P}(\mathbb{A})$ the set of probability measures on $\mathbb{A}$.

Definition 4 *A $\Phi-$Recursive Competitive Equilibrium is price function $R(\phi)$, a law of motion $\phi' = \Phi(\phi)$, a measurable map $G : R_+ \times \mathcal{P}(\mathbb{A})$,value functions $W(\theta, b, \phi; \Phi)$, policy functions $l'(\theta, l, \phi)$ such that*

1. *Given R and Φ, the value functions and policy functions solve the households' problems and*
2. *the sequence of distributions generated by G is such that markets clear*

$$\int_{\mathbb{A}\times\Theta} l'(\theta, l, \phi)\, d\lambda(l, \Theta) = 0$$

where

$$\lambda' = G(\phi, \lambda)$$

Definition 5 *A NTT$-$Recursive Competitive Equilibrium is price function $R(\phi)$, a law of motion $\phi' = \Phi(\phi)$, a measurable map $G : R_+ \times \mathcal{P}(\mathbb{A})$,value functions $W(\theta, l, \phi; \Phi)$, policy functions $b'(\theta, l, \phi)$ such that*

1. *Given R and Φ, the value functions and policy functions solve the agents problems and*
2. *the sequence of distributions generated by G is such that markets clear*

$$\int_{\mathbb{A}\times\Theta} l'(\theta, b, \phi)\, d\lambda(l, \Theta) = 0 = 0$$

where

$$\lambda' = G(\phi, \lambda)$$

3. If $\phi' \in \Phi(\phi)$ then

$$W\left(\theta, -\phi', \phi'; \Phi\right) \geq V^d(\theta) \text{ for all } \theta \in \Theta$$
$$W\left(\theta^*, -\phi', \phi'; \Phi\right) = V^d(\theta^*) \text{ for some } \theta^* \in \Theta$$

Define $\eta = \int_{\theta \in \Theta} u'(\theta)\, dF(\theta)$ and let

$$\kappa = \min_{\theta} \frac{u'(\theta) + \beta\eta}{u'(\theta) + \beta\eta + \beta^2\eta}$$

Theorem 2 (Existence: No banishment) *Under the following sufficient condition*

$$\frac{u'(\bar{\theta})}{\beta\eta} < \kappa$$

there exist multiple NTT−Recursive Competitive Equilibria.

Proof. See Appendix A. ■

The first step in the proof is to show that given a measurable map Φ, a $\Phi-$RCE always exists. Next, it always true that a $\Phi-$RCE with Φ being the zero map is NTT-RCE. The reason for this is clear. If debt constraints are zero each period, then in equilibrium agents consume their endowment which trivially implies that the voluntary participation constraint binds for each period and each type. The final and key proposition that completes the proof of Theorem 2 is to show that there exists a NTT-RCE with $\Phi \neq 0$. The idea is to show that for each θ, there exists Φ^θ, such that debt constraints are ϕ^θ each period and

$$W\left(\theta, -\phi^\theta, \phi^\theta; \Phi^\theta\right) = V^d(\theta)$$

Then setting $\phi = \min_\theta \phi^\theta$ given us a $\Phi-$RCE with debt constraints that are not too tight.

The above result along 1 shows that the intermediary game with no-banishment contracts has multiple equilibria. There exists an equilibrium of the decentralized contracting environment in which all intermediaries offer null contracts to households.. A simple way of understanding this result is to notice a *strategic complementarity* in the actions of intermediaries. In particular, if an intermediary believes that no future intermediary is willing to lend to households, it will be unwilling to lend since the household will choose to default in subsequent periods.

On the surface this might seem a surprising result since one would expect a intermediary to always be able to construct a deviating contract that offers some insurance and hence make positive profits. To see why this is not possible, consider a $\hat{T}$ lived intermediary born at date $t+1$. In the last period of the contract, $\hat{T}$ it must be that ${}_{t+1}\tau_{\hat{T}}\left(\theta^{\hat{T}}\right) \geq 0$ since no intermediary in the future is offering any insurance. If ${}_{t+1}\tau_{\hat{T}}\left(\theta^{\hat{T}}\right) < 0$ for any $\theta^{\hat{T}}$ that household will strictly prefer to default. Now consider $\hat{T}-1$. For any $\theta^{\hat{T}-1}$ it must be that ${}_{t+1}\tau_{\hat{T}-1}\left(\theta^{\hat{T}-1}\right) \leq 0$ since if is strictly positive then in order to preserve incentive compatibility and make positive profits the intermediary will have to set transfers negative for some type in $\hat{T}$. Therefore the only feasible perturbation in $\hat{T}-1$ must be ${}_{t+1}\tau_{\hat{T}-1}\left(\theta^{\hat{T}-1}\right) < 0$ and ${}_{t+1}\tau_{\hat{T}}\left(\theta^{\hat{T}-1}\right) > 0$. Note again that if ${}_{t+1}\tau_{\hat{T}}\left(\theta^{\hat{T}-1}\right)$ depended on $\theta_{\hat{T}}$ incentive compatibility would be violated. The perturbation resembles a savings contract. However if the interest rates are such that $R_{\hat{T}} \leq \frac{u'(\bar{\theta})}{\beta \mathbb{E} u'(\theta)} b$ such a contract would have to offer a return on savings $> R_{\hat{T}}$ which would mean that the intermediary makes negative profits. For any $R \leq R_{\hat{T}}$ the household prefers the transfer schedule ${}_{t+1}\tau_{\hat{T}-1}\left(\theta^{\hat{T}-1}\right) = 0$, ${}_{t+1}\tau_{\hat{T}}\left(\theta^{\hat{T}-1}\right) = 0$ to the one offered by the deviating contract. Therefore in $\hat{T}-1$ it must be that ${}_{t+1}\tau_{\hat{T}-1}\left(\theta^{\hat{T}-1}\right) \geq 0$. A similar argument works in $\hat{T}-2$ and hence for previous periods.

4 Contracts with Banishment

In this section, I allow intermediaries to use the banishment technology in equilibrium. The main result in this section shows that under sufficient conditions, intermediaries will choose to banish households in equilibrium. As a result, equilibria will feature periods in which households are in financial autarky.

Proposition 4 *For $\pi(\underline{\theta})$ small and $u(\underline{\theta}) + \frac{\beta}{1-\beta^2}\mathbb{E}u(\theta')$ large enough, any non-autarkic equilibrium features banishment on path.*

Proof. See Appendix A. ■

The idea behind the proof is to show that given any equilibrium with no banishment, a deviating intermediary can offer a contract with temporary banishment in some states and make strictly positive profits while making some household strictly better off. As we saw in the previous section (Theorem 1), any equilibrium contract with no banishment takes the form of a simple uncontingent borrowing and lending subject to history independent debt constraints. Consider any such equilibrium and a period t, and a type $\left(\theta^{t-1}, \theta\right)$ who is Euler-constrained (borrowing constrained). Given that there is no banishment, we can work with the more familiar type spaces Θ^t. One can show (Lemma 7) that this implies that in period $t+1$, the voluntary participation constraint for type $\left(\theta^{t-1}, \theta, \underline{\theta}\right)$ is binding. A deviating intermediary can modify the original contract as follows

$$
\begin{aligned}
{}_t\tilde{\tau}_t\left(\theta^t\right) &= {}_t\tau_t\left(\theta^t\right) + \varepsilon \\
{}_t\tilde{\tau}_{t+1}\left(\theta^t, \theta\right) &= -\frac{\left[{}_t\tau_{t+1}\left(\theta^t, \theta\right) - R_{t+1}\varepsilon\right]}{1-\pi\left(\underline{\theta}\right)} \text{ for all } \theta \neq \underline{\theta}
\end{aligned}
$$

where τ corresponds to the original equilibrium contract and $\varepsilon > 0$. Under the deviating contract, type $\left(\theta^{t-1}, \theta, \underline{\theta}\right)$ is banished and in each subsequent period is allowed back into the contracting environment with probability $\mu = 0$ and hence receives value $V_t^d\left(\underline{\theta}\right)$ equal to the value under the original contract. The change in welfare for the household in t is given by

$$\Delta\left(\theta^{t-1},\theta\right) = u\left(\theta + {}_{t-1}\tau_t + {}_t\tilde{\tau}_t\right) + \beta\sum_{\theta'>\underline{\theta}} u\left(\theta + {}_t\tilde{\tau}_{t+1} + {}_{t+1}\tau_{t+1}\right) - u\left(\theta + {}_{t-1}\tau_t + {}_t\tau_t\right) - \beta\sum_{\theta'\in\Theta} u\left(\theta + {}_t\tau_{t+1} + {}_{t+1}\tau_{t+1}\right)$$

One can use a Taylor approximation to show that $\text{sgn}\left(\Delta\left(\theta^{t-1},\theta\right)\right) \geq \text{sgn}\left(\tilde{\Delta}\left(\theta^{t-1},\theta\right)\right)$ where

$$\tilde{\Delta}\left(\theta^{t-1},\theta\right) \approx u'\left(\theta + {}_{t-1}\tau_t + {}_t\tau_t\right) - \frac{\beta R_{t+1}}{1-\pi\left(\underline{\theta}\right)}\sum_{\theta'\in\Theta}\pi\left(\theta'\right)u'\left(\theta' + {}_t\tau_{t+1} + {}_{t+1}\tau_{t+1}\right) + \beta\sum_{\theta'\in\Theta}\pi\left(\theta'\right)\left[u\left(\theta' + \frac{{}_t\tau_{t+1}}{1-\pi\left(\underline{\theta}\right)} + {}_{t+1}\tau_{t+1}\right) - u\left(\theta' + {}_t\tau_{t+1} + {}_{t+1}\tau_{t+1}\right)\right]$$

which is strictly positive if $\pi\left(\underline{\theta}\right)$ is small enough since the type is Euler-constrained in period t. Moreover, the intermediary is as well off as before. As a result a contract can be constructed that makes both the intermediary and the household strictly better off.

The above result might seem surprising since the intermediary can implement the outcomes associated with banishment without actually having to banish the household. For example, consider the following contract

$$\tilde{\tau}_t\left(\theta^{t-1},\theta\right) = \tau_t\left(\theta^{t-1},\theta\right) + \varepsilon$$
$$\tilde{\tau}_{t+1}\left(\theta^{t-1},\theta,\theta'\right) = \frac{\tau_{t+1}\left(\theta^{t-1},\theta,\theta'\right) - R_{t+1}\varepsilon}{1-\pi\left(\underline{\theta}\right)} \text{for all } \theta' \neq \underline{\theta}$$
$$\tilde{\tau}_{t+1}\left(\theta^{t-1},\theta,\underline{\theta}\right) = \tau_{t+1}\left(\theta^{t-1},\theta,\underline{\theta}\right)$$

Such a contract also gives type $\left(\theta^{t-1},\theta,\underline{\theta}\right)$, the value associated with banishment. However, Proposition 17 implies that such a contract is not incentive compatible since here the present discounted value of transfers to type $\left(\theta^{t-1},\theta,\underline{\theta}\right)$ is larger than that for types $\left(\theta^{t-1},\theta,\theta'\right)$, $\theta' > \underline{\theta}$. As a result, these types will strictly prefer to lie downwards and save with another intermediary.

Suppose we had *exclusive contracts* in that households can only sign contracts with one intermediary at a time. Then the above perturbation can be implemented without banishment on path using the following transfer scheme

$${}_t\tilde{\tau}_t\left(\theta^t\right) = {}_t\tau_t\left(\theta^t\right) + \varepsilon$$
$${}_t\tilde{\tau}_{t+1}\left(\theta^t,\theta\right) = -\frac{\left[{}_t\tau_{t+1}\left(\theta^t,\theta\right) - R_{t+1}\varepsilon\right]}{1-\pi\left(\underline{\theta}\right)} \text{ for all } \theta \neq \underline{\theta}$$
$${}_t\tilde{\tau}_{t+1}\left(\theta^t,\underline{\theta}\right) = 0 \text{ and } 0 \text{ in all future periods}$$

As before such a scheme gives type $\left(\theta^t,\underline{\theta}\right)$ a value equal to autarky. Since we know that under the original contract, the present discounted value of transfers to any type must be 0 and

$\tau_{t+1}\left(\theta^{t-1},\theta,\theta'\right) < 0$, it must that the under the above contract, the present discounted value of transfers to these types is less than zero. However, unlike the environment with non-exclusive contracts, these types will not strictly prefer to lie since they cannot save and borrow in a hidden fashion. In particular, given a type $\left(\theta^{t-1},\theta,\theta'\right),\ \theta' \neq \underline{\theta}$, the value of lying is

$$W_{\theta^t}\left(\theta',\underline{\theta}\right) = V^d\left(\theta'\right) \leq W_{\theta^t}\left(\theta',\theta'\right)$$

and so this perturbation preserves incentives. To summarize, the crucial difference in the case with hidden trading (non-exclusive contracts) is that banished households are unable to sign contracts with other intermediaries, which allows banishment to incentivize truthful revelation of types. With hidden trading, banishing a household is equivalent to a contract with transfers equal to zero in all future periods.

A more stark way to distinguish contracts with exclusivity to those without it is to consider an environment in which intermediaries face an exogenous cost of banishment. One way to interpret this cost is to assume that intermediaries need pay an outside regulatory authority to monitor households and make sure that they don't sign contracts with other intermediaries while banished. With exclusive contracts, any equilibrium in which households are being banished is Pareto-inferior to one in which they are not since intermediaries can provide the autarkic value to households on path and save the cost. However, with hidden trading, equilibria in which intermediaries pay this cost and banish households may Pareto-dominate all equilibria without banishment. This is the sense in which the hidden trading assumption is necessary to get banishment/default on path.

Next, I provide a characterization of the equilibrium in some special cases. First suppose that intermediaries live for two periods.[15] Then I show that an equilibrium outcome of this environment is also an equilibrium outcome of an Eaton-Gersovitz like environment with short-term defaultable debt and suitably chosen re-entry probabilities. An equilibrium of the intermediary game was defined in the previous section. Next, I define the equivalent environment.

There are continuum of infinitely lived households $i \in I$. Households begin each period t, with asset holdings l_t. The timing within a period is as follows:

1. At the beginning of period t, θ_t is realized
2. Households choose whether to default or pay back l_t.
    - If it pays back, the household can issue new debt, l_{t+1}, at corresponding price schedule $\mathcal{Q}_{t+1}\left(l_{t+1}\right)$
    - If the household defaults, it consumes its endowment in the current period and in future periods is allowed to trade in financial markets with probability $\lambda\left(l_t\right)$
3. Households consume

[15] A full characterization of the equilibrium with longer lived intermediaries is in progress.

In each period, given state (θ_t, l_t) households choose (c_t, l_{t+1}) to maximize

$$V_t^R(\theta_t, l_t; \mathcal{Q}_t) = u(c_t) + \beta \mathbb{E}_t V_{t+1}^0(\theta_{t+1}, l_{t+1}; \mathcal{Q}_{t+1})$$

subject to a budget constraint

$$c_t + \mathcal{Q}_t(l_{t+1}) \leq \theta_t + l_t \tag{4.1}$$

Here $\mathcal{Q}_t(l_{t+1})$ is the debt pricing schedule which is taken as given by households. If a household defaults, it consumes its endowment that period and in subsequent periods it can regain access to financial markets with probability $\lambda(l)$. Notice that the re-entry probability only depends on the level of debt l that was defaulted on and is independent of the household's endowment. Therefore, the value of default is given by

$$V_t^D(\theta_t; \lambda(l)) = u(\theta_t) + \beta \mathbb{E}_t \left[\lambda(l) V_{t+1}^R(\theta_{t+1}, l_{t+1}; \mathcal{Q}_{t+1}) + (1 - \lambda(l)) V_{t+1}^D(\theta_{t+1}; \lambda)\right]$$

At the beginning of each period, households choose whether to default or not, $d_t = \{0, 1\}$ with $d_t = 0$ implying default,

$$V_t^0(\theta_t, l_t; \mathcal{Q}_t, \lambda) = \max_{d_t \in \{0,1\}} d_t V_t^R(\theta_t, l_t; \mathcal{Q}_t) + [1 - d_t] V_t^D(\theta_t; \lambda(l_t))$$

Households borrow and lend with a continuum of risk-neutral lenders who have an outside option that yields return $R_{t+1} = \frac{1}{q_t}$ in period $t+1$. Therefore, in order to break even, the price of debt is determined by

$$\begin{aligned} &- \left[1 - \Pr\left[V_{t+1}^D > V_{t+1}^R\right]\right] \frac{l}{\mathcal{Q}_t(b)} \geq -R_{t+1} l \\ &\Rightarrow \mathcal{Q}_t(l) = \frac{\left[1 - \Pr\left[V_{t+1}^D > V_{t+1}^R\right]\right] l}{R_{t+1}} = q_t \left[1 - \Pr\left[V_{t+1}^D > V_{t+1}^R\right]\right] l \end{aligned}$$

where

$$\Pr\left[V_{t+1}^D > V_{t+1}^R\right] = \sum_{\theta \in \Theta} \pi(\theta) \mathbf{1}_{V_t^D(\theta; \lambda(l)) > V_t^R(\theta, l; \mathcal{Q}_t)}$$

determines the probability that the household will default the next period.

Definition 6 *Given a sequence $\{q_t\}$ and a function $\lambda(l)$, a competitive equilibrium consists of value functions V^0, V^D, V^R, policy functions, d, s, c and a pricing schedule $\mathcal{Q}(l)$ such that*

1. *Given the pricing schedule, the value functions and policy functions solve the household's problem*

2. *For all t, $\mathcal{Q}_t(l) = q_t \left[1 - \Pr\left[V_{t+1}^D > V_{t+1}^R\right]\right] l$*

The next result states that if intermediaries live for two periods, then an equilibrium outcome of the intermediary game (environment with PI, LC and HT) is also an equilibrium outcome of the Eaton-Gersovitz environment defined above.

Proposition 5 *Suppose intermediaries live for two periods. Then there exists a function $\lambda(l)$ such that an equilibrium outcome of the environment with PI, LC and HT is an equilibrium outcome of the EG economy with re-entry probabilities given by $\lambda(l)$.*

A sketch of the proof is as follows. Given a sequence $\{q_t\}_{t\geq 1}$ and a function $\lambda(l)$, we know that a sequence of outcomes $\{\mathcal{Q}(s), d, l, c\}$ is an equilibrium of the Eaton-Gersovitz economy if and only if $\exists$ functions V^0, V^D, V^R s.t.

1. $(c_t, l_{t+1}) \in \arg\max_{c,b} V_t^R(\theta_t, l_t; \mathcal{Q}_t) = u(c) + \beta \mathbb{E}_t V_{t+1}^0(\theta_{t+1}, l_{t+1}; \mathcal{Q}_{t+1})$ subject to (4.1)
2. $d \in \arg\max_{\hat{d}_t \in \{0,1\}} \hat{d}_t V_t^R(\theta_t, l_t; \mathcal{Q}_t) + \left[1 - \hat{d}_t\right] V_t^D(\theta_t; \lambda)$
3. $\mathcal{Q}_t(b) = q_t\left[1 - \Pr\left[V_{t+1}^D > V_{t+1}^R\right]\right] l$

The idea is to show that given an equilibrium outcome of the intermediary game we can construct such functions and outcomes that satisfy the above conditions. The four main results required to prove Proposition 5 are Lemma 1, Proposition 6, Proposition 7 and Proposition 8. The first establishes that each contract must make zero profits and intermediaries cannot cross-subsidize between types. The second result shows that households can never be savings constrained in equilibrium, the third that there exists the corresponding price function depends only on the level of debt and the last establishes that the re-entry probability is a function of the level of debt defaulted on.

Given a contract $B_t(z^t) = \left\{B_t^{h^t}(z^t) : h^t \in H^t\right\}$, let ${}_t\mathcal{P}_t(h^t)$ denote the expected present discounted value of transfers associated with contract $B_t^{h^t}(z^t)$ from period t onwards. Therefore,

$$
{}_t\mathcal{P}_t(h^t) = \left[1 - {}_t\delta_t(h^t)\right]\left[{}_t\tau_t(h^t) + q_t \sum_{h_{t+1} \in H_{t+1}} \zeta_{t+1}(h^t, h_{t+1})\, {}_t\mathcal{P}_{t+1}(h^t, h_{t+1})\right]
$$

where ${}_tb_t(h^t) = 0$ if ${}_t\delta_t(h^t) = 1$.

Lemma 1 *In any equilibrium, for any t and any contract offered by an intermediary born at date t, ${}_t\mathcal{P}_t(h^t) = 0$ for all $h^t \in H^t$.*

Proof of Lemma 1. See Appendix A. ■

In particular, when intermediaries live for two periods, perfect competition implies that each two period contract must make zero profits in equilibrium.

To compute properties of the equilibria in the intermediary game, we will consider the limit of a sequence of truncated economies. In particular, I assume that there exists a finite date T, such that from $0 \leq t \leq T$, intermediaries offer contracts and for all $t > T$, those agents who have not defaulted in the past trade a risk free bond subject to exogenous debt constraints $\{\phi_t^e\}_{t>T}$. The claim that we can take such limits is formalized in the appendix.

In the intermediary game, given that types are being banished, we can define *banishment sets* as follows, $D_t\left(h^{t-1}\right)=\left\{h_t \in H_t: \ \delta_t\left(h^{t-1},h_t\right)=1\right\}$, i.e. the set of types being banished in equilibrium. Similarly, let $D_t^c\left(h^{t-1}\right)$ denote the complement of that set. Given a random variable $x\left(\omega^t\right)$, define $\mathbb{E}_{D_t^c(h^{t-1})}x\left(\omega^t\right) \equiv \sum_{\omega^t\in\Omega^t}\wp\left(\omega^t\right)\left[1-\delta_t\left(h^t\right)\right]x\left(h^t\right)$. The first main result required to prove Proposition 5 says that in equilibrium, households can never be savings constrained.

Proposition 6 *In any equilibrium of the intermediary game, for all t and $h^t \in H^t$,*

$$q_t \geq \frac{\beta\mathbb{E}_{D_t^c(h^{t-1},h)}u'\left(c_{t+1}\left(\omega^{t+1}\right)\right)}{u'\left(c_t\left(\omega^t\right)\right)}$$

Proof. See Appendix A. ■

If a household is savings constrained, a deviating intermediary has an incentive to offer it an uncontingent savings contract. Such a contract is always incentive compatible and trivially satisfies voluntary participation constraints.

The next key result required to prove Proposition 5 states that for all types not being banished, their continuation utility depends only on the sum $\theta + {}_{t-1}\tau_t\left(h^t\right)$.

Proposition 7 *In equilibrium with two period lived intermediaries, for any t and $h^t, \hat{h}^t$ such that $\delta_t\left(h^t\right)=\delta_t\left(\hat{h}^t\right)=0$, if $\theta + {}_{t-1}\tau_t\left(h^t\right)=\hat{\theta}+{}_{t-1}\tau_t\left(\hat{h}^t\right)$, then*

$$V_t\left(h^t\right)=V_t\left(\hat{h}^t\right)$$

Proof. See Appendix A. ■

The result states that in equilibrium, the continuation value for any two types not being banished h^t and $\tilde{h}^t$ such that $\theta_t + {}_{t-1}b_t = \tilde{\theta}_t + {}_{t-1}\tilde{\tau}_t$ is identical. Notice that here ${}_{t-1}\tau_t$ corresponds to the transfer in period t from a contract signed in period $t-1$. I prove this using an induction argument. Given that we are working in a truncated economy, consider the last period T in which intermediaries are operational. Since from period T onwards households trade a risk-free bond, the household's value going forward depends on only its current endowment and transfer. Next, suppose the hypothesis is true from period $t+1$ onwards and so we want to establish that it is true in period t. For contradiction, suppose we have two histories such that $\theta_t + {}_{t-1}\tau_t = \theta_t + {}_{t-1}\tilde{\tau}_t$ but $V_t\left(h^t\right) > V_t\left(\tilde{h}^t\right)$. The idea of the proof is to show that a deviating intermediary can give agent $\tilde{h}^t$ a contract similar to type h^t, which makes both the household and it strictly better off while still satisfying incentives. The key condition that needs to be checked is that such a contract does not incentivize default the following period. Notice that household $\tilde{h}^t s$ incentives to default in period $t+1$ are exactly the same as household h^t if they receive the same transfers since the value of the two households going forward is identical by the induction assumption.

An important consequence of the previous result is that the probability of re-entry after banishment in period t is independent of current period reports and depends at most on ${}_{t-1}\tau_t$, the period t transfer from the contract signed in period $t-1$. This result will be important when we study an application of the framework to bankruptcy policy.

Proposition 8 *In any equilibrium,*

1. *For all* t *and* h^{t-1}, *if* $\exists$ h_t *and* $\hat{h}_t$ *such that* $\delta_t\left(h^{t-1}, h_t\right) = \delta_t\left(h^{t-1}, \hat{h}_t\right) = 1$ *then* $\mu_t\left(h^{t-1}, h_t\right) = \mu_t\left(h^{t-1}, \hat{h}_t\right)$.

2. *If* ${}_{t-1}\tau_{t-1}\left(h^{t-1}\right) = {}_{t-1}\tau_{t-1}\left(\hat{h}^{t-1}\right)$ *for any two histories* h^{t-1} *and* $\hat{h}^{t-1}$ *then* $D_t\left(h^{t-1}\right) = D_t\left(\hat{h}^{t-1}\right)$ *and* $\mu_t\left(h^{t-1}\right) = \mu_t\left(\hat{h}^{t-1}\right)$

3. *If* ${}_{t-1}\tau_{t-1}\left(h^{t-1}\right) \geq {}_{t-1}\tau_{t-1}\left(\hat{h}^{t-1}\right)$ *for any two histories* h^{t-1} *and* $\hat{h}^{t-1}$ *then* $D_t\left(h^{t-1}\right) \supseteq D_t\left(\hat{h}^{t-1}\right)$ *and* $\mu_t\left(h^{t-1}\right) \leq \mu_t\left(\hat{h}^{t-1}\right)$

Proof. See Appendix A. ■

The proposition implies that re-entry probabilities for any history h^t depend only on ${}_{t-1}\tau_t\left(h^{t-1}\right)$ for types not being banished in period t. In particular, the probability of re-entry after being banished in $t+1$ is decreasing in the transfer ${}_t\tau_t\left(h^t\right)$.

Since the value of not being banished depends only on the current shock θ and τ, the banishment sets $D_t\left(h^{t-1}\right) = D_t\left(\tau_t^{t-1}\left(h^{t-1}\right)\right)$ and so in any equilibrium contract, the incentives to banish are only affected by the current shock θ_t and the transfer ${}_{t-1}\tau_t\left(h^t\right)$. Since transfers are bounded, there exists $\bar{\phi}, \phi^D$ such that

1. ${}_{t-1}\tau_t\left(h^t\right) < \bar{\phi}$

2. If ${}_{t-1}\tau_t\left(h^t\right) < \phi^D$, $D_t\left({}_{t-1}\tau_t\left(h^t\right)\right) = \varnothing$ and $\hat{R}_t\left({}_{t-1}\tau_t\left(h^t\right)\right) = R_t = \frac{1}{q_{t-1}}$

3. If ${}_{t-1}\tau_t\left(h^t\right) \geq \phi^D$, $D_t\left({}_{t-1}\tau_t\left(h^t\right)\right) \neq \varnothing$ and $\hat{R}_t\left({}_{t-1}\tau_t\left(h^t\right)\right) = \frac{R_t\, {}_{t-1}\tau_t\left(\theta^t\right)}{\sum_{h \notin D_t\left({}_{t-1}\tau_t\left(h^{t-1}\right)\right)} \wp\left(h^{t-1}, h\right)}$

In particular, the intermediary only banishes households in states in which transfers are low and households have some incentive to voluntarily default. Since the contracts for those not being banished are still simple borrowing and lending contracts of the form $(l, -Rl)$ where Rl is independent of current announced type, the household being banished always has the option of lying and not being banished. This idea is made more concrete in the proof of the equivalence result where we see that in a formal sense, banished is equivalent to default by the household.

Using these results, we can now prove the equivalence result, Proposition 5 (see Appendix A. for the proof). The result is a consequence of the characterization results proved earlier. In summary, we showed that equilibrium contracts of the intermediary game when banished is allowed and intermediaries are two period lived, resembled short term defaultable debt. These turn out to be exactly the types of contracts that households are assumed to be able to sign in an EG model. Since in the contracting environment, I allow intermediaries to stochastically allow households back in after being banished, the actual contract resembles short term defaultable debt with stochastic re-entry. In particular, if intermediaries are not allowed to bring households back, then the equivalence would hold for an environment in which after default, households are in autarky forever.

