Online Appendix to *"Stability, Strategy-Proofness, and Cumulative Offer Mechanisms"*

John William Hatfield
McCombs School of Business
University of Texas at Austin

Scott Duke Kominers
Harvard Business School &
Department of Economics
Harvard University

Alexander Westkamp
Department of Management,
Economics, and Social Sciences
University of Cologne

December 1, 2017

Abstract

This appendix contains a proof of the sufficiency result (Theorem 4) in Section 3 and all proofs for Section 4.

JEL Classification: C62; C78; D44; D47

Keywords: Matching with contracts, Stability, Strategy-proofness, Substitutability, Size monotonicity, Cumulative offer mechanism

A Preliminary observations

We can write the combination of two offer processes $\mathbf{y} = (y^1, \ldots, y^M)$ and $\mathbf{z} = (z^1, \ldots, z^N)$ as $(\mathbf{y}, \mathbf{z}) = (w^1, \ldots, w^K)$ where

- $w^k = y^k$ for all $k \leq M$ and
- $w^k = z^{\ell_k}$ for $k > M$, where $\ell_k \equiv \min\{\ell \in 1, \ldots, N : z^\ell \notin \{w^1, \ldots, w^{k-1}\}\}$.

Our first lemma establishes a condition under which we can combine two different weakly observable offer processes to obtain another weakly observable offer process.

Lemma A.1. *Suppose that the choice function of every hospital is observably substitutable across doctors. Let* $\mathbf{y}$ *and* $\mathbf{z}$ *be two weakly observable offer processes that are both weakly compatible with the same preference profile* $\succ$*. Then* $(\mathbf{y}, \mathbf{z})$ *is a weakly observable offer process.*

Proof. Consider any weakly observable offer process $\mathbf{y} = (y^1, \ldots, y^M)$. We will prove the statement by induction on the length of $\mathbf{z} = (z^1, \ldots, z^N)$, showing at each step that $(\mathbf{y}, \mathbf{z})$ and $(\mathbf{z}, \mathbf{y})$ are weakly observable. If $N = 0$, the statement is trivially true. Hence, suppose that $(\mathbf{y}, (z^1, \ldots, z^{N-1}))$ and $((z^1, \ldots, z^{N-1}), \mathbf{y})$ are weakly observable.

We first show that $(\mathbf{y}, \mathbf{z})$ is weakly observable. There are two cases:

1. If $z^N \in \mathsf{c}(\mathbf{y})$, then $(\mathbf{y}, (z^1, \ldots, z^{N-1})) = (\mathbf{y}, \mathbf{z})$ and so $(\mathbf{y}, \mathbf{z})$ is weakly observable by the inductive assumption.
2. If $z^N \notin \mathsf{c}(\mathbf{y})$, we first note that $(\mathsf{c}(\mathbf{y}) \smallsetminus \mathsf{c}(\mathbf{z})) \cap \left(X_{\mathsf{d}(z^N)} \cap X_{\mathsf{h}(z^N)}\right) = \varnothing$;[1] that is, no contract between $\mathsf{d}(z^N)$ and $\mathsf{h}(z^N)$ is suggested in offer process $\mathbf{y}$ unless it was also suggested during $(z^1, \ldots, z^{N-1})$. Since $\mathbf{z}$ is weakly observable, we must have $\mathsf{d}(z^N) \notin \mathsf{d}(C^{\mathsf{h}(z^N)}(\{z^1, \ldots, z^{N-1}\}))$. By the inductive assumption, $((z^1, \ldots, z^{N-1}), \mathbf{y})$ is weakly observable. Since $C^{\mathsf{h}(z^N)}$ is observably substitutable across doctors, we then obtain that $\mathsf{d}(z^N) \notin \mathsf{d}(C^{\mathsf{h}(z^N)}(\{z^1, \ldots, z^{N-1}\} \cup \mathsf{c}(\mathbf{y})))$ given that $(\mathsf{c}(\mathbf{y}) \smallsetminus \mathsf{c}(\mathbf{z})) \cap \left(X_{\mathsf{d}(z^N)} \cap X_{\mathsf{h}(z^N)}\right) = \varnothing$; therefore, $(\mathbf{y}, \mathbf{z})$ is weakly observable by definition.

[1] Since $z^N \notin \mathsf{c}(\mathbf{y})$, we have that for all $z \in \mathsf{c}(\mathbf{y}) \cap \left(X_{\mathsf{d}(z^N)} \cap X_{\mathsf{h}(z^N)}\right)$ it must be the case that $z \succ_{\mathsf{d}(z^N)} z^N$. Hence, if there existed $w \in (\mathsf{c}(\mathbf{y}) \smallsetminus \mathsf{c}(\mathbf{z})) \cap \left(X_{\mathsf{d}(z^N)} \cap X_{\mathsf{h}(z^N)}\right)$, then $\mathbf{z}$ and $\mathbf{y}$ would not be weakly compatible with the same preference profile.

We now show by induction on m that, for all $m \leq M$, $(\mathbf{z}, (y^1, \ldots, y^m))$ is weakly observable. Suppose that for some $\bar{m} \leq M-1$, the statement has already been shown for all $m' \leq \bar{m}$. We will show that the statement holds for $\bar{m}+1$. There are two cases:

1. If $y^{\bar{m}+1} \in \mathsf{c}(\mathbf{z})$, then $(\mathbf{z}, (y^1, \ldots, y^{\bar{m}+1})) = (\mathbf{z}, (y^1, \ldots, y^{\bar{m}}))$ and $(\mathbf{z}, (y^1, \ldots, y^{\bar{m}+1}))$ is weakly observable by the inductive assumption.

2. If $y^{\bar{m}+1} \notin \mathsf{c}(\mathbf{z})$, we first note that $(\mathsf{c}(\mathbf{z}) \smallsetminus \mathsf{c}(\mathbf{y})) \cap \left(X_{\mathsf{d}(y^{\bar{m}+1})} \cap X_{\mathsf{h}(y^{\bar{m}+1})}\right) = \varnothing$;[2] that is, no contract between $\mathsf{d}(y^{\bar{m}+1})$ and $\mathsf{h}(y^{\bar{m}+1})$ is suggested in offer process $\mathbf{z}$ unless it was also suggested during $\mathbf{y}$. Since $\mathbf{y}$ is weakly observable, we must have $\mathsf{d}(y^{\bar{m}+1}) \notin \mathsf{d}(C^{\mathsf{h}(y^{\bar{m}+1})}(\{y^1, \ldots, y^{\bar{m}}\}))$. We have already established that $((y^1, \ldots, y^{\bar{m}}), \mathbf{z})$ is weakly observable. Since $C^{\mathsf{h}(y^{\bar{m}+1})}$ is observably substitutable across doctors, we then obtain that $\mathsf{d}(y^{\bar{m}+1}) \notin \mathsf{d}(C^{\mathsf{h}(y^{\bar{m}+1})}(\{y^1, \ldots, y^{\bar{m}}\} \cup \mathsf{c}(\mathbf{z})))$ given that $(\mathsf{c}(\mathbf{z}) \smallsetminus \mathsf{c}(\mathbf{y})) \cap \left(X_{\mathsf{d}(y^{\bar{m}+1})} \cap X_{\mathsf{h}(y^{\bar{m}+1})}\right) = \varnothing$; therefore, $(\mathbf{z}, (y^1, \ldots, y^{\bar{m}+1}))$ is weakly observable by definition.

This completes the proof of Lemma A.1. $\square$

Our second preliminary lemma shows that

- if the choice function of each hospital is observably substitutable then the set of rejected contracts expands monotonically along combined offer processes that are weakly observable, and

- if the choice function of each hospital is observably size monotonic then the set of chosen contracts grows weakly larger along combined offer processes that are weakly observable.

Lemma A.2. *Let $\mathbf{y}$ and $\mathbf{z}$ be two offer processes such that $(\mathbf{y}, \mathbf{z})$ is a weakly observable offer process. If the choice function of each hospital is observably substitutable, then $R^H(\mathsf{c}(\mathbf{y})) \subseteq R^H(\mathsf{c}(\mathbf{y}) \cup \mathsf{c}(\mathbf{z}))$. If the choice function of each hospital is observably size monotonic, then $|C^H(\mathsf{c}(\mathbf{y}))| \leq |C^H(\mathsf{c}(\mathbf{y}) \cup \mathsf{c}(\mathbf{z}))|$.*

Proof. Define ℓ_k for $k \geq 1$ inductively as $\ell_k \equiv \min\{\ell \in 1, \ldots, N : z^\ell \notin \mathsf{c}(\mathbf{y}) \cup \{z^{\ell_1}, \ldots, z^{\ell_{k-1}}\}\}$. The proof is by induction on k.

[2] Since $y^{\bar{m}+1} \notin \mathsf{c}(\mathbf{z})$, we have that for all $z \in \mathsf{c}(\mathbf{y}) \cap \left(X_{\mathsf{d}(y^{\bar{m}+1})} \cap X_{\mathsf{h}(y^{\bar{m}+1})}\right)$ it must be the case that $z \succ_{\mathsf{d}(y^{\bar{m}+1})} y^{\bar{m}+1}$. Hence, if there existed $w \in (\mathsf{c}(\mathbf{z}) \smallsetminus \mathsf{c}(\mathbf{y})) \cap \left(X_{\mathsf{d}(y^{\bar{m}+1})} \cap X_{\mathsf{h}(y^{\bar{m}+1})}\right)$, then $\mathbf{z}$ and $\mathbf{y}$ would not be weakly compatible with the same preference profile.

We first show the result for observable substitutability: Suppose that we have already established that $R^H(\mathsf{c}(\mathbf{y})) \subseteq R^H(\mathsf{c}(\mathbf{y}) \cup \{z^{\ell_1}, \ldots, z^{\ell_{k-1}}\})$; we will show that $R^H(\mathsf{c}(\mathbf{y})) \subseteq R^H(\mathsf{c}(\mathbf{y}) \cup \{z^{\ell_1}, \ldots, z^{\ell_k}\})$. Since $(\mathbf{y}, \mathbf{z})$ is a weakly observable offer process, $(\mathbf{y}, z^{\ell_1}, \ldots, z^{\ell_k})$ is a weakly observable process. Let $\mathbf{w} = (w^1, \ldots, w^M)$ be the offer process for $\mathsf{h}(z^{\ell_k})$ constructed by taking the subsequence of $(\mathbf{y}, z^{\ell_1}, \ldots, z^{\ell_k})$ that consists of contracts with $\mathsf{h}(z^{\ell_k})$; since $(\mathbf{y}, z^{\ell_1}, \ldots, z^{\ell_k})$ is weakly observable, we have that $\mathbf{w}$ is observable. As the choice function of $C^{\mathsf{h}(z^{\ell_k})}$ is observably substitutable, we have that $R^{\mathsf{h}(z^{\ell_k})}(\{w^1, \ldots, w^{M-1}\}) \subseteq R^{\mathsf{h}(z^{\ell_k})}(\mathbf{w})$. Moreover, since $[\{w^1, \ldots, w^{M-1}\}]_h = [\mathsf{c}(\mathbf{w})]_h$ for all $h \in H \smallsetminus \{\mathsf{h}(z^{\ell_k})\}$, we have that $R^h(\mathsf{c}(\mathbf{y}) \cup \{z^{\ell_1}, \ldots, z^{\ell_{k-1}}\}) = R^h(\mathsf{c}(\mathbf{y}) \cup \{z^{\ell_1}, \ldots, z^{\ell_k}\})$ for all $h \in H \smallsetminus \{\mathsf{h}(z^{\ell_k})\}$. Combining these last two observations, we obtain that

$$R^H(\mathsf{c}(\mathbf{y}) \cup \{z^{\ell_1}, \ldots, z^{\ell_{k-1}}\}) \subseteq R^H(\mathsf{c}(\mathbf{y}) \cup \{z^{\ell_1}, \ldots, z^{\ell_k}\});$$

combining this with our inductive hypothesis that $R^H(\mathsf{c}(\mathbf{y})) \subseteq R^H(\mathsf{c}(\mathbf{y}) \cup \{z^{\ell_1}, \ldots, z^{\ell_{k-1}}\})$, we obtain the desired result that $R^H(\mathsf{c}(\mathbf{y})) \subseteq R^H(\mathsf{c}(\mathbf{y}) \cup \{z^{\ell_1}, \ldots, z^{\ell_k}\})$.

We now show the result for observable size monotonoicity: Suppose that we have already established that $|C^H(\mathsf{c}(\mathbf{y}))| \leq |C^H(\mathsf{c}(\mathbf{y}) \cup \{z^{\ell_1}, \ldots, z^{\ell_{k-1}}\})|$; we will show that $|C^H(\mathsf{c}(\mathbf{y}))| \leq |C^H(\mathsf{c}(\mathbf{y}) \cup \{z^{\ell_1}, \ldots, z^{\ell_k}\})|$. Since $(\mathbf{y}, \mathbf{z})$ is a weakly observable offer process, $(\mathbf{y}, z^{\ell_1}, \ldots, z^{\ell_k})$ is a weakly observable process. Let $\mathbf{w} = (w^1, \ldots, w^M)$ be the offer process for $\mathsf{h}(z^{\ell_k})$ constructed by taking the subsequence of $(\mathbf{y}, z^{\ell_1}, \ldots, z^{\ell_k})$ that consists of contracts with $\mathsf{h}(z^{\ell_k})$; since $(\mathbf{y}, z^{\ell_1}, \ldots, z^{\ell_k})$ is weakly observable, we have that $\mathbf{w}$ is observable. As the choice function of $C^{\mathsf{h}(z^{\ell_k})}$ is observably size monotonic, we have that $|C^{\mathsf{h}(z^{\ell_k})}(\{w^1, \ldots, w^{M-1}\})| \leq |C^{\mathsf{h}(z^{\ell_k})}(\mathbf{w})|$. Moreover, since $[\{w^1, \ldots, w^{M-1}\}]_h = [\mathsf{c}(\mathbf{w})]_h$ for all $h \in H \smallsetminus \{\mathsf{h}(z^{\ell_k})\}$, we have that $C^h(\mathsf{c}(\mathbf{y}) \cup \{z^{\ell_1}, \ldots, z^{\ell_{k-1}}\}) = C^h(\mathsf{c}(\mathbf{y}) \cup \{z^{\ell_1}, \ldots, z^{\ell_k}\})$ for all $h \in H \smallsetminus \{\mathsf{h}(z^{\ell_k})\}$. Combining these last two observations, we obtain that

$$|C^H(\mathsf{c}(\mathbf{y}) \cup \{z^{\ell_1}, \ldots, z^{\ell_{k-1}}\})| \leq |C^H(\mathsf{c}(\mathbf{y}) \cup \{z^{\ell_1}, \ldots, z^{\ell_k}\})|;$$

combining this with our inductive hypothesis that $|C^H(\mathsf{c}(\mathbf{y}))| \leq |C^H(\mathsf{c}(\mathbf{y}) \cup \{z^{\ell_1}, \ldots, z^{\ell_{k-1}}\})|$, we obtain the desired result that $|C^H(\mathsf{c}(\mathbf{y}))| \leq |C^H(\mathsf{c}(\mathbf{y}) \cup \{z^{\ell_1}, \ldots, z^{\ell_k}\})|$. □

B Proof of Theorem 4

Theorem 4. *If the choice function of every hospital is observably substitutable, observably size monotonic, and not manipulable via contractual terms, then the cumulative offer mechanism is stable and strategy-proof.*

Proof. By Proposition 3,[3] which does not rely on Theorem 4, observable substitutability of each hospital's choice function is sufficient for the cumulative offer mechanism to produce a stable outcome. Hence, we only need to establish that if each hospital's choice function is observably substitutable, observably size monotonic, and non-manipulable via contractual terms (absent other hospitals), then the cumulative offer mechanism is strategy-proof.[4]

Consider a profile of choice functions $C = (C^h)_{h\in H}$ such that, for each $h \in H$, C^h is observably substitutable and observably size monotonic. Suppose that the cumulative offer mechanism is not strategy-proof, so that there exists a preference profile $\succ$, a doctor $\hat{d}$, and a preference relation $\hat{\succ}_{\hat{d}}$ such that $\mathcal{C}(\hat{\succ}_{\hat{d}}, \succ_{D\setminus\{\hat{d}\}}) \succ_{\hat{d}} \mathcal{C}(\succ)$. Let $\hat{x} \in [\mathcal{C}(\hat{\succ}_{\hat{d}}, \succ_{D\setminus\{\hat{d}\}})]_{\hat{d}}$ be the contract that $\hat{d}$ obtains under $\hat{\succ} \equiv (\hat{\succ}_{\hat{d}}, \succ_{D\setminus\{\hat{d}\}})$ and let $\hat{h} \equiv \mathsf{h}(\hat{x})$. We will show that $C^{\hat{h}}$ is manipulable via contractual terms.

As a first step of the proof, we introduce several assumptions about the preference profiles $\succ$ and $\hat{\succ}$ and show that these assumptions are without loss of generality. Let $\mathbf{x} = (x^1, \dots, x^K)$ be a complete offer process with respect to $\succ$ and let $\hat{\mathbf{x}}$ be a complete offer process with respect to $\hat{\succ}$. Note that $\mathcal{C}(\succ) = C^H(\mathsf{c}(\mathbf{x}))$ and $\mathcal{C}(\hat{\succ}) = C^H(\mathsf{c}(\hat{\mathbf{x}}))$ by Lemma 1. By Proposition 4, it is without loss of generality to assume that (1) all contracts in $X \setminus (\mathsf{c}(\mathbf{x}) \cup \mathsf{c}(\hat{\mathbf{x}}))$ are unacceptable to the associated doctors under $\succ$ and $\hat{\succ}$, and (2) $\hat{x}$ is the lowest ranked acceptable contract under $\succ_{\hat{d}}$ and $\hat{\succ}_{\hat{d}}$.[5] Finally, note that by Lemma 1 we can assume without loss of generality that $\mathbf{x}$ is the offer process

[3] Unless mentioned otherwise, all references to definitions, examples, sections, and results are with respect to Hatfield et al. (2017).