As a final point about the equivalence result, it is worth noting that the first-order condition of the household's problem in the EG environment is also satisfied in the equilibrium of the intermediary game. In the case with a continuous[16] type space, in EG, the first order condition for the household's problem is[17]

$$u'\left(c\left(\theta,l\right)\right)Q'\left(l'\right)=\beta\mathbb{E}_{t}d_{t+1}\left(\theta',l'\right)u'\left(c\left(\theta',l'\right)\right) \tag{4.2}$$

While in the contracting environment the object $Q'(s')$ doesn't show up directly, it is captured by the multipliers on the incentive/participation constraints. In particular, the first order condition is

$$u'\left(c_t\left(h^t\right)\right)q_t=\beta\mathbb{E}_t\left[1-\delta_{t+1}\left(h^{t+1}\right)\right]u'\left(c_{t+1}\left(h^{t+1}\right)\right)+\mathbb{E}_t\nu_{t+1}\left(h^{t+1}\right)u'\left(c_{t+1}\left(h^{t+1}\right)\right)$$

where $\nu_{t+1}\left(h^{t+1}\right)$ denotes the incentive constraints on incentive compatibility constraints in period $t+1$. The above equation can be rewritten in the form of (4.2) with $Q'=q_t\left(1+\frac{\mathbb{E}_t\nu_{t+1}\left(h^{t+1}\right)u'\left(c_{t+1}\left(h^{t+1}\right)\right)}{\beta\mathbb{E}_t\left[1-\delta_{t+1}\left(h^{t+1}\right)\right]u'\left(c_{t+1}\left(h^{t+1}\right)\right)}\right)^{-1}$.

Existence: The presence of re-entry choices on the part of intermediaries make the general existence problem quite hard. In Appendix A, using recent results by Auclert and Rognlie (2014), I prove a more limited existence theorem in the case in which intermediaries cannot allow households to re-renter. However, it easy to show that autarky is always an equilibrium of the intermediary game. This suggests that in general, the environment has multiple equilibria.

Lemma 2 *There exists an equilibrium in which for all t and $h^t\in H^t$, $\tau_t\left(h^t\right)=0$*

Proof. Suppose all intermediaries offer null contracts i.e. contracts in which transfers are zero in all dates and for all histories. Suppose further that the price in each period $q_t\geq\max_{\theta\in\Theta}\frac{\mathbb{E}u'(\theta')}{u(\theta)}$. It is easy to see that this is an equilibrium of the intermediary game. No intermediary has an incentive to lend to households since given that there is no borrowing and lending in the future, they will choose to default at some future date. On the other hand since $q_t\geq\max_{\theta\in\Theta}\frac{\mathbb{E}u'(\theta')}{u(\theta)}$, no household wishes to save at interest rates $\frac{1}{q_t}$ and hence there exists no profitable savings contract that an intermediary can offer. ■

Empirical estimates of re-entry probabilities: In a recent paper, Cruces and Trebesch (2013) construct a new database of haircut estimates for sovereign debt restructurings from 1970 until 2010. A key finding of their paper is that higher haircuts are associated with larger spreads and longer duration of banishment from capital markets. In particular, their data analysis shows that partial re-access to capital markets takes 2.3 years on average for a haircut size less than 30 percent while for haircuts larger than 30 percent, the average duration of banishment more than doubles to 6.1 years. While most models of sovereign default assume a constant probability of

[16] With a discrete state-space like the one assumed in this paper, the above condition might not always be well defined.

[17] Note that $Q(s)$ denotes the price times the debt. If $\tilde{Q}(s)$ was the price, then the term on the left hand side would be $\tilde{Q}'(s)\,s+\tilde{Q}(s)$.

re-entry after default, in the environment I consider, re-entry probabilities that depend on the level of defaulted debt arise as part of the profit maximizing contract. While this model features haircuts of a 100 percent, it is still true that the probability of re-entry is weakly decreasing in the level of defaulted debt.

4.1 Long Lived Intermediaries

I now consider the case in which intermediaries are long lived. We saw in the case with two period lived intermediaries that the banishment technology allowed intermediaries to introduce state-contingency into otherwise uncontingent contracts. With long lived intermediaries and banishment, contracts can now be history contingent as well as state contingent. As a result, contracts can feature more insurance than that associated with self insurance. In particular, the expected present discounted value of transfers can be larger for lower types than higher types. Recall that in standard private information environments such transfer schemes were incentive compatibility since the continuation value associated with positive transfers today were lower than that associated with negative transfers. With hidden trading opportunities and no-banishment, this is no longer incentive compatible. However, banishment allows intermediaries to credibly lower future continuation values. Since banished individuals cannot sign new contracts, incentive compatibility can be satisfied even though lower types receive a larger expected present discounted value of transfers. Clearly, such an equilibrium outcome can no longer be equivalent to a standard Eaton-Gersovitz environment. However, I show by means of an example that under some conditions, the equilibrium outcomes can be identical to environments with both short and long term defaultable debt.

The example I consider is a three period partial equilibrium example with an exogenous cost of banishment/default. $\Theta = \left\{\theta^h, \theta^l\right\}$ and households sign contracts with intermediaries in period 1 and can potentially sign new hidden contracts in period 2. The set up and definition of equilibrium is identical to that in the previous sections, except that the voluntary participation constraints are

$$\left[1 - {}_1\delta_3\left(h^3\right)\right] V_3\left(h^3\right)\left(\sigma^*\right) \geq \left[1 - {}_1\delta_3\left(h^3\right)\right]\left[u\left(\theta_3\right) - \psi\right]$$

where ψ is an exogenous cost faced by banished or defaulting households.

The corresponding environment with defaultable debt is similar to the one studied earlier except that in addition to the short term defaultable debt security, households can also trade a 2 period defaultable debt security in period 1. The security offers a household $Q_1^l(l)$ units of the consumption good in period 1 in return for $-l$ units in periods 2 and 3. In addition, households can also trade short term defaultable debt securities in periods 1 and 2 whose price is denoted by $Q_1^s(l)$ and $Q_2^s(l)$. The long term security is priced in a similar fashion so that lenders make zero profits. Given that the environment is 3 periods we can write the sequential problem for the household:

$$\max_{c,d,b} u(c_1) + \beta \mathbb{E}[d_2[u(c_2) + \beta \mathbb{E}[d_3 u(c_3) + [1-d_3](u(c_3)-\psi)]] + [1-d_2]([u(\theta_2)-\psi] + \beta[u(\theta_3)-\psi])]$$

$s.t.$

$c_1 + Q_1^s\left(l_1^s, l_1^l\right) l_1^s + Q_1^l\left(l_1^s, l_1^l\right) l_1^l \leq \theta$

$c_2 + Q_2^s\left(l_1^l, l_2^s\right) l_2^s \leq \theta_2 + l_1^s + l_1^l$ if $d_2 = 1$

$c_3 \leq \theta_3 + l_2^s + l_1^l$ if $d_2 = 1$ and $d_3 = 1$

Proposition 9 *For q small enough, any equilibrium outcome of the contracting environment is an equilibrium outcome of the above defaultable debt environment.*

Consider an equilibrium of the contracting game. The relevant incentive constraints in period 2 will concern the high type pretending to be the low type. We know from an earlier result that under some sufficient conditions that any non-autarkic equilibrium will feature banishment on path. Here the only histories that can be banished are the continuation histories associated with the low type in period 2. Moreover, if there is banishment, it must be in the subsequent low state only. Thus we have equilibrium in which the only type being banished is the one associated with low endowments in both period 2 and period 3.

There are two relevant deviations that the high type in period 2 can undertake. The first is to lie in period 2 and then tell the truth in 3 and the second is to lie in both periods. Since any of these deviations will have the high type signing a hidden uncontingent savings contract in period 2, which of these is binding will depend on the interest rate and the punishment associated with punishment. For q low enough the relative constraint will be the one in which the high type lies in period 2 and always claims to be the high type in period 3. In this case, the incentive compatibility constraint implies that (with slight abuse of notation)

$$\tau_2\left(h^1, \theta^l\right) + q\tau_3\left(h^1, \theta^l, \theta^h\right) = \tau_2\left(h^1, \theta^h\right) + q\tau_2\left(h^1, \theta^h, \theta_3\right)$$

To prove the result, as before, set the default strategy of the household to coincide with the banishment feature of the contract. We need to show that are long and short security holdings that replicate the outcomes associated with the contracting equilibrium. In particular, we need to construct $\left\{l_1^l, l_1^s, l_2^{s,h}, l_2^{s,l}\right\}$ to match the transfer schedule $\tau_t\left(h^t\right)$ which in this case is 5 numbers. However notice that the above condition places an additional restriction on transfers which implies that are 4 independent transfers. We can then use budget constraints to back out the security holdings. For example, the initial short and long term security holdings l_1^s, l_1^l and the new short term issuance by the high type in period 2, $l_2^{s,h}$ can be constructed using

$$\begin{bmatrix} -Q_1^s & -Q_1^l & 0 \\ 1 & 1 & -Q_2^{sh} \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} l_1^s \\ l_1^l \\ l_2^{s,h} \end{bmatrix} = \begin{bmatrix} \tau_1 \\ \tau_2\left(h^1,\theta^h\right) \\ \tau_3\left(h^1,\theta^h\right) \end{bmatrix}$$

where notice that in period 3, transfers will be independent of announced type. If the above is well defined then the short term holdings of the low type in period 2 can be backed out using the budget constraint. To see that there is unique solution to the above system of equations we need to check that the determinant of the first matrix is $\neq 0$. Notice that (since default equals banishment)

$$\begin{aligned} \det A &= -Q_1^s + Q_1^l - Q_1^s Q_2^{sh} \\ &= -q\mathbb{E}\left[1-\delta_2\right] + \left[q\mathbb{E}\left[1-\delta_2\right] + q^2\mathbb{E}\left[1-\delta_2\right]\left[1-\delta_3\right]\right] - q^2\mathbb{E}\left[1-\delta_2\right]\mathbb{E}\left[1-\delta_3\right] \\ &= q^2\left[\mathbb{E}\left[1-\delta_2\right]\left[1-\delta_3\right] - \mathbb{E}\left[1-\delta_2\right]\mathbb{E}\left[1-\delta_3\right]\right] \end{aligned}$$

But notice that $\mathbb{E}\left[1-\delta_2\right]\left[1-\delta_3\right] - \mathbb{E}\left[1-\delta_2\right]\mathbb{E}\left[1-\delta_3\right] \neq 0$ if there is some banishment on path. In particular, in the above example

$$\begin{aligned} \mathbb{E}\left[1-\delta_2\right]\mathbb{E}\left[1-\delta_3\right] &= \pi^h \\ \mathbb{E}\left[1-\delta_2\right]\left[1-\delta_3\right] &= \pi^h + \pi^l\pi^h \end{aligned}$$

and so

$$\mathbb{E}\left[1-\delta_2\right]\left[1-\delta_3\right] - \mathbb{E}\left[1-\delta_2\right]\mathbb{E}\left[1-\delta_3\right] = \pi^l\pi^h$$

This implies the a solution exists and one can find security holdings that replicate the equilibrium outcomes from the contracting environment.

An implication of the above is that expected present discounted value of transfers taking into account banishment/default is different for high and low types. In particular,

$$\tau_2\left(h^1,\theta^h\right) + q\tau_2\left(h^1,\theta^h,\theta_3\right) = \tau_2\left(h^1,\theta^l\right) + q\tau_3\left(h^1,\theta^l,\theta^h\right) < \tau_2\left(h^1,\theta^l\right) + q\pi^h\tau_3\left(h^1,\theta^l,\theta^h\right)$$

since $\tau_3\left(h^1,\theta^l,\theta^h\right)$ is a negative number. As a result banishment allows for insurance to re-introduced into contracts even in the presence of hidden trading.

5 Efficiency

The first step in asking whether the equilibria characterized in the previous sections are efficient is to define the right notion of constrained-efficiency. In environments with private information and limited commitment this is well understood and has been studied by Prescott and Townsend (1984) and Kehoe and Levine (1993). However, the definition of constrained-efficiency is less clear in environments with non-exclusive contracts in which interest rates are endogenously determined.

To begin, I consider a setup with a fictitious social planner and continuum of infinitely lived households who receive an unobservable perishable endowment each period. An important feature of the planning environment is that as in the intermediary game, I will allow the planner to temporarily banish households from the mechanism. As a result, we use the same expanded type space to take into account periods of banishment.

An allocation for the planner consists of a sequence $\left\{\delta_t\left(h^t\right), \mu_t\left(h^t\right), c_t\left(\omega^t\right), \tau_t\left(h^t\right)\right\}_{t\geq 0, h^t \in H^t}$. The first term $\delta_t\left(h^t\right) \in \{0,1\}$ corresponds to an banishment index which indicates if the household is part of the mechanism or not. If the household is not in some period t, it cannot receive any transfers from the planner and is also banished from trading in any hidden markets, which will be defined shortly. The planner still keeps track of banished households and can let them back into the mechanism at some future date. The next term $\mu_t\left(h^t\right) \in [0,1]$ corresponds to the re-entry probability chosen by the planner after the agent has been banished. Note that after banishment there is no reporting of types and so the re-entry probability can only depend on the last type reported before banishment. The next two terms correspond to the consumption and transfer sequences to households in the mechanism.

An allocation is *incentive-feasible* if it satisfies the following conditions. First, it must be resource feasible; for each t,

$$\begin{aligned} \sum_{h^t \in H^t} \wp\left(\omega^t\right) c_t\left(\omega^t\right) &= \sum_{h^t \in H^t} \wp\left(\omega^t\right) \theta_t \\ \delta_t\left(h^t\right) c_t\left(\omega^t\right) &= \delta_t\left(h^t\right) \theta_t \end{aligned} \tag{5.1}$$

Here the second equation $\delta_t\left(h^t\right) c_t\left(h^t\right) = \delta_t\left(h^t\right) \theta_t$ corresponds to the restriction that all banished households consume their endowment.

Next, the contract must satisfy voluntary participation constraints: for all t and $h \in H^t$,

$$\left[1-\delta_t\left(h^t\right)\right] V_t\left(h^t\right) \geq \left[1-\delta_t\left(h^t\right)\right] V_t^d\left(h^t\right) \tag{5.2}$$

I assume that at the beginning of each date, each household can voluntarily default on the planner and consequently live in autarky forever. In autarky, the household consumes its endowment each period. Note that without loss of generality we can restrict attention to outcomes in which the planner does all the banishment and household never voluntarily defaults. Next, the allocation must be incentive compatible

$$V_t\left(h^t\right)\left(\sigma^*\right) \geq \hat{V}_t\left(h^t, \{\tau\}, \{q\}\right)(\sigma), \tag{5.3}$$

Here $V_t\left(h^t\right)\left(\sigma^*\right)$ denotes the value of the contract to type h^t of following truth-telling strategy σ^*. $\hat{V}_t\left(h^t, \{\tau\}, \{q\}\right)(\sigma)$ denotes the value to the household of using reporting strategy σ and trading in a *hidden market*. We need to consider two types of hidden markets depending on whether the planner is allowed to banish households or not.

5.1 Efficiency without Banishment

First, as in the intermediary environment, I restrict the planner to only offer contracts without banishment. Given this, we can restrict ourselves to the usual type spaces Θ^t. In this case, I consider a hidden market in which households can trade a risk free bond subject to *endogenous* debt constraints. Therefore,

$$\hat{V}_t\left(\theta^t;\{\tau\},\{q\}\right)(\sigma)=\max\sum_{s=0}^{\infty}\beta^s\sum_{\theta^{t+s}\in\Theta^{t+s}}\pi\left(\theta^t\right)u\left(x_{t+s}\left(\theta^{t+s}\right)\right)$$
$$\text{subject to for all } s\geq 0,\ h^{t+s}$$
$$x_{t+s}\left(\theta^{t+s}\right)+q_{t+s}s_{t+s+1}\left(\theta^{t+s}\right)\geq\theta_{t+s}+\tau_{t+s}\left(\sigma_{t+s}\left(\theta^{t+s}\right)\right)+s_{t+s}\left(\theta^{t+s-1}\right)$$
$$s_{t+s+1}\left(\theta^{t+s}\right)\geq-\phi_{t+s}$$

Here $\tau_{t+s}\left(\sigma_{t+s}\left(\theta^{t+s}\right)\right)$ denotes the transfer from the planner when type θ^{t+s} reports $\sigma_{t+s}\left(\theta^{t+s}\right)$, $s_{t+s+1}\left(\theta^{t+s}\right)$, the amount the household saves in period $t+s$ and ϕ_{t+s}, the debt constraints. We can rewrite $\hat{V}\left(\theta^t;\{\tau\},\{q\}\right)$ as

$$J_t\left(\theta^t,s_t;\{\tau\},\{q\},\Phi_t\right)=\max u\left(x_t\right)+\beta\mathbb{E}_tJ_t\left(\theta^{t+1},s_{t+1};\{\tau\},\{q\},\Phi_t\right)$$
$$\text{subject to}$$
$$x_t+q_ts_{t+1}\geq\theta_t+\tau_t\left(\theta^t\right)+s_t$$
$$s_{t+1}\geq-\phi_t$$

where Φ_t denotes the sequence of current and future debt constraints which each household takes as given.

Definition 7 *An equilibrium in the hidden market given a transfer sequence $\{\tau\}$ consists of prices $\{q_t\}$, allocations $\{x_t,s_t\}$ and debt constraints $\{\phi_t\}$ such that*

1. *Households solve their problem defined above,*

2. *Markets clear: for all t,*

$$\sum_{h^t\in H^t}\pi\left(\theta^t\right)x_t\left(\theta^t\right)=\sum_{h^t\in H^t}\pi\left(\theta^t\right)\left[\theta_t+\tau_t\left(\theta^t\right)\right]$$

3. *Debt constraints are chosen to be Not-Too-Tight, i.e.*

$$J_t\left(\theta^t,-\phi_t;\{\tau\},\{q\},\Phi\right)\geq V_t^d\left(\theta^t\right)\ \text{for all }\theta^t$$
$$J_t\left(\hat{\theta}^t,-\phi_t;\{\tau\},\{q\},\Phi\right)=V_t^d\left(\hat{\theta}^t\right)\ \text{for some }\hat{\theta}^t$$

The definition of the hidden market is similar in spirit to Golosov and Tsyvinski (2007). In their model, agents traded a risk free bond with the interest rate determined in equilibrium. Here, households trade these bonds subject to debt constraints which along with the interest rates are also determined in equilibrium. I assume that households can also default on their hidden debt obligations. As in the intermediary game, default in the hidden markets is publicly observable and consequently households live in autarky in all future periods. Debt constraints are chosen in equilibrium so that all households weakly prefer not to default on their debt if they have borrowed up to the debt limit the previous period while some household is indifferent between the two options. It is clear that in any constrained-efficient allocation, there will be no trade in these markets. In particular, the efficient allocation will have the property that for any Euler-constrained household, borrowing more in the hidden market will incentivize default the next period. Moreover the price q_t will be such that no household will wish to save in these markets and as a result we a have a well defined equilibrium with no hidden trades.

The idea behind modelling the hidden market this way is as follows: suppose after receiving transfers from the planner, households could sign contracts in a hidden fashion with a continuum of hidden intermediaries subject to incentive and voluntary participation constraints. This game is identical to the one studied in the previous sections and we know that in the case in which intermediaries are not allowed to banish agents, equilibrium contracts are equivalent to trading an uncontingent bond subject to debt constraints.

The main result in this subsection is that the efficient allocation in the case with no banishment can be decentralized as an equilibrium of the intermediary game with no banishment.

Theorem 3 (Efficiency: No banishment) *The constrained efficient allocation without banishment can be implemented as an equilibrium of the intermediary game without banishment.*

To prove this is result, I first prove properties that any efficient allocation must satisfy. In particular, I show that the planner cannot do better than simple borrowing and lending contracts. Then, I show that if all intermediaries are offering the efficient contract, no incumbent or new intermediary has any incentive to offer a deviating contract.

As in the intermediary game I consider limits of $T-$period truncated environments in which from period 1 to T, the planner provides transfers and after T those households that have not defaulted can trade a risk-free bond subject to exogenous debt constraints.

Proposition 10 *Any $T-$period truncated incentive feasible allocation must satisfy*

$$q_t \geq \beta \frac{\mathbb{E}_{t+1} u'\left(c_{t+1}\left(\theta^{t+1}\right)\right)}{u'\left(c_t\left(\theta^t\right)\right)} \text{ for all } t, \theta^t \in \Theta^t$$

and

$$\sum_{t=1}^{T} \left(\prod_{s=1}^{t} q_s \right) \tau_t \left(\theta^T\right) = 0 \text{ for all } \theta^T \in \Theta^T \tag{5.4}$$

Proof. See Appendix A. ■

Notice that the proposition says that in the case with no banishment, the efficient contract is also a simple borrowing and lending contract subject to debt constraints. In particular, the presence of the hidden markets prevents the planner from introducing state-contingency in contracts. The intuition for this is exactly the same as in the intermediary game. If lower types receive a larger present discounted value of transfers, then higher types will lie and use the hidden markets to save. On the other hand if higher types receive a larger present discounted value of transfers then redistribution is welfare increasing. Given that voluntary participation constraints induce some agents to be Euler-constrained in the efficient allocation, the planner will allow agents to borrow the largest amount consistent with no default in the subsequent period. As a result the voluntary participation constraints will be binding for some type in the following period.

These two conditions imply that as in the intermediary game, the efficient contract are simple uncontingent borrowing and lending contracts. The next result provides necessary and sufficient conditions for an allocation to induce an equilibrium of the hidden market with no trades.

Lemma 3 *A no-banishment allocation induces no trades in the hidden market if and only if for all $t, \theta^t \in \Theta^t$*

$$q_t \geq \beta \frac{\mathbb{E}_{t+1} u'\left(c_{t+1}\left(\theta^{t+1}\right)\right)}{u'\left(c_t\left(\theta^t\right)\right)} \text{ for all } t, \theta^t \in \Theta^t \tag{5.5}$$

and

$$\left[q_t - \frac{\beta \mathbb{E}_t u'\left(c_{t+1}\left(\theta^{t+1}\right)\right)}{u'\left(c_t\left(\theta^t\right)\right)}\right] \min_{\tilde{\theta}^{t+1} \in \tilde{\Theta}^{t+1}} \left[V_{t+1}\left(\tilde{\theta}^{t+1}\right) - V_{t+1}^d\left(\tilde{\theta}^{t+1}\right)\right] = 0 \tag{5.6}$$

Proof. See Appendix A. ■

The second condition says that if a household is Euler-constrained in period t, then it must be that in the following period, the voluntary participation for some type binds. The reason for this is if not then, debt constraints in the hidden market will satisfy the Not-too-tight property. In other words intermediaries will be willing to lend more to agents without fearing default in the subsequent periods.

Next, as in Golosov and Tsyvinski (2007) we can re-write the planner's problem with no banishment as one in which the planner also chooses the prices in the hidden markets subject to additional conditions. Given that we are first restricting the planner to offer allocations without banishment/default, and $\delta_t\left(h^t\right) = 0$ for all $t, h^t \in H^t$, an allocation in this case is a sequence of transfers $\left\{\tau_t\left(\theta^t\right)\right\}_{t\geq 0, h^t \in H^t}$ and prices $\{q_t\}_{t\geq 0}$. In this case, (5.1), (5.2) and (5.3) simplify to

$$\sum_{\theta^t \in \Theta^t} \pi\left(\theta^t\right) c_t\left(\theta^t\right) = \sum_{\theta^t \in \Theta^t} \pi\left(\theta^t\right) \theta_t \text{ for all } t \tag{5.7}$$

$$V_t\left(\theta^t\right)\left(\sigma^*\right) \geq \hat{V}_t\left(\theta^t, \{\tau\}, \{q\}\right)(\sigma) \text{ for all } t, \theta^t \in \Theta^t \tag{5.8}$$

$$V_t\left(\theta^t\right) \geq V_t^d\left(\theta_t\right) \text{ for all } t, \theta^t \in \Theta^t \tag{5.9}$$

Lemma 4 *The constrained efficient allocation $\left\{c_t\left(h^t\right), \tau_t\left(h^t\right)\right\}_{t\geq 0, h^t \in H^t}$ and prices $\{q_t\}_{t\geq 0}$ is a* *solution to the following programming problem*

$$\max_{\{c,\tau,q\}} \sum_{t=1}^{T} \beta^{t-1} \sum_{\theta^t \in \Theta^t} \pi\left(\theta^t\right) u\left(c_t\left(\theta^t\right)\right)$$

subject to (5.7), (5.4), (5.9), (5.5), and (5.6).

To prove Theorem 3, I show that if all intermediaries are offering the efficient contract, no individual intermediary has an incentive to deviate and offer a different contract. In particular, it will not be able to offer some Euler-constrained individuals the option to borrow more since they will default the following period. This establishes that the efficient allocation can be decentralized as an equilibrium of the intermediary game. Note that even though as in Golosov and Tsyvinski (2007), the planner controls the price in the hidden market, he is unable to achieve outcomes better than the best competitive equilibrium. The planner chooses q_t consistent with best competitive equilibrium from the set of equilibria which we know is not a singleton. The reason for this is that incentive compatibility dictates that in any incentive feasible allocation no state contingency is possible. As a result, the best the planner can do is to choose the allocation that corresponds to loosest borrowing constraints which in turn corresponds to the best competitive equilibrium. Unlike Golosov and Tsyvinski (2007), in this model output is not publicly observable. Therefore, the planner cannot use incentives to work to provide state-contingency in contracts as in their paper.

5.2 Efficiency with Banishment

Consider a planner who is also allowed to banish households and set re-entry probabilities in all future periods. I restrict the planner to only offer two period contracts as in the intermediary game.[18] I assume that households can sign two period contracts with intermediaries in a hidden market. The equilibrium of the hidden market is identical to that described in section 3, taking into account the transfers from the planner. In particular, I assume that intermediaries can offer contracts that *banish households from the hidden market.* If such a household is not banished by the planner, it can continue to receive transfers from the planner but cannot take part in the hidden market. From Proposition 5, we know that any equilibrium contract in the hidden market will be a short-term defaultable debt contract where default constitutes banishment from the hidden market with a chosen re-entry probability. As a result, we can restrict to deviating contracts of the form $\mathbb{D}_t\left(h^t\right) = \left(z_t\left(h^t\right), z_{t+1}\left(h^t\right), \delta^H_{t+1}\left(h^{t+1}\right), \mu^H\left(h^t\right)\right)$ which consists of transfers in period t, $t+1$, banishment indices for the hidden market and re-entry probabilities.

Define $\mathcal{V}_t\left(h^t\right)$ to be continuation value for a household of type h^t when it has access to the planner's transfers and the hidden market, $\mathcal{V}^E_t\left(h^t; \mu, \mu^H\right)$ the value for a household banished from

[18]Without this restriction the planner can do better. However, the point of this exercise is to illustrate that the standard pecuniary externality agrument when markets are exogenously incomplete no longer holds with hidden markets of the form described in this section.

the planning problem, and $\mathcal{V}_t^N\left(h^t;\mu^H\right)$, the continuation value for h^t if it is only banished from the hidden market (and not by the planner). In particular

$$\begin{aligned}&\mathcal{V}_t^E\left(h^t;\mu,\mu^H\right)\\&=u\left(\theta_t\right)+\beta\mathbb{E}_t\left[\mu\left[\mu^H\mathcal{V}_{t+1}\left(h_t\right)+\left(1-\mu^H\right)\mathcal{V}_{t+1}^N\left(h_{t+1};\mu^H\right)\right]+\left(1-\mu\right)\mathcal{V}_{t+1}^E\left(h^{t+1};\mu,\mu^H\right)\right]\end{aligned} \tag{5.10}$$

and

$$\mathcal{V}_t^N\left(h^t;\mu^H\right)=u\left(\theta_t+\tau_t\left(h^t\right)\right)+\beta\mathbb{E}_t\left[\mu^H\mathcal{V}_{t+1}\left(h^{t+1}\right)+\left(1-\mu^H\right)\mathcal{V}_{t+1}^N\left(h^{t+1};\mu^H\right)\right] \tag{5.11}$$

Notice that $\mathcal{V}_t\left(h^t\right)$ will only differ from $V_t\left(h^t\right)$ when the household is trading in the hidden market.

The main result of this subsection is that the efficient allocation with banishment can be implemented as an equilibrium of the intermediary game with two period lived intermediaries and banishment.

Proposition 11 (Efficiency: With Banishment) *The constrained efficient two-period allocation with banishment can be implemented as an equilibrium of the intermediary game with two period lived intermediaries and banishment.*

To prove the result, I first prove characterization results about the efficient contract and then show that if intermediaries offer such a contract no profitable deviation exists. As in the intermediary game I consider limits of $T-$period truncated environments in which from period 1 to T, the planner provides transfers and after T those households that have not defaulted can trade a risk-free bond subject to exogenous debt constraints. In the appendix I show that we can take such limits.

Proposition 12 *Any $T-$period truncated incentive feasible allocation must satisfy*

$$q_t\geq\beta\frac{\mathbb{E}_{t+1}\left[1-\delta_{t+1}\left(h^{t+1}\right)\right]u'\left(c_{t+1}\left(\omega^{t+1}\right)\right)}{u'\left(c_t\left(\omega^t\right)\right)}\text{ for all }t,h^t\in H^t$$

and

$$\sum_{t=1}^{T}\left(\prod_{s=1}^{t}\hat{q}_s\left(h^s\right)\left[1-\delta_s\left(h^s\right)\right]\right)\tau_t\left(h^T\right)=0\text{ for all }\theta^T\in\Theta^T \tag{5.12}$$

where

$$\hat{q}_s\left(h^s\right)=q_s\sum_{h_{s+1}\in H_{s+1}}\zeta_{s+1}\left(h^s,h_{s+1}\right)\delta_{s+1}\left(h^s,h_{s+1}\right)$$

Proof. See Appendix A. ■

The proposition says that the efficient contract is also a simple short term defaultable debt contract. In particular, the presence of the hidden markets and the fact that the planner can only offer two period contracts prevents the planner from introducing state-contingency in contracts beyond banishment. The intuition for this is similar to that in the intermediary game.