[4] As we show in Appendix C.1, irrelevance of rejected contracts is necessary for the stability of the cumulative offer mechanism. Our proof that the cumulative offer mechanism is strategy-proof when each hospital's choice function is observably substitutable, observably size monotonic, and not manipulable via contractual terms does not depend on the irrelevance of rejected contracts condition.

[5] For all doctors $d \in D \setminus \{\hat{d}\}$, Proposition 4 implies that truncating $\succ_d = \hat{\succ}_d$ below d's least preferred contract in $\mathsf{c}(\mathbf{x}) \cup \mathsf{c}(\hat{\mathbf{x}})$ can neither improve nor worsen d's assignment under $\succ$ and $\hat{\succ}$. Similarly, for doctor $\hat{d}$, Proposition 4 implies that truncating $\hat{\succ}_{\hat{d}}$ below $\hat{x}$ can neither improve nor worsen $\hat{d}$'s assignment when others submit preferences according to $\succ_{D\setminus\{\hat{d}\}}$; in either case, $\hat{d}$ obtains $\hat{x}$. Finally, since $\hat{x} \succ_{\hat{d}} \mathcal{C}(\succ)$, Proposition 4 implies that $\hat{d}$ must remain unassigned when he truncates $\succ_{\hat{d}}$ below $\hat{x}$ and others submit preferences according to $\succ_{D\setminus\{\hat{d}\}}$.

with respect to an ordering $\vdash$ such that, for all $x \in X \setminus X_{\hat{d}}$ and all $y \in X_{\hat{d}}$, $x \vdash y$. This implies that the cumulative offer process corresponding to $\mathbf{x}$ ends with the rejection of $\hat{x}$, i.e., that $\hat{x}$ is the unique element of $R^H(\{x^1, \ldots, x^K\}) \setminus R^H(\{x^1, \ldots, x^{K-1}\})$.[6]

Now set $\succ' \equiv \succ^{X_{\hat{h}}}$ and $\hat{\succ}' \equiv \hat{\succ}^{X_{\hat{h}}}$. Let $\mathbf{x}'$ be a complete offer process with respect to $\succ'$, and let $\hat{\mathbf{x}}'$ be a complete offer process with respect to $\hat{\succ}'$. By Lemma 1, we must have that $\mathcal{C}(\succ') = C^H(\mathsf{c}(\mathbf{x}'))$ and $\mathcal{C}(\hat{\succ}') = C^H(\mathsf{c}(\hat{\mathbf{x}}'))$. To show that the choice function of $\hat{h}$ is manipulable via contractual terms, it is thus sufficient to establish that $\mathcal{C}_{\hat{d}}(\succ') = \varnothing$, i.e., $\hat{d}$ does not obtain a contract under $\succ'$, and that $\hat{x} \notin R^{\hat{h}}(\mathsf{c}(\hat{\mathbf{x}}'))$, i.e., $\hat{d}$ obtains an acceptable contract under $\succ'$; Claim 1 (which is easy) shows the former fact while Claim 2 (which is more difficult) shows the latter fact.

Claim 1. *Doctor $\hat{d}$ does not obtain a contract under $\succ'$, i.e., $\mathcal{C}_{\hat{d}}(\succ') = \varnothing$.*

Proof. Let $y^1, \ldots, y^M \in X_{\hat{h}}$ be contracts such that $(y^1, \ldots, y^M)$ is the subsequence of $\mathbf{x} = (x^1, \ldots, x^K)$ that consists of all and only contracts with $\hat{h}$. Let $\bar{m} = \min\{m : \hat{x} \in R^{\hat{h}}(\{y^1, \ldots, y^m\})\}$.[7] Now consider an ordering $\vdash$ such that $y^m \vdash y^{m+1}$, for all $m \in \{1, \ldots, M-1\}$, and $y^M \vdash y$, for all $y \in X \setminus \{y^1, \ldots, y^M\}$. By the construction of $\succ'$, the first $\bar{m}$ contracts in the complete offer process with respect to $\succ'$ and $\vdash$ are $y^1, \ldots, y^{\bar{m}}$. Given that $C^{\hat{h}}$ is observably substitutable and $\hat{x} \in R^{\hat{h}}(\{y^1, \ldots, y^{\bar{m}}\})$, $\hat{x}$ must be rejected by $\hat{h}$ when $\hat{h}$ has access to all contracts in the complete offer process with respect to $\succ'$ and $\vdash$. By Lemma 1, this implies $\hat{x} \in R^{\hat{h}}(\mathsf{c}(\mathbf{x}'))$. Since $\hat{x}$ is the least-preferred acceptable contract for doctor $\hat{d}$ under $\succ'$, this implies that $\mathcal{C}_{\hat{d}}(\succ') = \varnothing$. □

The remainder of the proof of Theorem 4 is devoted to showing Claim 2.

Claim 2. *The contract $\hat{x}$ is not rejected under $\hat{\succ}'$, i.e., $\hat{x} \notin R^{\hat{h}}(\mathsf{c}(\hat{\mathbf{x}}'))$.*

Before proving Claim 2, we introduce some auxiliary concepts that are useful in the argument. Consider an arbitrary preference profile $\tilde{\succ}$ and an arbitrary offer process $\mathbf{z}$. A *pre-run rejection chain at $\mathbf{z}$ under $\tilde{\succ}$* is a non-empty sequence of contracts $\mathbf{y} = (y^1, \ldots, y^N)$ such that the following conditions are satisfied:

1. For doctor $d^1 \equiv \mathsf{d}(y^1)$,

[6] To see this, note that, as each hospital's choice function is observably size monotonic, at most one contract is rejected in each step of the cumulative offer process with respect to $\succ$ and $\vdash$. Since $\hat{x}$ is the least preferred contract with respect to $\succ_{\hat{d}}$, the cumulative offer process with respect to $\succ$ and $\vdash$ ends as soon as $\hat{x}$ is rejected.

[7] Such an integer $\bar{m}$ has to exist as $\hat{x} \in R^{\hat{h}}(\mathsf{c}(\mathbf{x}))$ and $[\mathsf{c}(\mathbf{x})]_{\hat{h}} = \{y^1, \ldots, y^M\}$.

(a) $d^1 \in \mathsf{d}(C^H(\mathsf{c}(\mathbf{z})))$,

(b) $d^1 \notin \mathsf{d}(C^{\mathsf{h}(y^1)}(\mathsf{c}(\mathbf{z})))$, and

(c) for all $y \in [(X_{\mathsf{h}(y^1)} \cap X_{d^1}) \cup \{\emptyset\}] \smallsetminus \mathsf{c}(\mathbf{z})$, $y^1 \tilde{\succ}_{d^1} y$.

2. For all $n \in \{2, \ldots, N\}$, for doctor $d^n \equiv \mathsf{d}(y^n)$,

(a) $d^n \neq d^1$,

(b) $d^n \notin \mathsf{d}(C^H(\mathsf{c}(\mathbf{z}) \cup \{y^1, \ldots, y^{n-1}\}))$,

(c) $d^n \in \mathsf{d}(R^H(\mathsf{c}(\mathbf{z}) \cup \{y^1, \ldots, y^{n-1}\}) \smallsetminus R^H(\mathsf{c}(\mathbf{z}) \cup \{y^1, \ldots, y^{n-2}\}))$, and

(d) $y^n \tilde{\succ}_{d^n} y$ for all $y \in (X_{d^n} \cup \{\emptyset\}) \smallsetminus (\mathsf{c}(\mathbf{z}) \cup \{y^1, \ldots, y^{n-1}\})$.

3. $d^1 \in \mathsf{d}(R^H(\mathsf{c}(\mathbf{z}) \cup \mathsf{c}(\mathbf{y})) \smallsetminus R^H(\mathsf{c}(\mathbf{z}) \cup \{y^1, \ldots, y^{N-1})\}))$.

Essentially, a pre-run rejection chain is an offer process whose first element is a contract with a doctor who is employed by some hospital at $\mathbf{z}$ (part (a) of Condition 1) and with a hospital different from the hospital that currently employs that doctor (part (b) of Condition 1); moreover, that contract is the doctor's favorite contract at that hospital that has not yet been proposed (part (c) of Condition 1). In each subsequent step n of the pre-run rejection chain, a doctor other than d^1 (part (a) of Condition 2) who is not employed by any hospital after $(\mathbf{z}, y^1, \ldots, y^{n-1})$ (part (b) of Condition 2) and, in fact, just had a contract rejected after y^{n-1} was proposed (part (c) of Condition 2) proposes his favorite contract y^n that has not yet been proposed (part (d) of Condition 2). This process continues until the doctor d^1 has a contract rejected (Condition 3).

Pre-run rejection chains will prove useful in determining whether $\hat{x}$, which was not rejected under $\hat{\succ}$, will be rejected under $\hat{\succ}'$, i.e., after we remove hospitals other than $\hat{h}$ from the economy. Pre-running rejection chains after the complete offer process with respect to $\hat{\succ}_{D \smallsetminus \{\hat{d}\}}$ (where each chain begins with an element of $\mathsf{c}(\hat{\mathbf{x}}') \smallsetminus \mathsf{c}(\hat{\mathbf{x}})$) allows us to show that the additional proposals to $\hat{h}$ under $\hat{\succ}'$ will not induce $\hat{h}$ to reject $\hat{x}$.

A *generalized pre-run rejection chain at* $\mathbf{z}$ *under* $\tilde{\succ}$ is an offer process $\mathbf{y} = (\mathbf{y}^1, \ldots, \mathbf{y}^L)$ such that for each $\ell \in \{1, \ldots, L\}$, $\mathbf{y}^\ell$ is a pre-run rejection chain at $(\mathbf{z}, \mathbf{y}^1, \ldots, \mathbf{y}^{\ell-1})$ under $\tilde{\succ}$. An offer process $\mathbf{w}$ can be *obtained from* $\mathbf{z}$ *by pre-running rejection chains under* $\tilde{\succ}$ if $\mathbf{w} = (\mathbf{z}, \mathbf{y})$ for some generalized pre-run rejection chain $\mathbf{y}$ at $\mathbf{z}$ under $\tilde{\succ}$.

If $\mathbf{z}$ is weakly observable and weakly compatible with $\tilde{\succ}$, it follows immediately from the definition of a generalized pre-run rejection chain $\mathbf{y}$ at $\mathbf{z}$ that $(\mathbf{z}, \mathbf{y})$ is weakly observable and weakly compatible with $\tilde{\succ}$. We state this fact as a lemma here.

Lemma B.1. *If $\mathbf{z}$ is weakly observable and weakly compatible with $\tilde{\succ}$ and $\mathbf{y}$ is a generalized pre-run rejection chain at $\mathbf{z}$ under $\tilde{\succ}$, then $(\mathbf{z}, \mathbf{y})$ is weakly observable and weakly compatible with $\tilde{\succ}$.*

Proof of Claim 2. Let $\check{\mathbf{x}}$ be a complete offer process with respect to $\succ_{D \setminus \{\hat{d}\}}$. Note that $\mathsf{c}(\check{\mathbf{x}}) \subseteq (\mathsf{c}(\mathbf{x}) \cap \mathsf{c}(\hat{\mathbf{x}})) \setminus X_{\hat{d}}$. This follows from Lemma 1 since any complete offer process for $\succ$ or $\hat{\succ}$ has to contain all contracts that are contained in a complete offer process with respect to an ordering $\vdash$ such that, for all $y \in X \setminus X_{\hat{d}}$ and all $x \in X_{\hat{d}}$, $y \vdash x$. The key step of our proof lies in the construction of an offer process that can be obtained from $\check{\mathbf{x}}$ by constructing a generalized pre-run rejection chain from $\check{\mathbf{x}}$ that satisfies four specific properties.

Claim 3. *There exists an offer process $\mathbf{y}^*$ such that*

1. *$\mathbf{y}^*$ can be obtained from $\check{\mathbf{x}}$ by pre-running rejection chains under $\hat{\succ}$,*
2. *$\mathsf{c}(\mathbf{y}^*) \subseteq X \setminus X_{\hat{d}}$,*
3. *$\mathsf{c}(\hat{\mathbf{x}}') \setminus \mathsf{c}(\hat{\mathbf{x}}) \subseteq \mathsf{c}(\mathbf{y}^*)$, and*
4. *$R^H(\mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\mathbf{y}^*)) \setminus R^H(\mathsf{c}(\hat{\mathbf{x}})) \subseteq R^H(\mathsf{c}(\mathbf{y}^*))$.*

Condition 1 ensures in particular that $\mathbf{y}^*$ is weakly observable (by Lemma B.1). Condition 2 requires that no contract in $\mathsf{c}(\mathbf{y}^*)$ names doctor $\hat{d}$. Condition 3 ensures that $\mathsf{c}(\mathbf{y}^*)$ contains all the contracts that are proposed in the cumulative offer process for $\hat{\succ}'$ that are *not* in the cumulative offer process for $\hat{\succ}$. Condition 4 ensures that all the contracts that are rejected when contracts in $\mathsf{c}(\mathbf{y}^*)$ become available to hospitals in addition to contracts in $\mathsf{c}(\hat{\mathbf{x}})$ are contracts that are also rejected when hospitals have access to the contracts in $\mathsf{c}(\mathbf{y}^*)$.

Before proceeding to the proof of Claim 3, we argue why it implies Claim 2, i.e., $\hat{x} \notin R^H(\mathsf{c}(\hat{\mathbf{x}}'))$. We take an offer process $\mathbf{y}^*$ that satisfies the four conditions of Claim 3 and proceed in two steps:

Step 1: $\hat{x} \notin R^H(\mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\mathbf{y}^*))$. To show that $\hat{x} \notin R^H(\mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\mathbf{y}^*))$, note that, by the fourth condition of Claim 3, $R^H(\mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\mathbf{y}^*)) \setminus R^H(\mathsf{c}(\hat{\mathbf{x}})) \subseteq R^H(\mathsf{c}(\mathbf{y}^*))$.

Since $\mathsf{c}(\mathbf{y}^*) \subseteq X \smallsetminus X_{\hat{d}}$ by the second condition of Claim 3, we must have $R^H(\mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\mathbf{y}^*)) \smallsetminus R^H(\mathsf{c}(\hat{\mathbf{x}})) \subseteq X \smallsetminus X_{\hat{d}}$; combining this with the fact that $\hat{x} \in C^H(\mathsf{c}(\hat{\mathbf{x}}))$ and thus $\hat{x} \notin R^H(\mathsf{c}(\hat{\mathbf{x}}))$, it then follows that $\hat{x} \notin R^H(\mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\mathbf{y}^*))$.

Step 2: $R^H(\mathsf{c}(\hat{\mathbf{x}}')) \subseteq R^H(\mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\mathbf{y}^*))$. Since $\mathbf{y}^*$ can be obtained from $\check{\mathbf{x}}$ by constructing a generalized pre-run rejection chain at $\check{\mathbf{x}}$ by the first condition of Claim 3, Lemma B.1 implies that $\mathbf{y}^*$ is weakly observable and weakly compatible with $\hat{\succ}$. Since $\hat{\mathbf{x}}$ and $\hat{\mathbf{x}}'$ are also both weakly observable and weakly compatible with $\hat{\succ}$, $(\hat{\mathbf{x}}', \hat{\mathbf{x}}, \mathbf{y}^*)$ is weakly observable by Lemma A.1. Lemma A.2 then implies that $R^H(\mathsf{c}(\hat{\mathbf{x}}')) \subseteq R^H(\mathsf{c}(\hat{\mathbf{x}}') \cup \mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\mathbf{y}^*))$. By the third condition of Claim 3, $\mathsf{c}(\hat{\mathbf{x}}') \smallsetminus \mathsf{c}(\hat{\mathbf{x}}) \subseteq \mathsf{c}(\mathbf{y}^*)$, and hence $\mathsf{c}(\hat{\mathbf{x}}') \cup \mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\mathbf{y}^*) = \mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\mathbf{y}^*)$. Combining these last two observations yields $R^H(\mathsf{c}(\hat{\mathbf{x}}')) \subseteq R^H(\mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\mathbf{y}^*))$.

Combining the results of Steps 1 and 2 yields $\hat{x} \notin R^H(\mathsf{c}(\hat{\mathbf{x}}'))$.

We now proceed with the proof of Claim 3.

Proof of Claim 3. In the proof of Claim 3, we will iteratively construct an offer process $\mathbf{y}^*$ that satisfies Conditions 1–4 of Claim 3 from $\check{\mathbf{x}}$. A key step of the construction involves extending a given generalized pre-run rejection chain at $\check{\mathbf{x}}$. To do this, we first show that the set of doctors employed is unchanged when we pre-run rejection chains.

Claim 4. *Suppose that* $\mathbf{z}$ *is weakly observable and weakly compatible with some preference profile* $\tilde{\succ}$ *and that the offer process* $\mathbf{w}$ *can be obtained from* $\mathbf{z}$ *by pre-running rejection chains under* $\tilde{\succ}$*. Then* $\mathsf{d}(C^H(\mathsf{c}(\mathbf{w}))) = \mathsf{d}(C^H(\mathsf{c}(\mathbf{z})))$.

Proof. As $\mathbf{w}$ is an offer process that can be obtained from $\mathbf{z}$ by pre-running rejection chains under $\tilde{\succ}$, we have that $\mathbf{w} = (\mathbf{z}, \mathbf{y})$ for some generalized pre-run rejection chain $\mathbf{y}$ at $\mathbf{z}$ under $\tilde{\succ}$. Thus, $\mathbf{y}$ is an offer process $\mathbf{y} = (\mathbf{y}^1, \ldots, \mathbf{y}^L)$ such that for each $\ell \in \{1, \ldots, L\}$, $\mathbf{y}^\ell$ is a pre-run rejection chain at $(\mathbf{z}, \mathbf{y}^1, \ldots, \mathbf{y}^{\ell-1})$ under $\tilde{\succ}$. It suffices to show that $\mathsf{d}(C^H(\mathsf{c}(\mathbf{z}, \mathbf{y}^1, \ldots, \mathbf{y}^\ell))) = \mathsf{d}(C^H(\mathsf{c}(\mathbf{z}, \mathbf{y}^1, \ldots, \mathbf{y}^{\ell-1})))$ for all ℓ.

Let $\mathbf{y}^\ell = (y^1, \ldots, y^N)$ and $\tilde{\mathbf{z}} = (\mathbf{z}, \mathbf{y}^1, \ldots, \mathbf{y}^{\ell-1})$. Since $(\mathbf{y}^1, \ldots, \mathbf{y}^\ell)$ is a generalized pre-run rejection chain under $\tilde{\succ}$ at $\mathbf{z}$ and $\mathbf{z}$ is weakly observable and weakly compatible with $\tilde{\succ}$, Lemma B.1 implies that $(\tilde{\mathbf{z}}, y^1, \ldots, y^N)$ is weakly observable. Thus, by Lemma A.2, we have that

$$|R^H(\mathsf{c}(\tilde{\mathbf{z}}) \cup \{y^1, \ldots, y^n\}) \smallsetminus R^H(\mathsf{c}(\tilde{\mathbf{z}}) \cup \{y^1, \ldots, y^{n-1}\})| \leq 1 \qquad (1)$$

for all $n \in \{1, \dots N\}$.

If $N = 1$, Condition 3 of the definition of a pre-run rejection chain implies that there exists some contract w such that $d^1 \equiv \mathsf{d}(w)$ and $w \in R^H(\mathsf{c}(\tilde{\mathbf{z}}) \cup \{y^1\}) \smallsetminus R^H(\mathsf{c}(\tilde{\mathbf{z}}))$. Combining this with (1) yields that $\{w\} = R^H(\mathsf{c}(\tilde{\mathbf{z}}) \cup \{y^1\}) \smallsetminus R^H(\mathsf{c}(\tilde{\mathbf{z}}))$. If $w = y^1$, then we have that $C^H(\mathsf{c}(\tilde{\mathbf{z}}) \cup \{y^1\}) = C^H(\mathsf{c}(\tilde{\mathbf{z}}))$ and so the desired result $\mathsf{d}(C^H(\mathsf{c}(\tilde{\mathbf{z}}) \cup \{y^1\})) = \mathsf{d}(C^H(\mathsf{c}(\tilde{\mathbf{z}})))$ follows; if $w \neq y^1$, then $C^H(\mathsf{c}(\tilde{\mathbf{z}}) \cup \{y^1\}) = (C^H(\mathsf{c}(\tilde{\mathbf{z}})) \smallsetminus \{w\}) \cup \{y^1\}$, and so the desired result $\mathsf{d}(C^H(\mathsf{c}(\tilde{\mathbf{z}}) \cup \{y^1\})) = \mathsf{d}(C^H(\mathsf{c}(\tilde{\mathbf{z}})))$ follows as $\mathsf{d}(y^1) = \mathsf{d}(w)$.

If $N > 1$, it is enough to show that:

1. The doctor associated with y^2 is the unique doctor rejected after y^1 is proposed:

$$\mathsf{d}(C^H(\mathsf{c}(\tilde{\mathbf{z}}) \cup \{y^1\})) = \mathsf{d}(C^H(\mathsf{c}(\tilde{\mathbf{z}}))) \smallsetminus \{\mathsf{d}(y^2)\}.$$

2. For $n \in \{2, \dots, N-1\}$, the doctor associated with y^n is now chosen and the doctor associated with y^{n+1} is the unique doctor rejected after y^n is proposed:

$$\mathsf{d}(C^H(\mathsf{c}(\tilde{\mathbf{z}}) \cup \{y^1, \dots, y^n\})) = \Big(\mathsf{d}(C^H(\mathsf{c}(\tilde{\mathbf{z}}) \cup \{y^1, \dots, y^{n-1}\})) \cup \{\mathsf{d}(y^n)\}\Big) \smallsetminus \{\mathsf{d}(y^{n+1})\}.$$

3. The doctor associated with y^N is now chosen after y^N is proposed:

$$\mathsf{d}(C^H(\mathsf{c}(\tilde{\mathbf{z}}) \cup \{y^1, \dots, y^N\})) = \mathsf{d}(C^H(\mathsf{c}(\tilde{\mathbf{z}}) \cup \{y^1, \dots, y^{N-1}\})) \cup \{\mathsf{d}(y^N)\}.$$

To see the first point, note that Condition 2c of the definition of a pre-run rejection chain implies there exists some contract w such that $d^2 = \mathsf{d}(w)$ and $w \in R^H(\mathsf{c}(\tilde{\mathbf{z}}) \cup \{y^1\}) \smallsetminus R^H(\mathsf{c}(\tilde{\mathbf{z}}))$. Combining this with (1) yields that $\{w\} = R^H(\mathsf{c}(\tilde{\mathbf{z}}) \cup \{y^1\}) \smallsetminus R^H(\mathsf{c}(\tilde{\mathbf{z}}))$. Condition 2b of the definition of a pre-run rejection chain implies that $d^2 \notin \mathsf{d}(C^H(\mathsf{c}(\tilde{\mathbf{z}}) \cup \{y^1\}))$. Combining these last two results yields the desired result $\mathsf{d}(C^H(\mathsf{c}(\tilde{\mathbf{z}}) \cup \{y^1\})) = \mathsf{d}(C^H(\mathsf{c}(\tilde{\mathbf{z}}))) \smallsetminus \{\mathsf{d}(y^2)\}$. However, note also, that as $d^1 \equiv \mathsf{d}(y^1) \in \mathsf{d}(C^H(\mathsf{c}(\tilde{\mathbf{z}})))$ and $d^1 \notin \mathsf{d}(R^H(\mathsf{c}(\tilde{\mathbf{z}}) \cup \{y^1\}) \smallsetminus R^H(\mathsf{c}(\tilde{\mathbf{z}})))$, we have that

$$[C^H(\mathsf{c}(\tilde{\mathbf{z}}) \cup \{y^1\})]_{d_1} = 2. \tag{2}$$

To see the second point, note that Condition 2c of the definition of a pre-run rejection chain implies that there exists some contract w such that $\mathsf{d}(y^{n+1}) = \mathsf{d}(w)$

and

$$w \in R^H(\mathsf{c}(\tilde{\mathbf{z}}) \cup \{y^1, \ldots, y^n\}) \smallsetminus R^H(\mathsf{c}(\tilde{\mathbf{z}}) \cup \{y^1, \ldots, y^{n-1}\}).$$

Combining this with (1) yields that

$$\{w\} = R^H(\mathsf{c}(\tilde{\mathbf{z}}) \cup \{y^1, \ldots, y^n\}) \smallsetminus R^H(\mathsf{c}(\tilde{\mathbf{z}}) \cup \{y^1, \ldots, y^{n-1}\})$$

and so

$$C^H(\mathsf{c}(\tilde{\mathbf{z}}) \cup \{y^1, \ldots, y^n\}) = (C^H(\mathsf{c}(\tilde{\mathbf{z}}) \cup \{y^1, \ldots, y^{n-1}\}) \cup \{y^n\}) \smallsetminus \{w\}.$$

The desired result thus follows as $\mathsf{d}(y^{n+1}) \notin \mathsf{d}(C^H(\mathsf{c}(\tilde{\mathbf{z}}) \cup \{y^1, \ldots, y^n\}))$ by Condition 2b of the definition of a pre-run rejection chain. However, note also, that as $d^1 \neq \mathsf{d}(y^{n+1})$ by Condition 2a of the defintion of a pre-run rejection chain, we have that

$$[C^H(\mathsf{c}(\tilde{\mathbf{z}}) \cup \{y^1, \ldots, y^n\})]_{d_1} = [C^H(\mathsf{c}(\tilde{\mathbf{z}}) \cup \{y^1, \ldots, y^{n-1}\})]_{d_1}. \tag{3}$$

To see the third point, note that Condition 3 of the definition of a pre-run rejection chain implies that there exists some contract w such that $d^1 = \mathsf{d}(w)$ and

$$w \in R^H(\mathsf{c}(\tilde{\mathbf{z}}) \cup \{y^1, \ldots, y^N\}) \smallsetminus R^H(\mathsf{c}(\tilde{\mathbf{z}}) \cup \{y^1, \ldots, y^{N-1}\}).$$

Combining this with (1) yields that

$$\{w\} = R^H(\mathsf{c}(\tilde{\mathbf{z}}) \cup \{y^1, \ldots, y^N\}) \smallsetminus R^H(\mathsf{c}(\tilde{\mathbf{z}}) \cup \{y^1, \ldots, y^{N-1}\}).$$

Thus,

$$C^H(\mathsf{c}(\tilde{\mathbf{z}}) \cup \{y^1, \ldots, y^N\}) = (C^H(\mathsf{c}(\tilde{\mathbf{z}}) \cup \{y^1, \ldots, y^{N-1}\}) \cup \{y^N\}) \smallsetminus \{w\}$$

and so the desired result $\mathsf{d}(C^H(\mathsf{c}(\tilde{\mathbf{z}}) \cup \{y^1, \ldots, y^N\})) = \mathsf{d}(C^H(\mathsf{c}(\tilde{\mathbf{z}}) \cup \{y^1, \ldots, y^{N-1}\})) \cup \{\mathsf{d}(y^N)\}$ follows as

- $\mathsf{d}(y^N) \neq \mathsf{d}(w)$ by Condition 2a of the definition of a pre-run rejection chain, so that $\mathsf{d}(y^N) \in \mathsf{d}(C^H(\mathsf{c}(\tilde{\mathbf{z}}) \cup \{y^1, \ldots, y^N\}))$, and
- (2) and (3) together imply that $|C^H(\mathsf{c}(\tilde{\mathbf{z}}) \cup \{y^1, \ldots, y^{N-1}\})| = 2$, so that there exists some contract $\hat{w} \in (C^H(\mathsf{c}(\tilde{\mathbf{z}}) \cup \{y^1, \ldots, y^{N-1}\}) \cup \{y^N\}) \smallsetminus \{w\}$ such that

$\mathsf{d}(\hat{w}) = d^1$.

This completes the proof of Claim 4. $\square$

The next claim provides a condition that assures us, for an offer process $\tilde{\mathbf{z}}$ obtained from $\check{\mathbf{x}}$ by pre-running rejection chains, another pre-run rejection chain exists using only contracts in $\mathsf{c}(\mathbf{x}) \smallsetminus X_{\hat{d}}$.

Claim 5. *Let $\tilde{\mathbf{z}}$ be an offer process that can be obtained from $\check{\mathbf{x}}$ by pre-running rejection chains under $\hat{\succ}$ such that $\mathsf{c}(\tilde{\mathbf{z}}) \subseteq \mathsf{c}(\mathbf{x}) \smallsetminus X_{\hat{d}}$. If there exists*

1. *a doctor $\bar{d} \in D \smallsetminus \{\hat{d}\}$,*

2. *a hospital $\bar{h}$, and*

3. *a contract $y \in X_{\bar{h}} \cap X_{\bar{d}} \cap (\mathsf{c}(\mathbf{x}) \smallsetminus \mathsf{c}(\tilde{\mathbf{z}}))$ such that $\bar{d} \notin \mathsf{d}(C^{\bar{h}}(\mathsf{c}(\tilde{\mathbf{z}})))$ and $y \succ_{\bar{d}} \emptyset$,*

then there exists a pre-run rejection chain $\tilde{\mathbf{y}}$ at $\tilde{\mathbf{z}}$ such that $\mathsf{c}(\tilde{\mathbf{y}}) \subseteq \mathsf{c}(\mathbf{x}) \smallsetminus X_{\hat{d}}$.

If, in addition to the hypotheses 1–3 we have that $(\mathsf{c}(\tilde{\mathbf{z}}) \cup \{y\}) \subseteq \mathsf{c}(\hat{\mathbf{x}})$, then $\mathsf{c}(\tilde{\mathbf{y}}) \subseteq \mathsf{c}(\mathbf{x}) \cap \mathsf{c}(\hat{\mathbf{x}})$.

Proof of Claim 5. We proceed in 3 steps:

Step 1: We first show that $\bar{d} \in \mathsf{d}(C^h(\mathsf{c}(\tilde{\mathbf{z}})))$ for some $h \in H \smallsetminus \{\bar{h}\}$.

Since $X_{\bar{h}} \cap X_{\bar{d}} \cap (\mathsf{c}(\mathbf{x}) \smallsetminus \mathsf{c}(\tilde{\mathbf{z}})) \neq \varnothing$ by our third hypothesis, we must have that $X_{\bar{h}} \cap X_{\bar{d}} \cap (\mathsf{c}(\mathbf{x}) \smallsetminus \mathsf{c}(\check{\mathbf{x}})) \neq \varnothing$. Thus, $\bar{d} \in \mathsf{d}(C^H(\mathsf{c}(\check{\mathbf{x}})))$ since if $\bar{d}$ was not employed after $\check{\mathbf{x}}$ then $\bar{d}$ must have already proposed all of his acceptable contracts during the offer process $\check{\mathbf{x}}$. Therefore, Claim 4 implies that $\bar{d} \in \mathsf{d}(C^H(\mathsf{c}(\tilde{\mathbf{z}})))$. Moreover, since $\bar{d} \notin \mathsf{d}(C^{\bar{h}}(\mathsf{c}(\tilde{\mathbf{z}})))$ by our third hypothesis, we have that $\bar{d} \in \mathsf{d}(C^h(\mathsf{c}(\tilde{\mathbf{z}})))$ for some $h \in H \smallsetminus \{\bar{h}\}$.

Step 2: We now construct a pre-run rejection chain $\tilde{\mathbf{y}}$ at $\tilde{\mathbf{z}}$ such that $\mathsf{c}(\tilde{\mathbf{y}}) \subseteq \mathsf{c}(\mathbf{x}) \smallsetminus X_{\hat{d}}$.

Let $\tilde{y}^1$ be the highest-ranked contract in $(X_{\bar{d}} \cap X_{\bar{h}}) \smallsetminus \mathsf{c}(\tilde{\mathbf{z}})$ with respect to $\succ_{\bar{d}}$; such a contract must exist by our third hypothesis. Clearly, $\tilde{y}^1$ satisfies Condition 1 of the definition of a pre-run rejection chain at $\tilde{\mathbf{z}}$ by Step 1. Furthermore, given that $y \in \mathsf{c}(\mathbf{x})$, $\tilde{y}^1 \mathrel{\hat{\succeq}_{\bar{d}}} y$, and that $\mathbf{x}$ is compatible with $\succ$, it has to be the case that $\tilde{y}^1 \in \mathsf{c}(\mathbf{x})$.

Proceeding inductively, suppose that we have defined a sequence of $n \geq 1$ distinct contracts $\tilde{y}^1, \ldots, \tilde{y}^n \in \mathsf{c}(\mathbf{x}) \smallsetminus (\mathsf{c}(\tilde{\mathbf{z}}) \cup X_{\hat{d}})$ such that $(\tilde{y}^1, \ldots, \tilde{y}^n)$ satisfies

Conditions 1 and 2 of the definition of a pre-run rejection chain at $\tilde{\mathbf{z}}$. We will show that either $\bar{d} \in \mathsf{d}(R^H(\mathsf{c}(\tilde{\mathbf{z}}) \cup \{\tilde{y}^1, \ldots, \tilde{y}^n\}) \smallsetminus R^H(\mathsf{c}(\tilde{\mathbf{z}}) \cup \{\tilde{y}^1, \ldots, \tilde{y}^{n-1}\}))$, so that $(\tilde{y}^1, \ldots, \tilde{y}^n)$ is a pre-run rejection chain at $\tilde{\mathbf{z}}$, or there exists a contract $\tilde{y}^{n+1} \in (\mathsf{c}(\mathbf{x}) \smallsetminus (\mathsf{c}(\tilde{\mathbf{z}}) \cup \{\tilde{y}^1, \ldots, \tilde{y}^n\}) \smallsetminus X_{\hat{d}})$ that satisfies Condition 2 of the definition of a pre-run rejection chain at $\tilde{\mathbf{z}}$.