Since we are modelling the hidden market as one in which households can transact with intermediaries, given earlier results about the nature of the equilibrium contracts we need only consider short-term deviating contracts of the form $\left(z_t\left(h^t\right), z_{t+1}\left(h^t\right), \delta^H_{t+1}\left(h^{t+1}\right), \mu^H\left(h^{t+1}\right)\right)$ where $z_t\left(h^t\right)$ and $z_{t+1}\left(h^t\right)$ denote the transfers specified by the hidden contracts, $\delta^H_{t+1}\left(h^{t+1}\right)$, whether the household is banished from the hidden market and μ^H denotes the probability of re-entry to the hidden market. As I will show, there are two types of deviating contracts to consider. The first is a simple savings contract. The second is a short term defaultable debt contract.

Proposition 13 *An allocation induces no trades in the hidden market if and only if for all* $t, h^t \in h^t$

$$q_t \geq \beta \frac{\mathbb{E}_{t+1}\left[1-\delta_{t+1}\left(h^{t+1}\right)\right] u'\left(c_{t+1}\left(h^{t+1}\right)\right)}{u'\left(c_t\left(h^t\right)\right)} \text{ for all } t, h^t \in H^t \tag{5.13}$$

and

$$u'\left(\theta_t+\tau_t\left(h^t\right)\right)\hat{q}_t\left(h^t\right) \leq \beta u'\left(\hat{\theta}_{t+1}+\tau_{t+1}\left(\hat{h}^{t+1}\right)\right) \text{ for all } t, h^t \in H^t \tag{5.14}$$

where $\hat{q}_t\left(h^t\right) = q_t \sum_{t+1}\left(1-\delta_{t+1}\left(h^{t+1}\right)\right)$ *and* $\hat{h}^{t+1}$ *is such that*

$$V_{t+1}\left(\hat{h}^{t+1}\right) - \mathcal{V}^E_t\left(\hat{h}^{t+1};\mu\right) \leq V_{t+1}\left(h^{t+1}\right) - \mathcal{V}^E_t\left(h^{t+1};\mu\right) \ \forall h^{t+1} \text{ such that } \delta_{t+1}\left(h^{t+1}\right) = 0$$

Proof. See Appendix A. ■

We can use the fact that in equilibrium these deviations must be short-term contracts to greatly simplify the types of deviating contracts. As mentioned earlier, the first a simple savings contract that ensures that the in the efficient allocation, no household can be savings constrained. The second type of deviation involves a debt contract which allows the household to borrow a little more in the current period and in the following period, some types are banished from the hidden markets. To understand the intuition for this result, consider the case in which as part of the efficient allocation, for some type h^t, there exists a set of h^{t+1} that is being banished by the planner in $t+1$. Let $\hat{h}^{t+1}$ correspond to the type with the smallest θ_{t+1} not being banished. Suppose that the planner's allocation satisfies $u'\left(\theta_t+\tau_t\left(h^t\right)\right)\hat{q}_t\left(h^t\right) > \beta u'\left(\hat{\theta}_{t+1}+\tau_{t+1}\left(\hat{h}^{t+1}\right)\right)$. Then I prove that there exists such a deviating contract that gives the household strictly higher utility and the deviating intermediary breaks even. In this contract, households receive a positive transfer $\hat{q}_t\varepsilon$ in period t, a negative transfer in all non-banished states, and and a probability μ^H that that the household will be allowed to trade in the hidden market even after it can receive transfers from the planner.

Given these characterization results, as in the no-banishment case we can simplify the constrained-efficient programming problem.

Proposition 14 *The constrained efficient allocation with banishment is the solution to*

$$\max_{\{\delta,\mu,c,\tau,q\}} \sum_{t=1}^{T} \beta^{t-1} \sum_{h^t \in H^t} \wp\left(\omega^t\right) u\left(c_t\left(\omega^t\right)\right)$$

subject to (5.1), (5.2), (5.12),(5.13) and (5.14).

To prove Proposition 11, I ask if there exists a profitable deviation if all intermediaries are offering the efficient contract. Since these deviating contracts cannot be state-contingent, the two types of deviating contracts we need to consider are simple savings contracts and one in which intermediary transfers more in the current period and potentially banishes more types the following period. However, given that the efficient contract satisfies the conditions in Proposition 13, both these deviations can never be profitable.

While in general, characterizing the efficient re-entry decisions is difficult, under some sufficient conditions, if the planner banishes a household, it is never let back in. To show this I first introduce stochastic re-entry after *voluntary default*. We can easily modify the voluntary participation constraints in the planning problem to allow for this. In particular, if households default, they are allowed back into the mechanism with probability λ each period. Note that this is feature of the technology and in general different from the re-entry probabilities the planner sets after banishment. Denote above planning problem when the default punishment is parameterized by λ^D by $\mathbb{P}\left(\lambda^D\right)$ and the set of feasible allocation-price pairs as $Feas\left(\lambda^D\right)$. Let $x^*\left(\lambda^D\right) \in Feas\left(\lambda^D\right)$ denote the constrained-efficient allocation-price pair when the punishment is λ^D and $W\left(x^*\left(\lambda^D\right)\right)$, the ex-ante welfare of the planning problem. The following result is immediate.

Lemma 5 *If* $\lambda^{D\prime} > \lambda^D$ *then* $Feas\left(\lambda^{D\prime}\right) \subseteq Feas\left(\lambda^D\right)$ *and* $W\left(x^*\left(\lambda^D\right)\right) \geq W\left(x^*\left(\lambda^{D\prime}\right)\right)$

Proof. It is straightforward to notice that any $x \in Feas\left(\lambda^{D\prime}\right)$, satisfies all the constraints in $Feas\left(\lambda^D\right)$ since $V_t^d\left(\theta_t;\lambda^D\right) < V_t^d\left(\theta_t;\lambda^{D\prime}\right)$. Therefore $x \in Feas\left(\lambda^D\right)$. It is follows that $W\left(x^*\left(\lambda^D\right)\right) \geq W\left(x^*\left(\lambda^{D\prime}\right)\right)$ ■

In particular the solution to the constrained-efficient planning problem must satisfy $W\left(x^*\left(0\right)\right) \geq W\left(x^*\left(\lambda^D\right)\right)$ for all $\lambda^D \in [0,1]$.

Definition 8 *An allocation-price pair* $x\left(\lambda^D\right) \in Feas\left(\lambda^D\right)$ *is* $\mathcal{E}$*-constrained if for all* t *and histories* h^t*, if* $\delta_t\left(h^{t-1}\right) = 0$ *and* $\delta_t\left(h^t\right) = 1$*, then there exists* $\tilde{h}$ *such that* $\delta_t\left(h^{t-1},\tilde{h}\right) = 0$ *and* $V_t\left(h^{t-1},\tilde{h}\right) = V_t^d\left(h^{t-1},\tilde{h};\lambda^D\right)$

$\mathcal{E}$-constrained allocations are important to the subsequent results since we will show that it is exactly these allocations which can be decentralized as equilibria of an Eaton and Gersovitz (1981) environment. They key argument will rely on the fact that in any $\mathcal{E}$-constrained allocation, the planner will not bring back any banished agent with probability greater than λ^D. The next result shows that if the allocation is $\mathcal{E}$-constrained then if the planner banishes a household, it is never let back in with probability greater than λ^D.

Proposition 15 *If a solution to* $\mathcal{P}\left(\lambda^D\right)$ *is* $\mathcal{E}$*-constrained , then for any* t *and history* h^t*, if* $\delta_{t-1}\left(h^{t-1}\right)=0$ *and* $\delta_t\left(h^t\right)=1$*,* $\mu_{t+s}\left(h^t\right)\leq\lambda^D$ *for all* $s>1$

Proof. Proof: Suppose we have such an $\mathcal{E}$-constrained solution and consider some t and type h^t such that $\delta_t\left(h^t\right)=1$. By assumption, it must be that for some $\left(h^{t-1},\tilde{h}\right)$ such that $\delta_t\left(h^{t-1},\tilde{h}\right)=0$, $V_t\left(h^{t-1},\tilde{h}\right)=V_t^d\left(h^{t-1},\tilde{h};\lambda^D\right)$. Suppose now that $\mu_{t+s}\left(h^t\right)>\lambda^D$ for some $s>1$. Notice that type $\left(h^{t-1},\tilde{h}\right)$ will strictly prefer to lie and pretend to be type $\left(\tilde{\theta}^{t-2},\underline{\theta},\tilde{\theta}_t\right)$ since it will receive a value greater than that of defaulting, $V_t^d\left(h^{t-1},\tilde{h};\lambda^D\right)$. As a result, incentive compatibility constraints are violated. ■

The conditions guaranteeing that an allocation is $\mathcal{E}$-constrained can be seen more intuitively in a simple two state example with $\Theta=\left\{\theta^l,\theta^h\right\}$. Here, an allocation-price pair is $\mathcal{E}$-constrained if for all t, θ^{t-1}

$$c_t\left(\theta^{t-1},\theta^l\right)\leq c_{t+1}\left(\theta^{t-1},\theta^l,\theta^h\right)$$

and

$$R_t=\frac{u'\left(c_t\left(\theta^{t-1},\theta^h\right)\right)}{\beta\left[\pi u'\left(c_{t+1}\left(\theta^{t-1},\theta^h,\theta^h\right)\right)+(1-\pi)\,u'\left(c_{t+1}\left(\theta^{t-1},\theta^h,\theta^l\right)\right)\right]}<\frac{1}{\beta}$$

Suppose $\delta_{t+1}\left(\theta^{t-1},\theta^l,\theta^l\right)=1$ so that the planner banishes type $\left(\theta^{t-1},\theta^l,\theta^l\right)$ in period $t+1$. It must be that

$$u'\left(c_t\left(\theta^{t-1},\theta^l\right)\right)\geq\beta\frac{R_{t+1}}{\pi}\pi u'\left(c_{t+1}\left(\theta^{t-1},\theta^l,\theta^h\right)\right)$$

Notice that if the above equation held with an equality then

$$u'\left(c_t\left(\theta^{t-1},\theta^l\right)\right)=\beta R_{t+1}u'\left(c_{t+1}\left(\theta^{t-1},\theta^l,\theta^h\right)\right)<u'\left(c_{t+1}\left(\theta^{t-1},\theta^l,\theta^h\right)\right)$$

so that $c_t\left(\theta^{t-1},\theta^l\right)>c_{t+1}\left(\theta^{t-1},\theta^l,\theta^h\right)$ which is a contradiction. Therefore an $\mathcal{E}$-constrained allocation if $\delta_{t+1}\left(\theta^{t+1}\right)=1$, then there must exist some $\tilde{\theta}$ such that $\delta_{t+1}\left(\theta^t,\tilde{\theta}\right)=0$ and $V_{t+1}\left(\theta^t,\tilde{\theta}\right)=V_{t+1}^d\left(\theta^t,\tilde{\theta}\right)$. Suppose now that $\mu_{t+s}\left(\theta^{t+s}\right)>\lambda^D$ for some $s>1$. Then notice that type $\left(\theta^t,\tilde{\theta}\right)$ will strictly prefer to lie and pretend to be type $\left(\theta^t,\theta\right)$ since he will receive a value greater than that of default which violated incentive compatibility.

Recall that $x^*(0)$ is the solution to the constrained-efficient planning problem when $\lambda^D=0$. The next result shows that if the solution to $\mathbb{P}\left(\lambda^D\right)$ is $\mathcal{E}$-constrained, then it can be decentralized as an equilibrium of the standard EG environment where after default households live in autarky forever.

Lemma 6 *If the solution* $x^*(0)$ *is* $\mathcal{E}$*-constrained, then it can be implemented as an equilibrium of the intermediary game with default punishment being autarky with re-entry probability* $\lambda^D=0$.

Proof. Follows directly from Proposition 11 ■

The following corollary is immediate from the previous result and Proposition 5.

Corollary 4 *If the solution $x^*(0)$ is $\mathcal{E}$-constrained, then it can be implemented as an equilibrium of the EG environment default punishment being autarky with re-entry probability $\lambda^D = 0$.*

As in the intermediary game, the presence of hidden trading opportunities severely limits the amount of insurance the planner can provide. As a result, the planner will choose to banish some types in order to sustain greater ex-ante risk sharing. The planner always has the option of bring this household back in a later period. However we know that this is never the case in any $\mathcal{E}$-constrained allocation from Proposition 15. Therefore, the planner will only choose to banish some type θ^t in period t if $V_t(\theta^t) < V_t^d(\theta_t; \lambda^D)$. This allows us to implement the allocation in a decentralized environment in which the household voluntarily chooses to default in period t and consequently live in autarky forever.

5.3 Efficiency with Exogenous Incompleteness

A general result when markets are exogenously incomplete is that equilibrium outcomes are constrained inefficient. This literature considers a planner who restricted from making state-contingent transfers to agents but internalizes the effect of its allocations on prices. Geanakoplos and Polemarchakis (1986) find that equilibrium outcomes are generically inefficient in an exchange economy with multiple goods. In particular, they find that aggregate welfare can be increased if households are induced to save different amounts. More recently, Dávila et al. (2012) find that the equilibria in the model studied by Aiyagari (1994) are also constrained inefficient. Consumers do not internalize the effects of their choices on factor prices which in a model with uninsurable risk implies that there can be oversaving or undersaving relative to the constrained efficient equilibrium.

While the environment I consider is observationally equivalent to a large class of exogenously incomplete models, the approach to efficiency I take in the case with no banishment is substantially different. Rather than exogenously restrict the set of instruments available to the planner, I derive the incompleteness as a consequence of informational and commitment frictions. In this section, I explore whether for two observationally equivalent models, the two notions have different implications for whether the competitive equilibria are efficient. As I show using a simple example, it is possible that outcomes that are considered inefficient when markets are exogenously incomplete are no longer so when they are endogenously incomplete.

Consider a simple two period environment with $t = 1, 2$ and a continuum of households. In period 1, households can receive endowment shocks $\theta_i \in \Theta = (\theta_h, \theta_l)$ with probability π_i, $i \in \{h, l\}$. In period 2, households receive endowment shocks $x_i \in \mathbb{X} = (x_h, x_l)$ with probability κ_j, $j \in \{h, l\}$. The shocks are i.i.d over time and across households. As in previous sections, there are a large number of intermediaries who sign 2 period contracts with households. The timing of the game is follows:

1. Households can sign a contract with a single intermediary before period 1 types are realized
2. In period 1, after types are realized, households receive transfers from original the intermediary

3. Next, households can sign a contract with another intermediary. This contract is unobservable to the original intermediary and vice-versa.

4. At the beginning period 2, households can default on their obligations to the intermediary and receive utility

$$u\left(x_j\right)-\psi$$

Note here that since the horizon is finite I need to assume an exogenous cost of default. If $\psi=0$, no household would ever have an incentive to pay back in period 2. A contract for the date 0 intermediary is $B=\{\tau_1\left(i\right),\tau_2\left(i\right)\}$. While the equilibrium contract is derived in **??**, it suffices to notice from Proposition 5 that the equilibrium is equivalent to one in which households trade a risk free bond subject to debt constraints ϕ. In particular households choose $s\geq-\phi$ to maximize

$$u\left(\theta_i-qs_i\right)+\beta\mathbb{E}u\left(x_j+s_i\right)$$

where q and ϕ are chosen to clear markets and satisfy not-too-tight restrictions respectively. Moreover from Theorem 3 we know that given ψ, the equilibrium outcome is efficient. Under the following parametrization, $\beta=.9$; $\pi_i=1/2, \kappa_h=.8,\ \theta_l=.3,\ \theta_h=2,\ x_l=.5,\ x_h=1.4,$ in Figure 1, I plot the change in the ex-ante welfare and debt levels for $\psi\in[0,2]$.

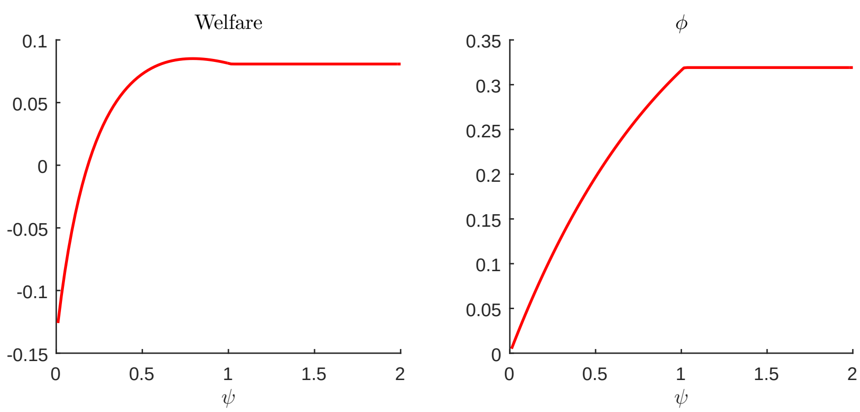

Figure 1: Welfare and Debt Levels

As one would expect, initially, as ψ increases, welfare increases and for ψ large enough, the change in welfare is zero after the low type ceases to be Euler-constrained. In addition, the endogenous debt levels ϕ increase and eventually flatten out. The key portion of Figure 1 to notice is the downward sloping part of the welfare plot. In a region around $\psi=1$, welfare *decreases* as ψ increases. The reason for this is a price effect which redistributes wealth from the period 1 low to the high type. This can be seen easily in the example by computing how the ex-ante welfare

$$W\left(\psi\right)=\pi_h\left[u\left(\theta_h-qs\right)+\beta\mathbb{E}u\left(x_j+s\right)\right]+\pi_l\left[u\left(\theta_l+qs\right)+\beta\mathbb{E}u\left(x_j-s\right)\right]$$

changes with ψ. One can show using simple algebra that

$$W'(\psi) = q'(\psi) s \left[-\pi_h u'(\theta_h - qs) + \pi_l u'(\theta_l + qs)\right] + \nu(\psi)$$

where $q'(\psi)$ is the change in price as a function of ψ and $\nu(\psi)$ is the multiplier on the borrowing constraint for the low type. Since risk sharing is imperfect, in general, $u'(\theta_h - qs) \leq u'(\theta_l + qs)$. Further, $q'(\psi) \leq 0$ since interest rates need to rise to clear markets as ψ increases. Given that the multiplier $\nu(\psi) \geq 0$, the change in welfare as ψ is increases is ambiguous. For ψ small enough, s will be small and so the multiplier effect will dominate and hence $W'(\psi) > 0$. However, as we can see from the picture as ψ get larger, s gets larger and $\nu(\psi)$ smaller, which causes the redistribution effect to dominate and $W'(\psi) < 0$.

Suppose we were to take as given the exogenously incomplete market structure and ask if the debt-constrained economy is efficient by considering a planning problem similar to Diamond (1967). For ϕ corresponding to the downward sloping portion of the welfare plot, we would conclude that outcomes are inefficient. In this case, imposing additional borrowing limits will implement the desired allocation.

As we have seen, when markets are endogenously incomplete, the outcome is efficient. This is because of hidden trading and in particular the fact that if the planner tried to transfer an amount smaller than ϕ to the low type in period 1, its voluntary participation constraints in period 2 would be slack. Therefore, it would use the hidden markets to borrow which would make these additional limits ineffective. In other words, the allocation would no longer satisfy the no-hidden-trades condition in Lemma 3.

The key difference between these two environments is presence of hidden markets. If in the exogenously incomplete world, the assumption is that contracts are observable and exclusivity can be enforced, then the planner should be able do much better than offer uncontingent transfers. However, if we think that the assumption of non-exclusivity is reasonable, then the outcomes are efficient.

5.4 Unique Implementation

The results in this section so far have two important implications for policy in the context of models with incomplete markets. The first is that interventions which may be desirable when markets are exogenously incomplete, might be ineffective in this environment. In addition, policies like ex-post bailouts will in general reduce welfare by lowering the amount of ex-ante risk-sharing that is possible. The second important message is that there is a role for policy to uniquely implement the best equilibrium. This motivates the use of credible off-equilibrium policies that will ensure that the best outcome will occur on path. To this end, I consider the effect of simple lender of last resort policies. In particular, I introduce a third strategic player into the game, namely a government.

Consider the intermediary game without banishment. Recall that the public history at the beginning of each period was denoted by $\hat{z}^{t-1} = \left(q^{t-1}, \mathcal{B}^{t-1}, \left(\gamma^{i,t-1}\right)_{i \in I}\right)$. Note that I am assuming that signed contracts between private agents are still unobservable to any outside authority. A *lender of last resort* policy is vector $G_t = \left\{q_t^G, \phi_t^G\right\}$ which consists of an interest rate $\frac{1}{q_t^G}$ and debt constraint ϕ_t^G for all $t \geq 1$.[19] In particular, under such a policy

1. Households can borrow and lend with the government at prices q_t^G subject to debt constraints ϕ_t^G
2. Intermediaries can borrow and lend with the government at prices q_t^G in an unconstrained fashion.

Given government policy G_t, we can define a competitive equilibrium given $\{G_t\}_{t\geq 1}$ in an analogous fashion to section 2.

Notice that a lender of last resort policy does not in general depend on the public history $\hat{z}^t$. Using the language of Atkeson, Chari, and Kehoe (2010) we can define a *sophisticated lender of last resort* policy to be a vector $\mathcal{G}_t\left(\hat{z}^t\right) = \left(q_t^G\left(\hat{z}^t\right), \phi_t^G\left(\hat{z}^t\right)\right)$ that depends on the public history $\hat{z}^t$. Given that we are including a third player into the game, the government, we need to modify the structure of the game. The timing within a period is identical to section 2, except that after private transactions have taken place, the government implements a policy $\mathcal{G}_t\left(\hat{z}^t\right)$ and finally private agents transact with the government. I now define the strategies of the players in this game. Given any history we can define a *continuation competitive equilibrium* as one that requires optimality by intermediaries and households. An equilibrium outcome is a collection $a_t = \{B_t, q_t, \mathcal{G}_t\}$ of contracts offered by intermediaries, prices q_t and government policy $\mathcal{G}_t$. Denote the government's strategy by σ^G. After any history, these strategies induce continuation outcomes in a standard fashion. Given this setup, we can define a *sophisticated equilibrium* as in Atkeson, Chari, and Kehoe (2010).

Definition 9 *A sophisticated equilibrium is a collection of strategies* $\left(\sigma^{HH}, \sigma^{INT}, \sigma^G\right)$ *such that after all histories, the continuation outcomes induced by* $\left(\sigma^{HH}, \sigma^{INT}, \sigma^G\right)$ *constitute a continuation competitive equilibrium.*

We can define a sophisticated outcome to be the equilibrium outcome associated with a sophisticated equilibrium. A policy σ_G^* *uniquely implements* a desired competitive equilibrium $a_t^* = \{B_t^*, q_t^*, \mathcal{G}_t^*\}$ if the sophisticated outcome associated with any sophisticated equilibrium of the form $\left(\sigma^{HH}, \sigma^{INT}, \sigma^{*G}\right)$ coincides with the desired competitive equilibrium. The main result in this section is that there exists a sophisticated lender of last resort policy that uniquely implements the best equilibrium.

Proposition 16 *Given a desired competitive equilibrium* a^**, there exists a sophisticated policy that uniquely implements it.*

Proof. We know from Theorem 1 that the equilibrium contract B_t^* is a simple borrowing and lending contract with debt constraints ϕ_t^*.Consider a history $\hat{z}^t$ with $\tilde{q}_t \neq q_t^*$ where q_t^* is the

[19] The government uses lump-sum taxes to balance its budget.

interest rate associated with ϕ_t^*. In this case $\tilde{B}_t \neq B_t^*$. $\tilde{B}_t$ is also an uncontingent contract and is characterized by debt constraints $\tilde{\phi}_t$. As a result to each price $\tilde{q}_t$ we can associate a private debt constraint $\phi_t^{\tilde{q}_t}$. Consider the following lender of last resort policy: for all $t \geq 0$

$$\mathcal{G}_t^* \left(\hat{z}^{t-1}, \hat{z}_t\right) = (0,0) \text{ if } q_t = q_t^*$$
$$\mathcal{G}_t^* \left(\hat{z}^{t-1}, \hat{z}_t\right) = \left(q_t^*, \max\left(\phi_t^* - \phi_t^{\tilde{q}_t}, 0\right)\right) \text{ if } q_t \neq q_t^* \text{ and } q_{t-j} = q_{t-j}^* \text{ for all } j \geq 1$$
$$\mathcal{G}_{t+s}^* \left(\hat{z}^{t-1}, \hat{z}_t, \cdot\right) = \left(q_{t+s}^*, \phi_{t+s}^*\right) \text{ for all } s \geq 1 \text{ if } q_t \neq q_t^* \text{ and } q_{t-j} = q_{t-j}^* \text{ for all } j \geq 1$$

where (q_t^*, ϕ_t^*) correspond to the price and debt constraint associated with the desired equilibrium. Given strategy σ_G^* and associated policy, $\{\mathcal{G}_t^*\}_{t\geq 0}$, it is easy to see that a^* is an equilibrium outcome of the game. We want to show that it is the unique outcome. Given a period t, consider whether outcome $\left\{\tilde{B}_t, \tilde{q}_t, \mathcal{G}_t^*\right\}$ with $\tilde{q}_t \neq q_t$ can ever occur on the equilibrium path. It is easy to see that if $\tilde{q}_t \neq q_t$ then arbitrage opportunities exist and so in any equilibrium, it must be that $\tilde{q}_t = q_t^*$. As a result, since the only equilibrium contract consistent with q^* is B^*, it must be that $\tilde{B}_t = \tilde{B}_t^*$. Finally, we need to show that the continuation outcomes after any history constitute continuation competitive equilibria. In this case, after an undesirable history, Euler-constrained households will borrow from the government. In following periods, given $\mathcal{G}_{t+s}^* \left(\hat{z}^{t-1}, \hat{z}_t, \cdot\right)$, market prices will be q_{t+s}^* and private intermediaries will only offer uncontingent savings contracts, and households will only transact with the government. Consider the incentives for any household to default in $t+1$ given this policy. Since the equilibrium outcome $\left(q_{t+s}^*, \phi_{t+s}^*\right)$ is consistent with no default, all households will weakly prefer to pay the government back in all future periods. ■

The policies that uniquely implement the desired equilibrium are simple. After any undesired history, the government announces a sophisticated lender of last resort policy that allows private agents to borrow and lend with it at prices $\left\{q_{t+s}^*\right\}_{s\geq 0}$. In period t, households can borrow up to an amount so that the total debt is at most ϕ_t^* while in all future periods, they can borrow the full amount ϕ_t^* from the government. After any undesired history, in the continuation equilibrium, households will only transact with the government while intermediaries will offer uncontingent savings contracts. As a result the policy is well defined. It is then easy to see that the only equilibrium consistent with this policy is the desired one since no-arbitrage will ensure that $\tilde{q}_t = q_t^*$.

6 Application: Optimal Bankruptcy Policies

In this section I present a simple example to illustrate how the framework with endogenously incomplete markets can be useful for thinking about a variety of policy questions. The short term defaultable debt model with stochastic re-entry has been known to match several key aspects of bankruptcy and unsecured credit in the United States. Further, these models have been used to study the effects of changing bankruptcy laws on welfare. The environment with private information, limited commitment and hidden trading has sharp implications for how to design these policies. First Proposition 11 implies that the efficient allocation can be decentralized as in equilibrium in which intermediaries choose the re-entry probabilities after default. More importantly it suggests that the optimal re-entry probabilities should be functions of the *level of debt defaulted on*. In particular, the probability of re-entry after default should be smaller if the level of debt defaulted on is larger. One can re-interpret the environment as follows: intermediaries can only choose whether or not to banish a household while an independent regulatory authority, the government chooses the probability of re-entry. It is straightforward to see that the efficient allocation in both environments are identical and in particular, the efficient allocation can be decentralized as an equilibrium with short-term defaultable debt in which the government optimally chooses the re-entry probability to be a function of the level of debt defaulted upon. While in general, making default punishments a function of the level of debt is always weakly better, in the simple example below, I demonstrate how moving from a system with constant punishments to one in which these are functions of the level of defaulted debt can strictly increase overall welfare in the economy.

Consider a simple two period environment identical to that in subsection 5.3 except that in period 1, households can receive endowment shocks $\theta_i \in \Theta = (\theta_h, \theta_m, \theta_l)$ with probability π_i, $i \in \{h, m, l\}$. In particular, there are three initial types rather than two. The timing is identical, except that analogously to the environment with banishment, I will allow intermediaries to control the severity of punishment after banishment. Therefore, if a household is banished in period 2, it receives

$$u(x_j) - \psi(i,j)$$

A contract for the date 0 intermediary is $B = \{\tau_1(i), \tau_2(i), \delta_2(i,j), \psi(i,j)\}$. Since this is a two-period environment, intermediaries cannot choose a re-entry probability. However, I allow them to choose the level of default punishment $\psi(i,j)$.[20] One can show using the results in the previous sections that the best equilibrium is the equilibrium of the following game: Period 0 intermediaries choose B to minimize

$$\sum_i [\tau_1(i) + q\mathbb{E}_j [1 - \delta_2(i,j)] \tau_2(i,j)]$$

subject to $\forall i$,

$$c_1(i) = \theta_i + \tau_1(i) \ ,$$
$$c_2(i,j) = x_j + \tau_2(i) \text{ if } \delta_2(i,j) = 0,$$
$$c_2(i,j) = x_j \text{ if } \delta_2(i,j) = 1$$

$$\tau_1(i) + q\mathbb{E}_j [1 - \delta_2(i,j)] \tau_2(i) = 0$$

and for all (i,j)

$$[1 - \delta_2(i,j)] u(x_j + \tau_2(i)) \geq [1 - \delta_2(i,j)] \left(u(x_j) - \psi(i,j')\right)$$

[20]The example is constructed to be simple in order to illustrate that default punishments should depend on the level of defaulted debt. This intuition carries over to the infinite hoirzon game with re-entry probabilities.

$$\sum_i \pi_i \left[u\left(c_1\left(i\right)\right) + \beta \mathbb{E}_j \left[1 - \delta_2\left(i,j\right)\right] c_2\left(i,j\right)\right] \geq \underline{u}$$

In equilibrium q is chosen so that markets clear,

$$\sum_i \tau_1\left(i\right) = 0$$

I will first restrict intermediaries to choose $\psi\left(i,j\right) = \psi$ and next relax this assumption. Clearly welfare under the latter will always be weakly greater but in a simple numerical illustration I show that ex-ante welfare can be strictly higher. Under the following parametrization, $\beta = .9$, $\theta_h = 2$, $\theta_m = .5$, $\theta_l = .3$, $x_h = 1.4$, $x_l = .5$, $\kappa = .8$, $\pi_i = 1/3$, ex-ante welfare is approximately 5.7% larger when the default costs are allowed to be different. In the best equilibrium, in both cases, $\delta_2\left(m,l\right) = \delta_2\left(l,l\right)$ and so these types are banished. Using Proposition 5 we can construct an equilibrium of an Eaton-Gersovitz economy with price function, $\mathcal{Q}\left(l\right)$ and default cost function $\psi^E\left(l\right)$. Figure 2 plots these functions for the above example. In this figure, the red lines correspond

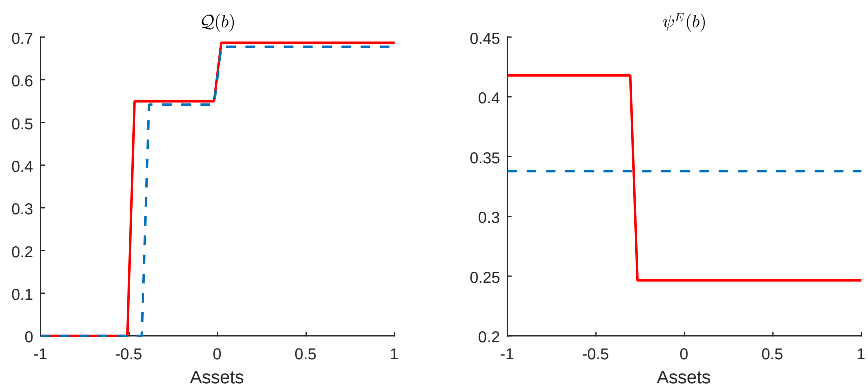

Figure 2: Pricing and Default Cost Functions

to variables in the efficient allocation while the blue dashed lines corresponds to the equilibrium when the punishment is restricted to be independent of the level of debt. As we can see, the optimal default punishment is a function of the level of debt defaulted on. The larger the amount of debt defaulted on, the more stringent the punishment. The intuition for this result is simple. Since types are hidden, the punishment only needs to be large enough to prevent the high type in period 2 from pretending to be the low type. Since the voluntary participation constraints are $u\left(x_h - \tau_1\left(i\right)\right) \geq u\left(x_h\right) - \psi\left(i,l\right)$, a larger amount of debt implies a larger punishment required to keep the high type indifferent. The efficient allocation trades off the benefit of higher consumption in period 1 at the cost of imposing a harsher punishment in period 2. Since the marginal benefit is larger for θ_l as compared to θ_m in the efficient allocation, type θ_l consumes more in period 1 and suffers a larger punishment in low state in period 2 when it defaults.

7 Discussion of Assumptions

In this section, I discuss the role of some of the assumptions in the model.

1. *Finitely lived intermediaries*: The reason for this is an existence problem. In the model, tighter debt constraints imply lower interest rates or higher q_t. In particular, it may be that the value of default is large enough so that the equilibrium debt constraints imply an interest rate that is less than 1. In this case, the present discounted value of transfers to the household is ∞ and as a result we cannot have infinitely lived intermediaries in the model. One example of such an environment is Hellwig and Lorenzoni (2009).

2. *i.i.d endowments*: I have assumed that the endowment shocks are independently and identically distributed across time and households. The reason is tractability. Introducing persistent endowment complicates the environment further but would be an interesting extension of the model.

3. *Banishment/Default assumptions*: The assumption that banished or defaulting households cannot sign with a new intermediary is important to the results in the model. Banishment is useful precisely because banished households cannot sign with intermediaries. However note that the problem is well defined even if banished/defaulting households can sign a restricted set of contracts. For example, one can assume that defaulting households are allowed only to save and not borrow, as in Hellwig and Lorenzoni (2009). The equilibrium will still be equivalent to an incomplete markets environment subject to endogenous debt constraints. However, in this case banishment will no longer be a useful tool to introduce state-contingency in contracts.

4. *Restriction to signing with only one new intermediary at a time*: While the environment allows households to sign multiple contracts in a hidden fashion, I only allow them to sign at most one new contract each period. The reason for this is that if all intermediaries posted identical contracts and households could sign multiple hidden contracts, households could in theory borrow an infinite large amount and default the next period.

8 Conclusion

Models with exogenously incomplete markets have been widely used to study a variety of quantitative questions in macroeconomics and international economics. The purpose of this paper is to complement this literature by providing a framework to think about policy questions in the context of these models. The main advantage of my approach is that unlike the majority of the contracting literature, the resulting contracts are identical to the ones assumed by the applied literature. In particular, I show that both uncontingent contracts with debt constraints and defaultable debt contracts endogenously arise under appropriate assumptions from a contracting environment with private information, limited commitment and hidden trading. I show that the best equilibrium outcome in both cases are efficient but that there are multiple equilibria. This paper has three important implications for policy. The first is that outcomes that might appear inefficient with exogenously incomplete markets may not be so when we explicitly model the underlying frictions. The second is that there is an important role for policy to implement the best equilibrium. The third and final implication is that by explicitly modeling the informational and commitment frictions, we can better understand what kinds of policies will be welfare enhancing. For example, I show that in the context of bankruptcy models that allowing the punishment after default to depend on the level of defaulted debt is welfare improving.

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A Appendix: Proofs From the Main Text

This appendix contains proofs from the main text.

A.1 Proofs from Section 3

Proof of Proposition 1. Suppose that in some equilibrium, for some t and history θ^t, $u'\left(c_t\left(\theta^t\right)\right)q_t <$

$\beta\mathbb{E}_t u'\left(c_{t+1}\left(\theta^{t+1}\right)\right)$. Recall that $\tau_t\left(\theta^t\right) = \tau_t^{old}\left(\theta^t\right) + {}_t\tau_t\left(\theta^t\right)$ and so in equilibrium $c_t\left(\theta^t\right) = \theta_t + \tau_t\left(\theta^t\right)$.

Since we are considering symmetric equilibria in which ex-ante identical intermediaries offer the same contract in equilibrium, we consider the incentives for a deviating intermediary to offer a different contract and make strictly positive profits. Consider an intermediary offering a $\varepsilon\delta-$savings contract $S_t^{\varepsilon,\delta}$ for some $\varepsilon > 0$ and $\delta < 1$. Notice that the intermediary makes positive profits whenever this contract is accepted. Since $u'\left(c_t\left(\theta^t\right)\right)q_t < \beta\mathbb{E}_t u'\left(c_{t+1}\left(\theta^{t+1}\right)\right)$, there exists $\varepsilon > 0, \delta < 1$ such that type θ^t will strictly prefer to sign an $\varepsilon\delta$ savings contract if offered. These contracts are by construction incentive compatible and satisfy voluntary participation constraints. As a result an intermediary offering such a contract will make positive profits which is a contradiction.

The proof of part 2 is also straightforward. Suppose for contradiction we have a household who is Euler-constrained in period t, and in period $t+1$, for all $\tilde{\theta}^{t+1}$,

$$V_{t+1}\left(\tilde{\theta}^{t+1}\right) > V_{t+1}^{d}\left(\tilde{\theta}^{t+1}\right)$$

Consider the following deviating contract

$$\begin{aligned} {}_t\tilde{\tau}_t &= \delta\varepsilon \\ {}_t\tilde{\tau}_{t+1} &= -\frac{\varepsilon}{q_t} \end{aligned}$$

where $\varepsilon > 0, \delta < 1$ and the contract is not contingent on reported type. Clearly we can find an ε, δ such a constrained household accepting is made strictly better off. Moreover for ε, δ small, incentives are preserved for all households since the voluntary participation constraints are assumed to be slack. Since intermediaries make a strictly positive contract by offering such a contract, we have a contradiction. ■

Proof of Proposition 2: The proof requires a series of preliminary results.

The first intermediate lemma tells us that we need only consider a relaxed problem and drop all voluntary participation constraints besides those for the lowest type.

Lemma 7 *In any equilibrium, for at any date t and history θ^{t-1}, if the voluntary participation constraint for type $\left(\theta^{t-1}, \underline{\theta}\right)$ is satisfied, then it is satisfied for all types $\left(\theta^{t-1}, \theta\right)$, $\theta \in \Theta$.*

Proof of Lemma 7. Let $W_{\theta^{t-1}}\left(\theta, \hat{\theta}\right) = u\left(\theta + \tau_t\left(\theta^{t-1}, \hat{\theta}\right)\right) + \beta \mathbb{E}_t V_{t+1}\left(\theta^{t-1}, \hat{\theta}\right)$ be the equilibrium value for type $\left(\theta^{t-1}, \theta\right)$ pretending to be $\left(\theta^{t-1}, \hat{\theta}\right)$. Suppose first that the VP constraint for type $\left(\theta^{t-1}, \underline{\theta}\right)$ is satisfied and that $\tau_t\left(\theta^{t-1}, \underline{\theta}\right) \leq 0$. Then

$$\begin{aligned} W_{\theta^{t-1}}\left(\theta, \theta\right) &= u\left(c_t\left(\theta^{t-1}, \theta\right)\right) + \beta \mathbb{E}_t V_{t+1}\left(\theta^{t-1}, \theta\right) \\ &\geq W_{\theta^{t-1}}\left(\theta, \underline{\theta}\right) \\ &= W_{\theta^{t-1}}\left(\underline{\theta}, \underline{\theta}\right) + u\left(\theta + \tau_t\left(\theta^{t-1}, \underline{\theta}\right)\right) - u\left(\underline{\theta} + \tau_t\left(\theta^{t-1}, \underline{\theta}\right)\right) \\ &= V^d\left(\underline{\theta}\right) + u\left(\theta + \tau_t\left(\theta^{t-1}, \underline{\theta}\right)\right) - u\left(\underline{\theta} + \tau_t\left(\theta^{t-1}, \underline{\theta}\right)\right) \\ &= V^d\left(\theta\right) + u\left(\underline{\theta}\right) - u\left(\theta\right) + u\left(\theta + \tau_t\left(\theta^{t-1}, \underline{\theta}\right)\right) - u\left(\underline{\theta} + \tau_t\left(\theta^{t-1}, \underline{\theta}\right)\right) \end{aligned}$$

Since $\tau_t\left(\theta^{t-1}, \underline{\theta}\right) \leq 0$,

$$\begin{aligned} & u\left(\underline{\theta}\right) - u\left(\theta\right) + u\left(\theta + \tau_t\left(\theta^{t-1}, \underline{\theta}\right)\right) - u\left(\underline{\theta} + \tau_t\left(\theta^{t-1}, \underline{\theta}\right)\right) \\ &= -u'\left(x\right)\left[\theta - \underline{\theta}\right] + u'\left(y\right)\left[\theta - \underline{\theta}\right] \\ &= \left[\theta - \underline{\theta}\right]\left(u'\left(y\right) - u'\left(x\right)\right) \\ &\geq 0 \end{aligned}$$

where $x \in \left[\underline{\theta}, \theta\right]$ and $y \in \left[\underline{\theta} + \tau_t\left(\theta^{t-1}, \underline{\theta}\right), \theta + \tau_t\left(\theta^{t-1}, \underline{\theta}\right)\right]$. Next, suppose that $\tau_t\left(\theta^{t-1}, \underline{\theta}\right) > 0$. Then the VP constraint for type $\left(\theta^{t-1}, \underline{\theta}\right)$ is slack. Suppose that the the VP constraint binds for some other type $\left(\theta^{t-1},\theta\right)$. Then

$$\begin{aligned}
W_{\theta^{t-1}}\left(\underline{\theta},\underline{\theta}\right) &\leq W_{\theta^{t-1}}\left(\theta,\theta\right)+u\left(\underline{\theta}+\tau_t\left(\theta^{t-1},\underline{\theta}\right)\right)-u\left(\theta+\tau_t\left(\theta^{t-1},\underline{\theta}\right)\right) \\
&= V^d\left(\theta\right)+u\left(\underline{\theta}+\tau_t\left(\theta^{t-1},\underline{\theta}\right)\right)-u\left(\theta+\tau_t\left(\theta^{t-1},\underline{\theta}\right)\right) \\
&= u\left(\underline{\theta}+\tau_t\left(\theta^{t-1},\underline{\theta}\right)\right)+EV^d\left(\theta'\right)+u\left(\theta\right)-u\left(\theta+\tau_t\left(\theta^{t-1},\underline{\theta}\right)\right) \\
&< u\left(\underline{\theta}+\tau_t\left(\theta^{t-1},\underline{\theta}\right)\right)+EV^d\left(\theta'\right) \\
&\leq W_{\theta^{t-1}}\left(\underline{\theta},\underline{\theta}\right)
\end{aligned}$$

which is a contradiction. In particular, if $\tau_t\left(\theta^{t-1},\underline{\theta}\right)>0$ the the VP constraints for all types $\left(\theta^{t-1},\theta\right)$ are slack. ■

Recall that $\tau_t\left(\theta^t\right)=\tau_t^{old}\left(\theta^t\right)+{}_t\tau_t\left(\theta^t\right)$. The result states that in general, the voluntary participation constraints will bind for the lowest type $\underline{\theta}$. This is true generally in models with private information and limited commitment, for example in Dovis (2014). Note the binding pattern of these constraints is the opposite of models with only limited commitment such as Kehoe and Levine (1993) and Alvarez and Jermann (2000).

For any household define $A_t\left(\theta^{t-1},\theta\right)$ to be the equilibrium expected present discounted value of future transfers for type $\left(\theta^{t-1},\theta\right)$

$$A_t\left(\theta^{t-1},\theta\right)\equiv\tau_t\left(\theta^{t-1},\theta_t\right)+q_t\sum_{\theta'\in\Theta}\pi\left(\theta'\right)A_{t+1}\left(\theta^{t-1},\theta,\theta'\right)$$

Similarly, given a contract B_t, let

$${}_t\mathcal{P}_s\left(\theta^s\right)\equiv{}_t\tau_s\left(\theta^s\right)+q_t\sum_{\theta'\in\Theta}\pi\left(\theta^s,\theta'\right){}_t\mathcal{P}_{s+1}\left(\theta^s,\theta'\right)$$

denote the expected present discounted value of transfers associated with contract B_t from period s onwards.

The next set of results will be used to prove that in any equilibrium, the expected present discounted value of transfers to households with the same history θ^{t-1}, is independent of their period t reports.

Lemma 8 *In any equilibrium, for any t and any contract offered by an intermediary born at date t, ${}_t\mathcal{P}_t\left(\theta^t\right)=0$ for all θ^t.*

Proof of Lemma 8. Suppose not. Clearly, ${}_t\mathcal{P}_t\left(\theta^t\right)>0$ for all θ^t is not possible since the intermediary would making negative profits. On the other hand if ${}_t\mathcal{P}_t\left(\theta^t\right)\leq 0$ for all θ^t with strict inequality for some, then a deviating intermediary can offer a contract which transfers a little more to some types and still continue to make positive profits. As a result these types will strictly prefer to sign with the deviating intermediary. Finally, suppose that there exists θ,θ' such that ${}_t\mathcal{P}_t\left(\theta^{t-1},\theta\right)>0$ and ${}_t\mathcal{P}_t\left(\theta^{t-1},\theta'\right)<0$. Then at the beginning of period t, consider a deviating intermediary offering the following contract,

$$\begin{aligned}
{}_t\tilde{\mathcal{P}}_t\left(\theta^{t-1},\theta'\right) &= {}_t\mathcal{P}_t\left(\theta^{t-1},\theta'\right)+\varepsilon \\
{}_t\tilde{\tau}_{t+s}\left(\theta^{t-1},\hat{\theta}\right) &= 0 \text{ for all } s\geq 0,\ \text{for } \hat{\theta}\neq\theta'
\end{aligned}$$

where $\varepsilon>0$ and small. Notice that types θ strictly prefer the original contract while types θ' strictly prefer ${}_t\tilde{\mathcal{P}}_t$ to ${}_t\mathcal{P}_t$. As a result, these households will strictly prefer to sign with the deviating intermediary who makes a positive profit. ■

The lemma shows that all contracts offered by intermediaries must make zero profits and as a result there is no cross subsidization between contracts. The result is a direct consequence of perfect competition among intermediaries. If there is cross subsidization between initial types, a deviating intermediary can offer only the contract that yields positive profits and make strictly positive profits in equilibrium.

In an environment with 2 period lived intermediaries for any $t \geq 0$,

$$
\begin{aligned}
{}_t\mathcal{P}_t\left(\theta^t\right) &= {}_t\tau_t\left(\theta^t\right) + q_t \sum_{\theta' \in \Theta} \pi\left(\theta^t, \theta'\right) {}_t\tau_{t+1}\left(\theta^t, \theta'\right) \\
{}_t\mathcal{P}_{t+1}\left(\theta^{t+1}\right) &= {}_t\tau_{t+1}\left(\theta^{t+1}\right)
\end{aligned}
$$

The final result required for the proof of Proposition 2 shows that higher types will always strictly prefer transfer sequences with a larger present discounted value even if they are Euler-constrained. Given an equilibrium transfer sequence A, define

$$
\begin{aligned}
A_{\varepsilon_+}\left(\theta^{t-1}, \theta_t\right) &\equiv \tau_t\left(\theta^{t-1}, \theta_t\right) + \varepsilon + q_t \sum_{\theta'} \pi\left(\theta'\right)\left[A_{t+1}\left(\theta^{t-1}, \theta_t, \theta'\right) - a\varepsilon\right] \\
A_{\varepsilon_-}\left(\theta^{t-1}, \theta_t\right) &\equiv \tau_t\left(\theta^{t-1}, \theta_t\right) - \varepsilon + q_t \sum_{\theta'} \pi\left(\theta'\right)\left[A_{t+1}\left(\theta^{t-1}, \theta_t, \theta'\right) + a\varepsilon\right]
\end{aligned}
$$

Notice that if $A_{\varepsilon_+}\left(\theta^{t-1}, \theta_t\right) > A_t\left(\theta^{t-1}, \theta_t\right)$ then it must be that $a < R_{t+1} = \frac{1}{q_t}$ and if $A_{\varepsilon_-}\left(\theta^{t-1}, \theta_t\right) > A_t\left(\theta^{t-1}, \theta_t\right)$ then $a > R_{t+1}$.

Given a transfer schedule A and the associated transfer sequence, define

$$
\begin{aligned}
Z_t\left(\theta^t, s_t, \tau_t; A\right) = \max_{s_{t+1}} & \, u\left(c_t\right) + \beta \mathbb{E}_t Z_{t+1}\left(\theta^{t+1}, s_{t+1}, \tau_{t+1}; A\right) \\
& s.t. \\
& c_{t'} + s_{t'+1} \leq \theta_{t'} + \tau_{t'}\left(\theta^{t'}\right) + R_{t'} s_{t'}, \ \forall t' \geq t \\
& s_{t'+1} \geq 0, \ \forall t' \geq t
\end{aligned}
$$

where $R_{t'} = \frac{1}{q_{t'}}$. Here $Z_t\left(\theta^t, s_t, \tau; A\right)$ denotes the continuation value for a household of type θ^t who receives transfers according to A and can save at rate R_{t+s}, $s \geq 0$. The reason this will be useful is that in general, deviating intermediaries are always willing to provide savings contracts since they have no fear of default the following period. Therefore, if it is true that a household can do strictly better by lying and and saving, there exists a deviating contract that makes both the intermediary and household strictly better off. This will be particularly useful in the proof of Proposition 2.

Lemma 9 *1. If $A_{\varepsilon_+}\left(\theta^{t-1}, \theta_t\right) > A\left(\theta^{t-1}, \theta_t\right)$ then $Z\left(\left(\theta^{t-1}, \theta'\right), 0, \tau_t + \varepsilon; A_{\varepsilon_+}\right) > Z\left(\left(\theta^{t-1}, \theta'\right), 0, \tau_t; A\right)$ for all $\theta' > \theta$*

2. If $A_{\varepsilon_-}\left(\theta^{t-1}, \theta_t\right) > A\left(\theta^{t-1}, \theta_t\right)$ and $Z\left(\left(\theta^{t-1}, \theta'\right), 0, \tau_t - \varepsilon; A_{\varepsilon_-}\right) \geq Z\left(\left(\theta^{t-1}, \theta'\right), 0, \tau_t; A\right)$ then $Z\left(\left(\theta^{t-1}, \theta'\right), 0, \tau_t - \varepsilon; A_{\varepsilon_-}\right) > Z\left(\left(\theta^{t-1}, \theta'\right), 0, \tau_t; A\right)$ for $\theta' > \theta$

Proof. Part 1. To prove this I show that $\frac{\partial}{\partial \varepsilon} Z\left(\left(\theta^{t-1}, \theta'\right), 0, \tau_t + \varepsilon; A_{\varepsilon_+}\right)\Big|_{\varepsilon=0} > 0$.

We have that

$$
\begin{aligned}
&\frac{\partial}{\partial \varepsilon} Z_t\left(\left(\theta^{t-1}, \theta'\right), 0, \tau_t+\varepsilon ; A_{\varepsilon_+}\right)=u'\left(\theta'-s_{t+1}+\tau_t+\varepsilon\right)\left[-\frac{\partial}{\partial \varepsilon} s_{t+1}+1\right] \\
&+\beta \frac{\partial}{\partial \varepsilon} \mathbb{E}_t Z_{t+1}\left(\left(\theta^{t-1}, \theta', \theta_{t+1}\right), s_{t+1}, \tau_{t+1}-a \varepsilon\right) \\
&=u'\left(\theta'-s_{t+1}+\tau_t\left(\theta^t\right)+\varepsilon\right)\left[-\frac{\partial}{\partial \varepsilon} s_{t+1}+1\right]+\beta\left[\mathbb{E}_t Z_{2, t+1} \frac{\partial}{\partial \varepsilon} s_{t+1}-a \mathbb{E}_t Z_{3, t+1}\right] \\
&=u'\left(c_t\left(\theta^{t-1}, \theta'\right)\right)\left[-\frac{\partial}{\partial \varepsilon} s_{t+1}+1\right]+\beta\left[\begin{array}{c}
R_{t+1} \mathbb{E}_t u'\left(c_{t+1}\left(\theta^{t-1}, \theta', \theta_{t+1}\right)\right) \frac{\partial}{\partial \varepsilon} s_{t+1} \\
-a \mathbb{E}_t u'\left(c_{t+1}\left(\theta^{t-1}, \theta', \theta_{t+1}\right)\right)
\end{array}\right] \\
&=\frac{\partial}{\partial \varepsilon} s_{t+1}\left[-u'\left(c_t\left(\theta^{t-1}, \theta'\right)\right)+\beta R_{t+1} \mathbb{E}_t u'\left(c_{t+1}\left(\theta^{t-1}, \theta', \theta_{t+1}\right)\right)\right] \\
&+u'\left(c_t\left(\theta^{t-1}, \theta'\right)\right)-\beta a \mathbb{E}_t u'\left(c_{t+1}\left(\theta^{t-1}, \theta', \theta_{t+1}\right)\right) \\
&=-\frac{\partial}{\partial \varepsilon} s_{t+1} \mu_t\left(\theta^{t-1}, \theta'\right)+u'\left(c_t\left(\theta^{t-1}, \theta'\right)\right)-\beta a \mathbb{E}_t u'\left(c_{t+1}\left(\theta^{t-1}, \theta', \theta_{t+1}\right)\right)
\end{aligned}
$$

$$
\begin{aligned}
&>-\frac{\partial}{\partial \varepsilon} s_{t+1} \mu_t\left(\theta^{t-1}, \theta'\right)+u'\left(c_t\left(\theta^{t-1}, \theta'\right)\right)-\beta R_{t+1} \mathbb{E}_t u'\left(c_{t+1}\left(\theta^{t-1}, \theta', \theta_{t+1}\right)\right) \qquad \text{(A.1)} \\
&=-\frac{\partial}{\partial \varepsilon} s_{t+1} \mu_t\left(\theta^{t-1}, \theta'\right)+\mu_t\left(\theta^{t-1}, \theta'\right) \\
&\geq 0
\end{aligned}
$$

where $\mu_t\left(\theta^{t-1}, \theta'\right)$ is the multiplier on the non-negative savings constraint. The strict inequality in (A.1) follows since $a<R$ and

$$
\begin{aligned}
& c_t+s_{t+1}=\theta_t+\tau_t+\varepsilon \\
& \Rightarrow \frac{\partial}{\partial \varepsilon} c_t+\frac{\partial}{\partial \varepsilon} s_{t+1}=1 \\
& \Rightarrow \frac{\partial}{\partial \varepsilon} s_{t+1}<1
\end{aligned}
$$

Part 2. Notice that

$$
\begin{aligned}
&\frac{\partial}{\partial \varepsilon} Z\left(\left(\theta^{t-1}, \theta'\right), 0, \tau_t-\varepsilon ; A_{\varepsilon_-}\right)=u'\left(\theta'-s_{t+1}+\tau_t\left(\theta^t\right)-\varepsilon\right)\left[-\frac{\partial}{\partial \varepsilon} s_{t+1}-1\right] \\
&+\beta \frac{\partial}{\partial \varepsilon} E Z_{t+1}\left(\left(\theta^{t-1}, \theta', \theta_{t+1}\right), s_{t+1}, \tau_{t+1}+a \varepsilon\right) \\
&=u'\left(c_t\left(\theta^{t-1}, \theta'\right)\right)\left[-\frac{\partial}{\partial \varepsilon} s_{t+1}-1\right]+\beta\left[E Z_{2, t+1} \frac{\partial}{\partial \varepsilon} s_{t+1}+a E Z_{3, t+1}\right] \\
&=u'\left(c_t\left(\theta^{t-1}, \theta'\right)\right)\left[-\frac{\partial}{\partial \varepsilon} s_{t+1}-1\right]+\beta\left[\begin{array}{c}
R E u'\left(c_{t+1}\left(\theta^{t-1}, \theta', \theta_{t+1}\right)\right) \frac{\partial}{\partial \varepsilon} s_{t+1} \\
+a E u'\left(c_{t+1}\left(\theta^{t-1}, \theta', \theta_{t+1}\right)\right)
\end{array}\right] \\
&=\frac{\partial}{\partial \varepsilon} s_{t+1}\left[-u'\left(c_t\left(\theta^{t-1}, \theta'\right)\right)+\beta R E u'\left(c_{t+1}\left(\theta^{t-1}, \theta', \theta_{t+1}\right)\right)\right] \\
&-u'\left(c_t\left(\theta^t\right)\right)+\beta a E u'\left(c_{t+1}\left(\theta^{t-1}, \theta', \theta_{t+1}\right)\right) \\
&=-\frac{\partial}{\partial \varepsilon} s_{t+1} \mu_t\left(\theta^{t-1}, \theta'\right)-u'\left(c_t\left(\theta^{t-1}, \theta'\right)\right)+\beta a E u'\left(c_{t+1}\left(\theta^{t-1}, \theta', \theta_{t+1}\right)\right)
\end{aligned}
$$

If type $\left(\theta^{t-1},\theta'\right)$ is Euler-unconstrained then clearly

$$\begin{aligned}&-\frac{\partial}{\partial\varepsilon}s_{t+1}\mu_t\left(\theta^{t-1},\theta'\right)-u'\left(c_t\left(\theta^{t-1},\theta'\right)\right)+\beta a\mathbb{E}_t u'\left(c_{t+1}\left(\theta^{t-1},\theta',\theta_{t+1}\right)\right)\\&=-u'\left(c_t\left(\theta^{t-1},\theta'\right)\right)+\beta a\mathbb{E}_t u'\left(c_{t+1}\left(\theta^{t-1},\theta',\theta_{t+1}\right)\right)>0\end{aligned}$$

since $a>R$. Suppose however that the type $\left(\theta^{t-1},\theta'\right)$ is Euler-constrained at $\varepsilon=0$. Then $s_{t+1}=0$ and so $\beta aEu'\left(c_{t+1}\left(\theta^{t-1},\theta',\theta_{t+1}\right)\right)=\beta aEu'\left(c_{t+1}\left(\theta^{t+1}\right)\right)$ since type $\left(\theta^{t-1},\theta\right)$ will also be Euler-constrained. Moreover since $\theta'>\theta$, we must have that $\mu_t\left(\theta^{t-1},\theta'\right)<\mu_t\left(\theta^{t-1},\theta\right)$. Therefore

$$\begin{aligned}&=-\frac{\partial}{\partial\varepsilon}s_{t+1}\mu_t\left(\theta^{t-1},\theta'\right)-u'\left(c_t\left(\theta^{t-1},\theta'\right)\right)+\beta a\mathbb{E}_t u'\left(c_{t+1}\left(\theta^{t-1},\theta',\theta_{t+1}\right)\right)\\&>-\mu_t\left(\theta^{t-1},\theta'\right)-u'\left(c_t\left(\theta^{t-1},\theta'\right)\right)+\beta a\mathbb{E}_t u'\left(c_{t+1}\left(\theta^{t-1},\theta',\theta_{t+1}\right)\right)\\&\geq-\mu_t\left(\theta^{t}\right)-u'\left(c_t\left(\theta^{t}\right)\right)+\beta a\mathbb{E}_t u'\left(c_{t+1}\left(\theta^{t-1},\theta',\theta_{t+1}\right)\right)\\&=-\mu_t\left(\theta^{t}\right)-u'\left(c_t\left(\theta^{t}\right)\right)+\beta a\mathbb{E}_t u'\left(c_{t+1}\left(\theta^{t+1}\right)\right)\\&\geq 0\end{aligned}$$

since by assumption $\frac{\partial}{\partial\varepsilon}Z\left(\left(\theta^{t-1},\theta\right),0,\tau_t-\varepsilon;A_{\varepsilon-}\right)\big|_{\varepsilon=0}\geq 0$. ■

Lemma 10 *Given an equilibrium transfer sequence A, if for any date and history θ^{t-1},*

$$Z_t\left(\left(\theta^{t-1},\theta\right),s_t,\tau_t\left(\theta^{t-1},\tilde{\theta}\right);A\right)>Z_t\left(\left(\theta^{t-1},\theta\right),s_t,\tau_t\left(\theta^{t-1},\theta\right);A\right)$$

then there exists a deviating contract that makes both the intermediary and type $\left(\theta^{t-1},\theta\right)$ strictly better off.