Substep 1: $R^H(\mathsf{c}(\tilde{\mathbf{z}}) \cup \{\tilde{y}^1, \ldots, \tilde{y}^n\}) \smallsetminus R^H(\mathsf{c}(\tilde{\mathbf{z}}) \cup \{\tilde{y}^1, \ldots, \tilde{y}^{n-1}\}) \neq \varnothing$. First, we show that

$$|C^H(\mathsf{c}(\check{\mathbf{x}}))| = |C^H(\mathsf{c}(\mathbf{x}))|. \tag{4}$$

Note first that by Lemma 1 we can think of $\mathbf{x}$ as a combined offer process $\mathbf{x} \equiv (\check{\mathbf{x}}, \hat{z}^1, \ldots, \hat{z}^M)$, where $\hat{z}^1$ is the highest ranked acceptable contract in $X_{\hat{d}}$ with respect to $\succ_{\hat{d}}$ and, for all $m \in \{2, \ldots, M\}$, we have that $\mathsf{d}(\hat{z}^m) \in \mathsf{d}(R^H(\mathsf{c}(\check{\mathbf{x}}) \cup \{\hat{z}^1, \ldots, \hat{z}^{m-1}\}) \smallsetminus R^H(\mathsf{c}(\check{\mathbf{x}}) \cup \{\hat{z}^1, \ldots, \hat{z}^{m-2}\}))$ and $\hat{z}^m$ is the highest ranked contract in $X_{\mathsf{d}(\hat{z}^m)} \smallsetminus (\mathsf{c}(\check{\mathbf{x}}) \cup \{\hat{z}^1, \ldots, \hat{z}^{m-1}\})$ with respect to $\succ_{\mathsf{d}(\hat{z}^m)}$. Moreover, as each hospital's choice function is observably substitutable, we have that, for all m, $C^H(\mathsf{c}(\check{\mathbf{x}}) \cup \{\hat{z}^1, \ldots, \hat{z}^m\}) \subseteq C^H(\mathsf{c}(\check{\mathbf{x}}) \cup \{\hat{z}^1, \ldots, \hat{z}^{m-1}\}) \cup \{\hat{z}^m\}$. In particular, $|C^H(\mathsf{c}(\check{\mathbf{x}}) \cup \{\hat{z}^1, \ldots, \hat{z}^m\})| \leq |C^H(\mathsf{c}(\check{\mathbf{x}}) \cup \{\hat{z}^1, \ldots, \hat{z}^{m-1}\})| + 1$. If there were an m such that $|C^H(\mathsf{c}(\check{\mathbf{x}}) \cup \{\hat{z}^1, \ldots, \hat{z}^m\})| = |C^H(\mathsf{c}(\check{\mathbf{x}}) \cup \{\hat{z}^1, \ldots, \hat{z}^{m-1}\})| + 1$, we would have that $R^H(\mathsf{c}(\check{\mathbf{x}}) \cup \{\hat{z}^1, \ldots, \hat{z}^m\}) \smallsetminus R^H(\mathsf{c}(\check{\mathbf{x}}) \cup \{\hat{z}^1, \ldots, \hat{z}^{m-1}\}) = \varnothing$, contradicting the fact that $\mathbf{x}$ ends with the rejection of contract $\hat{x}$.[8] Hence, we must have $|C^H(\mathsf{c}(\check{\mathbf{x}}) \cup \{\hat{z}^1, \ldots, \hat{z}^m\})| \leq |C^H(\mathsf{c}(\check{\mathbf{x}}) \cup \{\hat{z}^1, \ldots, \hat{z}^{m-1}\})|$ for all m and thus $|C^H(\mathsf{c}(\check{\mathbf{x}}) \cup \{\hat{z}^1, \ldots, \hat{z}^M\})| \leq |C^H(\mathsf{c}(\check{\mathbf{x}}))|$. Since $\mathbf{x} = (\check{\mathbf{x}}, \hat{z}^1, \ldots, \hat{z}^M)$, we obtain $|C^H(\mathsf{c}(\mathbf{x}))| \leq |C^H(\mathsf{c}(\check{\mathbf{x}}))|$. By observable size monotonicity, we must have, for all $h \in H$, $|C^h(\mathsf{c}(\mathbf{x}))| \geq |C^h(\mathsf{c}(\check{\mathbf{x}}))|$. Combining these last two observations, we obtain $|C^h(\mathsf{c}(\mathbf{x}))| = |C^h(\mathsf{c}(\check{\mathbf{x}}))|$.

Second, we show the desired result

$$R^H(\mathsf{c}(\tilde{\mathbf{z}}) \cup \{\tilde{y}^1, \ldots, \tilde{y}^n\}) \smallsetminus R^H(\mathsf{c}(\tilde{\mathbf{z}}) \cup \{\tilde{y}^1, \ldots, \tilde{y}^{n-1}\}) \neq \varnothing.$$

Note that $\check{\mathbf{x}}$, $(\tilde{\mathbf{z}}, \tilde{y}^1, \ldots, \tilde{y}^n)$, and $\mathbf{x}$ are all weakly observable and weakly compatible with $\succ$. Hence, $(\check{\mathbf{x}}, \tilde{\mathbf{z}}, \tilde{y}^1, \ldots, \tilde{y}^n, \mathbf{x})$ is weakly observable by

[8] See Footnote 6.

Lemma A.1. Since we also have that $\mathsf{c}(\check{\mathbf{x}}) \subseteq \mathsf{c}(\tilde{\mathbf{z}}) \cup \{\tilde{y}^1, \ldots, \tilde{y}^n\} \subseteq \mathsf{c}(\mathbf{x})$, observable size monotonicity implies that

$$|C^h(\mathsf{c}(\mathbf{x}))| \geq |C^h(\mathsf{c}(\tilde{\mathbf{z}}) \cup \{\tilde{y}^1, \ldots, \tilde{y}^n\})| \geq |C^h(\mathsf{c}(\tilde{\mathbf{z}}) \cup \{\tilde{y}^1, \ldots, \tilde{y}^{n-1}\})| \geq |C^h(\mathsf{c}(\check{\mathbf{x}}))|$$

for all $h \in H$. Since $|C^h(\mathsf{c}(\mathbf{x}))| = |C^h(\mathsf{c}(\check{\mathbf{x}}))|$ by (4), we must have $|C^h(\mathsf{c}(\tilde{\mathbf{z}}) \cup \{\tilde{y}^1, \ldots, \tilde{y}^n\})| = |C^h(\mathsf{c}(\tilde{\mathbf{z}}) \cup \{\tilde{y}^1, \ldots, \tilde{y}^{n-1}\})|$ and thus $R^H(\mathsf{c}(\tilde{\mathbf{z}}) \cup \{\tilde{y}^1, \ldots, \tilde{y}^n\}) \smallsetminus R^H(\mathsf{c}(\tilde{\mathbf{z}}) \cup \{\tilde{y}^1, \ldots, \tilde{y}^{n-1}\}) \neq \varnothing$.

Substep 2: $|R^H(\mathsf{c}(\tilde{\mathbf{z}}) \cup \{\tilde{y}^1, \ldots, \tilde{y}^n\}) \smallsetminus R^H(\mathsf{c}(\tilde{\mathbf{z}}) \cup \{\tilde{y}^1, \ldots, \tilde{y}^{n-1}\})| \leq 1$. The desired result follow immediately, as Lemma A.2 implies that

$$|R^H(\mathsf{c}(\tilde{\mathbf{z}}) \cup \{\tilde{y}^1, \ldots, \tilde{y}^n\}) \smallsetminus R^H(\mathsf{c}(\tilde{\mathbf{z}}) \cup \{\tilde{y}^1, \ldots, \tilde{y}^{n-1}\})| \leq 1.$$

Combining the results of Substeps 1 and 2, we obtain that there has to be a unique contract $\bar{y}^{n+1} \in R^H(\mathsf{c}(\tilde{\mathbf{z}}) \cup \{\tilde{y}^1, \ldots, \tilde{y}^n\}) \smallsetminus R^H(\mathsf{c}(\tilde{\mathbf{z}}) \cup \{\tilde{y}^1, \ldots, \tilde{y}^{n-1}\})$.

If $\mathsf{d}(\bar{y}^{n+1}) = \bar{d}$, we are done since $(\tilde{y}^1, \ldots, \tilde{y}^n)$ is a pre-run rejection chain at $\tilde{\mathbf{z}}$. If $\mathsf{d}(\bar{y}^{n+1}) \neq \bar{d}$, let $d^{n+1} \equiv \mathsf{d}(\bar{y}^{n+1})$; as the contract $\bar{y}^{n+1} \in R^H(\mathsf{c}(\tilde{\mathbf{z}}) \cup \{\tilde{y}^1, \ldots, \tilde{y}^n\}) \smallsetminus R^H(\mathsf{c}(\tilde{\mathbf{z}}) \cup \{\tilde{y}^1, \ldots, \tilde{y}^{n-1}\})$ and $\mathsf{c}(\check{\mathbf{x}}) \subseteq \mathsf{c}(\tilde{\mathbf{z}}) \cup \{\tilde{y}^1, \ldots, \tilde{y}^{n-1}\}$, Lemma A.2 implies that

$$\bar{y}^{n+1} \notin R^H(\mathsf{c}(\check{\mathbf{x}})). \tag{5}$$

Next, note that $(\tilde{\mathbf{z}}, \tilde{y}^1, \ldots, \tilde{y}^n)$ and $\mathbf{x}$ are both weakly observable and weakly compatible with $\succ$. Hence, $(\tilde{\mathbf{z}}, \tilde{y}^1, \ldots, \tilde{y}^n, \mathbf{x})$ is weakly observable by Lemma A.1. By Lemma A.2, we must have $R^H(\mathsf{c}(\tilde{\mathbf{z}}) \cup \{\tilde{y}^1, \ldots, \tilde{y}^n\}) \subseteq R^H(\mathsf{c}(\tilde{\mathbf{z}}) \cup \{\tilde{y}^1, \ldots, \tilde{y}^n\} \cup \mathsf{c}(\mathbf{x})) = R^H(\mathsf{c}(\mathbf{x}))$, where the equality follows from the facts that $\mathsf{c}(\mathbf{z}) \subseteq \mathsf{c}(\mathbf{x})$ and $\{\tilde{y}^1, \ldots, \tilde{y}^n\} \subseteq \mathsf{c}(\mathbf{x})$. Since $\bar{y}^{n+1} \notin R^H(\mathsf{c}(\check{\mathbf{x}}))$ by (5), there must thus exist an $m \geq 1$ such that $\bar{y}^{n+1}$ is the unique contract in $R^H(\mathsf{c}(\check{\mathbf{x}}) \cup \{\hat{z}^1, \ldots, \hat{z}^m\}) \smallsetminus R^H(\mathsf{c}(\check{\mathbf{x}}) \cup \{\hat{z}^1, \ldots, \hat{z}^{m-1}\})$.

Now, note that the offer process $\mathbf{x}$ can be written as $(\check{\mathbf{x}}, \hat{z}^1, \ldots, \hat{z}^M)$ where, for $m < M$, the unique element $y \in R^H(\mathsf{c}(\check{\mathbf{x}}) \cup \{\hat{z}^1, \ldots, \hat{z}^m\}) \smallsetminus R^H(\mathsf{c}(\check{\mathbf{x}}) \cup \{\hat{z}^1, \ldots, \hat{z}^{m-1}\})$ is not the lowest-ranked acceptable contract for $\mathsf{d}(y)$;[9] moreover,

[9] To see this, let $\mathbf{x} \equiv (\check{\mathbf{x}}, \hat{z}^1, \ldots, \hat{z}^M)$, where $\hat{z}^1$ is the highest ranked acceptable contract in $X_{\hat{d}}$ with respect to $\succ_{\hat{d}}$ and, for all $m \in \{2, \ldots, M\}$, $\mathsf{d}(\hat{z}^m) \in \mathsf{d}(R^H(\mathsf{c}(\check{\mathbf{x}}) \cup \{\hat{z}^1, \ldots, \hat{z}^{m-1}\}) \smallsetminus R^H(\mathsf{c}(\check{\mathbf{x}}) \cup$

$\hat{x}$ is the unique element of $R^H(\mathsf{c}(\check{\mathbf{x}}) \cup \{\hat{z}^1, \ldots, \hat{z}^M\}) \setminus R^H(\mathsf{c}(\check{\mathbf{x}}) \cup \{\hat{z}^1, \ldots, \hat{z}^{M-1}\})$. Thus, $\bar{y}^{n+1}$ is not the lowest-ranked acceptable contract for $\mathsf{d}(\bar{y}^{n+1})$ as $d^{n+1} \neq \hat{d}$.

Thus there must be a contract in $\mathsf{c}(\mathbf{x}) \setminus (\mathsf{c}(\tilde{\mathbf{z}}) \cup \{\tilde{y}^1, \ldots, \tilde{y}^n\})$ that is acceptable to d^{n+1}. Hence, we can let $\tilde{y}^{n+1}$ be the favorite contract of d^{n+1} in $\mathsf{c}(\mathbf{x}) \setminus (\mathsf{c}(\tilde{\mathbf{z}}) \cup \{\tilde{y}^1, \ldots, \tilde{y}^n\})$ and proceed.

Since the set of contracts is finite, there must exist a smallest integer $N \geq 1$ such that $\bar{d} \in \mathsf{d}(R^H(\mathsf{c}(\tilde{\mathbf{z}}) \cup \{\tilde{y}^1, \ldots, \tilde{y}^N\}) \setminus R^H(\mathsf{c}(\tilde{\mathbf{z}}) \cup \{\tilde{y}^1, \ldots, \tilde{y}^{N-1}\}))$ and $\tilde{\mathbf{y}} = (\tilde{y}^1, \ldots, \tilde{y}^N)$ is a pre-run rejection chain at $\tilde{\mathbf{z}}$ such that $\mathsf{c}(\tilde{\mathbf{y}}) \subseteq \mathsf{c}(\mathbf{x}) \setminus X_{\hat{d}}$.

Step 3: Finally, we establish that $(\mathsf{c}(\tilde{\mathbf{z}}) \cup \{y\}) \subseteq \mathsf{c}(\hat{\mathbf{x}})$ implies $\mathsf{c}(\tilde{\mathbf{y}}) \subseteq \mathsf{c}(\mathbf{x}) \cap \mathsf{c}(\hat{\mathbf{x}})$.

We will prove by induction on n that $\{\tilde{y}^1, \ldots, \tilde{y}^n\} \subseteq \mathsf{c}(\hat{\mathbf{x}})$. For $n = 1$, the facts that $y \in \mathsf{c}(\hat{\mathbf{x}})$, $\tilde{y}^1 \mathrel{\hat{\succ}_{\bar{d}}} y$, and the compatibility of $\hat{\mathbf{x}}$ with $\hat{\succ}$, imply that $\tilde{y}^1 \in \mathsf{c}(\hat{\mathbf{x}})$. Now assume that, for some $n < N$, we had already shown that $\{\tilde{y}^1, \ldots, \tilde{y}^n\} \subseteq \mathsf{c}(\hat{\mathbf{x}})$. Since $\tilde{\mathbf{z}}, (\tilde{y}^1, \ldots, \tilde{y}^n), \hat{\mathbf{x}}$ are all weakly observable and weakly compatible with $\hat{\succ}$, Lemma A.1 implies that $(\tilde{\mathbf{z}}, (\tilde{y}^1, \ldots, \tilde{y}^n), \hat{\mathbf{x}})$ is weakly observable. By Lemma A.2, we must have that $R^H(\mathsf{c}(\tilde{\mathbf{z}}) \cup \{\tilde{y}^1, \ldots, \tilde{y}^n\}) \subseteq R^H(\mathsf{c}(\tilde{\mathbf{z}}) \cup \{\tilde{y}^1, \ldots, \tilde{y}^n\} \cup \mathsf{c}(\hat{\mathbf{x}}))$. Since $\mathsf{c}(\tilde{\mathbf{z}}) \subseteq \mathsf{c}(\hat{\mathbf{x}})$ by hypothesis, and $\{\tilde{y}^1, \ldots, \tilde{y}^n\} \subseteq \mathsf{c}(\hat{\mathbf{x}})$ by the inductive assumption, we must have

$$R^H(\mathsf{c}(\tilde{\mathbf{z}}) \cup \{\tilde{y}^1, \ldots, \tilde{y}^n\} \cup \mathsf{c}(\hat{\mathbf{x}})) = R^H(\mathsf{c}(\hat{\mathbf{x}})). \tag{6}$$

As $(\tilde{y}^1, \ldots, \tilde{y}^N)$ is a pre-run rejection chain at $\tilde{\mathbf{z}}$, we must have that $\tilde{y}^{n+1}$ is the highest ranked contract in $X_{\mathsf{d}(\tilde{y}^{n+1})} \setminus R^H(\mathsf{c}(\tilde{\mathbf{z}}) \cup \{\tilde{y}^1, \ldots, \tilde{y}^n\})$ by Condition 2d of the definition of a pre-run rejection chain. Since $R^H(\mathsf{c}(\tilde{\mathbf{z}}) \cup \{\tilde{y}^1, \ldots, \tilde{y}^n\}) \subseteq R^H(\mathsf{c}(\hat{\mathbf{x}}))$ by (6) and since $\hat{\mathbf{x}}$ is a complete offer process with respect to $\hat{\succ}$, we must have $\tilde{y}^{n+1} \in \mathsf{c}(\hat{\mathbf{x}})$.