Proof. It is clear from the definition of Z that such a contract will be savings contract. In particular the deviating intermediary can offer an $\varepsilon\delta$ savings contract that make both it and the household strictly better off. Such a contract will always be incentive compatible and satisfy voluntary participation constraints. ■

Proof of Proposition 2. Without loss of generality, we can just consider the truncated $T-$ period economy with a T lived intermediaries. Suppose we have an equilibrium in this environment. Let the equilibrium transfer sequence for the households be denoted by $\left\{\zeta_t\left(\theta^t\right)\right\}_{t,\theta^t}$ where in each period $\zeta_t\left(\theta^t\right)=\zeta_t^{old}\left(\theta^t\right)+{}_t\zeta_t\left(\theta^t\right)$ for all $\theta^t\in\Theta^t$. Let $R_t=\frac{1}{q_t}$ and construct a sequence of contracts for 2 period intermediaries $\left({}_t\tau_t\left(\theta^t\right),{}_t\tau_{t+1}\left(\theta^{t+1}\right)\right)$ as follows

$$\begin{aligned}{}_1\tau_1\left(\theta_1\right)&=\zeta_1\left(\theta_1\right)\\&\vdots\\{}_{t-1}\tau_t\left(\theta^t\right)&=-R_t\,{}_{t-1}\tau_{t-1}\left(\theta^t\right)\\{}_t\tau_t\left(\theta^t\right)&=\zeta_t\left(\theta^t\right)-{}_{t-1}\tau_t\left(\theta^t\right)\\&\vdots\\{}_{T-1}\tau_{T-1}\left(\theta^{T-1}\right)&=\zeta_{T-1}\left(\theta^{T-1}\right)-{}_{T-2}\tau_{T-1}\left(\theta^{T-1}\right)\\{}_{T-1}\tau_T\left(\theta^T\right)&=\zeta_T\left(\theta^T\right)\end{aligned}$$

We know from Lemma 8 that the expected present discounted value of transfers associated with the sequence $\left\{\zeta_t\left(\theta^t\right)\right\}_{t,\theta^t}$, $A_1\left(\theta^1\right)=0$. By construction[21],

$$
\begin{aligned}
A_1\left(\theta_1\right) &= \zeta_1\left(\theta_1\right)+q_1\sum_{\theta^2}\pi\left(\theta^2\right)A_2\left(\theta^2\right)\\
&= \zeta_1\left(\theta_1\right)+q_1\sum_{\theta_2}\pi\left(\theta_2\right)\left[\zeta_2\left(\theta_0,\theta_1\right)+\ldots\left[\ldots+\sum_{\theta_{T-1}}\pi\left(\theta_{T-1}\right)\left[\zeta_{T-1}\left(\theta^{T-1}\right)+q_{T-1}\sum_{\theta_T}\pi\left(\theta_T\right)\zeta_T\left(\theta^T\right)\right]\right]\right]\\
&= {}_1\tau_1\left(\theta_1\right)+q_1\sum_{\theta_2}\pi\left(\theta_2\right)\left[{}_1\tau_2+{}_1\tau_2+\ldots\left[\ldots+\sum_{\theta_{T-1}}\pi\left(\theta_{T-1}\right)\left[..+{}_{T-1}\tau_{T-1}+q_{T-1}\sum_{\theta_T}\pi\left(\theta_T\right){}_{T-1}\tau_T\right]\right]\right]\\
&= q_1\sum_{\theta_2}\pi\left(\theta_2\right)\left[\left[\ldots+\ldots\left[\ldots+\sum_{\theta_{T-1}}\pi\left(\theta_{T-1}\right)\left[{}_{T-2}\tau_{T-1}+{}_{T-1}\tau_{T-1}+q_{T-1}\sum_{\theta_T}\pi\left(\theta_T\right){}_{T-1}\tau_T\right]\right]\right]\right]\\
&= \prod_{s=1}^{T-2}q_s\sum_{\theta^{T-2}}\pi\left(\theta^{T-2}\right)\sum_{\theta_{T-1}}\pi\left(\theta_{T-1}\right)\left[{}_{T-1}\tau_{T-1}\left(\theta^{T-2},\theta_{T-1}\right)+q_{T-1}\sum_{\theta_T}\pi\left(\theta_T\right){}_{T-1}\tau_T\left(\theta^{T-2},\theta_{T-1},\theta_T\right)\right]
\end{aligned}
$$

Since $A_1^1\left(\theta_1\right)=0$,

$$
\sum_{\theta^{T-2}}\pi\left(\theta^{T-2}\right)\sum_{\theta_{T-1}}\pi\left(\theta_{T-1}\right)\left[{}_{T-1}\tau_{T-1}\left(\theta^{T-2},\theta_{T-1}\right)+q_{T-1}\sum_{\theta_T}\pi\left(\theta_T\right){}_{T-1}\tau_T\left(\theta^{T-2},\theta_{T-1},\theta_T\right)\right]=0
$$

We want to show that for all θ^{T-2}, $\left[{}_{T-1}\tau_{T-1}\left(\theta^{T-2},\theta_{T-1}\right)+q_{T-1}\sum_{\theta_T}\pi\left(\theta_T\right){}_{T-1}\tau_T\left(\theta^{T-2},\theta_{T-1},\theta_T\right)\right]$ is independent of θ_{T-1}. By construction, this is equivalent to showing that

$$
A_{T-1}\left(\theta^{T-2},\theta_{T-1}\right)=\zeta_{T-1}\left(\theta^{T-2},\theta_{T-1}\right)+q_{T-1}\sum_{\theta_T}\pi\left(\theta_T\right)\zeta_T\left(\theta^{T-2},\theta_{T-1},\theta_T\right)
$$

is independent of θ_{T-1}. It is easy to see that $\zeta_T\left(\theta^{T-2},\theta_{T-1},\theta_T\right)$ must be independent of θ_T else households would always announce the type consistent with the largest transfer. Therefore, ${}_{T-1}\tau_{T-1}\left(\theta^{T-2},\theta_{T-1},\theta_T\right)$ must also be independent of θ_T.

Suppose for some history θ^{T-2} and $\theta,\theta'\in\Theta$,

$$
A_{T-1}\left(\theta^{T-2},\theta\right)>A_{T-1}\left(\theta^{T-2},\theta'\right)
$$

First suppose that $\theta<\theta'$. There exists some $\delta>0$

$$
A_{T-1}\left(\theta^{T-2},\theta\right)=A_{T-1}\left(\theta^{T-2},\theta'\right)+\delta
$$

Since this excess transfer can either be front or back-loaded, we need to consider two cases. If the transfer is front loaded then

$$
A_{T-1}\left(\theta^{T-2},\theta\right)=\zeta_{T-1}\left(\theta^{T-2},\theta'\right)+\varepsilon+q_{T-1}\sum_{\theta'\in\Theta}\pi\left(\theta'\right)\left[\zeta_T\left(\theta^{T-2},\theta'\right)-a\varepsilon\right]
$$

[21]Note that I have dropped some of the history dependence, wherever clear, for ease of notation.

where $\varepsilon > 0, a < R_T$ and $\varepsilon - q_{T-1}a\varepsilon = \delta$. Similarly if the transfers are back-loaded then

$$A_{T-1}\left(\theta^{T-2},\theta\right) = \zeta_{T-1}\left(\theta^{T-2},\theta'\right) - \varepsilon + q_1 \sum_{\theta'\in\Theta} \pi\left(\theta'\right)\left[\zeta_T\left(\theta^{T-2},\theta'\right) + a\varepsilon\right]$$

where $\varepsilon > 0, a > R_T$ and $\varepsilon - q_{T-1}a\varepsilon = \delta$.

In the first case, the first part of Lemma 9 along with Lemma 10 tells us that type $\left(\theta^{T-2},\theta'\right)$ would strictly prefer to lie and pretend to be type $\left(\theta^{T-2},\theta\right)$ and save with another intermediary and so incentive compatibility constraints are violated. In the second case notice that in any equilibrium type $\left(\theta^{T-2},\theta\right)$ must weakly prefer to tell the truth than announce $\left(\theta^{T-2},\theta'\right)$. As a result this type must weakly prefer transfer scheme $A_{T-1}\left(\theta^{T-2},\theta\right)$ to $A_{T-1}\left(\theta^{T-2},\theta'\right)$. Then part 2 of Lemma 9 along with Lemma 10 implies that the incentive compatibility constraint for $\left(\theta^{t-1},\theta'\right)$ is violated again.

Next, suppose $\theta > \theta'$. We know from Lemma 7 that if the voluntary participation constraint binds, it does so for the lowest type and hence

$$V_{T-1}\left(\theta^{T-2},\theta\right) > V_{T-1}^{d}\left(\theta\right)$$

so that the household with a larger present discounted value of transfers strictly prefers the existing contract to defaulting. In this case consider an intermediary modifying the original contract as follows; for some $\delta > 0$, small

$$\tilde{\zeta}_{T-1}\left(\theta^{T-2},\theta\right) = \zeta_{T-1}\left(\theta^{T-2},\theta\right) - \delta$$
$$\tilde{\zeta}_{1}\left(\theta^{T-2},\hat{\theta}\right) = \zeta_{T-1}\left(\theta^{T-2},\hat{\theta}\right) + \frac{\delta}{\sum_{\theta'<\theta}\pi\left(\theta^{T-2},\theta'\right)} \text{ for all } \hat{\theta} < \theta$$

Since this provides more insurance in period $T-1$, it increases the expected welfare of agent θ^{T-2}. The perturbation continues to satisfy incentive compatibility and also the participation constraints for δ small enough.

Clearly, the sequence of constructed transfers is budget feasible and satisfies incentive compatibility and participation constraints. Moreover as demonstrated above, each two-period contract makes 0 profits and hence there is no cross-subsidization. We only need to check that a particular two period intermediary cannot do strictly better. But this is clear since if it could then a T intermediary could just modify its contract and also make positive profits. ■

The next lemma is consequence of the above characterization.

Lemma 11 *Given a sequence of two-period contracts, for each t, ${}_t\mathcal{P}_t\left(\theta^{t-1},\theta\right) = {}_t\mathcal{P}_t\left(\theta^{t-1},\theta'\right)$ and ${}_t\mathcal{P}_{t+1}\left(\theta^{t-1},\theta\right) = {}_t\mathcal{P}_{t+1}\left(\theta^{t-1},\theta'\right)$ for all $\theta,\theta' \in \Theta$*

Proof of Lemma 11. In the last period T of the truncated economy, clearly

$${}_{T-1}\tau_T\left(\theta^{T-1},\theta\right) = {}_{T-1}\tau_T\left(\theta^{T-1},\theta'\right)$$

for otherwise the household would always announce the type with the highest transfer. In $T-1$, for the $T-1$ intermediary,

$${}_{T-1}\mathcal{P}_{T-1}\left(\theta^{T-2},\theta\right) = {}_{T-1}\tau_{T-1}\left(\theta^{T-2},\theta\right) + q_{T-1}\,{}_{T-1}\tau_T\left(\theta^{T-1}\right)$$

We know from ?? that competition implies that

$$
\begin{aligned}
&{}_{T-1}\mathcal{P}_{T-1}\left(\theta^{T-2},\theta\right)=0\\
&\Rightarrow {}_{T-1}\tau_{T-1}\left(\theta^{T-2},\theta\right)+q_{T-1}\,{}_{T-1}\tau_{T}\left(\theta^{T-1}\right)=0
\end{aligned}
$$

And so ${}_{T-1}\mathcal{P}_{T-1}\left(\theta^{T-2},\theta\right) = {}_{T-1}\mathcal{P}_{T-1}\left(\theta^{T-2},\theta'\right)$. Competition similarly implies that for any t, ${}_{t}\mathcal{P}_{t}\left(\theta^{t-1},\theta\right) = {}_{t}\mathcal{P}_{t}\left(\theta^{t-1},\theta'\right)$. Lastly consider $t+1$ and

$$
{}_{t}\mathcal{P}_{t+1}\left(\theta^{t-1},\theta\right) = {}_{t}\tau_{t+1}\left(\theta^{t-1},\theta\right)
$$

Since

$$
c_t\left(\theta^t,\theta\right)=\theta+{}_{t}\tau_{t+1}\left(\theta^t\right)+{}_{t+1}\tau_{t+1}\left(\theta^t,\theta\right)
$$

and future sequence of transfers ${}_{t+1}\mathcal{P}_{t+1}\left(\theta^t,\theta\right)$ is independent of θ, incentive compatibility implies that ${}_{t}\mathcal{P}_{t+1}\left(\theta^{t-1},\theta\right) = {}_{t}\mathcal{P}_{t+1}\left(\theta^{t-1},\theta'\right)$ for all $\theta,\theta'\in\Theta$. ■

The following result follows from the previous two results.

Proposition 17 *In any equilibrium with $\hat{T}$ lived intermediaries, $2\le\hat{T}<\infty$, for all t and θ^{t-1}, $A_t\left(\theta^{t-1},\theta\right)=A_t\left(\theta^{t-1},\theta'\right)$ for all $\theta,\theta'\in\Theta$*

Proof of Proposition 17. The result follows from the previous lemmas. We know that the above is true in any model with 2 period lived intermediaries. Moreover since any model with $\hat{T}+1$ lived intermediaries is equivalent to one with the 2 period lived ones, the above must be true. ■

Proof of Proposition 3: The proof requires the following result

Proposition 18 *In any equilibrium with two period lived intermediaries, for any t and $\theta^t,\hat{\theta}^t$ such that $\theta_t+{}_{t-1}\tau_t\left(\theta^t\right)=\hat{\theta}_t+{}_{t-1}\tau_t\left(\hat{\theta}^t\right)$,*

$$
V_t\left(\theta^t\right)=V_t\left(\hat{\theta}^t\right)
$$

Proof of Proposition 18. Because of the assumption that after period T, households can only trade a risk free bond subject to exogenous debt constraints, it is easy to see that the statement holds in period T, since all that matters for the households' choices is the sum $\theta_T+{}_{T-1}\tau_T\left(\theta^T\right)$. In period $T-1$ suppose θ^{T-1} and $\hat{\theta}^{T-1}$ such that $\theta_{T-1}+{}_{T-2}\tau_{T-1}\left(\theta^{T-1}\right)=\hat{\theta}_{T-1}+{}_{T-2}\tau_{T-1}\left(\hat{\theta}^{T-1}\right)$ and

$$
V_{T-1}\left(\theta^{T-1}\right)>V_{T-1}\left(\hat{\theta}^{T-1}\right)
$$

For ease of notation denote the corresponding transfers by ${}_{T-2}\tau_{T-1}$ and ${}_{T-2}\hat{\tau}_{T-1}$ We need to consider a few cases. Suppose first that for both $\theta^{T-1},\hat{\theta}^{T-1}$

$$
u'\left(\theta_{T-1}+{}_{T-2}\tau_{T-1}+{}_{T-1}\tau_{T-1}\right)=\beta R_T\mathbb{E}_{T-1}u'\left(\theta_T+{}_{T-1}\tau_T+\psi_{T+1}\left(\theta_T+{}_{T-1}\tau_T\right)\right) \tag{A.2}
$$

$$
u'\left(\hat{\theta}_{T-1}+{}_{T-2}\hat{\tau}_{T-1}+{}_{T-1}\hat{\tau}_{T-1}\right)=\beta R_T\mathbb{E}_{T-1}u'\left(\theta_T+{}_{T-1}\hat{\tau}_T+\psi_{T+1}\left(\theta_T+{}_{T-1}\hat{\tau}_T\right)\right) \tag{A.3}
$$

where $\psi_{T+1}\left(\theta_T+{}_{T-1}\tau_T\right)$ is the savings choice for the household (given that it is subject to debt constraint ϕ^e_{T+1}). Since ${}_{T-1}\tau_{T-1}+\frac{{}_{T-1}\tau_T}{q_{T-1}}={}_{T-1}\hat{\tau}_{T-1}+\frac{{}_{T-1}\hat{\tau}_T}{q_{T-1}}=0$ and the savings choice ψ_{T+1} depends only on the sum $\theta_T+{}_{T-1}\tau_T$, it must be that ${}_{T-1}\tau_{T-1}={}_{T-1}\hat{\tau}_{T-1}$ and $V\left(\theta^t\right)=V\left(\hat{\theta}^{T-1}\right)$ and so we have a contradiction. Suppose on the other hand that (A.2) holds with equality and (A.3) with strictly inequality. Again, since ${}_{T-1}\tau_{T-1}+\frac{{}_{T-1}\tau_T}{q_{T-1}}={}_{T-1}\hat{\tau}_{T-1}+\frac{{}_{T-1}\hat{\tau}_T}{q_{T-1}}=0$, it must be that ${}_{T-1}\tau_{T-1} > {}_{T-1}\hat{\tau}_{T-1}$. Since the household is Euler-constrained, assume that ${}_{T-1}\hat{\tau}_{T-1} > 0$. It is easy to see that that giving type $\hat{\theta}^{T-1}$ the contract associated with θ^{T-1} makes it strictly better off. Consider modifying the original contract so that

$$\begin{aligned} {}_{T-1}\tilde{\tau}_{T-1} &= {}_{T-1}\hat{\tau}_{T-1} + \varepsilon \\ {}_{T-1}\tilde{\tau}_{T} &= {}_{T-1}\hat{\tau}_{T} - \delta\varepsilon \end{aligned}$$

where ε chosen so that ${}_{T-1}\hat{\tau}_T - \delta\varepsilon \geq {}_{T-1}\tau_T$ and

$$\frac{u'\left(\hat{\theta}_{T-1} + {}_{T-2}\hat{\tau}_{T-1} + {}_{T-1}\hat{\tau}_{T-1}\right)}{\beta\mathbb{E}_{T-1}u'\left(\theta_T + {}_{T-1}\hat{\tau}_T + \psi_{T+1}\left(\theta_T + {}_{T-1}\hat{\tau}_T\right)\right)} > \delta > R_T$$

This perturbation makes type $\hat{\theta}^{T-1}$ strictly better off. To see that voluntary participation constraints continue to hold for type $\hat{\theta}^{T-1}$ in period t, notice that this household's value in period T is exactly the same as θ^{T-1}. Since the original transfer scheme was incentive compatible and satisfied voluntary participation constraints in period T, it must be that for all $\theta \in \Theta$

$$u\left(\theta + {}_{T-1}\tilde{\tau}_T + \psi_{T+1}\left(\theta + {}_{T-1}\tilde{\tau}_T\right)\right) + \beta\mathbb{E}_T V_{T+1}\left(\theta, \psi_{T+1}\right) \geq V_T^d\left(\theta\right)$$

As a result, these constraints continue to hold under this deviation. Finally since $\delta > R_T$, the deviating intermediary makes strictly positive profits. Therefore it must be that ${}_{T-1}\tau_{T-1} = {}_{T-1}\hat{\tau}_{T-1}$. Note that a similar argument holds if both (A.2) and (A.3) hold with inequality and $V_{T-1}\left(\theta^T\right) > V_{T-1}\left(\hat{\theta}^{T-1}\right)$.Given that the property holds for $\hat{T}-1$, assume that this property holds for some $t+1 < \hat{T}-1$. Our goal is to show that the property holds in t. Suppose for contradiction we have some $\theta^t, \hat{\theta}^t$ such that $\theta_t + {}_{t-2}\tau_{t-1}\left(\hat{\theta}^{t-1}\right) = \hat{\theta}_t + {}_{t-2}\tau_{t-1}\left(\hat{\theta}^{t-1}\right)$ and

$$V_t\left(\hat{\theta}^t\right) > V_t\left(\hat{\theta}^t\right)$$

Again, denote the transfers by ${}_{t-2}\tau_{t-1}$ and ${}_{t-2}\hat{\tau}_{t-1}$. As before, first consider the case in which both type's Euler equations hold with equality. Suppose ${}_t\tau_t < {}_t\hat{\tau}_t$. Then it is easy to see that an intermediary can offer an $\varepsilon\delta$ savings contract which will be accepted by this agent making both intermediary strictly better off. To see why notice that since ${}_t\tau_t + \frac{{}_t\tau_{t+1}}{q_t} = {}_t\hat{\tau}_t + \frac{{}_t\hat{\tau}_{t+1}}{q_t} = 0$ and $V\left(\theta^t\right) > V\left(\hat{\theta}^t\right)$ it must be that there exists some $\varepsilon > 0$ such that the transfer scheme ${}_t\hat{\tau}_t - \varepsilon + \frac{{}_t\hat{\tau}_{t+1} + \varepsilon}{q_t}$ makes this type strictly better off. Next suppose that

$$\begin{aligned} u'\left(\theta_t + {}_{t-1}\tau_t + {}_t\tau_t\right) &= \beta R_{t+1}\mathbb{E}_t u'\left(\theta_{t+1} + {}_t\tau_{t+1} + {}_{t+1}\tau_{t+1}\right) \\ u'\left(\hat{\theta}_t + {}_{t-1}\hat{\tau}_t + {}_t\hat{\tau}_t\right) &> \beta R_{t+1}\mathbb{E}_t u'\left(\theta_{t+1} + {}_t\hat{\tau}_{t+1} + {}_{t+1}\hat{\tau}_{t+1}\right) \end{aligned}$$

As in the period $T-1$ case, consider modifying the original contract

$$\begin{aligned} {}_t\tilde{\tau}_t &= {}_t\hat{\tau}_t + \varepsilon \\ {}_t\tilde{\tau}_{t+1} &= {}_t\hat{\tau}_{t+1} - \delta\varepsilon \end{aligned}$$

where

$$\frac{u'\left(\hat{\theta}_t + {}_{t-1}\hat{\tau}_t + {}_t\hat{\tau}_t\right)}{\beta \mathbb{E}_t u'\left(\theta_{t+1} + {}_t\hat{\tau}_{t+1} + {}_{t+1}\hat{\tau}_{t+1}\right)} > \delta > R$$

independently of reported type. To see that no agent would choose to default on this intermediary notice that for any type that signs this contract will have the same value in $t+1$ by the induction assumption. Therefore since type θ^{t+1} preferred not to default under the original contract, type $\hat{\theta}^t$ will not want to default under the deviating contract. If the original contract was incentive compatible, the deviating one will be as well.

Finally, since there exists a type, $\hat{\theta}^t$ who is made strictly better off for some $\delta < 1$, the deviating intermediary makes strictly positive profits. Therefore by induction the claim must hold in period t and by induction for all previous periods as well. ■

Proof of Proposition 3. Note that the proposition is written in terms of the equivalent 2 period contracts. We know from Proposition 18 that for all θ^t, $V_t\left(\theta^t\right)$ only depends on $\theta_t + {}_{t-1}\tau_t\left(\theta^t\right)$. Given the nature of these two period contracts, we consider transfers of the form $\left({}_t\tau_t\left(\theta^t\right), {}_t\tau_{t+1}\left(\theta^t\right)\right) = \left(\varphi\left(\theta^t\right), \frac{-\varphi\left(\theta^t\right)}{q_t}\right)$. Let φ^* be largest such $\varphi\left(\theta^t\right)$ given to all households that are Euler-constrained and denote the corresponding history by θ^{*t}. Given some $\varphi\left(\theta^t\right)$ define

$$R^{\varphi\left(\theta^t\right)} = \frac{u'\left(\theta_t + {}_{t-1}\tau_t\left(\theta^t\right) + \varphi\left(\theta^t\right)\right)}{\beta \mathbb{E}_t u'\left(\theta + \frac{-\varphi\left(\theta^t\right)}{q_t} + {}_{t+1}\tau_{t+1}\left(\theta^{t+1}\right)\right)}$$

Since this household is Euler-constrained, $R^{\varphi\left(\theta^t\right)} > R_{t+1}$. Suppose there exists an Euler-constrained household $\tilde{\theta}^t$ such that $\varphi\left(\tilde{\theta}^t\right) < \varphi^*$. In this case it must also be that $R^{\varphi\left(\tilde{\theta}^t\right)} > R_{t+1}$.

Consider modifying the original contract as follows

$$ {}_t\tilde{\tau}_t\left(\tilde{\theta}^t\right) = \varphi\left(\tilde{\theta}^t\right) + \varepsilon$$

$$ {}_t\tilde{\tau}_{t+1}\left(\tilde{\theta}^t\right) = -\frac{\varphi\left(\tilde{\theta}^t\right)}{q_t} - \frac{\varepsilon}{\hat{q}_t}$$

where

$$R_{t+1} = \frac{1}{q_t} < \frac{1}{\hat{q}_t} < R^{\varphi\left(\tilde{\theta}^t\right)}$$

Notice that for ε small, type $\tilde{\theta}^t$ will be made strictly better off by signing such a contract since $R^{\varphi\left(\tilde{\theta}^t\right)} > \frac{1}{\hat{q}_t}$ and the household is Euler-constrained. For ε small enough, ${}_t\tilde{\tau}_{t+1}\left(\tilde{\theta}^t\right) \geq \frac{-\varphi^*}{q_t}$. Since we have shown earlier that equilibrium continuation value for any agent going forward only depends on the sum $\theta + {}_t\tau_{t+1}\left(\theta^t\right)$, if

$$u\left(\theta^* + \frac{-\varphi^*}{q_t} + {}_{t+1}\tau_{t+1}\left(\theta^{*t+1}\right)\right) + \beta \mathbb{E}_{t+1} V_{t+2}\left(\theta^{*t+2}\right) \geq V^d_{t+1}\left(\theta^{*t+1}\right)$$

then all households accepting the deviating contract will also prefer not to default. To check incentive compatibility, notice that if the original contract was incentive compatible and all other types preferred their transfers to $\left(\varphi^*, \frac{-\varphi^*}{q_t}\right)$, clearly the modified transfer sequence will be incentive compatible as well. Finally, since $\frac{1}{q_t} < \frac{1}{\hat{q}_t}$, the deviating intermediary is also made strictly better off. ■

Using these results, we can proceed to the proof of the equivalence theorem..

Proof of Theorem 1. Given an equilibrium of the decentralized contracting problem with equilibrium transfer schedules $s_t\left(\theta^t\right)$, construct the equivalent 2 period contracts (which we proved exists earlier). As a result we have a sequence of transfers $\left\{{}_t\zeta_t\left(\theta^t\right), {}_t\zeta_{t+1}\left(\theta^{t+1}\right)\right\}_{\theta^t,t}$. Construct bond holdings after each history for the agent as follows (assume that agents start off with 0 initial wealth)

$$s_2\left(\theta_1\right) = -\,{}_1\zeta_1\left(\theta_1\right)$$
$$\vdots$$
$$s_{t+1}\left(\theta^t\right) = -\,{}_t\zeta_t\left(\theta^t\right)$$
$$\vdots$$

Let the interest rates $\{R_t\}$ be defined such that $R_{t+1} = \frac{1}{q_t}$. Given that the sequence of transfers satisfies the zero profit condition we know that $-R_t s_t\left(\theta^{t-1}\right) = {}_{t-1}\zeta_t\left(\theta^{t-1}\right)$ and therefore the constructed bond holdings satisfy the household's budget constraints. To construct the sequence of debt constraints recall that we showed that in contracting environment, that for any t, and θ^t such that

$$u'\left(\theta_t + {}_{t-1}\zeta_t\left(\theta^{t-1}\right) + {}_t\zeta_t\left(\theta^t\right)\right) > \beta R_{t+1}\mathbb{E}_t u'\left(\theta_{t+1} + {}_t\zeta_{t+1}\left(\theta^t\right) + {}_{t+1}\zeta_{t+1}\left(\theta^{t+1}\right)\right)$$

it must be that ${}_t\zeta_t\left(\theta^t\right) = \varphi_t$ where φ_t is independent of the agent's history. Let

$$\phi_{t+1} = \varphi_t$$

for all t. The necessary and sufficient conditions for agent optimality in the bond trading economy are

$$u'\left(c_t\left(\theta^t\right)\right) \geq \beta R_{t+1}\mathbb{E}_t u'\left(c_{t+1}\left(\theta^{t+1}\right)\right)$$

with strict inequality if

$$s_{t+1}\left(\theta^t\right) = -\phi_{t+1}$$

along with budget feasibility (which we have already established). We know from earlier results that any allocation from the decentralized contracting environment satisfies exactly these conditions which shows that the constructed allocation is optimal for all agents. It only remains to show that these debt constraints are not-too-tight which follows from Proposition 1 and Proposition 3.

For part 2, consider an equilibrium of the debt constrained environment. Construct transfer schedules for $\hat{T}$ period lived intermediaries as follows

$${}_1\zeta_1\left(\theta_1\right) = -s_2\left(\theta_1\right)$$
$${}_1\zeta_2\left(\theta^1\right) = Rs_2\left(\theta_1\right) - s_3\left(\theta^2\right)$$
$$\vdots$$
$${}_1\zeta_{\hat{T}}\left(\theta^{\hat{T}}\right) = R_{\hat{T}}s_{\hat{T}}\left(\theta^{\hat{T}-1}\right)$$
$${}_{\hat{T}}\zeta_{\hat{T}}\left(\theta^{\hat{T}}\right) = -s_{\hat{T}+1}\left(\theta^{\hat{T}}\right)$$
$$\vdots$$

And let $q_t = \frac{1}{R_{t+1}}$. Note that we are constructing an equilibrium in which each intermediary born at date $1, \hat{T}, 2\hat{T} - 1, ...$ offers a single contract with transfers as constructed. All intermediaries born at other dates offer simple uncontingent savings contracts. While these will never be signed in equilibrium, a deviating contract that offers some state-contingency will never be profitable since households can always lie and use these savings contracts to smooth any excess transfers. This similar to the "latent contracts" used by Ales and Maziero (2014) to sustain their equilibrium. Suppose these contracts and prices did not constitute an equilibrium. There are two cases to consider:

1. Given prices and the contract offered by this intermediary, no new intermediary has an incentive to offer a contract and make strictly positive profits.

2. The existing intermediary has no incentive to modify its contract and make strictly positive profits.

Consider the first case. Suppose that this was a $\tilde{T}$ period contract that spanned dates $t \to \tilde{T}-1$. Notice that the only way in which households will strictly prefer to sign with such a deviating contract and the intermediary make a positive profit is if it increases increases insurance in some period.

First consider the last period $\tilde{T} - 1$. It is easy to see that in this period, the transfers from the intermediary to the household cannot depend on $\theta_{\tilde{T}-1}$ else the household would always announce the type consistent with the highest transfer. Next consider period $\tilde{T} - 2$. Suppose the contract made a positive transfer to some type who is Euler constrained in period $\tilde{T} - 1$. Incentive compatibility requires that this type must receive a negative uncontingent transfer in period $\tilde{T}$ otherwise households would lie to get this increased transfer. Since the household is Euler constrained and debt constraints are chosen to be Not-too-tight, we know that some type's voluntary participation constraint holds with equality in $\tilde{T}-1$ and so such a perturbation is not possible. If the household is unconstrained in this state, a perturbation that makes both the intermediary and the agent strictly better off is not possible.

On the other hand, suppose the contract made a negative transfer to some type. Again incentive compatibility dictates that a positive uncontingent transfer be made to this type in period $\tilde{T} - 1$. However, this is exactly a pure savings contract and since the households are not savings constrained, this will never be profitable.

Now consider period $\tilde{T}-3$. First, consider a state contingent positive transfer to some type who is constrained. This must be compensated for by a negative transfer in period $\tilde{T} - 2$. This transfer cannot be independent of state since some household's voluntary participation constraint binds. It also cannot be state contingent by the previous argument. As before, a negative transfer followed by an uncontingent transfer at date $\tilde{T} - 2$ can never make both the intermediary and agent strictly better off. A similar argument holds for all previous periods by induction.

Finally, consider a positive transfer to some type in $\tilde{T} - 3$ who is not constrained. Incentive compatibility requires that a negative uncontingent transfer be made in period $\tilde{T}-2$. However such a perturbation can never be welfare enhancing if the present discounted value of transfers is less than zero and so the intermediary can never make a positive profit on this particular deviation.

Next, we need to check that the *existing* intermediary has no incentive to modify its contract given prices. As above, the only such modifications will involve providing some type in some period a little more insurance. Consider a period t, and a type θ^t who is Euler-constrained under the original contract. Given our equilibrium definition, we know that there exists some θ^c such that the voluntary participation constraint for type $\left(\theta^t, \theta^c\right)$ holds with equality in $t + 1$

We consider a deviation in which the intermediary increases the transfer to this household by some $\varepsilon > 0$. It is easy to see that incentive compatibility requires that the intermediary make a negative transfer at some future date, say $t+1$. So there exists some $\left(\theta^t, \theta^*\right)$ who receives a negative transfer δ in $t+1$. Note that the negative transfer cannot be uncontingent since for some type in $t+1$, the voluntary participation constraint holds with equality. Therefore, the transfer δ must be state contingent. We can group states into two classes; the first $Con_{t+1}\left(\theta^t\right)$ are those that are Euler-constrained at $t+1$, i.e.

$$u'\left(c_{t+1}\left(\theta^t, \theta\right)\right) > \beta R_{t+2}\mathbb{E}_{t+1}u'\left(c_{t+2}\left(\theta^t, \theta, \theta'\right)\right)$$

and the second $Uncon_{t+1}\left(\theta^t\right)$, those that are not, i.e.

$$u'\left(c_{t+1}\left(\theta^t, \theta\right)\right) = \beta R_{t+2}\mathbb{E}_{t+1}u'\left(c_{t+2}\left(\theta^t, \theta, \theta'\right)\right)$$

We know that $\theta^c \in Con_{t+1}\left(\theta^t\right)$. Also, from Proposition 17, it must be that $A_{t+1}\left(\theta^t, \theta\right) = A_{t+1}\left(\theta^t, \theta'\right)$ for any $\theta, \theta' \in Uncon_{t+1}\left(\theta^t\right)$. Therefore a negative transfer δ' will have to be imposed on all such types. However, since under the original contract, $A_{t+1}\left(\theta^t, \theta\right)$ is independent of θ, the perturbation implies that $A_{t+1}\left(\theta^t, \theta\right) > A_{t+1}\left(\theta^t, \hat{\theta}\right)$ for any $\theta, \hat{\theta}$ in $Con_{t+1}\left(\theta^t\right)$ and $Uncon_{t+1}\left(\theta^t\right)$ respectively.

Therefore, all types in $Uncon_{t+1}\left(\theta^t\right)$ will strictly prefer to lie and announce some type in $Con_{t+1}\left(\theta^t\right)$ and save with some other intermediary. ■

Proofs from Section 3.1

Proof of Theorem 2: The first step in the proof is to show that given a measurable map Φ, a $\Phi-$RCE always exists.

Proposition 19 *For any finite measurable map $\Phi : \mathbb{R}_+ \to \mathbb{R}_+$, a $\Phi-RCE$ exists*

Proof of Proposition 19. The first step of the proof is to show that given continuous pricing functions $R(\phi)$, there exists a unique list of value functions W and policy functions $l'(\theta, l, \phi)$ that solve the individual household's problems. This part of the proof uses arguments developed in Miao (2006).

Let $\mathbb{A} \subset \mathbb{R}$ be the compact feasible asset space, $\mathbb{D} \subset \mathbb{R}_+$ the compact space of debt constraints and the $\mathbb{V}$ denote the set of uniformly bounded and continuous real valued functions on $\Theta \times \mathbb{A} \times \mathbb{D}$.

Define operator $\mathbb{T}$ as follows: Given some $w \in \mathbb{V}$,

$$(\mathbb{T}w)(\theta, l, \phi) = \max_{l' \in \Gamma(l,\phi)} u\left(\theta - Rl - l'\right) + \beta\mathbb{E}w\left(\theta', l', \phi'; \Phi'\right)$$

where $\Gamma(l, \phi) = [-\phi, \theta + Rl]$. In order to apply the contraction mapping theorem I first show $\mathbb{T}w \in \mathbb{V}$. Boundedness follows. To show continuity, consider a sequence $(\theta, l, \phi)^n \to (\theta, l, \phi)$. Given our restriction to continuous pricing functions, $R(\phi^n) \to R(\phi)$. As a result correspondence Γ is continuous. Then first term on the right hand side of the above dynamic program is continuous since u is continuous. Consider second term.

We want to show that

$$|\mathbb{E}w(\theta^n, l^n, \phi^n) - \mathbb{E}w(\theta, l, \phi)| \to 0$$

Since $\mathbb{A} \times \mathbb{D} \times \Theta$ is compact by Tychonoff's theorem, w is uniformly continuous and as a result $w(\theta^n, l^n, \phi^n) \to w(\theta, l, \phi)$ uniformly. As a result we can interchange the limit and integrals.

Therefore by Maximum theorem, $\mathbb{T}w$ is also continuous and hence $\mathbb{T}w \in \mathbb{V}$. It is easy to see that the operator satisfies Blackwell's sufficiency conditions. As a result operator $\mathbb{T}$ is a contraction and so by the Contraction Mapping Theorem we have unique sequence of functions w^* and corresponding policy functions l'^*. Next, we can use the individual policy function to compute the aggregate distribution

$$\lambda (A \times B) = \mu \left(i \in I : \left(l' (i) , \theta (i)\right) \in A \times B,\ A \times B = \mathcal{B} (A) \times \mathcal{B} (\Theta)\right)$$

Consequently

$$\lambda' (A \times B) = \int \mu \left(i \in I, \theta' (i) \in A, l' (\theta, l, \phi) \in B\right) d\lambda (\theta, l)$$

which defines the measurable mapping G.

Next, it is straightforward to note that the policy functions $l' (\theta, l, \phi)$ are strictly increasing in R for all $b' > -\phi$ and that $l' (\theta, l, \phi) = -\phi$ for R small enough. As a result given ϕ, for $R (\phi)$ large enough

$$\int_{\mathbb{A}\times\Theta} l' (\theta, l, \phi) \, d\lambda (l, \Theta) > 0$$

and for $R (\phi)$ small enough

$$\int_{\mathbb{A}\times\Theta} l' (\theta, l, \phi) \, d\lambda (l, \Theta) = -\phi < 0$$

As a result continuity implies that there exists $R (\phi)$ such that

$$\int_{\mathbb{A}\times\Theta} l' (\theta, l, \phi) \, d\lambda (l, \Theta) = 0$$

■

Next, it always true that a $\Phi-$RCE with Φ being the zero map is NTT-RCE

Lemma 12 *There exists an NTT-RCE in which* $\Phi = 0$

Proof of Lemma 12. Consider the $\Phi-$RCE in which Φ is the zero map i.e. $\phi = 0$ and $\Phi (\phi) = 0$. We know that such an equilibrium exists from the previous lemma.

To show that these also constitutes a NTT-RCE we also need to show that

$$W \left(\theta, 0, 0; \Phi^0\right) = V^d (\theta)$$

which is straightfoward since

$$W \left(\theta, 0, 0; \Phi^0\right) = u (\theta) + \mathbb{E}u \left(\theta'\right) = V^d (\theta)$$

■

The reason for this is clear. If debt constraints are zero each period, then in equilibrium agents consume their endowment which trivially implies that the voluntary participation constraint binds for each period and each type. The final and main proposition that completes the proof of Theorem 2 is to show that there exists a NTT-RCE with $\Phi \neq 0$.

Proposition 20 *If*

$$\frac{u'\left(\bar{\theta}\right)}{\beta\eta} < \kappa$$

then there exists a NTT-RCE in which $\Phi > 0$

Proof of Proposition 20. Define $\phi_\varepsilon = \phi + \varepsilon$ and Φ_ε such that $\Phi_\varepsilon\left(\phi + \varepsilon\right) = \phi' + \varepsilon$.

The first step in the proof is to compute the sign of the following object

$$\lim_{\substack{\phi\to 0\\ \Phi\to\mathbf{0}}} \frac{\partial}{\partial\varepsilon} W\left(\theta, -\phi_\varepsilon, \phi_\varepsilon; \Phi_\varepsilon\right)\bigg|_{\varepsilon=0}$$

In words, this measures the change in equilibrium welfare of the $\Phi-$RCE as we change Φ from zero the something positive.

In equilibrium we must have from the agent's problem

$$W\left(\theta, -\phi_\varepsilon, \phi_\varepsilon; \Phi_\varepsilon\right) = u\left(z - R\phi - R\varepsilon - l'\left(\theta, -\phi_\varepsilon, \phi_\varepsilon\right)\right) + \beta\mathbb{E}W\left(\theta', l'\left(\theta, -\phi_\varepsilon, \phi_\varepsilon\right), \phi'; \Phi'_\varepsilon\right)$$

where $l'\left(\theta, -\phi_\varepsilon, \phi_\varepsilon, R\right)$ denote the policy function for bond holdings. We can then compute the following derivative (which is well defined)

$$\begin{aligned}\frac{\partial}{\partial\varepsilon} W\left(\theta, -\phi_\varepsilon, \phi_\varepsilon; \Phi_\varepsilon\right) &= u'\left(\theta - R\phi - R\varepsilon - l'\right)\left[-R - R_\varepsilon\varepsilon - l'_\varepsilon\right] \\ &\quad + \beta E W_1\left(\theta', l', \phi'_\varepsilon; \Phi'_\varepsilon\right) l'_\varepsilon + \beta\mathbb{E}W_2\left(\theta', l'\left(\theta, -\phi_\varepsilon, \phi_\varepsilon\right), \phi'_\varepsilon; \Phi'_\varepsilon\right)\end{aligned}$$

This implies that

$$\begin{aligned}\frac{\partial}{\partial\varepsilon} W\left(\theta, -\phi_\varepsilon, \phi_\varepsilon; \Phi_\varepsilon\right)\bigg|_{\varepsilon=0} &= u'\left(\theta - R\phi - l'\left(\theta, -\phi, \phi\right)\right)\left[-R - l'_\varepsilon\left(\theta, -\phi, \phi\right)\right] \\ &\quad + \beta\mathbb{E}W_1\left(\theta', l'\left(\theta, -\phi, \phi\right), \phi'; \Phi'\right) l'_\varepsilon\left(\theta, -\phi, \phi\right) + \beta\mathbb{E}W_2\left(l'\left(\theta, -\phi, \phi\right), \phi'; \Phi'\right)\end{aligned}$$

Given the continuity of the policy and price functions

$$\begin{aligned}\lim_{\substack{\phi\to 0\\ \Phi\to\mathbf{0}}} \frac{\partial}{\partial\varepsilon} W\left(\theta, -\phi_\varepsilon, \phi_\varepsilon; \Phi_\varepsilon\right)\bigg| &= u'\left(\theta\right)\left[-R - l'_\varepsilon\right] + \beta\mathbb{E}W_1\left(\theta', 0, 0; 0\right) l'_\varepsilon + \beta\mathbb{E}W_2\left(\theta', 0, 0; 0\right) \\ &= -Ru'\left(\theta\right) - u'\left(\theta\right) l'_\varepsilon + \beta\mathbb{E}W_1\left(\theta', 0, 0; 0\right) l'_\varepsilon + \beta\mathbb{E}W_2\left(\theta', 0, 0; 0\right)\end{aligned}$$

From the first order conditions of the above problem where $\mu\left(\theta, l, \phi\right)$ is the multiplier on the debt constraint, we have

$$\beta\mathbb{E}W_1\left(\theta', l'\left(\theta, l, \phi\right), \phi'; \Phi\right) = u'\left(c\left(\theta, l, \phi\right)\right) - \mu\left(\theta, l, \phi\right)$$

we see that

$$\lim_{\substack{\phi\to 0\\ \Phi\to\mathbf{0}}} \frac{\partial}{\partial\varepsilon} W\left(\theta, -\phi_\varepsilon, \phi_\varepsilon; \Phi_\varepsilon\right)\bigg|_{\varepsilon=0} = -Ru'\left(\theta\right) - \mu\left(\theta, 0, 0\right) l'_\varepsilon + \beta\mathbb{E}W_2\left(\theta', l'\left(\theta, 0, 0\right), 0; 0\right)$$

From the complementary slackness condition we have that

$$
\begin{aligned}
&\mu(z,0,0)\left[l'(\theta,-\phi,\phi)+\phi\right]=0\\
&\Rightarrow \mu_\varepsilon(\theta,0,0)\left[l'(\theta,-\phi,\phi)+\phi\right]+\mu(\theta,0,0)\left[l'_\varepsilon(\theta,-\phi,\phi)+\phi_\varepsilon\right]=0\\
&\Rightarrow \mu_\varepsilon(\theta,0,0)\left[l'(\theta,-\phi,\phi)+\phi\right]+\mu(\theta,0,0)\left[l'_\varepsilon(\theta,-\phi,\phi)+1\right]=0
\end{aligned}
$$

As $\phi,\Phi\to 0$ we have

$$
\begin{aligned}
&\mu(\theta,0,0)\left[l'_\varepsilon(\theta,-\phi,\phi)+1\right]=0\\
&\Rightarrow \mu(\theta,0,0)\,l'_\varepsilon(\theta,0,0)=-\mu(\theta,0,0)
\end{aligned}
$$

Therefore

$$
\lim_{\substack{\phi\to 0\\ \mathbf{\Phi}\to\mathbf{0}}}\frac{\partial}{\partial\varepsilon}W(\theta,-\phi_\varepsilon,\phi_\varepsilon;\Phi_\varepsilon)\bigg|_{\varepsilon=0}=-Ru'(\theta)+\mu(\theta,0,0)+\beta\mathbb{E}W_2\left(\theta',l'(\theta,0,0),0;0\right)
$$

From the first order conditions we have

$$
\begin{aligned}
\mu(\theta,0,0)&=u'(\theta)-\beta\mathbb{E}W_1\left(\theta',0,0;0\right)\\
&=u'(\theta)-\beta R\sum_{\theta'\in\Theta}\pi\left(\theta'\right)u'\left(\theta'\right)
\end{aligned}
$$

Define $\eta=\sum_{\theta'\in\Theta}\pi(\theta')\,u'(\theta')$. Therefore

$$
\mu(\theta,0,0)=u'(\theta)-\beta R\eta
$$

and

$$
\begin{aligned}
\mathbb{E}W_2\left(\theta',0,0;0\right)&=\sum_{\theta'\in\Theta}\pi\left(\theta'\right)\mu\left(\theta',0,0\right)\\
&=\sum_{\theta'\in\Theta}\pi\left(\theta'\right)\left[u'\left(\theta'\right)-\beta R\eta\right]\\
&=\eta-\beta R\eta
\end{aligned}
$$

As a result

$$
\begin{aligned}
\lim_{\substack{\phi\to 0\\ \mathbf{\Phi}\to\mathbf{0}}}\frac{\partial}{\partial\varepsilon}W(\theta,-\phi_\varepsilon,\phi_\varepsilon;\Phi_\varepsilon)\bigg|_{\varepsilon=0}&=-Ru'(\theta)+u'(\theta)-\beta R\eta+\beta\left[\eta-\beta R\eta\right]\\
&=-R\left[u'(\theta)+\beta\eta+\beta^2\eta\right]+u'(\theta)+\beta\eta \qquad \text{(A.4)}
\end{aligned}
$$

Notice that if

$$
R<\frac{u'(\theta)+\beta\eta}{\left[u'(\theta)+\beta\eta+\beta^2\eta\right]}
$$

then (A.4)>0 since $\eta>0$.

In any $\Phi-$RCE, when $\phi = 0$ the interest rate must satisfy

$$u'\left(\bar{\theta}\right) \geq \beta R \eta$$
$$\Rightarrow R \leq \frac{u'\left(\bar{\theta}\right)}{\beta\eta}$$

Therefore if

$$\frac{u'\left(\bar{\theta}\right)}{\beta\eta} < \frac{u'\left(\theta\right)+\beta\eta}{\left[u'\left(\theta\right)+\beta\eta+\beta^2\eta\right]}$$

then we know that

$$\lim_{\substack{\phi\to 0\\ \Phi\to\mathbf{0}}} \frac{\partial}{\partial\varepsilon} W\left(\theta,-\phi_\varepsilon,\phi_\varepsilon;\Phi_\varepsilon\right)\bigg|_{\varepsilon=0} > 0$$

But since $\frac{u'(\theta)+\beta\eta}{[u'(\theta)+\beta\eta+\beta^2\eta]} \geq \kappa$ by assumption the property is true.

Next notice because of Inada conditions that

$$\lim_{\substack{\phi\to\infty\\ \Phi\to\infty}} W\left(\theta,-\phi,\phi;\Phi\right) \to -\infty$$

since eventually, the debt constraints cease to bind for all agents. And so continuity implies that there exists ϕ^θ such that

$$W\left(\theta,-\phi^\theta,\phi^\theta;\Phi^\theta\right) = V^d\left(\theta\right)$$

with $\phi > 0$ and $\Phi\left(\phi^\theta\right) = \phi^\theta$. If there are many such we pick the one closest to 0. In this equilibrium all agents are subject to debt constraints ϕ^θ in each period. However it might be that for some $\tilde{\theta}$

$$W\left(\tilde{\theta},-\phi^\theta,\phi^\theta;\Phi^\theta\right) < V^d\left(\tilde{\theta}\right)$$

and as a result this would cease to be a NTT-RCE. Using a similar procedure, we can construct debt constraints $\phi^{\tilde{\theta}}$ for any $\tilde{\theta}$ such that the above constraint holds with equality. Consider $\phi = \min_\theta \phi^\theta$. By continuity it must be that

$$\frac{\partial}{\partial\varepsilon} W\left(\theta,-\phi-\varepsilon,\phi+\varepsilon\right)\bigg|_{\varepsilon=0} \leq 0$$

Therefore for all $\theta\in\Theta$,

$$W\left(\theta,-\phi,\phi;\Phi\right) \geq W\left(\theta,-\phi^\theta,\phi^\theta\right) = V^d\left(\theta\right)$$

which proves the claim. ■

A.2 Proofs from Section 4

Proof of Proposition 4. As we saw in Theorem 1, the equilibrium contract when $\delta_t\left(\theta^t\right)=0$, for all t,θ^t looks like a simple borrowing contract with exogenous debt limits ϕ. If $u\left(\underline{\theta}\right)+\frac{\beta}{1-\beta^2}\mathbb{E}u\left(\theta'\right)$ large enough, some household will be borrowing constrained in equilibrium. We now construct a feasible contract with banishment on path that makes both the intermediary and some household strictly better off. Consider the continuation contract at date $t-1$. Suppose there exists some type

θ^{t-1} who is Euler-constrained in $t-1$.In particular,

$$u'\left(c_{t-1}\left(\theta^{t-1}\right)\right) > \beta R_t \mathbb{E}_{t-1} u'\left(c_t\left(\theta^{t-1},\theta\right)\right)$$

We saw in Lemma 7 that in period t,

$$V_t\left(\theta^{t-1},\underline{\theta}\right) = V^d\left(\underline{\theta};\lambda\right)$$

Given that we can transform any $\hat{T}$ period contract into an equivalent 2 period contract, consider the following deviating contract

$$\begin{aligned}
{}_{t-1}\tilde{\tau}_{t-1}\left(\theta^{t-1}\right) &= {}_{t-1}\tau_{t-1}\left(\theta^{t-1}\right)+\varepsilon \\
{}_{t-1}\tilde{\tau}_{t}\left(\theta^{t-1},\theta\right) &= -\frac{\left[{}_{t-1}\tilde{\tau}_{t}\left(\theta^{t-1},\theta\right)-R_t\varepsilon\right]}{1-\pi\left(\underline{\theta}\right)} \text{ for all } \theta\neq\underline{\theta}
\end{aligned}$$

and the intermediary chooses to banish type $\left(\theta^{t-1},\underline{\theta}\right)$ .The following is true of the deviating contract:

1. The value to type $\left(\theta^{t-1},\underline{\theta}\right)$ is identical under both contracts since its participation constraint was binding
2. The present discounted value of transfers for types $\left(\theta^{t-1},\theta\right)$ is identical for all $\theta\in\Theta$, staying

   in the contract (and not being banished).
3. The equilibrium present discounted value of transfers for type θ^{t-1} is identical to the one under the original contract since

$$\begin{aligned}
&{}_{t-1}\tilde{\tau}_{t-1}\left(\theta^{t-1}\right)+\frac{1}{R_t}\left[\sum_{\theta>\underline{\theta}}\pi\left(\theta\right)\left[{}_{t-1}\tilde{\tau}_{t}\left(\theta^{t-1},\theta\right)\right]+\pi\left(\underline{\theta}\right)\cdot 0\right] \\
&= {}_{t-1}\tilde{\tau}_{t-1}\left(\theta^{t-1}\right)+\frac{1}{R_t}\sum_{\theta\in\Theta}\pi\left(\theta\right){}_{t-1}\tau_{t}\left(\theta^{t-1},\theta\right)
\end{aligned}$$

Fact 2 implies incentive compatibility in period t,continues to hold. The three facts together imply that if the contract is perturbed is a similar fashion for all such constrained types, incentive compatibility also holds in $t-1$.