This completes the proof of Claim 5. □

Claim 5 allows us to extend pre-run rejection chains at $\check{\mathbf{x}}$ so as to consider how

$\{\hat{z}^1, \ldots, \hat{z}^{m-2}\}))$ and $\hat{z}^m$ is the highest ranked contract in $X_{\mathsf{d}(\hat{z}^m)} \setminus (\mathsf{c}(\check{\mathbf{x}}) \cup \{\hat{z}^1, \ldots, \hat{z}^{m-1}\})$ with respect to $\succ_{\mathsf{d}(\hat{z}^m)}$. Since all choice functions are observably substitutable and observably size monotonic, we must have that, for all $m \leq M$, $|R^H(\mathsf{c}(\check{\mathbf{x}}) \cup \{\hat{z}^1, \ldots, \hat{z}^m\}) \setminus R^H(\mathsf{c}(\check{\mathbf{x}}) \cup \{\hat{z}^1, \ldots, \hat{z}^{m-1}\})| \leq 1$. If m is such that either $R^H(\mathsf{c}(\check{\mathbf{x}}) \cup \{\hat{z}^1, \ldots, \hat{z}^m\}) \setminus R^H(\mathsf{c}(\check{\mathbf{x}}) \cup \{\hat{z}^1, \ldots, \hat{z}^{m-1}\}) = \varnothing$, or such that there exists a contract $y \in R^H(\mathsf{c}(\check{\mathbf{x}}) \cup \{\hat{z}^1, \ldots, \hat{z}^m\}) \setminus R^H(\mathsf{c}(\check{\mathbf{x}}) \cup \{\hat{z}^1, \ldots, \hat{z}^{m-1}\})$ for which, for all $z \in X_{\mathsf{d}(y)} \setminus R^H(\mathsf{c}(\check{\mathbf{x}}) \cup \{\hat{z}^1, \ldots, \hat{z}^m\})$, $\emptyset \succ_{\mathsf{d}(y)} z$, $(\check{\mathbf{x}}, \hat{z}^1, \ldots, \hat{z}^m)$ is a complete offer process and $m = M$.

the additional contracts that will be proposed by doctors (other than $\hat{d}$) under $\hat{\succ}'$, as opposed to under $\hat{\succ}$, will affect whether $\hat{d}$ will obtain $\hat{x}$ when $\hat{h}$ is the only hospital.

Our next claim shows that each doctor $d \neq \hat{d}$ prefers any contract in $[\mathsf{c}(\hat{\mathbf{x}})]_d$ to any contract in $[\mathsf{c}(\mathbf{x}) \smallsetminus \mathsf{c}(\hat{\mathbf{x}})]_d$.

Claim 6. *For all doctors $d \in D \smallsetminus \{\hat{d}\}$, for all contracts $y \in [\mathsf{c}(\hat{\mathbf{x}})]_d$, and all contracts $z \in [\mathsf{c}(\mathbf{x}) \smallsetminus \mathsf{c}(\hat{\mathbf{x}})]_d$, we have that $y \succ_d z$.*

Proof of Claim 6. Let $d \in D \smallsetminus \{\hat{d}\}$ be arbitrary. There are two cases to consider:

Case 1: $[\mathsf{c}(\mathbf{x})]_d \subseteq [\mathsf{c}(\hat{\mathbf{x}})]_d$**.** In this case, $[\mathsf{c}(\mathbf{x}) \smallsetminus \mathsf{c}(\hat{\mathbf{x}})]_d = \varnothing$ and so the result is vacuously satisfied.

Case 2: $[\mathsf{c}(\mathbf{x})]_d \nsubseteq [\mathsf{c}(\hat{\mathbf{x}})]_d$**.** In this case, there exists a contract $z \in [\mathsf{c}(\mathbf{x}) \smallsetminus \mathsf{c}(\hat{\mathbf{x}})]_d$. But since $\succ_d = \hat{\succ}_d$, this implies that every contract $y \in [\mathsf{c}(\hat{\mathbf{x}})]_d$ is proposed in the offer process $\mathbf{x}$ before z; hence, $y \succ_d z$.

This completes the proof of Claim 6. □

With the help of the just established Claims 5 and 6, we now finish our proof of Claim 3. Let $\tilde{\mathbf{x}}$ be an offer process such that

1. $\tilde{\mathbf{x}}$ can be obtained from $\check{\mathbf{x}}$ by pre-running rejection chains under $\hat{\succ}$,
2. $\mathsf{c}(\tilde{\mathbf{x}}) \subseteq (\mathsf{c}(\mathbf{x}) \cap \mathsf{c}(\hat{\mathbf{x}})) \smallsetminus X_{\hat{d}}$, and
3. $\tilde{\mathbf{x}}$ is *maximal*, i.e., there is no other offer process $\mathbf{w}$ that can be obtained from $\check{\mathbf{x}}$ by pre-running rejection chains under $\hat{\succ}$ such that $\mathsf{c}(\tilde{\mathbf{x}}) \subsetneq \mathsf{c}(\mathbf{w}) \subseteq (\mathsf{c}(\mathbf{x}) \cap \mathsf{c}(\hat{\mathbf{x}})) \smallsetminus X_{\hat{d}}$.[10]

That is, $\tilde{\mathbf{x}}$ is an offer process that can be obtained from $\check{\mathbf{x}}$ by pre-running rejection chains under $\hat{\succ}$ using only contracts in $\mathsf{c}(\mathbf{x}) \cap \mathsf{c}(\hat{\mathbf{x}})$ not involving $\hat{d}$—and it is "maximal" in the sense that there is no other pre-run rejection chain at $\tilde{\mathbf{x}}$ using only contracts in $\mathsf{c}(\mathbf{x}) \cap \mathsf{c}(\hat{\mathbf{x}})$ not involving $\hat{d}$. In the remainder of the proof we will establish that there exists a generalized pre-run rejection chain $\mathbf{z}^*$ at $\tilde{\mathbf{x}}$ such that the combined offer process $\mathbf{y}^* \equiv (\tilde{\mathbf{x}}, \mathbf{z}^*)$ satisfies all four conditions of Claim 3.

Claim 7. *If there exists a generalized pre-run rejection chain $\mathbf{z}^*$ at $\tilde{\mathbf{x}}$ under $\hat{\succ}$ such that*

[10] Note that an offer process such as $\tilde{\mathbf{x}}$ must exist given that the set of contracts is finite.

1. $\mathsf{c}(\hat{\mathbf{x}}') \smallsetminus \mathsf{c}(\hat{\mathbf{x}}) \subseteq \mathsf{c}(\mathbf{z}^*)$*,*

2. $\mathsf{c}(\mathbf{z}^*) \subseteq \mathsf{c}(\mathbf{x}) \smallsetminus X_{\hat{d}}$*, and*

3. $R^H(\mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}^*)) \smallsetminus R^H(\mathsf{c}(\hat{\mathbf{x}})) \subseteq R^H(\mathsf{c}(\tilde{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}^*))$*,*

then $\mathbf{y}^* \equiv (\tilde{\mathbf{x}}, \mathbf{z}^*)$ *satisfies all four conditions of Claim 3.*

The first condition of Claim 7 requires that any contract proposed under $\hat{\succ}'$ that was not proposed under $\hat{\succ}$ is proposed in $\mathbf{z}^*$. The second condition of Claim 7 ensures that $\mathbf{z}^*$ only includes contracts in $\mathbf{x}$ with doctors other than $\hat{d}$. The third condition of Claim 7 requires that any contract rejected during the combined offer process $(\hat{\mathbf{x}}, \mathbf{z}^*)$ that is not rejected during the offer process $\hat{\mathbf{x}}$ is also rejected during the combined offer process $(\tilde{\mathbf{x}}, \mathbf{z}^*)$.

Proof of Claim 7. Since $\tilde{\mathbf{x}}$ is obtained from $\check{\mathbf{x}}$ by pre-running rejection chains under $\hat{\succ}$ and since $\mathbf{z}^*$ is a generalized pre-run rejection chain at $\tilde{\mathbf{x}}$ under $\hat{\succ}$, the combined offer process $\mathbf{y}^* \equiv (\tilde{\mathbf{x}}, \mathbf{z}^*)$ can be obtained from $\check{\mathbf{x}}$ by pre-running rejection chains under $\hat{\succ}$, satisfying Condition 1 of Claim 3. Given that $\mathsf{c}(\tilde{\mathbf{x}}) \subseteq X \smallsetminus X_{\hat{d}}$ by the construction of $\tilde{\mathbf{x}}$ and given that $\mathsf{c}(\mathbf{z}^*) \subseteq X \smallsetminus X_{\hat{d}}$ by the second condition of Claim 7, we obtain $\mathsf{c}((\tilde{\mathbf{x}}, \mathbf{z}^*)) \subseteq X \smallsetminus X_{\hat{d}}$, satisfying Condition 2 of Claim 3. Since $\mathsf{c}(\hat{\mathbf{x}}') \smallsetminus \mathsf{c}(\hat{\mathbf{x}}) \subseteq \mathsf{c}(\mathbf{z}^*)$ by the first condition of Claim 7, we must have that

$$\mathsf{c}(\hat{\mathbf{x}}') \smallsetminus \mathsf{c}(\hat{\mathbf{x}}) \subseteq \mathsf{c}(\mathbf{z}^*) \subseteq \mathsf{c}(\tilde{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}^*) = \mathsf{c}(\mathbf{y}^*),$$

so that $\mathbf{y}^* = (\tilde{\mathbf{x}}, \mathbf{z}^*)$ satisfies Condition 3 of Claim 3. Finally, since $\mathbf{z}^*$ satisfies the third condition of Claim 7, we must have

$$R^H(\mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}^*)) \smallsetminus R^H(\mathsf{c}(\hat{\mathbf{x}})) \subseteq R^H(\mathsf{c}(\tilde{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}^*)) = R^H(\mathsf{c}(\mathbf{y}^*)),$$

which implies that $\mathbf{y}^* = (\tilde{\mathbf{x}}, \mathbf{z}^*)$ satisfies Condition 4 of Claim 3. □

To prove Claim 7 is true, we show how to extend a generalized pre-run rejection chain $\mathbf{z}$ at $\tilde{\mathbf{x}}$ that satisfies only the second and third conditions of Claim 7 to a strictly longer generalized pre-run rejection chain $\mathbf{z}'$ that still satisfies the second and third conditions.[11]

[11]We show below that $\tilde{\mathbf{x}}$ satisfies the second and third conditions of Claim 7.

Let $\tilde{D} \equiv \{d \in D \smallsetminus \{\hat{d}\} : [C^H(\mathsf{c}(\hat{\mathbf{x}}))]_d \neq \varnothing\}$, i.e., the set of doctors other than $\hat{d}$ who are employed at the end of $\hat{\mathbf{x}}$.

Claim 8. *Suppose there exists a generalized pre-run rejection chain* $\mathbf{z}$ *at* $\tilde{\mathbf{x}}$ *under* $\hat{\succ}$ *that satisfies the following conditions:*

(C1) $\mathsf{c}(\hat{\mathbf{x}}') \smallsetminus (\mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z})) \neq \varnothing$.

(C2) $\mathsf{c}(\mathbf{z}) \subseteq \mathsf{c}(\mathbf{x}) \smallsetminus X_{\hat{d}}$.

(C3) $R^H(\mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z})) \smallsetminus R^H(\mathsf{c}(\hat{\mathbf{x}})) \subseteq R^H(\mathsf{c}(\tilde{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}))$.

(C4) For all $d \in \tilde{D}$*, if* $[\mathsf{c}(\hat{\mathbf{x}})]_d \nsubseteq R^H(\mathsf{c}(\tilde{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}))$*, then* $[C^H(\mathsf{c}(\hat{\mathbf{x}}))]_d \nsubseteq R^H(\mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}))$.

(C5) For all $d \in D \smallsetminus \{\hat{d}\}$*, if* $[C^H(\mathsf{c}(\tilde{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}))]_d \neq \varnothing$*, then* $[C^H(\mathsf{c}(\tilde{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}))]_d$ *contains the highest ranking contract in* $X_d \smallsetminus R^H(\mathsf{c}(\tilde{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}))$ *with respect to* $\succ_d$.

Then there exists a non-empty generalized pre-run rejection chain $\mathbf{y}$ *at* $(\tilde{\mathbf{x}}, \mathbf{z})$ *such that* $\hat{\mathbf{z}} = (\mathbf{z}, \mathbf{y})$ *satisfies Conditions (C2)–(C5) with respect to* $\hat{\mathbf{z}}$.

Condition C1 of Claim 8 implies that there are contracts proposed during $\hat{\mathbf{x}}'$ that are not proposed during $(\hat{\mathbf{x}}, \mathbf{z})$ and thus $\mathbf{z}$ does *not* satisfy Condition 1 of Claim 7. Condition C2 of Claim 8 corresponds to the second condition of Claim 7 and ensures that $\mathbf{z}$ only includes contracts in $\mathsf{c}(\mathbf{x}) \smallsetminus X_{\hat{d}}$. Condition C3 of Claim 8 corresponds to the third condition of Claim 7 and requires that any contract rejected during the combined offer process $(\hat{\mathbf{x}}, \mathbf{z})$ that is not rejected during the offer process $\hat{\mathbf{x}}$ is also rejected during the combined offer process $(\tilde{\mathbf{x}}, \mathbf{z})$. Condition C4 states that for each doctor other than $\hat{d}$ employed after the offer process $\hat{\mathbf{x}}$, if there is some contract in $\hat{\mathbf{x}}$ with that doctor that is not rejected during the combined offer process $(\tilde{\mathbf{x}}, \mathbf{z})$, then the contract that doctor obtains after $\hat{\mathbf{x}}$ is not rejected during the combined offer process $(\hat{\mathbf{x}}, \mathbf{z})$. Finally, Condition C5 ensures that each doctor other than $\hat{d}$ employed after the offer process $(\tilde{\mathbf{x}}, \mathbf{z})$ obtains his highest-ranked contract not yet rejected.

To show that Claim 8 implies Claim 7, we first show that $(\tilde{\mathbf{x}}, \mathbf{z})$ satisfies Conditions (C2)–(C5) when $\mathsf{c}(\mathbf{z}) = \varnothing$:

- Condition C2 is immediate when $\mathsf{c}(\mathbf{z}) = \varnothing$.
- Condition C3 is satisfied as, when $\mathsf{c}(\mathbf{z}) = \varnothing$, we have that $R^H(\mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z})) \smallsetminus R^H(\mathsf{c}(\hat{\mathbf{x}})) = \varnothing$.

- Condition C4 is satisfied since there exists a contract $y \in [C^H(\mathsf{c}(\hat{\mathbf{x}}))]_d$ (as $d \in \tilde{D}$), so that $y \notin R^H(\mathsf{c}(\hat{\mathbf{x}}))$ and, hence, $[C^H(\mathsf{c}(\hat{\mathbf{x}}))]_d \not\subseteq R^H(\mathsf{c}(\hat{\mathbf{x}}))$.

- Finally, if Condition C5 was not satisfied, there would be a doctor $d \in D \setminus \{\hat{d}\}$ and a contract $\tilde{z} \in X_d \setminus R^H(\mathsf{c}(\tilde{\mathbf{x}}))$ such that $\tilde{z} \succ_d C^H(\mathsf{c}(\tilde{\mathbf{x}}))$. We can assume without loss of generality that $\tilde{z}$ is the highest ranked contract in $X_d \setminus R^H(\mathsf{c}(\tilde{\mathbf{x}}))$ with respect to $\succ_d$. We must have $d \notin \mathsf{d}(C^{\mathsf{h}(\tilde{z})}(\mathsf{c}(\tilde{\mathbf{x}})))$: Otherwise, the contract in $[C^H(\mathsf{c}(\tilde{\mathbf{x}}))]_d$ would have been proposed before $\tilde{z}$ so that $\tilde{\mathbf{x}}$ would not be weakly compatible with $\succ_d$. Now note that $\tilde{\mathbf{x}}$ and $\mathbf{x}$ are both weakly observable and weakly compatible with $\succ$. Hence, $(\tilde{\mathbf{x}}, \mathbf{x})$ is weakly observable by Lemma A.1. By Lemma A.2, we have $R^H(\mathsf{c}(\tilde{\mathbf{x}})) \subseteq R^H(\mathsf{c}(\mathbf{x})) = R^H(\mathsf{c}(\mathbf{x}) \cup \mathsf{c}(\tilde{\mathbf{x}}))$. Since $\tilde{z}$ is the highest ranked contract in $X_d \setminus R^H(\mathsf{c}(\tilde{\mathbf{x}}))$ and $\mathbf{x}$ is compatible with $\succ$, we must have $\tilde{z} \in \mathsf{c}(\mathbf{x})$. A completely analogous argument shows that $\tilde{z} \in \mathsf{c}(\hat{\mathbf{x}})$.[12] Since $\tilde{z} \in (\mathsf{c}(\mathbf{x}) \cap \mathsf{c}(\hat{\mathbf{x}})) \setminus X_{\hat{d}}$ and $d \notin \mathsf{d}(C^{\mathsf{h}(\tilde{z})}(\mathsf{c}(\tilde{\mathbf{x}})))$, we obtain a contradiction to the maximality of $\tilde{\mathbf{x}}$ given that Claim 5 implies that there exists a pre-run rejection chain $\tilde{\mathbf{y}}$ at $\tilde{\mathbf{x}}$ such that $\mathsf{c}(\tilde{\mathbf{y}}) \subseteq (\mathsf{c}(\mathbf{x}) \cap \mathsf{c}(\hat{\mathbf{x}})) \setminus X_{\hat{d}}$.

If $\mathsf{c}(\hat{\mathbf{x}}') \setminus \mathsf{c}(\hat{\mathbf{x}}) = \varnothing$, then the empty offer process satisfies all the requirements of Claim 7 for $\mathbf{z}^*$. Otherwise, $\mathsf{c}(\hat{\mathbf{x}}') \setminus \mathsf{c}(\hat{\mathbf{x}}) \neq \varnothing$, and so the empty offer process at $\tilde{\mathbf{x}}$ satisfies all the requirements of Claim 8 for $\mathbf{z}$. Thus, using Claim 8, we can sequentially construct offer processes $\mathbf{y}^1, \ldots, \mathbf{y}^\ell$ such that $\mathbf{y}^\ell$ is a generalized pre-run rejection chain at $(\tilde{\mathbf{x}}, \mathbf{y}^1, \ldots, \mathbf{y}^{\ell-1})$ under $\hat{\succ}$ that satisfies Conditions C2–C5 and contains at least one contract in $\mathsf{c}(\hat{\mathbf{x}}') \setminus (\mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\mathbf{y}^1) \cup \cdots \cup \mathsf{c}(\mathbf{y}^{\ell-1}))$. If $\mathsf{c}(\hat{\mathbf{x}}') \setminus (\mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\mathbf{y}^1) \cup \cdots \cup \mathsf{c}(\mathbf{y}^\ell)) = \varnothing$, then $(\tilde{\mathbf{x}}, \mathbf{y}^1, \ldots, \mathbf{y}^\ell)$ satisfies all three conditions of Claim 7 for $\mathbf{z}^*$; otherwise, if $\mathsf{c}(\hat{\mathbf{x}}') \setminus (\mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\mathbf{y}^1) \cup \cdots \cup \mathsf{c}(\mathbf{y}^\ell)) \neq \varnothing$, then $(\tilde{\mathbf{x}}, \mathbf{y}^1, \ldots, \mathbf{y}^\ell)$ satisfies all five conditions of Claim 8 for $\mathbf{z}$. Since the set of contracts is finite, there must exist some L such that $\mathsf{c}(\hat{\mathbf{x}}') \setminus (\mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\mathbf{y}^1) \cup \cdots \cup \mathsf{c}(\mathbf{y}^L)) = \varnothing$, and so $(\tilde{\mathbf{x}}, \mathbf{y}^1, \ldots, \mathbf{y}^L)$ satisfies all three conditions of Claim 7 for $\mathbf{z}^*$.

To construct $\mathbf{y}^\ell$ from $(\tilde{\mathbf{x}}, \mathbf{y}^1, \ldots, \mathbf{y}^{\ell-1})$, let $\mathbf{z} \equiv (\mathbf{y}^1, \ldots, \mathbf{y}^{\ell-1})$, let y^1 be the first contract in the sequence $\hat{\mathbf{x}}'$ such that $y^1 \notin \mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z})$, and let $d^1 \equiv \mathsf{d}(y^1)$. Note that $d^1 \neq \hat{d}$ as $\mathsf{c}(\hat{\mathbf{x}}') \setminus \mathsf{c}(\hat{\mathbf{x}}) \subseteq X \setminus X_{\hat{d}}$ and $\mathsf{h}(y^1) = \hat{h}$ since $\mathsf{c}(\hat{\mathbf{x}}') \setminus \mathsf{c}(\hat{\mathbf{x}}) \subseteq X_{\hat{h}}$. We will now show that there exists a generalized pre-run rejection chain $\mathbf{y} = (y^1, \ldots, y^N)$ at $(\tilde{\mathbf{x}}, \mathbf{z})$

[12] Now note that $\tilde{\mathbf{x}}$ and $\hat{\mathbf{x}}$ are both weakly observable and weakly compatible with $\hat{\succ}$. Hence, $(\tilde{\mathbf{x}}, \hat{\mathbf{x}})$ is weakly observable by Lemma A.1. Lemma A.2 then implies that $R^H(\mathsf{c}(\tilde{\mathbf{x}})) \subseteq R^H(\mathsf{c}(\hat{\mathbf{x}})) = R^H(\mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\tilde{\mathbf{x}}))$. Since $\tilde{z}$ is the highest ranked contract in $X_d \setminus R^H(\mathsf{c}(\tilde{\mathbf{x}}))$ and $\hat{\mathbf{x}}$ is compatible with $\hat{\succ}$, we must have $\tilde{z} \in \mathsf{c}(\hat{\mathbf{x}})$.

under $\hat{\succ}$ such that $(\mathbf{z}, \mathbf{y})$ satisfies Conditions (C2)–(C5) of Claim 8 in three steps:

Step 1: We show that for all $\tilde{y} \in X_{\hat{h}} \cap X_{d^1}$ such that $\tilde{y} \succ_{d^1} y^1$ we have $\tilde{y} \in R^H(\mathsf{c}(\tilde{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}))$.

Suppose, by way of contradiction, that there exists some $\tilde{y} \in X_{\hat{h}} \cap X_{d^1}$ such that

$$\tilde{y} \succ_{d^1} y^1 \text{ and } \tilde{y} \notin R^H(\mathsf{c}(\tilde{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z})). \tag{7}$$

For this step, it is useful to define $\hat{\mathbf{x}}''$ to be the offer process that is obtained from $\hat{\mathbf{x}}'$ by deleting y^1 and all contracts that are proposed after y^1. We note here two facts that will prove useful later in Step 1. First, since by definition y^1 is the contract in $\mathsf{c}(\hat{\mathbf{x}}') \setminus (\mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}))$ that appears first in the sequence $\hat{\mathbf{x}}'$, we must have

$$\mathsf{c}(\hat{\mathbf{x}}'') \subseteq \mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}). \tag{8}$$

Moreover, since $\hat{\mathbf{x}}''$ is weakly compatible with $\hat{\succ}$ and since $\tilde{y} \hat{\succ}_{d^1} y^1$ by (7),[13] the definition of $\hat{\mathbf{x}}''$ implies

$$\tilde{y} \in R^H(\mathsf{c}(\hat{\mathbf{x}}'')). \tag{9}$$

Substep 1: We first show that $\tilde{y} \notin R^H(\mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}))$. There are two cases to consider:

Case 1: $[C^H(\mathsf{c}(\hat{\mathbf{x}}))]_{d^1} \cap X_{\hat{h}} = \{\tilde{y}\}$**.** In this case, it is immediate that $\tilde{y} \notin R^H(\mathsf{c}(\hat{\mathbf{x}}))$. Since $\mathbf{z}$ satisfies Condition C3 of Claim 8, we must have $R^H(\mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z})) \setminus R^H(\mathsf{c}(\hat{\mathbf{x}})) \subseteq R^H(\mathsf{c}(\tilde{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}))$. Combining the last two observations with the assumption (7) that $\tilde{y} \notin R^H(\mathsf{c}(\tilde{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}))$, we must have $\tilde{y} \notin R^H(\mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}))$.

Case 2: $[C^H(\mathsf{c}(\hat{\mathbf{x}}))]_{d^1} \cap X_{\hat{h}} \neq \{\tilde{y}\}$**.** We will show first that we must have that $[C^H(\mathsf{c}(\hat{\mathbf{x}}))]_{d^1} \cap X_{\hat{h}} = \varnothing$ in this case. Suppose by way of contradiction that there is some $\tilde{y}' \neq \tilde{y}$ such that

$$\tilde{y}' \in [C^H(\mathsf{c}(\hat{\mathbf{x}}))]_{d^1} \cap X_{\hat{h}}. \tag{10}$$

[13] Remember that $d^1 \neq \hat{d}$ and that, for all $d \neq \hat{d}$, $\hat{\succ}_d = \succ_d$.

Since, by Claim 6, d^1 prefers every contract in $[\mathsf{c}(\hat{\mathbf{x}})]_{d^1}$ to every contract in $[\mathsf{c}(\hat{\mathbf{x}}') \smallsetminus \mathsf{c}(\hat{\mathbf{x}})]_{d^1}$, we must have $\tilde{y}' \mathrel{\hat{\succ}_{d^1}} y^1$ and therefore also $\{\tilde{y}, \tilde{y}'\} \subseteq R^H(\mathsf{c}(\hat{\mathbf{x}}''))$. Note that $\hat{\mathbf{x}}''$, $\hat{\mathbf{x}}$, and $\mathbf{z}$ are all weakly observable and weakly compatible with $\hat{\succ}$ and so $(\hat{\mathbf{x}}'', \hat{\mathbf{x}}, \mathbf{z})$ is weakly observable by Lemma A.1; since $\mathsf{c}(\hat{\mathbf{x}}'') \subseteq \mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z})$ by (8), we have that

$$\{\tilde{y}, \tilde{y}'\} \subseteq R^H(\mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z})) \tag{11}$$

by Lemma A.2. There are two subcases to consider:

Subcase 1: $\tilde{y} \mathrel{\hat{\succ}_{d^1}} \tilde{y}'$. Since $\mathbf{z}$ is a generalized pre-run rejection chain at $\tilde{\mathbf{x}}$ under $\hat{\succ}$, and $\tilde{\mathbf{x}}$ is weakly compatible with $\hat{\succ}$ by definition, Lemma B.1 implies that the combined offer process $(\tilde{\mathbf{x}}, \mathbf{z})$ is weakly observable. Since $\tilde{y} \notin R^H(\mathsf{c}(\tilde{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}))$ by the assumption (7), Lemma A.2 implies that $\tilde{y}$ is not rejected at any step of the combined offer process $(\tilde{\mathbf{x}}, \mathbf{z})$.

Therefore, if $\tilde{y} \mathrel{\hat{\succ}_{d^1}} \tilde{y}'$, we must have $\tilde{y}' \notin \mathsf{c}(\tilde{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z})$ since d^1 could not have proposed $\tilde{y}'$ before $\tilde{y}$ was rejected, and so $\tilde{y}' \notin R^H(\mathsf{c}(\tilde{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}))$. Given that $\tilde{y}' \notin R^H(\mathsf{c}(\hat{\mathbf{x}}))$ by the assumption (10) and that $R^H(\mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z})) \smallsetminus R^H(\mathsf{c}(\hat{\mathbf{x}})) \subseteq R^H(\mathsf{c}(\tilde{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}))$ by Condition C3 of Claim 8 with respect to $\mathbf{z}$, we obtain $\tilde{y}' \notin R^H(\mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}))$, contradicting (11).

Subcase 2: $\tilde{y}' \mathrel{\hat{\succ}_{d^1}} \tilde{y}$. Since $\tilde{y}' \notin R^H(\mathsf{c}(\hat{\mathbf{x}}))$ by the assumption (10), the observable substitutability of each hospital's choice function implies that $\tilde{y}'$ is not rejected at any step of the offer process $\hat{\mathbf{x}}$.

Therefore, if $\tilde{y}' \mathrel{\hat{\succ}_{d^1}} \tilde{y}$, we obtain $\tilde{y} \notin \mathsf{c}(\hat{\mathbf{x}})$ since d^1 could not have proposed $\tilde{y}$ before $\tilde{y}'$ was rejected and so $\tilde{y} \notin R^H(\mathsf{c}(\hat{\mathbf{x}}))$. Since $R^H(\mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z})) \smallsetminus R^H(\mathsf{c}(\hat{\mathbf{x}})) \subseteq R^H(\mathsf{c}(\tilde{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}))$ by Condition C3 of Claim 8 with respect to $\mathbf{z}$ and since $\tilde{y} \notin R^H(\mathsf{c}(\tilde{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}))$ by assumption (7), we obtain $\tilde{y} \notin R^H(\mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}))$, contradicting (11).

Hence, the assumption that $[C^H(\mathsf{c}(\hat{\mathbf{x}}))]_{d^1} \cap X_{\hat{h}} \neq \varnothing$ necessarily leads to a contradiction.

Now given that $[C^H(\mathsf{c}(\hat{\mathbf{x}}))]_{d^1} \cap X_{\hat{h}} = \varnothing$ and d^1 is associated with the contract y^1, which is in $\mathsf{c}(\hat{\mathbf{x}}') \smallsetminus \mathsf{c}(\hat{\mathbf{x}})$ by definition, there must be a hospital $\tilde{h} \neq \hat{h}$ such that

$$d^1 \in \mathsf{d}(C^{\tilde{h}}(\mathsf{c}(\hat{\mathbf{x}}))), \tag{12}$$

as otherwise $\hat{\mathbf{x}}$ would not be a complete offer process with respect to $\hat{\succ}$.

Next, we show that $d^1 \notin \mathsf{d}(C^{\hat{h}}(\mathsf{c}(\tilde{\mathbf{x}})))$; to do this, we assume by way of contradiction that

$$d^1 \in \mathsf{d}(C^{\hat{h}}(\mathsf{c}(\tilde{\mathbf{x}}))) \tag{13}$$

and show that (13) contradicts (12).

We first establish that (13) implies

$$(X_{\tilde{h}} \cap X_{d^1}) \cap \mathsf{c}(\hat{\mathbf{x}}) \subseteq \mathsf{c}(\tilde{\mathbf{x}}). \tag{14}$$

Note that, by Claim 6, d^1 prefers every contract in $[\mathsf{c}(\hat{\mathbf{x}})]_{d^1}$ to every contract in $[\mathsf{c}(\hat{\mathbf{x}}') \smallsetminus \mathsf{c}(\hat{\mathbf{x}})]_{d^1} \subseteq [\mathsf{c}(\mathbf{x}) \smallsetminus \mathsf{c}(\hat{\mathbf{x}})]_{d^1}$; moreover, since $y^1 \in \mathsf{c}(\hat{\mathbf{x}}') \smallsetminus \mathsf{c}(\hat{\mathbf{x}})$ by its definition, there is at least one contract in $\mathsf{c}(\mathbf{x})$—namely, y^1—that d^1 likes strictly less than every contract in $\mathsf{c}(\hat{\mathbf{x}})$. Thus, since $\mathbf{x}$ is compatible with $\succ$, we must have $[\mathsf{c}(\hat{\mathbf{x}})]_{d^1} \subseteq \mathsf{c}(\mathbf{x})$. Now, if there is a contract $\tilde{z} \in X_{\tilde{h}} \cap X_{d^1} \cap (\mathsf{c}(\hat{\mathbf{x}}) \smallsetminus \mathsf{c}(\tilde{\mathbf{x}}))$, we obtain a contradiction to the definition of $\tilde{\mathbf{x}}$: Since $[\mathsf{c}(\hat{\mathbf{x}})]_{d^1} \subseteq \mathsf{c}(\mathbf{x})$, $\tilde{z} \in X_{\tilde{h}} \cap X_{d^1} \cap (\mathsf{c}(\hat{\mathbf{x}}) \smallsetminus \mathsf{c}(\tilde{\mathbf{x}}))$ implies that $\tilde{z} \in \mathsf{c}(\mathbf{x}) \cap \mathsf{c}(\hat{\mathbf{x}})$; the assumption (13) and the feasibility of $C^H(\mathsf{c}(\tilde{\mathbf{x}}))$ imply that $d^1 \notin \mathsf{d}(C^{\tilde{h}}(\mathsf{c}(\tilde{\mathbf{x}})))$. Claim 5 then implies that there exists a pre-run rejection chain $\tilde{\mathbf{y}}$ at $\tilde{\mathbf{x}}$ starting with $\tilde{z}$ such that $\mathsf{c}(\tilde{\mathbf{y}}) \subseteq (\mathsf{c}(\mathbf{x}) \cap \mathsf{c}(\hat{\mathbf{x}})) \smallsetminus X_{\hat{d}}$, thus contradicting the maximality of $\tilde{\mathbf{x}}$. Hence, $X_{\tilde{h}} \cap X_{d^1} \cap (\mathsf{c}(\hat{\mathbf{x}}) \smallsetminus \mathsf{c}(\tilde{\mathbf{x}})) = \varnothing$, and thus (14) is satisfied.

Since $\tilde{\mathbf{x}}$ and $\hat{\mathbf{x}}$ are weakly observable and weakly compatible with $\hat{\succ}$, Lemma A.1 implies that $(\tilde{\mathbf{x}}, \hat{\mathbf{x}})$ is weakly observable. By Lemma A.2, we must have $R^H(\mathsf{c}(\tilde{\mathbf{x}})) \subseteq R^H(\mathsf{c}(\tilde{\mathbf{x}}) \cup \mathsf{c}(\hat{\mathbf{x}}))$. Since $\mathsf{c}(\tilde{\mathbf{x}}) \subseteq \mathsf{c}(\hat{\mathbf{x}})$ by the definition of $\tilde{\mathbf{x}}$, we have that $R^H(\mathsf{c}(\tilde{\mathbf{x}})) \subseteq R^H(\mathsf{c}(\hat{\mathbf{x}}))$. Furthermore, since $C^H(\mathsf{c}(\tilde{\mathbf{x}}))$ is feasible and $d^1 \in \mathsf{d}(C^{\hat{h}}(\mathsf{c}(\tilde{\mathbf{x}})))$ by (13), we have that (14) implies $(X_{\tilde{h}} \cap X_{d^1}) \cap \mathsf{c}(\hat{\mathbf{x}}) \subseteq R^H(\mathsf{c}(\tilde{\mathbf{x}}))$. Combining our last two findings, we obtain $(X_{\tilde{h}} \cap X_{d^1}) \cap \mathsf{c}(\hat{\mathbf{x}}) \subseteq R^H(\mathsf{c}(\hat{\mathbf{x}}))$, contradicting (12).