The change in welfare for this type under this proposed perturbation, $\Delta\left(\theta^{t-1}\right)$ is

$$\begin{aligned}
&u\left(\theta+{}_{t-2}\tau_{t-1}+{}_{t-1}\tilde{\tau}_{t-1}\right)+\beta\sum_{\theta'>\underline{\theta}}u\left(\theta+{}_{t-1}\tilde{\tau}_{t}+{}_{t}\tau_{t}\right) \\
&-u\left(\theta+{}_{t-2}\tau_{t-1}+{}_{t-1}\tau_{t-1}\right)-\beta\sum_{\theta'\in\Theta}u\left(\theta+{}_{t-1}\tau_{t}+{}_{t}\tau_{t}\right) \\
&=u\left(\theta+{}_{t-2}\tau_{t-1}+{}_{t-1}\tilde{\tau}_{t-1}\right)-u\left(\theta+{}_{t-2}\tau_{t-1}+{}_{t-1}\tau_{t-1}\right) \\
&\quad+\beta\sum_{\theta'>\underline{\theta}}u\left(\theta+{}_{t-1}\tilde{\tau}_{t}+{}_{t}\tau_{t}\right)-\beta\sum_{\theta'\in\Theta}u\left(\theta+{}_{t-1}\tau_{t}+{}_{t}\tau_{t}\right) \\
&\geq u\left(\theta+{}_{t-2}\tau_{t-1}+{}_{t-1}\tilde{\tau}_{t-1}\right)-u\left(\theta+{}_{t-2}\tau_{t-1}+{}_{t-1}\tau_{t-1}\right) \\
&\quad+\beta\sum_{\theta'\in\Theta}\pi\left(\theta'\right)u\left(\theta+{}_{t-1}\tilde{\tau}_{t}+{}_{t}\tau_{t}\right)-\beta\sum_{\theta'\in\Theta}u\left(\theta+{}_{t-1}\tau_{t}+{}_{t}\tau_{t}\right)
\end{aligned}$$

where the inequality in the second line follows since $u\left(\theta' + {}_{t-1}\tilde{\tau}_t + {}_t\tau_t\right) \geq u\left(\underline{\theta} + {}_{t-1}\tilde{\tau}_t + {}_t\tau_t\right)$. Consider the last two terms,

$$\begin{aligned}
&\beta \sum_{\theta'\in\Theta} \pi\left(\theta'\right) u\left(\theta + {}_{t-1}\tilde{\tau}_t + {}_t\tau_t\right) - \beta \sum_{\theta'\in\Theta} u\left(\theta + {}_{t-1}\tau_t + {}_t\tau_t\right) \\
&= \beta \sum_{\theta'\in\Theta} \pi\left(\theta'\right) u\left(\theta' + {}_{t-1}\tilde{\tau}_t + {}_t\tau_t\right) - \beta \sum_{\theta'\in\Theta} \pi\left(\theta'\right) u\left(\theta' + \frac{{}_{t-1}\tau_t}{1-\pi\left(\underline{\theta}\right)} + {}_t\tau_t\right) \\
&+ \beta \sum_{\theta'\in\Theta} \pi\left(\theta'\right) u\left(\theta' + \frac{{}_{t-1}\tau_t}{1-\pi\left(\underline{\theta}\right)} + {}_t\tau_t\right) - \beta \sum_{\theta'\in\Theta} \pi\left(\theta'\right) u\left(\theta' + {}_{t-1}\tau_t + {}_t\tau_t\right)
\end{aligned}$$

Therefore for ε small, the sign of the change in welfare is sign of the following expression

$$\begin{aligned}
&u'\left(\theta + {}_{t-2}\tau_{t-1} + {}_{t-1}\tau_{t-1}\right)\varepsilon - \frac{1}{1-\pi\left(\underline{\theta}\right)}\beta R_{t+1} \sum_{\theta'\in\Theta} \pi\left(\theta'\right) u'\left(\theta' + \frac{{}_{t-1}\tau_t\left(\theta^{t-1}\right)}{1-\pi\left(\underline{\theta}\right)} + \tau_t^t\right)\varepsilon \\
&+ \beta \sum_{\theta'\in\Theta} \pi\left(\theta'\right) u\left(\theta' + \frac{{}_{t-1}\tau_t\left(\theta^{t-1}\right)}{1-\pi\left(\underline{\theta}\right)} + {}_t\tau_t\right) - \beta \sum_{\theta'\in\Theta} \pi\left(\theta'\right) u\left(\theta' + {}_{t-1}\tau_t\left(\theta^{t-1}\right) + {}_t\tau_t\right)
\end{aligned}$$

Since this is strictly positive at $\pi\left(\underline{\theta}\right) = 0$, and so for $\pi\left(\underline{\theta}\right) > 0$ small, the change in welfare is strictly positive for this household. Clearly, the intermediary can construct a contract that makes both it and the household strictly better off. ■

Proof of Proposition 6. Suppose that in an equilibrium, for some t and history θ^t, $u'\left(c_t\left(\omega^t\right)\right) q_t < \beta \mathbb{E}_{D_t^c(h^{t-1},h)} u'\left(c_{t+1}\left(\omega^{t+1}\right)\right)$. Recall that $\tau_t\left(h^t\right) = \tau_t^{old}\left(h^t\right) + {}_t\tau_t\left(h^t\right)$ and so in equilibrium $c_t\left(\omega^t\right) = \theta_t + \tau_t\left(h^t\right)$. Since we are considering symmetric equilibria in which ex-ante identical intermediaries offer the same contract in equilibrium, we consider the incentives for a deviating intermediary to offer a deviating contract in period t and make strictly positive profits. Consider an intermediary offering a $\varepsilon\delta-$savings contract $S_t^{\varepsilon,\delta}$ (defined in section 2) for some $\varepsilon > 0$ and $\delta < 1$. Notice that the intermediary makes positive profits whenever this contract is accepted. Since $u'\left(c_t\left(\omega^t\right)\right) q_t < \beta \mathbb{E}_{D_t^c(h^{t-1},h)} u'\left(c_{t+1}\left(\omega^{t+1}\right)\right)$, there exists $\varepsilon > 0, \delta < 1$ such that type h^t will strictly prefer to sign an $\varepsilon\delta$ savings contract if offered. These contracts are incentive compatible and satisfy voluntary participation constraints. As a result an intermediary offering such a contract will make positive profits which is a contradiction. ■

For any history, define $A_t\left(h^{t-1}, h_t\right)$ to be the equilibrium expected present discounted value of future transfers for type $\left(h^{t-1}, h_t\right)$

$$A_t\left(h^t\right) \equiv \left(1 - \delta_t\left(h^t\right)\right)\left[\tau_t\left(h^t\right) + q_t \sum_{h_{t+1}\in H_{t+1}} \zeta\left(h^t, h_{t+1}\right) A_{t+1}\left(h^t, h_{t+1}\right)\right]$$

To compute properties of the equilibria of the intermediary game, we will consider the limit of a sequence of truncated economies. In particular, I assume that there exists a finite date T, such that from $0 \leq t \leq T$, intermediaries offer contracts and for all $t > T$, those agents who have not defaulted in the past trade a risk free bond subject to exogenous debt constraints $\{\phi_t^e\}_{t>T}$. The claim that we can take such limits is formalized later in the appendix.

Proof of Lemma 1. Suppose not. Clearly, ${}_t\mathcal{P}_t\left(h^t\right) > 0$ for all h^t is not possible since the intermediary would making negative profits. On the other hand if ${}_t\mathcal{P}_t\left(h^t\right) \leq 0$ for all θ^t with strict inequality for some, then a deviating intermediary can offer a contract which transfers a little more to some types and still continue to make positive profits. As a result these types will strictly prefer to sign with the deviating intermediary. Finally, suppose that there exists h^t and $\hat{h}^t$ such that ${}_t\mathcal{P}_t\left(h^t\right) > 0$ and ${}_t\mathcal{P}_t\left(\hat{h}^t\right) < 0$. Then at the beginning of period t, consider a deviating intermediary offering the following contract,

$$
\begin{aligned}
{}_t\tilde{\mathcal{P}}_t\left(\hat{h}^t\right) &= {}_t\mathcal{P}_t\left(\hat{h}^t\right) + \varepsilon \\
{}_t\tilde{\tau}_{t+s}\left(h^t\right) &= 0 \text{ for all } s \geq 0, \text{ for } h^t \neq \hat{h}^t
\end{aligned}
$$

where $\varepsilon > 0$ and small. Notice that types h^t strictly prefer the original contract while types $\hat{h}^t$ strictly prefer ${}_t\tilde{\mathcal{P}}_t$ to ${}_t\mathcal{P}_t$. As a result, these households will strictly prefer to sign with the deviating intermediary who makes a positive profit. ■

Proof of Proposition 7. Because of the assumption that after period T, households can only trade a risk free bond subject to exogenous debt constraints, it is easy to see that the statement holds in period T, since all that matters for the household's choice is the sum $\theta_T + \tau_T^{T-1}\left(h^T\right)$. In period $T-1$ suppose h^{T-1} and $\hat{h}^{T-1}$ such that $\theta_{T-1} + \tau_{T-1}^{T-2}\left(h^{T-1}\right) = \hat{\theta}_{T-1} + \tau_{T-1}^{T-2}\left(\hat{h}^{T-1}\right)$ and

$$
V_{T-1}\left(h^{T-1}\right) > V_{T-1}\left(\hat{h}^{T-1}\right)
$$

Suppose first that for both $h^{T-1}, \hat{h}^{T-1}$

$$
u'\left(\theta_{T-1} + {}_{T-2}\tau_{T-1} + {}_{T-1}\tau_{T-1}\right) = \beta R_T \mathbb{E}_{D_T^c\left(h^{T-1}\right)} u'\left(\theta_T + {}_{T-1}\tau_T + \psi_{T+1}\left(\theta_T + {}_{T-1}\tau_T\right)\right) \quad \text{(A.5)}
$$

$$
u'\left(\hat{\theta}_{T-1} + {}_{T-2}\hat{\tau}_{T-1} + {}_{T-1}\hat{\tau}_{T-1}\right) = \beta R_T \mathbb{E}_{D_T^c\left(h^{T-1}\right)} u'\left(\theta_T + {}_{T-1}\hat{\tau}_T + \psi_{T+1}\left(\theta_T + {}_{T-1}\hat{\tau}_T\right)\right) \quad \text{(A.6)}
$$

where as before, $\psi_{T+1}\left(\theta_T + {}_{T-1}\tau_T\right)$ is the savings choice for the household (given that it is subject to debt constraint ϕ^e_{T+1}). Since ${}_{T-1}\tau_{T-1} + \mathbb{E}_{D_T^c\left(h^{T-1}\right)} \frac{{}_{T-1}\tau_T}{q_{T-1}} = {}_{T-1}\hat{\tau}_{T-1} + \mathbb{E}_{D_T^c\left(h^{T-1}\right)} \frac{{}_{T-1}\hat{\tau}_T}{q_{T-1}} = 0$ and the savings choice ψ_{T+1} depends only on the sum $\theta_T + {}_{T-1}\tau_T$, it must be that ${}_{T-1}\tau_{T-1} = {}_{T-1}\hat{\tau}_{T-1}$ and $V_{T-1}\left(h^{T-1}\right) = V_{T-1}\left(\hat{h}^{T-1}\right)$ and so we have a contradiction. Suppose on the other hand that (A.5) held with an equality and (A.6) with strictly inequality. Since ${}_{T-1}\tau_{T-1} + \frac{{}_{T-1}\tau_T}{q_{T-1}} = {}_{T-1}\hat{\tau}_{T-1} + \frac{{}_{T-1}\hat{\tau}_T}{q_{T-1}} = 0$, it must be that ${}_{T-1}\tau_T > {}_{T-1}\hat{\tau}_T$ and $D_T^c\left(h^{T-1}\right) \subseteq D_T^c\left(\hat{h}^{T-1}\right)$. Since the agent is Euler-constrained, assume that ${}_{T-1}\tau_T > 0$. Consider modifying the original contract for type $\hat{h}^{T-1}$ so that

$$
\begin{aligned}
{}_{T-1}\tilde{\tau}_{T-1} &= {}_{T-1}\hat{\tau}_{T-1} + \varepsilon \\
{}_{T-1}\tilde{\tau}_T &= -\frac{R_T \; {}_{T-1}\tilde{\tau}_{T-1}}{\sum_{h_T \in \tilde{D}_T^c} \zeta\left(\hat{h}^{T-1}, h_T\right)}
\end{aligned}
$$

where ε chosen so that ${}_{T-1}\tilde{\tau}_T \geq {}_{T-1}\tau_T$. Notice that such a perturbation might incentivize some types to default in period T, and so intermediary can choose to banish these types and consequently $D_T^c\left(\hat{h}^{T-1}\right) \supseteq \tilde{D}_T^c$. However, since ${}_{T-1}\tilde{\tau}_T \geq {}_{T-1}\tau_T$ the transfer ${}_{T-1}\tau_T$ is associated with more banishment and hence $D_T^c\left(h^{T-1}\right) \subseteq \tilde{D}_T^c$.

For ε small enough, this perturbation makes type $\hat{h}^{T-1}$ strictly better off in period $T-1$ and the the intermediary is equally well off. Also by construction the present discounted value of transfers is unchanged for the agent in $T-1$. To check for incentive compatibility we need only consider the households who are constrained and who might lie to get the increased transfer in period $T-1$. However, if the original transfer sequence was incentive compatible, the perturbed one is as well. Moreover, the default incentives in period T, are exactly the same as the types considered above and so the intermediary is as well off while the agent is strictly better off. Therefore a contract can be constructed that makes both the intermediary and the household strictly better off. Hence, it must be that ${}_{T-1}\tau_{T-1} = {}_{T-1}\hat{\tau}_{T-1}$. Note that a similar argument holds if both (A.5) and (A.6) held with strict inequality. Given that the property holds for $\hat{T}-1$, assume that this property holds for some $t+1 < \hat{T}-1$. Our goal is to show that the property holds in t. Suppose for contradiction we have some $h, \hat{h}^t$ such that $\theta_t + {}_{t-1}\tau_t\left(h^t\right) = \hat{\theta}_t + {}_{t-1}\tau_t\left(\hat{h}^t\right)$ and

$$V_t\left(h^t\right) > V_t\left(\hat{h}^t\right)$$

As before, first consider the case where the Euler equations hold with equality for both types. Suppose ${}_t\tau_t\left(h^t\right) < {}_t\tau_t\left(\hat{h}^t\right)$. Then it is easy to see that an intermediary can offer an $\varepsilon\delta$ savings contract to type $\hat{h}^t$ which will be accepted by this household making the intermediary strictly better off. The argument is identical to the one for period $T-1$. Next suppose that

$$u'\left(\theta_t + {}_{t-1}\tau_t + {}_t\tau_t\right) = \beta R_{t+1}\mathbb{E}_{D_t^c(h^{t-1})}u'\left(\theta_{t+1} + {}_t\tau_{t+1} + {}_{t+1}\tau_{t+1}\right)$$
$$u'\left(\hat{\theta}_t + {}_{t-1}\hat{\tau}_t + {}_t\hat{\tau}_t\right) > \beta R_{t+1}\mathbb{E}_{D_t^c(h^{t-1})}u'\left(\theta_{t+1T} + {}_t\hat{\tau}_{t+1} + {}_{t+1}\hat{\tau}_{t+1}\right)$$

As in the period $T-1$ case, the transfer scheme to type $\hat{h}^t$ can be modified so that, it is made strictly better off. We can use a similar argument to show that the intermediary can construct an incentive compatible contract that makes it and the agent strictly better off. Therefore by induction the claim must hold in period t and by induction for all previous periods as well. ■

Proof of Proposition 8. Part 1 is a direct consequence of incentive compatibility. If household types are being banished, they will always announce the type consistent with the largest re-entry probability.

Part 2. Consider a period t, and $h^t, \hat{h}^t$ such that ${}_{t-1}\tau_{t-1}\left(h^{t-1}\right) = {}_{t-1}\tau_{t-1}\left(\hat{h}^{t-1}\right)$. First suppose that $D_t\left(h^{t-1}\right) \subset D_t\left(\hat{h}^{t-1}\right)$. Since ${}_{t-1}\tau_{t-1}\left(h^{t-1}\right) + q_t\mathbb{E}_{D_t^c(h^{t-1})}\,{}_{t-1}\tau_t\left(h^{t-1}\right) = {}_{t-1}\tau_{t-1}\left(\hat{h}^{t-1}\right) + q_t\mathbb{E}_{D_t^c(\hat{h}^{t-1})}\,{}_{t-1}\tau_t\left(\hat{h}^{t-1}\right) = 0$, it must be that ${}_{t-1}\tau_t\left(\hat{h}^{t-1}\right) < {}_{t-1}\tau_t\left(h^{t-1}\right)$. In this case the intermediary can set ${}_{t-1}\tau_t\left(\hat{h}^{t-1}\right) = {}_{t-1}\tau_t\left(h^{t-1}\right)$ and $D_t\left(h^{t-1}\right) = D_t\left(\hat{h}^{t-1}\right)$, which leaves the intermediary equally well off but type $\hat{h}^{t-1}$ strictly better off. As a result a deviating contract exists that makes both strictly better off. An identical argument applies if $D_t\left(\hat{h}^{t-1}\right) \subset D_t\left(h^{t-1}\right)$. Given that $D_t\left(\hat{h}^{t-1}\right) = D_t\left(h^{t-1}\right)$, a similar argument implies that $\mu_t\left(h^{t-1}\right) = \hat{\mu}_t\left(h^{t-1}\right)$. If not, the probability of re-entry can be increased for the type making it strictly better off, while still preserving incentives.

Part 3. Suppose that ${}_{t-1}\tau_{t-1}\left(h^{t-1}\right) \geq {}_{t-1}\tau_{t-1}\left(\hat{h}^{t-1}\right)$ for two histories h^{t-1} and $\hat{h}^{t-1}$. We prove this by contradiction. There are three cases to consider. First suppose that $D_t\left(h^{t-1}\right) \subset D_t\left(\hat{h}^{t-1}\right)$ and $\mu_t\left(h^{t-1}\right) > \mu_t\left(\hat{h}^{t-1}\right)$. Since ${}_{t-1}\tau_{t-1}\left(h^{t-1}\right) + \mathbb{E}_{D_t^c(h^{t-1})}\,{}_{t-1}\tau_t\left(h^{t-1}\right) = {}_{t-1}\tau_{t-1}\left(\hat{h}^{t-1}\right) + \mathbb{E}_{D_t^c(\hat{h}^{t-1})}\,{}_{t-1}\tau_t\left(\hat{h}^{t-1}\right) = 0$, $\mathbb{E}_{D_t^c(h^{t-1})}\,{}_{t-1}\tau_t\left(h^{t-1}\right) \leq \mathbb{E}_{D_t^c(\hat{h}^{t-1})}\,{}_{t-1}\tau_t\left(\hat{h}^{t-1}\right)$ implies that ${}_{t-1}\tau_t\left(\hat{h}^{t-1}\right) < {}_{t-1}\tau_t\left(h^{t-1}\right)$. In this case it is easy to see that the intermediary can increase ${}_{t-1}\tau_t\left(\hat{h}^{t-1}\right)$, reduce the banishment set and make the household strictly better off while preserving incentives and still making zero profits.

Next suppose that $D_t\left(h^{t-1}\right) \subset D_t\left(\hat{h}^{t-1}\right)$ and $\mu_t\left(h^{t-1}\right) \leq \mu_t\left(\hat{h}^{t-1}\right)$. In this a similar argument to the one above works. Finally suppose that $D_t\left(h^{t-1}\right) \supseteq D_t\left(\hat{h}^{t-1}\right)$ but $\mu_t\left(h^{t-1}\right) > \mu_t\left(\hat{h}^{t-1}\right)$. Since $D_t\left(h^{t-1}\right) \supseteq D_t\left(\hat{h}^{t-1}\right)$ it must be that ${}_{t-1}\tau_{t-1}\left(h^{t-1}\right) \leq {}_{t-1}\tau_{t-1}\left(\hat{h}^{t-1}\right)$. Therefore, the intermediary can increase $\mu_t\left(\hat{h}^{t-1}\right)$ and make the household strictly better off. From Proposition 7 we know that since there is no default after history h^{t-1}, this perturbation still preserves default incentives after history $\hat{h}^{t-1}$. ■

Proof of Proposition 5. Since we have shown that the types being banished depend only on the current endowment report and transfer, the banishment sets $D_t\left(h^{t-1}\right) = D_t\left(\tau_t^{t-1}\left(h^{t-1}\right)\right)$ in any equilibrium contract. Since transfers are bounded, there exists $\bar{\phi}, \phi^D$ such that

1. ${}_{t-1}\tau_t\left(h^t\right) < \bar{\phi}$
2. If ${}_{t-1}\tau_t\left(h^t\right) < \phi^D$, $D_t\left({}_{t-1}\tau_t\left(h^t\right)\right) = \varnothing$ and $\hat{R}_t\left({}_{t-1}\tau_t\left(h^t\right)\right) = R_t\, {}_{t-1}\tau_t\left(h^{t-1}\right)$
3. If ${}_{t-1}\tau_t\left(h^t\right) \geq \phi^D$, $D_t\left({}_{t-1}\tau_t\left(h^t\right)\right) \neq \varnothing$ and $\hat{R}_t\left({}_{t-1}\tau_t\left(h^t\right)\right) = \frac{R_t\, {}_{t-1}\tau_t\left(h^{t-1}\right)}{\sum_{h \notin D_t\left({}_{t-1}\tau_t\left(h^{t-1}\right)\right)} \zeta\left(h^{t-1}, h\right)}$

Given this, construct the equilibrium objects as follows: For any agent with history $h^t = \left(\theta^t, \gamma^t, B^{t-1}\right)$, let $s_t = {}_{t-1}\tau_t\left(h^{t-1}\right)$ and define

$$\begin{aligned} V_t^R\left(\theta_t, s_t\right) &\equiv V_t\left(h^t\right) \\ d_t\left(\theta_t, s_t\right) &\equiv 1 - \delta_t\left(h^t\right) \\ s_{t+1}\left(\theta_t, s_t\right) &\equiv \mathcal{Q}_t^{-1}\left(-\, {}_t\tau_t\left(h^t\right)\right) \\ \lambda_{t+s}\left(s_t\right) &\equiv \mu_{t+s}\left(h^t\right) \end{aligned}$$

where (since $s_t = {}_{t-1}\tau_t\left(h^{t-1}\right)$),

$$\begin{aligned} &\mathcal{Q}_t(s) = \frac{s}{R_t} \text{ if } {}_{t-1}\tau_t\left(h^{t-1}\right) > -\phi^D \\ &\mathcal{Q}_t(s) = -\sum_{h \notin D_t\left({}_{t-1}\tau_t\left(h^{t-1}\right)\right)} \zeta\left(h^{t-1}, h\right) \quad {}_{t-1}\tau_t\left(h^{t-1}\right) \text{ if } -\bar{\phi} < {}_{t-1}\tau_t\left(h^{t-1}\right) \leq -\phi^D \\ &\mathcal{Q}_t(s) = 0 \text{ if } {}_{t-1}\tau_t\left(h^{t-1}\right) < \bar{\phi} \end{aligned}$$

and $c\left(\theta_t, \tau_t\right)$ is determined residually from the budget constraint. Given any other type $\tilde{h}^t$ with $\theta\left(\tilde{h}^t\right) = \theta_t$ and ${}_{t-1}\tau_t\left(\tilde{h}^t\right) = {}_{t-1}\tau_t(h)$ we know from Proposition 7 that $V_t\left(h^t\right) = V_t\left(\tilde{h}^t\right)$ and so $V_t^R\left(\theta_t, s_t\right) = V_t^R\left(\tilde{\theta}_t, s_t\right)$ and $d_t\left(\theta_t, s_t\right) = d_t\left(\tilde{\theta}_t, s_t\right)$ since $\mu_{t+s}\left(h^t\right) = \mu_{t+s}\left(\tilde{h}^t\right)$.

Since the value of default is common to both problems it must be that

$$d_t\left(\theta_t, s_t\right) = \arg\max_d V_t^0\left(\theta_{t+1}, s_{t+1}; \mathcal{Q}_{t+1}, \lambda\right) = \arg\max_d\, [1-d]\, V_t^R\left(\theta_t, s_t; \mathcal{Q}_t\right) + d V_t^D\left(\theta_t; \lambda(s)\right)$$

In particular, if $d_t\left(\theta_t, s_t\right) = 0$ and $V_t^R\left(\theta_t, s_t; \mathcal{Q}_t\right) > V_t^D\left(\theta_t; \lambda(s)\right)$, the household will strictly prefer to lie and pretend to be a type that is not banished in the intermediary game.

Next, given the constructed interest rate schedule, it must be that the constructed policy functions solve

$$V_t^R(\theta_t, s_t; \mathcal{Q}_t) = \max u(c_t) + \beta \mathbb{E}_t V_{t+1}^0(\theta_{t+1}, s_{t+1}; \mathcal{Q}_{t+1}, \lambda)$$
$$s.t.$$
$$c_t + \mathcal{Q}_t(s_{t+1}) \le \theta_t + s_t$$

since we showed earlier that $\tau_t^{t-1}\left(h^t\right) = -\hat{R}_t\left({}_{t-1}\tau_{t-1}\left(h^{t-1}\right)\right)$. If there exists different choice of τ_{t+1} that gives the households larger utility, a deviating intermediary can offer such a contract in the original environment and make a strictly positive profit. Finally, by construction, for all t

$$\mathcal{Q}_t(s) = \frac{\left[1 - \Pr\left[V_{t+1}^D > V_{t+1}^R\right]\right] s}{R_{t+1}}$$

which proves the result. ■

Existence: Define a bounded interval for assets/debt $[\underline{s}, \bar{s}]$. Given functions $V(\theta, s)$ define operator $\mathcal{B}(V)$ that takes in value functions and outputs of a set of default thresholds D. For each θ, find $s^*(\theta)$ s.t. $V(\theta, s^*(\theta)) = V^d(\theta)$. If $\sup_s V(\theta, s) < V^d(\theta)$, set $s^*(\theta) = -\infty$ and if $\inf_s V(\theta, s) > V^d(\theta)$, set $s^*(\theta) = \infty$

Given $D = \{s^*(\theta)\}$, define functional $\mathcal{V}(D)$ that takes in a set of thresholds and outputs value functions V

$$V(\theta, s; \{s^*(\theta)\})$$

Given a set of thresholds $\{s^*(\theta)\}$ define $V(\theta, s; s^*)$ as follows

$$V(\theta, s; s^*) = \left(\max_{c,s'} u(c) + \beta\left[\sum_{\theta'} \pi(\theta') V(\theta', s') \mathbf{1}_{s' \le s^*(\theta')} + \sum_{\theta'} \pi(\theta') V^d(\theta') \mathbf{1}_{s' > s^*(\theta')}\right]\right) \mathbf{1}_{s \ge s^*(\theta)} + V^d(\theta) \mathbf{1}_{s < s^*(\theta)}$$

subject to

$$c + \left(\frac{\sum_{\theta'} \pi(\theta') \mathbf{1}_{s' \le s^*(\theta')} + \sum_{\theta'} \pi(\theta') \chi \mathbf{1}_{s' > s^*(\theta')}}{R_t}\right) s' = \theta + s$$

Then we can define an operator $T = \mathcal{B} \circ \mathcal{V}$ that maps sets of default thresholds into themselves. Using similar arguments to Auclert and Rognlie (2014), one can show that this operator has a fixed point B^*. Using this, we can construct an equilibrium debt price schedule as follows

$$Q(s') = \frac{1}{R_t} \sum_{\theta'} \pi(\theta') \mathbf{1}_{s' \le s^*(\theta')} s'$$

This says that given a re-entry probability sequence of interest rates $\{R_t\}$ an equilibrium of the EG environment always exists. Since these interest rates are equilibrium objects we need to find a sequence that clears markets at each date. Given the policy functions we can define a measure

$$\nu_t(A \times B) = \mu^L\left(i \in \tilde{I}_t : (b'(i), \theta(i)) \in A \times B,\ A \times B = \mathcal{B}(A) \times \mathcal{B}(\Theta)\right)$$

where μ^L is the lebesgue measure and $\tilde{I}_t$ is the set of households who haven't defaulted in time

t. Then by continuity we know for R_t large enough

$$\int_{\mathbb{A}\times\Theta} s'(\theta,s)\,d\nu_t(b,\theta) > 0$$

while for R_t small enough

$$\int_{\mathbb{A}\times\Theta} s'(\theta,s)\,d\nu_t(s,\theta) < 0$$

Therefore for each t, there exists R_t that clears the market.

A.3 Proofs from Section 5

This section contains proofs from section 5 of the main text.