We can now show that $\tilde{y} \notin R^H(\mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}))$. Since $\tilde{\mathbf{x}}$ is compatible with $\hat{\succ}$, and $\mathbf{z}$ is is a generalized pre-run rejection chain at $\tilde{\mathbf{x}}$ under $\hat{\succ}$, Lemma B.1 implies that $(\tilde{\mathbf{x}}, \mathbf{z})$ is weakly observable. Thus, by Lemma A.2, we have that $R^H(\mathsf{c}(\tilde{\mathbf{x}})) \subseteq R^H(\mathsf{c}(\tilde{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}))$. Since $\tilde{y} \notin R^H(\mathsf{c}(\tilde{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}))$ by (7), we must have that $\tilde{y} \notin R^H(\mathsf{c}(\tilde{\mathbf{x}}))$. Moreover, since $d^1 \notin \mathsf{d}(C^{\hat{h}}(\mathsf{c}(\tilde{\mathbf{x}})))$ (as we have just shown that (13) leads to a contradiction), we have that $\tilde{y} \notin C^{\hat{h}}(\mathsf{c}(\tilde{\mathbf{x}}))$. Combining these last two observations yields $\tilde{y} \notin \mathsf{c}(\tilde{\mathbf{x}})$. Furthermore, if $\tilde{y} \in \mathsf{c}(\hat{\mathbf{x}}) \cap \mathsf{c}(\mathbf{x})$, then $\tilde{\mathbf{x}}$ would not be maximal given that $\mathsf{h}(\tilde{y}) = \hat{h}$, $d^1 \notin \mathsf{d}(C^{\hat{h}}(\mathsf{c}(\tilde{\mathbf{x}})))$ (as we have just shown that (13) leads to a contradiction), and $\tilde{y} \notin \mathsf{c}(\tilde{\mathbf{x}})$ imply that there exists a pre-run rejection chain at $\tilde{\mathbf{x}}$ that lies entirely in $\mathsf{c}(\mathbf{x}) \cap \mathsf{c}(\hat{\mathbf{x}})$ by Claim 5; hence, $\tilde{y} \notin \mathsf{c}(\hat{\mathbf{x}}) \cap \mathsf{c}(\mathbf{x})$. Therefore, given that $[\mathsf{c}(\hat{\mathbf{x}})]_{d^1} \subseteq \mathsf{c}(\mathbf{x})$,[14] we must have $\tilde{y} \notin \mathsf{c}(\hat{\mathbf{x}})$ and, thus, $\tilde{y} \notin R^H(\mathsf{c}(\hat{\mathbf{x}}))$. Given that $\tilde{y} \notin R^H(\mathsf{c}(\tilde{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}))$ by (7) and $R^H(\mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z})) \smallsetminus R^H(\mathsf{c}(\hat{\mathbf{x}})) \subseteq R^H(\mathsf{c}(\tilde{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}))$ by Condition C3 of Claim 8 with respect to $\mathbf{z}$, we obtain that $\tilde{y} \notin R^H(\mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}))$.

Substep 2: We will now complete the proof of Step 1 by showing that the result of Substep 1, that $\tilde{y} \notin R^H(\mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}))$, necessarily leads to a contradiction. Since $\hat{\mathbf{x}}''$, $\hat{\mathbf{x}}$, and $\mathbf{z}$ are all weakly observable and weakly compatible with $\hat{\succ}$, $(\hat{\mathbf{x}}'', \hat{\mathbf{x}}, \mathbf{z})$ is weakly observable by Lemma A.1. Hence, $R^H(\mathsf{c}(\hat{\mathbf{x}}'')) \subseteq R^H(\mathsf{c}(\hat{\mathbf{x}}'') \cup \mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}))$ by Lemma A.2. Since $\mathsf{c}(\hat{\mathbf{x}}'') \subseteq \mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z})$ by (8), we obtain $R^H(\mathsf{c}(\hat{\mathbf{x}}'') \cup \mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z})) = R^H(\mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}))$. Combining the last two observations yields $R^H(\mathsf{c}(\hat{\mathbf{x}}'')) \subseteq R^H(\mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}))$. By (9) we have that $\tilde{y} \in R^H(\mathsf{c}(\hat{\mathbf{x}}''))$, and so $\tilde{y} \in R^H(\mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}))$. But we have established in Substep 1 that $\tilde{y} \notin R^H(\mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}))$, and hence obtain a contradiction to (7).

Step 2: We construct $\mathbf{y}$ so that the generalized pre-run rejection chain $(\mathbf{z}, \mathbf{y})$ satisfies both Conditions C2 and C5 of Claim 8.

We will argue first that $d^1 \notin \mathsf{d}(C^{\hat{h}}(\mathsf{c}(\tilde{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z})))$. Since $y^1 \in \mathsf{c}(\hat{\mathbf{x}}') \smallsetminus (\mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}))$ (by the definition of y^1) and $\mathsf{c}(\tilde{\mathbf{x}}) \subseteq \mathsf{c}(\mathbf{x}) \cap \mathsf{c}(\hat{\mathbf{x}})$ (by the definition of $\tilde{\mathbf{x}}$), we

[14]Recall that, by Claim 6, d^1 prefers every contract in $[\mathsf{c}(\hat{\mathbf{x}})]_{d^1}$ to every contract in $[\mathsf{c}(\hat{\mathbf{x}}') \smallsetminus \mathsf{c}(\hat{\mathbf{x}})]_{d^1} \subseteq [\mathsf{c}(\mathbf{x}) \smallsetminus \mathsf{c}(\hat{\mathbf{x}})]_{d^1}$; moreover, since $y^1 \in \mathsf{c}(\hat{\mathbf{x}}') \smallsetminus \mathsf{c}(\hat{\mathbf{x}})$ by its definition, there is at least one contract in $\mathsf{c}(\mathbf{x})$—namely, y^1—that d^1 likes strictly less than every contract in $\mathsf{c}(\hat{\mathbf{x}})$. Thus, since $\mathbf{x}$ is compatible with $\succ$, we must have $[\mathsf{c}(\hat{\mathbf{x}})]_{d^1} \subseteq \mathsf{c}(\mathbf{x})$.

obtain

$$y^1 \notin \mathsf{c}(\tilde{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}). \tag{15}$$

Since $\mathbf{z}$ is a generalized pre-run rejection chain at $\tilde{\mathbf{x}}$ under $\hat{\succ}$, and $\tilde{\mathbf{x}}$ is weakly compatible with $\hat{\succ}$ by definition, Lemma B.1 implies that the combined offer process $(\tilde{\mathbf{x}}, \mathbf{z})$ is weakly compatible with $\hat{\succ}$. Hence, (15) implies that, for all $\tilde{y}$ such that $y^1 \succ_{d^1} \tilde{y}$ and $\mathsf{h}(\tilde{y}) = \hat{h}$, we have that $\tilde{y} \notin \mathsf{c}(\tilde{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z})$. Finally, by Step 1, for any $\tilde{y} \in X_{\hat{h}} \cap X_{d^1}$ such that $\tilde{y} \succ_{d^1} y^1$, we have that $\tilde{y} \in R^H(\mathsf{c}(\tilde{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}))$. Combining these last two observations with (15), we find that $d^1 \notin \mathsf{d}(C^{\hat{h}}(\mathsf{c}(\tilde{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z})))$.

Thus, since $d^1 \neq \hat{d}$, Claim 5 implies that there exist $N_1 - 1$ contracts $y^2, \ldots, y^{N_1}$ such that $\mathbf{y}^1 \equiv (y^1, \ldots, y^{N_1})$ is a pre-run rejection chain at $(\tilde{\mathbf{x}}, \mathbf{z})$ under $\hat{\succ}$ and that $\mathsf{c}(\mathbf{y}^1) \subseteq \mathsf{c}(\mathbf{x}) \smallsetminus X_{\hat{d}}$ and so $(\mathbf{z}, \mathbf{y})$ at $\tilde{\mathbf{x}}$ satisfies Condition C2 (as $\mathbf{z}$ satisfies Condition C2 by our inductive assumption).

Note that the assumption that $\mathbf{z}$ satisfies Condition C5 and the fact that $\mathbf{y}^1$ is a pre-run rejection chain at $(\tilde{\mathbf{x}}, \mathbf{z})$ under $\hat{\succ}$ imply, for all $d \in D \smallsetminus \{d^1, \hat{d}\}$ such that $[C^H(\mathsf{c}(\tilde{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \mathsf{c}(\mathbf{y}^1))]_d \neq \varnothing$, that $[C^H(\mathsf{c}(\tilde{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \mathsf{c}(\mathbf{y}^1))]_d$ contains the highest-ranking acceptable contract in $X_d \smallsetminus R^H(\mathsf{c}(\tilde{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \mathsf{c}(\mathbf{y}^1))$ with respect to $\succ_d$. Thus, to see that we can satisfy Condition C5, there are two cases to consider:

Case 1: $y^1 \notin C^H(\mathsf{c}(\tilde{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \mathsf{c}(\mathbf{y}^1))$. In this case, let $\check{y}$ be the unique contract in $[C^H(\mathsf{c}(\tilde{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \mathsf{c}(\mathbf{y}^1))]_{d^1}$—note that such a $\check{y}$ must exist as $\mathbf{y}^1$ is a pre-run rejection chain. Note first that

$$\{\check{y}\} = [C^H(\mathsf{c}(\tilde{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}))]_{d^1} \tag{16}$$

as

1. $[(\mathsf{c}(\tilde{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \mathsf{c}(\mathbf{y}^1)) \smallsetminus (\mathsf{c}(\tilde{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}))]_{d^1} = \{y^1\}$,
2. $\mathsf{d}(C^H(\mathsf{c}(\tilde{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \mathsf{c}(\mathbf{y}^1))) = \mathsf{d}(C^H(\mathsf{c}(\tilde{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z})))$ by Claim 4, and
3. $R^H(\mathsf{c}(\tilde{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z})) \subseteq R^H(\mathsf{c}(\tilde{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \mathsf{c}(\mathbf{y}^1))$, by Lemma A.2, as $(\tilde{\mathbf{x}}, \mathbf{z}, \mathbf{y}^1)$ is weakly observable by Lemma B.1 (as $(\mathbf{z}, \mathbf{y}^1)$ is a generalized pre-run rejection chain at $\tilde{\mathbf{x}}$ under $\hat{\succ}$ and $\tilde{\mathbf{x}}$ is weakly compatible with $\hat{\succ}$ by

definition).

Moreover, $\check{y}$ is the highest-ranking acceptable contract in $X_{d^1} \smallsetminus R^H(\mathsf{c}(\tilde{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}))$ by Condition C5 with respect to $\mathbf{z}$. Since $R^H(\mathsf{c}(\tilde{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z})) \subseteq R^H(\mathsf{c}(\tilde{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \mathsf{c}(\mathbf{y}^1))$ (as explained in point 3 above), we have that $X_{d^1} \smallsetminus R^H(\mathsf{c}(\tilde{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z})) \subseteq X_{d^1} \smallsetminus R^H(\mathsf{c}(\tilde{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \mathsf{c}(\mathbf{y}^1))$. Finally, since $\{\check{y}\} = [C^H(\mathsf{c}(\tilde{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}))]_{d^1}$ by (16), we have that $\check{y}$ is the highest-ranking acceptable contract in $X_{d^1} \smallsetminus R^H(\mathsf{c}(\tilde{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \mathsf{c}(\mathbf{y}^1))$, so $(\mathbf{z}, \mathbf{y}^1)$ satisfies Condition C5 of Claim 8.

Case 2: $y^1 \in C^H(\mathsf{c}(\tilde{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \mathsf{c}(\mathbf{y}^1))$**.** If y^1 is the highest ranking acceptable contract in $X_d \smallsetminus R^H(\mathsf{c}(\tilde{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \mathsf{c}(\mathbf{y}^1))$, we are done. Otherwise, let $y^{N_1+1} \neq y^1$ be the highest ranking contract in $X_{d^1} \smallsetminus R^H(\mathsf{c}(\tilde{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \mathsf{c}(\mathbf{y}^1))$ with respect to $\succ_{d^1}$. Note that we must have $\mathsf{h}(y^{N_1+1}) \neq \hat{h}$ since $y^{N_1+1} \succ_{d^1} y^1$ and y^1 is the highest-ranking contract in $X_{\hat{h}} \smallsetminus R^H(\mathsf{c}(\tilde{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \mathsf{c}(\mathbf{y}^1))$ by Step 1. Hence, by Claim 5 we can start a new pre-run rejection $\mathbf{y}^2$ at $(\tilde{\mathbf{x}}, \mathbf{z}, \mathbf{y}^1)$ with y^{N_1+1} under $\hat{\succsim}$ that consists only of contracts in $\mathsf{c}(\mathbf{x}) \smallsetminus X_{\hat{d}}$ given that $\mathsf{d}(y^{N_1+1}) = d^1 \neq \hat{d}$. Iterating this procedure, as X is finite, we must eventually reach an integer M such that

$$\left(\mathbf{z}, y^1, \ldots, y^{N_1}, \ldots, y^{\left(\sum_{m=1}^{M-1} N_m\right)+1}, \ldots, y^{\left(\sum_{m=1}^{M} N_m\right)}\right)$$

is a generalized pre-run rejection at $\tilde{\mathbf{x}}$ that contains y^1 and satisfies both Conditions C2 and C5. Hence, we can set $\mathbf{y} \equiv \left(y^1, \ldots, y^{\left(\sum_{m=1}^{M} N_m\right)}\right)$ to obtain a new generalized pre-run rejection chain $(\mathbf{z}, \mathbf{y})$ at $\tilde{\mathbf{x}}$ under $\hat{\succsim}$ that satisfies both Conditions C2 and C5.

Step 3: We show that the extended generalized pre-run rejection chain $(\mathbf{z}, \mathbf{y})$ at $\tilde{\mathbf{x}}$ under $\hat{\succsim}$ satisfies Conditions C3 and C4.

Define the offer process $\mathbf{w}$ that lists the contracts in $\mathsf{c}(\mathbf{y}) \smallsetminus \mathsf{c}(\hat{\mathbf{x}})$ in order of appearance in $\mathbf{y}$ as follows: Set $w^1 \equiv y^1$ and $n_1 \equiv 1$. Now assuming that $w^1, \ldots, w^o$ and $n_1, \ldots, n_o$ have already been defined and that $\mathsf{c}(\mathbf{y}) \smallsetminus (\mathsf{c}(\hat{\mathbf{x}}) \cup \{w^1, \ldots, w^o\}) \neq \varnothing$, set $w^{o+1} \equiv y^{n_{o+1}}$, where

$$n_{o+1} \equiv \min\{n \in \{1, \ldots, N\} : y^n \notin \{w^1, \ldots, w^o\} \text{ and } y^n \in \mathsf{c}(\mathbf{y}) \smallsetminus \mathsf{c}(\hat{\mathbf{x}})\}.$$

Let O be such that $\{w^1, \ldots, w^O\} = \mathsf{c}(\mathbf{y}) \smallsetminus \mathsf{c}(\hat{\mathbf{x}})$. Note that the offer process

$\mathbf{w} = (w^1, \ldots, w^O)$ will not in general be observable. However, by the construction of $\mathbf{w}$, we must have, for all $o \leq O$, that $\mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \{w^1, \ldots, w^o\} = \mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \{y^1, \ldots, y^{n_o}\}$ and so $(\hat{\mathbf{x}}, \mathbf{z}, \mathbf{w}) = (\hat{\mathbf{x}}, \mathbf{z}, \mathbf{y})$. Moreover, as $(\tilde{\mathbf{x}}, \mathbf{z}, \mathbf{y})$ is weakly observable and weakly compatible with $\hat{\succ}$ by Lemma B.1 (as $(\mathbf{z}, \mathbf{y})$ is a generalized pre-run rejection chain at $\tilde{\mathbf{x}}$ under $\hat{\succ}$ and $\tilde{\mathbf{x}}$ is weakly compatible with $\hat{\succ}$ by definition), $\hat{\mathbf{x}}$ is compatible with $\hat{\succ}$, and $\mathsf{c}(\tilde{\mathbf{x}}) \subseteq \mathsf{c}(\hat{\mathbf{x}})$ by the definition of $\tilde{\mathbf{x}}$, we have by Lemma A.1 that $(\hat{\mathbf{x}}, \tilde{\mathbf{x}}, \mathbf{z}, \mathbf{y}) = (\hat{\mathbf{x}}, \mathbf{z}, \mathbf{y})$ is weakly observable and weakly compatible with $\hat{\succ}$. Combining these last two observations yields

$$(\hat{\mathbf{x}}, \mathbf{z}, \mathbf{y}) = (\hat{\mathbf{x}}, \mathbf{z}, \mathbf{w}) \text{ is weakly observable and weakly compatible with } \hat{\succ}. \tag{17}$$

We now proceed in four substeps.

Substep 1: We now show that, for all $o \leq O$,

$$|R^H(\mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \{w^1, \ldots, w^o\}) \smallsetminus R^H(\mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \{w^1, \ldots, w^{o-1}\})| \leq 1.$$

Since $(\hat{\mathbf{x}}, \mathbf{z}, \mathbf{w})$ is weakly observable by (17), by Lemma A.2, we must have, for all $o \leq O$, that $|R^H(\mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \{w^1, \ldots, w^o\}) \smallsetminus R^H(\mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \{w^1, \ldots, w^{o-1}\})| \leq 1$.

Substep 2: We now establish that, for all $o \in \{1, \ldots, O\}$, there exists a contract $\hat{w}^o$ with the following three attributes:

(A1) $\mathsf{d}(\hat{w}^o) = \mathsf{d}(w^{o+1})$ for $o < O$ and $\mathsf{d}(\hat{w}^O) = \mathsf{d}(w^1)$,

(A2) $\{\hat{w}^o\} = R^H(\mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \{w^1, \ldots, w^o\}) \smallsetminus R^H(\mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \{w^1, \ldots, w^{o-1}\})$, and

(A3) $\hat{w}^o \in R^H(\mathsf{c}(\tilde{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \{y^1, \ldots, y^{n_{o+1}-1}\})$ for $o < O$ and $\hat{w}^O \in R^H(\mathsf{c}(\tilde{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \{y^1, \ldots, y^N\})$.