A.3.1 Proofs from Section 5.1

Proof of Proposition 10. The first part is as Proposition 12. Next, given a date t and history θ^{t-1},

if $\theta > \theta'$ we can use an identical argument as in the intermediary game to show that it must be that $A_t\left(\theta^{t-1},\theta\right) \geq A_t\left(\theta^{t-1},\theta'\right)$ where $A_t\left(\theta^{t-1},\theta\right) = \tau_t\left(\theta^{t-1},\theta\right) + q_t\sum_{\theta_{t+1}}\pi\left(\theta^{t-1},\theta,\theta_{t+1}\right)A_{t+1}\left(\theta^{t-1},\theta,\theta_{t+1}\right)$ is the expected present discounted value of transfers to type $\left(\theta^{t-1},\theta\right)$. In particular, if this did not hold, type θ will strictly prefer to lie and pretend to be type θ' and use the hidden markets to save. Suppose that $A_t\left(\theta^{t-1},\theta\right) > A_t\left(\theta^{t-1},\theta'\right)$. There are two cases to consider. First suppose that $q_t > \frac{\beta\mathbb{E}_t u'\left(c_{t+1}\left(\theta^{t-1},\theta,\theta_{t+1}\right)\right)}{u'\left(c_t\left(\theta^{t-1},\theta\right)\right)}$. Then as in the intermediary game we can find a perturbation which involves a small transfer of wealth between type $\left(\theta^{t-1},\theta\right)$ and the types below that increases ex-ante welfare. The second case to consider is one in which $q_t = \frac{\beta\mathbb{E}_t u'\left(c_{t+1}\left(\theta^{t-1},\theta,\theta_{t+1}\right)\right)}{u'\left(c_t\left(\theta^{t-1},\theta\right)\right)}$. We want to consider a wealth transfer from type $\left(\theta^{t-1},\theta\right)$ that leaves this equation unchanged. Choose $(\varepsilon, a\varepsilon)$ where

$$u'\left(c_t\left(\theta^{t-1},\theta\right)-\varepsilon\right)q - \beta\mathbb{E}_t u'\left(c_{t+1}\left(\theta^{t-1},\theta,\theta_{t+1}\right)-a\varepsilon\right) = 0$$

Modify the transfer sequence as follows: $\tilde{\tau}_t\left(\theta^{t-1},\theta\right) = \tau_t\left(\theta^{t-1},\theta\right) - \varepsilon$ and $\tilde{\tau}_t\left(\theta^{t-1},\theta,\theta_{t+1}\right) = \tau_t\left(\theta^{t-1},\theta,\theta_{t+1}\right) - a\varepsilon$ for all θ_{t+1}. This constitutes a wealth transfer from type $\left(\theta^{t-1},\theta\right)$ which can be redistributed to lower types. For ε small, the voluntary participation constraints are still satisfied and the pricing equation is unchanged since given the choice of a. As a result any solution to the constrained-efficient problem must satisfy $A_t\left(\theta^{t-1},\theta\right) = A_t\left(\theta^{t-1},\theta'\right)$. Moreover, it must be that $A_1(\theta_1) = 0$ for all θ_1. These two conditions imply that $\sum_{t=1}^{T}\left(\prod_{s=1}^{t} q_s\right)\tau_t\left(\theta^T\right) = 0$ for all $\theta^T \in \Theta^T$. ■

Proof of Lemma 3. We have already established the first part in an earlier proposition. Next, suppose that

$$q_t > \frac{\beta\mathbb{E}_t u'\left(c_{t+1}\left(\theta^{t+1}\right)\right)}{u'\left(c_t\left(\theta^t\right)\right)}$$

for some type h^t and

$$V_{t+1}\left(\tilde{\theta}^{t+1}\right) - V_{t+1}^d\left(\tilde{\theta}^{t+1}\right) > 0 \text{ for all } \tilde{\theta}^{t+1}$$

In this case, zero debt constraints would no longer be be not-too-tight in the hidden market. More generally, intermediaries can find a deviating contract that makes both it and the household strictly better off. ■

Proof of Theorem 3. Given the previous result we know that constrained-efficient allocation with no banishment looks like uncontingent borrowing and lending subject to debt constraints. In particular we can decompose the sequence of efficient transfers $\left\{\tau_t\left(\theta^t\right)\right\}$ into $\tau_t\left(\theta^t\right)=\tilde{\tau}_t\left(\theta^{t-1}\right)+\tilde{\tau}_t\left(\theta^t\right)$. We can construct contracts for T period lived intermediaries as follows:

$$
\begin{aligned}
&{}_1\zeta_1\left(\theta_1\right)=\tau_1\left(\theta_1\right)\\
&{}_1\zeta_t\left(\theta^t\right)=\tau_t\left(\theta^t\right),\ t<T\\
&{}_1\zeta_T\left(\theta^T\right)=\tilde{\tau}_t\left(\theta^{t-1}\right)\\
&{}_T\zeta_T\left(\theta^T\right)=\tilde{\tau}_T\left(\theta^T\right)\\
&\vdots
\end{aligned}
$$

while all other intermediaries (for example those born in period $2, T+1, ..$) offer simple uncontingent savings contracts. Given the prices from the planning problem consider the incentives of any particular intermediary to deviate when all other intermediaries are offering the uncontingent contracts constructed above. Given that intermediaries are offering savings contracts, a deviating intermediary cannot offer a contract with state-contingency. Therefore, since the intermediary is restricted to offer no-banishment contracts, the best this intermediary can do is to offer an agent who is Euler constrained the opportunity to borrow more at date t. Consider some t and history $\theta^t\in\Theta^t$ such that $u'\left(c_t\left(\theta^t\right)\right)q_t>\beta E_{t+1}u'\left(c_{t+1}\left(\theta^{t+1}\right)\right)$. We know from (5.6) and Lemma 4 that in period $t+1$, $V_{t+1}\left(\theta^t,\theta\right)=V^d_{t+1}\left(\theta^t,\theta\right)$ for some $\theta\in\Theta$. Therefore, the deviating contract will violate voluntary participation constraints for the agent in some state at date $t+1$. Notice that offering a savings contract can never lead to positive profits for any deviating intermediary since it would have to offer a return $\tilde{R}_{t+1}<\frac{1}{q_t}$ no household will ever accept such a contract. ■

A.3.2 Proofs from Section 5.2

Proof of Proposition 12. The proof of the first part of the proposition is clear. It must be that

$$u'\left(c_t\left(\theta^t\right)\right)q_t\geq\beta\mathbb{E}_{t+1}u'\left(c_{t+1}\left(\theta^{t+1}\right)\right)$$

else, the households will use the hidden markets to save. Next, since we are restricting the planner to only offer two period contracts, it cannot cross-subsidize between types. This along with an initial zero profit condition implies (5.12) . ■

Proof of Proposition 13. The proof of the first part is as before. Households can never be savings constrained. Since we know that any equilibrium contract of the hidden market must be short-term it suffices to consider deviating contracts of the following form: intermediaries offer a vector $\mathbb{D}_t\left(h^t\right)=\left(z_t\left(h^t\right),z_{t+1}\left(h^t\right),\delta^H_{t+1}\left(h^{t+1}\right),\mu^H\left(h^t\right)\right)$ which consists of transfers in period $t,\ t+1$, banishment indices for the hidden market and re-entry probabilities. Given a period t and history h^t, the best such contract solves the following problem (note that $\delta_{t+1},\delta^H_{t+1}$, and τ_{t+1} are functions of h^{t+1}):

$$
\begin{aligned}
&\max_{\mathbb{D}_t(h^t)} u\left(\theta_t+\tau_t\left(h^t\right)+z_t\left(h^t\right)\right)\\
&+\beta\sum_{h^{t+1}}\zeta\left(h^{t+1}\right)\left[\left(1-\delta_{t+1}\right)\left[\left(1-\delta^H_{t+1}\right)u\left(\theta_{t+1}+\tau_{t+1}+z_{t+1}\left(h^{t+1}\right)\right)+\delta^H_{t+1}\mathcal{V}^N_{t+1}\left(h^{t+1};\mu^H\right)\right]\right.\\
&\left.+\delta_{t+1}\mathcal{V}^E_t\left(h^t;\mu_{t+1},\mu^H_{t+1}\right)\right]
\end{aligned}
$$

subject to for all h^{t+1}

$$\left[1-\delta_{t+1}^{H}\left(h^{t+1}\right)\right]\mathcal{V}_{t+1}\left(h^{t+1}\right)\geq\left[1-\delta_{t+1}\left(h^{t+1}\right)\right]\mathcal{V}_{t+1}^{N}\left(h^{t+1};\mu_{t}^{H}\left(h^{t-1}\right)\right) \tag{A.7}$$

and

$$z_{t}\left(h^{t}\right)+q_{t}\sum_{h^{t+1}}\left(1-\delta_{t+1}\left(h^{t+1}\right)\right)\left(1-\delta_{t+1}^{H}\left(h^{t+1}\right)\right)z_{t+1}\left(h^{t}\right)=0 \tag{A.8}$$

Notice that this contract contains all possible short-term deviating contracts intermediaries can offer. We can substitute (A.8) into the objective function and rewrite the problem as

$$\begin{aligned}&\max_{\tilde{\mathbb{D}}_{t}(h^{t})} u\left(\theta_{t}+\tau_{t}\left(h^{t}\right)+\tilde{q}_{t}\tilde{z}_{t}\left(h^{t}\right)\right)\\&+\beta\sum_{h^{t+1}}\zeta\left(h^{t+1}\right)\left[\left(1-\delta_{t+1}\right)\left[\left(1-\delta_{t+1}^{H}\right)u\left(\theta_{t+1}+\tau_{t+1}-\tilde{z}_{t}\left(h^{t}\right)\right)+\delta_{t+1}^{H}\mathcal{V}_{t+1}^{N}\left(h^{t+1};\mu^{H}\right)\right]\right.\\&\left.+\delta_{t+1}\mathcal{V}_{t}^{E}\left(h^{t};\mu_{t+1},\mu_{t+1}^{H}\right)\right]\end{aligned}$$

subject to (A.7). Here $\tilde{q}_t = q_t \sum_{t+1}\left(1-\delta_{t+1}\left(h^{t+1}\right)\right)$ and $\tilde{\mathbb{D}}_t\left(h^t\right)=\left(\tilde{z}_t\left(h^t\right),\delta_{t+1}^H\left(h^{t+1}\right),\mu^H\left(h^{t+1}\right)\right)$. We know from Lemma 7 that we need only consider the constraint for the lowest type θ_{t+1} such that $\delta_{t+1}\left(h^{t+1}\right)=0$. Denote this type by $\hat{h}^{t+1}$ and let η denote the multiplier on this constraint. The first order conditions for $\tilde{z}_t\left(h^t\right)$ and $\mu^H\left(h^t\right)$ are

$$\begin{aligned}&u'\left(\theta_{t}+\tau_{t}\left(h^{t}\right)+\tilde{q}_{t}\tilde{z}_{t}\left(h^{t}\right)\right)\tilde{q}_{t}\\&-\beta\sum_{h^{t+1}}\zeta\left(h^{t+1}\right)\left(1-\delta_{t+1}\left(h^{t+1}\right)\right)\left[\left(1-\delta_{t+1}^{H}\left(h^{t+1}\right)\right)u'\left(\theta_{t+1}+\tau_{t+1}\left(h^{t+1}\right)-\tilde{z}_{t}\left(h^{t}\right)\right)\right]\\&=-\eta\frac{\partial}{\partial\varepsilon}\mathcal{V}_{t+1}\left(\hat{h}^{t+1}\right)\end{aligned} \tag{A.9}$$

and

$$\begin{aligned}&\beta\sum_{h^{t+1}}\zeta\left(h^{t+1}\right)\left[\left(1-\delta_{t+1}\left(h^{t+1}\right)\right)\delta_{t+1}^{H}\left(h^{t+1}\right)\frac{\partial}{\partial\mu^{H}}\mathcal{V}_{t+1}^{N}\left(h^{t+1};\mu^{H}\right)+\delta_{t+1}\left(h^{t+1}\right)\frac{\partial}{\partial\mu^{H}}\mathcal{V}_{t}^{E}\left(h^{t};\mu_{t+1},\mu_{t+1}^{H}\right)\right]\\&=\eta\frac{\partial}{\partial\mu^{H}}\mathcal{V}_{t+1}^{N}\left(\hat{h}^{t+1};\mu^{H}\left(h^{t}\right)\right)\end{aligned}$$

where

$$\begin{aligned}\frac{\partial}{\partial\varepsilon}\mathcal{V}_{t+1}\left(h^{t+1}\right)&=u'\left(\theta_{t+1}+\tau_{t+1}\left(h^{t+1}\right)-\tilde{z}_{t}\left(h^{t}\right)\right)\\\frac{\partial}{\partial\mu^{H}}\mathcal{V}_{t+1}^{N}\left(h^{t+1};\mu^{H}\right)&=\beta\mathbb{E}_{t}\left[\mathcal{V}_{t+2}\left(h^{t+2}\right)-\mathcal{V}_{t+2}^{N}\left(h^{t+2};\mu^{H}\right)\right]\\\frac{\partial}{\partial\mu^{H}}\mathcal{V}_{t}^{E}\left(h^{t};\mu_{t+1},\mu_{t+1}^{H}\right)&=\beta\mathbb{E}_{t}\mu\left(h^{t}\right)\left[\mathcal{V}_{t+2}\left(h^{t+2}\right)-\mathcal{V}_{t+2}^{N}\left(h^{t+2};\mu^{H}\right)\right]\end{aligned}$$

and the last two equations follow from (5.11) and (5.10) respectively.

Therefore

$$\eta = \frac{\beta \sum_{h^{t+1}} \zeta\left(h^{t+1}\right)\left[\left(1-\delta_{t+1}\left(h^{t+1}\right)\right)\delta_{t+1}^{H}\left(h^{t+1}\right)\frac{\partial}{\partial\mu^{H}}\mathcal{V}_{t+1}^{N}\left(h^{t+1};\mu^{H}\right)+\delta_{t+1}\left(h^{t+1}\right)\frac{\partial}{\partial\mu^{H}}\mathcal{V}_{t}^{E}\left(h^{t};\mu_{t+1},\mu_{t+1}^{H}\right)\right]}{\frac{\partial}{\partial\mu^{H}}\mathcal{V}_{t+1}^{N}\left(\hat{h}^{t+1};\mu^{H}\left(h^{t}\right)\right)}$$

Substituting this into (A.9) we obtain

$$\begin{aligned}
&u'\left(\theta_t+\tau_t\left(h^t\right)+\tilde{q}_t\tilde{z}_t\left(h^t\right)\right)\tilde{q}_t = \\
&\beta\sum_{h^{t+1}}\zeta\left(h^{t+1}\right)\left(1-\delta_{t+1}\left(h^{t+1}\right)\right)\left(1-\delta_{t+1}^{H}\left(h^{t+1}\right)\right)u'\left(\theta_{t+1}+\tau_{t+1}\left(h^{t+1}\right)-\tilde{z}_t\left(h^t\right)\right)- \\
&\left[\frac{\beta\sum_{h^{t+1}}\zeta\left(h^{t+1}\right)\left[\left(1-\delta_{t+1}\left(h^{t+1}\right)\right)\delta_{t+1}^{H}\left(h^{t+1}\right)\frac{\partial}{\partial\mu^{H}}\mathcal{V}_{t+1}^{N}+\delta_{t+1}\left(h^{t+1}\right)\frac{\partial}{\partial\mu^{H}}\mathcal{V}_{t+1}^{E}\right]}{\frac{\partial}{\partial\mu^{H}}\mathcal{V}_{t+1}^{N}\left(\hat{h}^{t+1};\mu^{H}\left(h^{t}\right)\right)}\right]\frac{\partial}{\partial\varepsilon}\mathcal{V}_{t+1}\left(\hat{h}^{t+1}\right)
\end{aligned}$$

Suppose that $\delta_{t+1}^{H}\left(h^{t+1}\right)=0$ for all h^{t+1}. This implies that the deviating contract does not banish additional types from the hidden market. Then

$$\begin{aligned}
&u'\left(\theta_t+\tau_t\left(h^t\right)+\tilde{q}_t\tilde{z}_t\left(h^t\right)\right)\tilde{q}_t = \\
&\beta\sum_{h^{t+1}}\zeta\left(h^{t+1}\right)\left(1-\delta_{t+1}\left(h^{t+1}\right)\right)u'\left(\theta_{t+1}+\tau_{t+1}\left(h^{t+1}\right)-\tilde{z}_t\left(h^t\right)\right)- \\
&\left[\frac{\beta\sum_{h^{t+1}}\zeta\left(h^{t+1}\right)\left[\delta_{t+1}\left(h^{t+1}\right)\frac{\partial}{\partial\mu^{H}}\mathcal{V}_{t+1}^{E}\left(h^{t+1};\mu_{t+1},\mu_{t+1}^{H}\right)\right]}{\frac{\partial}{\partial\mu^{H}}\mathcal{V}_{t+1}^{N}\left(\hat{h}^{t+1};\mu^{H}\left(h^{t}\right)\right)}\right]\frac{\partial}{\partial\varepsilon}\mathcal{V}_{t+1}\left(\hat{h}^{t+1}\right)
\end{aligned} \tag{A.10}$$

Consider the last term of the above equation. We have

$$\begin{aligned}
&-\frac{\beta\sum_{h^{t+1}}\zeta\left(h^{t+1}\right)\left[\delta_{t+1}\left(h^{t+1}\right)\frac{\partial}{\partial\mu^{H}}\mathcal{V}_{t+1}^{E}\left(h^{t+1};\mu_{t+1},\mu_{t+1}^{H}\right)\right]}{\frac{\partial}{\partial\mu^{H}}\mathcal{V}_{t+1}^{N}\left(\hat{h}^{t+1};\mu^{H}\left(h^{t}\right)\right)}\frac{\partial}{\partial\varepsilon}\mathcal{V}_{t+1}\left(\hat{h}^{t+1}\right) \\
&=-\beta\sum_{h^{t+1}}\zeta\left(h^{t+1}\right)\left[\delta_{t+1}\left(h^{t+1}\right)\frac{\partial}{\partial\varepsilon}\mathcal{V}_{t+1}\left(\hat{h}^{t+1}\right)\right]\frac{\frac{\partial}{\partial\mu^{H}}\mathcal{V}_{t+1}^{E}\left(\hat{h}^{t+1};\mu_{t+1},\mu_{t+1}^{H}\right)}{\frac{\partial}{\partial\mu^{H}}\mathcal{V}_{t+1}^{N}\left(\hat{h}^{t+1};\mu^{H}\left(h^{t}\right)\right)} \\
&=\beta\sum_{h^{t+1}}\zeta\left(h^{t+1}\right)\left[\delta_{t+1}\left(h^{t+1}\right)u'\left(\hat{\theta}_{t+1}+\tau_{t+1}\left(\hat{h}^{t+1}\right)-\tilde{z}_t\left(h^t\right)\right)\right]\frac{\frac{\partial}{\partial\mu^{H}}\mathcal{V}_{t+1}^{E}\left(\hat{h}^{t+1};\mu_{t+1},\mu_{t+1}^{H}\right)}{\frac{\partial}{\partial\mu^{H}}\mathcal{V}_{t+1}^{N}\left(\hat{h}^{t+1};\mu^{H}\left(h^{t}\right)\right)} \\
&=\beta\sum_{h^{t+1}}\zeta\left(h^{t+1}\right)\left[\delta_{t+1}\left(h^{t+1}\right)u'\left(\hat{\theta}_{t+1}+\tau_{t+1}\left(\hat{h}^{t+1}\right)-\tilde{z}_t\left(h^t\right)\right)\right]\frac{\beta\mathbb{E}_t\mu\left(h^{t+1}\right)\left[\mathcal{V}_{t+2}\left(h^{t+2}\right)-\mathcal{V}_{t+2}^{N}\left(h^{t+2};\mu^{H}\right)\right]}{\beta\mathbb{E}_t\left[\mathcal{V}_{t+2}\left(h^{t+2}\right)-\mathcal{V}_{t+2}^{N}\left(h^{t+2};\mu^{H}\right)\right]} \\
&\leq\beta\sum_{h^{t+1}}\zeta\left(h^{t+1}\right)\left[\delta_{t+1}\left(h^{t+1}\right)u'\left(\hat{\theta}_{t+1}+\tau_{t+1}\left(\hat{h}^{t+1}\right)-\tilde{z}_t\left(h^t\right)\right)\right]
\end{aligned}$$

where the first equality follows since $\mathcal{V}_{t+1}^{E}\left(h^{t+1};\mu_{t+1},\mu_{t+1}^{H}\right)=\mathcal{V}_{t+1}^{E}\left(\hat{h}^{t+1};\mu_{t+1},\mu_{t+1}^{H}\right)$. Therefore, from (A.10) we know that

$$
\begin{aligned}
&u'\left(\theta_t+\tau_t\left(h^t\right)+\tilde{q}_t\tilde{z}_t\left(h^t\right)\right)\tilde{q}_t \leq \\
&\beta\sum_{h^{t+1}}\zeta\left(h^{t+1}\right)\left(1-\delta_{t+1}\left(h^{t+1}\right)\right)u'\left(\theta_{t+1}+\tau_{t+1}\left(h^{t+1}\right)-\tilde{z}_t\left(h^t\right)\right)+ \\
&\beta\sum_{h^{t+1}}\zeta\left(h^{t+1}\right)\left[\delta_{t+1}\left(h^{t+1}\right)u'\left(\hat{\theta}_{t+1}+\tau_{t+1}\left(\hat{h}^{t+1}\right)-\tilde{z}_t\left(h^t\right)\right)\right] \\
&\leq \beta\sum_{h^{t+1}}\zeta\left(h^{t+1}\right)\left(1-\delta_{t+1}\left(h^{t+1}\right)\right)u'\left(\hat{\theta}_{t+1}+\tau_{t+1}\left(\hat{h}^{t+1}\right)-\tilde{z}_t\left(h^t\right)\right)+ \\
&\beta\sum_{h^{t+1}}\zeta\left(h^{t+1}\right)\left[\delta_{t+1}\left(h^{t+1}\right)u'\left(\hat{\theta}_{t+1}+\tau_{t+1}\left(\hat{h}^{t+1}\right)-\tilde{z}_t\left(h^t\right)\right)\right] \\
&= \beta u'\left(\hat{\theta}_{t+1}+\tau_{t+1}\left(\hat{h}^{t+1}\right)-\tilde{z}_t\left(h^t\right)\right)
\end{aligned}
$$

Therefore, if (5.14) holds, there exists no hidden contract $\mathbb{D}_t\left(h^t\right)$ that makes the household strictly better off. Suppose that $u'\left(\theta_t+\tau_t\left(h^t\right)\right)\hat{q}_t\left(h^t\right)>\beta u'\left(\hat{\theta}_{t+1}+\tau_{t+1}\left(\hat{h}^{t+1}\right)\right)$.We have that

$$
\begin{aligned}
&u'\left(\theta_t+\tau_t\left(h^t\right)\right)\tilde{q}_t- \\
&-\beta\sum_{h^{t+1}}\zeta\left(h^{t+1}\right)\left(1-\delta_{t+1}\left(h^{t+1}\right)\right)u'\left(\theta_{t+1}+\tau_{t+1}\left(h^{t+1}\right)\right)- \\
&\left[\frac{\beta\sum_{h^{t+1}}\zeta\left(h^{t+1}\right)\left[\delta_{t+1}\left(h^{t+1}\right)\frac{\partial}{\partial\mu^H}\mathcal{V}^E_{t+1}\left(h^{t+1}\right)\right]}{\frac{\partial}{\partial\mu^H}\mathcal{V}^N_{t+1}\left(\hat{h}^{t+1};\mu^H\left(h^t\right)\right)}\right]u'\left(\hat{\theta}_{t+1}+\tau_{t+1}\left(\hat{h}^{t+1}\right)\right) \\
&\geq u'\left(\theta_t+\tau_t\left(h^t\right)\right)\tilde{q}_t- \\
&-\beta\sum_{h^{t+1}}\zeta\left(h^{t+1}\right)\left(1-\delta_{t+1}\left(h^{t+1}\right)\right)u'\left(\hat{\theta}_{t+1}+\tau_{t+1}\left(h^{t+1}\right)\right)-\beta\sum_{h^{t+1}}\zeta\left(h^{t+1}\right)\delta_{t+1}\left(h^{t+1}\right)u'\left(\hat{\theta}_{t+1}+\tau_{t+1}\left(\hat{h}^{t+1}\right)\right) \\
&= u'\left(\theta_t+\tau_t\left(h^t\right)\right)\tilde{q}_t-\beta u'\left(\hat{\theta}_{t+1}+\tau_{t+1}\left(\hat{h}^{t+1}\right)-\tilde{z}_t\left(h^t\right)\right)>0
\end{aligned}
$$

So there exists a deviating contract $\tilde{\mathbb{D}}_t\left(h^t\right)=\left(\varepsilon_t\left(h^t\right),\mu^H\left(h^{t+1}\right)\right)$ that makes the household strictly better off. ■

Given these characterization results, we can prove a version of the Second Welfare Theorem in this environment.

Proof of Proposition 11. As we showed in Proposition 12, any solution to the planning problem households can never be savings constrained. Second, using an argument identical to that in decentralized environment with banishment, we can split the sequence of optimal transfers $\tau_t\left(h^t\right)=-\hat{R}_t\left({}_{t-1}\tilde{\tau}_t\left(h^{t-1}\right)\right)+{}_t\tilde{\tau}_t\left(h^t\right)$. As a result we can construct the short term defaultable debt contracts for intermediaries born in period $1,T,2T-1,..$ as was done in the intermediary game. All intermediaries born at dates besides these offer simple uncontingent savings contract to deter deviating intermediaries from offering state-contingency contracts. We can show using similar arguments to show that if $\delta_t\left(h^{t-1},h_t\right)=1$, then the fact that savers must get a return $R_t=\frac{1}{q_t}$ and incentive-feasiblity imply that $\hat{R}_t\left({}_{t-1}\tilde{\tau}_t\left(h^{t-1}\right)\right)=\frac{R_t\,{}_{t-1}\tilde{\tau}_t\left(h^{t-1}\right)}{\sum_{h\notin D_t\left(\tau_t^{t-1}\left(h^{t-1}\right)\right)}\zeta\left(h^{t-1},h\right)}$. As before, if $D_t\left(\tau_t^{t-1}\left(h^{t-1}\right)\right)=\varnothing$, then $\hat{R}_t\left(\tau\right)=R_t\tau$. We can then define an allocation for the intermediary game as follows in exactly the same manner in which we constructed two period contracts in the decentralized environment and the banishment policy the same as the one chosen by the planner. By construction these contracts satisfy incentive compatibility, resource feasibility and voluntary participation constraints. Under these contracts, intermediaries make zero profits as well. Suppose that all intermediaries were offering these contracts. Let consider the incentive for a deviating intermediary to offer a contract that makes both it and the household strictly better off. Such a contract can never be a savings contract because of (5.13). As a result we need to consider if a deviating debt contract exists. Since the contract must be short-term i.e. $\mathbb{D}_t\left(h^t\right)=\left({}_t z_t\left(h^t\right), {}_t z_{t+1}\left(h^t\right), {}_t\tilde{\delta}_{t+1\,t+1}\left(h^{t+1}\right), {}_t\tilde{\mu}_{t+1}\left(h^t\right)\right)$ that solves

$$\begin{aligned}
&\max_{\mathbb{D}_t(h^t)} u\left(\theta_t+\tau_t\left(h^t\right)+{}_t z_t\left(h^t\right)\right)\\
&+\beta\sum_{h^{t+1}}\zeta\left(h^{t+1}\right)\left[(1-\delta_{t+1})\left[\left(1-{}_t\tilde{\delta}_{t+1}\right)u\left(\theta_{t+1}+\tau_{t+1}+{}_t z_{t+1}\right)+{}_t\tilde{\delta}_{t+1}\,\tilde{V}_{t+1}\left(h^{t+1};{}_t\tilde{\mu}_{t+1}\,\mu_{t+1}\right)\right]\right.\\
&\left.+\delta_{t+1}\tilde{V}_{t+1}\left(h^{t+1};\mu_{t+1}\right)\right]
\end{aligned}$$

where

$$\begin{aligned}
\tilde{V}_{t+1}\left(h^{t+1};{}_t\tilde{\mu}_{t+1}\,\mu_{t+1}\right)&=u\left(\theta_{t+1}\right)\\
&+\beta\mathbb{E}_{t+1}\left[{}_t\tilde{\mu}_{t+1}\,\mu_{t+1}V_{t+2}\left(h_{t+2}\right)+\left(1-{}_t\tilde{\mu}_{t+1}\,\mu_{t+1}\right)\tilde{V}_{t+2}\left(h^{t+2};{}_t\tilde{\mu}_{t+1}\,\mu_{t+1}\right)\right]
\end{aligned}$$

From (5.13), such a contract can never make the household strictly better off. Therefore, these contracts must constitute a competitive equilibrium. ■

A.4 Truncation argument

Let V_∞^* denote the ex-ante welfare for the planner under the constrained efficient allocation with no banishment. Now consider two truncated economies in which after period T, households trade a risk free bond subject to debt constraints. In the first, the debt constraints after period T, are $\phi+\varepsilon$ and the second, the constraints are $\phi-\varepsilon$ for $\phi,\varepsilon>0$. Let $V_\infty^{T,H}(\phi+\varepsilon)$ denote the welfare to the planner in the first truncated economy and note that $V_\infty^{T,H}(\phi+\varepsilon)=V_T^{T,H}(\phi+\varepsilon)+V_{T+}^{T,H}(\phi+\varepsilon)$ where $V_T^{T,H}(\phi+\varepsilon)$ denotes aggregate expected welfare from period 1 to T and $V_{T+}^{T,H}(\phi+\varepsilon)$ the welfare from periods after T . We can similarly define $V_\infty^{T,L}(\phi-\varepsilon)=V_T^{T,L}(\phi-\varepsilon)+V_{T+}^{T,L}(\phi-\varepsilon)$. We can always find a pair (ϕ,ε) such that

$$V_\infty^{T,L}(\phi-\varepsilon)\leq V_\infty^*\leq V_\infty^{T,H}(\phi+\varepsilon)$$

Since we have proved properties of the constrained efficient allocations in the truncated economies, if we can prove that $\lim_{\varepsilon\to 0}\lim_{T\to\infty}\left[V_\infty^{T,H}(\phi+\varepsilon)-V_\infty^{T,L}(\phi-\varepsilon)\right]=0$, then we know that V_∞^* inherits these properties as well. To show this limiting property I show that that for T large enough and ε small enough, the set of feasible allocations in both truncated economies are identical.

Consider the some $x\in Feas^{T,L}(\phi-\varepsilon)$. Since x satisfies feasibility and incentive compatibility in $Feas^{T,L}(\phi-\varepsilon)$, to show that $x\in Feas^{T,H}(\phi+\varepsilon)$ we just need to show that the voluntary participation constraints are satisfied. But since for all t and types h^t

$$V_{t-T}^{T,H}(\phi+\varepsilon)\left(x\left(h^t\right)\right)+V_{T+}^{T,H}(\phi+\varepsilon)\left(x\left(h^t\right)\right)\geq V_{t-T}^{T,L}(\phi-\varepsilon)\left(x\left(h^t\right)\right)+V_{T+}^{T,L}(\phi-\varepsilon)\left(x\left(h^t\right)\right)$$

where $V_{t-T}^{T,H}(\phi+\varepsilon)\left(x\left(h^t\right)\right)$ denotes the value for type h^t from period t to T, this is clearly satisfied. Hence $Feas^{T,L}(\phi-\varepsilon) \subseteq Feas^{T,H}(\phi+\varepsilon)$. Next consider some $x \in Feas^{T,H}(\phi+\varepsilon)$. As in the previous case we need to show that this satisfies voluntary participation constraints in $Feas^{T,L}(\phi-\varepsilon)$. Note that in general since $V_{T+}^{T,L}(\phi-\varepsilon)(x) < V_{T+}^{T,H}(\phi+\varepsilon)(x)$, this will not be satisfied. However as $T \to \infty$ and $\varepsilon \to 0$, $\left[V_{t-T}^{T,H}(\phi+\varepsilon)(x) + V_{T+}^{T,H}(\phi+\varepsilon)(x)\right] - \left[V_{t-T}^{T,L}(\phi-\varepsilon)(x) + V_{T+}^{T,L}(\phi-\varepsilon)(x)\right]$ 0. Therefore for T large enough and ε small enough, it must be that $x \in Feas^{T,L}(\phi-\varepsilon)$ and so eventually $Feas^{T,H}(\phi+\varepsilon) \subseteq Feas^{T,L}(\phi-\varepsilon)$.