Suppose that, for some $o \in \{1, \ldots, O\}$, the statement has been established for all $o' \in \{1, \ldots, o-1\}$.[15] Since $\mathbf{y}$ is a generalized pre-run rejection chain, we must have, for $o < O$,

$$R^H(\mathsf{c}(\tilde{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \{y^1, \ldots, y^{n_{o+1}-1}\}) \smallsetminus R^H(\mathsf{c}(\tilde{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \{y^1, \ldots, y^{n_{o+1}-2}\}) = \{\tilde{w}^o\}$$

[15] Note that, when $o = 1$, this assumption is vacuously satisfied as $\{1, \ldots, o-1\} = \varnothing$.

for some contract $\tilde{w}^o$ such that $\mathsf{d}(\tilde{w}^o) = \mathsf{d}(w^{o+1}) = \mathsf{d}(y^{n_{o+1}})$; similarly, we must have

$$R^H(\mathsf{c}(\tilde{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \{y^1, \ldots, y^N\}) \smallsetminus R^H(\mathsf{c}(\tilde{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \{y^1, \ldots, y^{N-1}\}) = \{\tilde{w}^O\}$$

for some contract $\tilde{w}^O$ such that $\mathsf{d}(\tilde{w}^O) = \mathsf{d}(w^1) = \mathsf{d}(y^1)$.[16]

There are two cases:

Case 1: $\tilde{w}^o \in \mathsf{c}(\mathbf{x}) \smallsetminus \mathsf{c}(\hat{\mathbf{x}})$**.** We will show that $\hat{w}^o \equiv \tilde{w}^o$ satisfies Attributes A1–A3.

We first show that $\tilde{w}^o \notin R^H(\mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \{w^1, \ldots, w^{o-1}\})$. By the inductive assumption, for each $o' \in \{1, \ldots, o-1\}$, we have by Attribute A2 that $\hat{w}^{o'}$ is the unique contract in $R^H(\mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \{w^1, \ldots, w^{o'}\}) \smallsetminus R^H(\mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \{w^1, \ldots, w^{o'-1}\})$. Hence, we must have

$$R^H(\mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \{w^1, \ldots, w^{o-1}\}) \smallsetminus R^H(\mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z})) = \{\hat{w}^1, \ldots, \hat{w}^{o-1}\}.$$

By the inductive assumption, for each $o' \in \{1, \ldots, o-1\}$, we have by Attribute A3 that

$$\hat{w}^{o'} \in R^H(\mathsf{c}(\tilde{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \{y^1, \ldots, y^{n_{o'+1}-1}\}).$$

Combining these last two observations, we obtain

$$R^H(\mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \{w^1, \ldots, w^{o-1}\}) \smallsetminus R^H(\mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z})) \subseteq R^H(\mathsf{c}(\tilde{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \{y^1, \ldots, y^{n_o-1}\}). \quad (18)$$

For $o < O$, since $n_{o+1} \geq n_o + 1$ and since $\tilde{w}^o \notin R^H(\mathsf{c}(\tilde{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \{y^1, \ldots, y^{n_{o+1}-2}\})$ by the construction of $\tilde{w}^o$, we obtain $\tilde{w}^o \notin R^H(\mathsf{c}(\tilde{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \{y^1, \ldots, y^{n_o-1}\})$ by Lemma A.2. Similarly, for $o = O$, since $N - 1 \geq n_O - 1$ and since $\tilde{w}^O \notin R^H\left(\mathsf{c}(\tilde{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \{y^1, \ldots, y^{N-1}\}\right)$, we obtain $\tilde{w}^O \notin R^H(\mathsf{c}(\tilde{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \{y^1, \ldots, y^{n_O-1}\})$ by Lemma A.2.

[16] The fact that $\tilde{w}^{o'}$ is unique for all $o' \in \{1, \ldots, O\}$ follows from Lemma A.2 and (17).

Combining the last two observations with (18) we obtain that

$$\tilde{w}^o \notin R^H(\mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \{w^1, \ldots, w^{o-1}\}) \setminus R^H(\mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z})). \tag{19}$$

Next, note that, by Condition C3 applied to $\mathbf{z}$, we must have $R^H(\mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z})) \setminus R^H(\mathsf{c}(\hat{\mathbf{x}})) \subseteq R^H(\mathsf{c}(\tilde{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}))$. For $o < O$, since (by the definition of $\tilde{w}^o$) $\tilde{w}^o \notin R^H(\mathsf{c}(\tilde{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \{y^1, \ldots, y^{n_{o+1}-2}\})$, $n_{o+1} - 2 \geq 0$, and $(\tilde{\mathbf{x}}, \mathbf{z}, \mathbf{y})$ is weakly observable, Lemma A.2 implies that $\tilde{w}^o \notin R^H(\mathsf{c}(\tilde{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}))$ and therefore also $\tilde{w}^o \notin R^H(\mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z})) \setminus R^H(\mathsf{c}(\hat{\mathbf{x}}))$; similarly, since $\tilde{w}^O \notin R^H\left(\mathsf{c}(\tilde{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \{y^1, \ldots, y^{N-1}\}\right)$, $N - 1 \geq 0$, and $(\tilde{\mathbf{x}}, \mathbf{z}, \mathbf{y})$ is weakly observable, Lemma A.2 implies that $\tilde{w}^O \notin R^H(\mathsf{c}(\tilde{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}))$ and therefore also $\tilde{w}^O \notin R^H(\mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z})) \setminus R^H(\mathsf{c}(\hat{\mathbf{x}}))$. Thus,

$$\tilde{w}^o \notin R^H(\mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z})) \setminus R^H(\mathsf{c}(\hat{\mathbf{x}})) \tag{20}$$

for all $o \leq O$.

Finally, we must have

$$\tilde{w}^o \notin R^H(\mathsf{c}(\hat{\mathbf{x}})) \tag{21}$$

as $\tilde{w}^o \notin \mathsf{c}(\hat{\mathbf{x}})$ by the Case 1 assumption.

Since

$$\begin{aligned} R^H(\mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \{w^1, \ldots, w^{o-1}\}) = \\ [R^H(\mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \{w^1, \ldots, w^{o-1}\}) \setminus R^H(\mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}))] \cup \\ [R^H(\mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z})) \setminus R^H(\mathsf{c}(\hat{\mathbf{x}}))] \cup R^H(\mathsf{c}(\hat{\mathbf{x}})), \end{aligned}$$

we may use (19), (20), and (21) to obtain the desired statement

$$\tilde{w}^o \notin R^H(\mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \{w^1, \ldots, w^{o-1}\}). \tag{22}$$

Next, we show that $\tilde{w}^o \in R^H(\mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \{w^1, \ldots, w^o\})$. Note that $\tilde{\mathbf{x}}$, $\mathbf{z}$, $\mathbf{y}$, and $\hat{\mathbf{x}}$ are all weakly observable and weakly compatible with $\hat{\succsim}$,[17] so that $(\tilde{\mathbf{x}}, \mathbf{z}, \mathbf{y}, \hat{\mathbf{x}})$ is weakly observable by Lemma A.1. By Lemma A.2,

[17] Remember that $\mathsf{c}(\tilde{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \mathsf{c}(\mathbf{y}) \subseteq X \setminus X_{\hat{d}}$.

we must have

$$R^H(\mathsf{c}(\tilde{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \{y^1, \ldots, y^{n_{o+1}-1}\}) \subseteq R^H(\mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\tilde{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \{y^1, \ldots, y^{n_{o+1}-1}\})$$

for $o < O$ and

$$R^H(\mathsf{c}(\tilde{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \{y^1, \ldots, y^N\}) \subseteq R^H(\mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\tilde{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \{y^1, \ldots, y^N\}).$$

By the construction of $\mathbf{w}$, we must have $\mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\tilde{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \{y^1, \ldots, y^{n_{o+1}-1}\} = \mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \{w^1, \ldots, w^o\}$ for $o < O$ and $\mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\tilde{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \{y^1, \ldots, y^N\} = \mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \{w^1, \ldots, w^O\}$.[18] Since $\tilde{w}^o \in R^H(\mathsf{c}(\tilde{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \{y^1, \ldots, y^{n_{o+1}-1}\})$ if $o < O$ and $\tilde{w}^O \in R^H(\mathsf{c}(\tilde{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \{y^1, \ldots, y^N\})$, we must therefore have that

$$\tilde{w}^o \in R^H(\mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \{w^1, \ldots, w^o\}) \tag{23}$$

for all $o \in \{1, \ldots, O\}$.

Hence, noting (22) and (23), we can let $\hat{w}^o \equiv \tilde{w}^o$ to obtain a contract in $R^H(\mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \{w^1, \ldots, w^o\}) \setminus R^H(\mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \{w^1, \ldots, w^{o-1}\})$. For $o < O$, since $\hat{w}^o = \tilde{w}^o$ is the unique contract in $R^H(\mathsf{c}(\tilde{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \{y^1, \ldots, y^{n_{o+1}-1}\}) \setminus R^H(\mathsf{c}(\tilde{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \{y^1, \ldots, y^{n_{o+1}-2}\})$ and since $\mathbf{y}$ is a generalized pre-run rejection chain, we must have $\mathsf{d}(\tilde{w}^o) = \mathsf{d}(w^{o+1}) = \mathsf{d}(y^{n_{o+1}})$, so that Attribute A1 is satisfied; similarly, for $o < O$, $\hat{w}^O = \tilde{w}^O$ is the unique contract in the set $R^H(\mathsf{c}(\tilde{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \{y^1, \ldots, y^N\}) \setminus R^H(\mathsf{c}(\tilde{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \{y^1, \ldots, y^{N-1}\})$ and since $\mathbf{y}$ is a generalized pre-run rejection chain, we must have $\mathsf{d}(\tilde{w}^O) = \mathsf{d}(w^1) = \mathsf{d}(y^1)$. Next, given that we have established in Substep 1 the fact that $|R^H(\mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \{w^1, \ldots, w^o\}) \setminus R^H(\mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \{w^1, \ldots, w^{o-1}\})| \leq 1$, we must have $R^H(\mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \{w^1, \ldots, w^o\}) \setminus R^H(\mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \{w^1, \ldots, w^{o-1}\}) = \{\hat{w}^o\}$, so that Attribute A2 is satisfied. Finally, Attribute A3 is satisfied since both $\hat{w}^o \in R^H(\mathsf{c}(\tilde{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \{y^1, \ldots, y^{n_{o+1}-1}\})$ for $o < O$ and $\hat{w}^O \in R^H(\mathsf{c}(\tilde{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \{y^1, \ldots, y^N\})$.

Case 2: Suppose that $\tilde{w}^o \in \mathsf{c}(\hat{\mathbf{x}})$. We will show that if $\hat{w}^o$ is the unique contract in $[C^H(\mathsf{c}(\hat{\mathbf{x}}))]_{\mathsf{d}(\tilde{w}^o)}$, then $\hat{w}^o$ satisfies Attributes A1–A3.

[18] Recall that $\tilde{\mathbf{x}} \subseteq \hat{\mathbf{x}}$ by construction.

Throughout the proof of Case 2, keep in mind that,

- for $o < O$, since $\tilde{w}^o$ is the unique contract in $R^H(\mathsf{c}(\tilde{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \{y^1, \dots, y^{n_{o+1}-1}\}) \smallsetminus R^H(\mathsf{c}(\tilde{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \{y^1, \dots, y^{n_{o+1}-2}\})$ and $\mathbf{y}$ is a generalized pre-run rejection chain, we must have $\mathsf{d}(\tilde{w}^o) = \mathsf{d}(w^{o+1}) = \mathsf{d}(y^{n_{o+1}})$, and
- for $o = O$, since $\tilde{w}^O$ is the unique contract in $R^H(\mathsf{c}(\tilde{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \{y^1, \dots, y^N\}) \smallsetminus R^H(\mathsf{c}(\tilde{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \{y^1, \dots, y^{N-1}\})$ and $\mathbf{y}$ is a generalized pre-run rejection chain, we must have $\mathsf{d}(\tilde{w}^O) = \mathsf{d}(w^1) = \mathsf{d}(y^1)$.

First, we establish that

$$[\mathsf{c}(\hat{\mathbf{x}})]_{\mathsf{d}(\tilde{w}^o)} = [\mathsf{c}(\hat{\mathbf{x}})]_{\mathsf{d}(y^{n_{o+1}})} \subseteq R^H(\mathsf{c}(\tilde{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \{y^1, \dots, y^{n_{o+1}-1}\}) \tag{24a}$$

for $o < O$ and

$$[\mathsf{c}(\hat{\mathbf{x}})]_{\mathsf{d}(\tilde{w}^O)} = [\mathsf{c}(\hat{\mathbf{x}})]_{\mathsf{d}(y^1)} \subseteq R^H(\mathsf{c}(\tilde{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \{y^1, \dots, y^N\}). \tag{24b}$$

There are two subcases to consider:

Subcase 1: $o < O$**.** In this case, the construction of $\mathbf{y}$ and the assumption that $\mathbf{z}$ satisfies Condition C5 ensure that $y^{n_{o+1}}$ is the highest ranking contract in $X_{\mathsf{d}(y^{n_{o+1}})} \smallsetminus R^H(\mathsf{c}(\tilde{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \{y^1, \dots, y^{n_{o+1}-1}\})$. Given that, by Claim 6, each doctor $d \in D \smallsetminus \{\hat{d}\}$ prefers any contract in $[\mathsf{c}(\hat{\mathbf{x}})]_d$ to any contract in $[\mathsf{c}(\mathbf{x}) \smallsetminus \mathsf{c}(\hat{\mathbf{x}})]_d$, and that $w^{o+1} = y^{n_{o+1}} \notin \mathsf{c}(\hat{\mathbf{x}})$, we must have $[\mathsf{c}(\hat{\mathbf{x}})]_{\mathsf{d}(y^{n_{o+1}})} \subseteq R^H(\mathsf{c}(\tilde{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \{y^1, \dots, y^{n_{o+1}-1}\})$.

Subcase 2: $o = O$**.** The assumption that $\mathbf{z}$ satisfies Condition C5 implies that $[C^H(\mathsf{c}(\tilde{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}))]_{d^1}$ contains the highest ranking contract in $X_{d^1} \smallsetminus R^H(\mathsf{c}(\tilde{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}))$ with respect to $\succ_{d^1}$. If there was an $n \le N$ such that $y^1 \in R^H(\mathsf{c}(\tilde{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \{y^1, \dots, y^n\}) \smallsetminus R^H(\mathsf{c}(\tilde{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \{y^1, \dots, y^{n-1}\})$, the construction of $\mathbf{y}$ in Step 2 would have therefore ensured that $n = N$. But in this case, we would have $y^1 = \tilde{w}^O$ and would hence obtain a contradiction to the assumption that $\tilde{w}^O \in \mathsf{c}(\hat{\mathbf{x}})$ as $y^1 \notin \mathsf{c}(\hat{\mathbf{x}})$. Hence, we must have $[C^H(\mathsf{c}(\tilde{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \mathsf{c}(\mathbf{y}))]_{d^1} = \{y^1\}$. But at the end of the combined offer process $(\tilde{\mathbf{x}}, \mathbf{z}, \mathbf{y})$, the set $[C^H(\mathsf{c}(\tilde{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \mathsf{c}(\mathbf{y}))]_{d^1}$ has to contain the highest ranking contract in $X_{d^1} \smallsetminus R^H(\mathsf{c}(\tilde{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \mathsf{c}(\mathbf{y}))$ with respect to $\succ_{d^1}$ by Condition C5 applied to $(\mathbf{z}, \mathbf{y})$. Since, by Claim 6, d^1 ranks each contract in $[\mathsf{c}(\hat{\mathbf{x}})]_{d^1}$ higher than each contract in $[\mathsf{c}(\mathbf{x}) \smallsetminus \mathsf{c}(\hat{\mathbf{x}})]_{d^1}$, this implies $[\mathsf{c}(\hat{\mathbf{x}})]_{d^1} \subseteq R^H(\mathsf{c}(\tilde{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \mathsf{c}(\mathbf{y}))$.

Second, we establish that

$$[C^H(\mathsf{c}(\hat{\mathbf{x}}))]_{\mathsf{d}(\tilde{w}^o)} \nsubseteq R^H(\mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \{w^1, \ldots, w^{o-1}\}). \tag{25}$$

Recall that $\mathsf{d}(\tilde{w}^o) = \mathsf{d}(y^{n_{o+1}})$ for $o < O$ and $\mathsf{d}(\tilde{w}^O) = \mathsf{d}(y^1)$, so that we need to establish $[C^H(\mathsf{c}(\hat{\mathbf{x}}))]_{\mathsf{d}(y^{n_{o+1}})} \nsubseteq R^H(\mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \{w^1, \ldots, w^{o-1}\})$ for $o < O$ and $[C^H(\mathsf{c}(\hat{\mathbf{x}}))]_{\mathsf{d}(y^1)} \nsubseteq R^H(\mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \{w^1, \ldots, w^{O-1}\})$. Note first that, since $\hat{\mathbf{x}}$ is a complete offer process with respect to $\hat{\succ}$, for all $d \in D \smallsetminus \{\hat{d}\}$ such that $[\mathsf{c}(\mathbf{x}) \smallsetminus \mathsf{c}(\hat{\mathbf{x}})]_d \neq \varnothing$, we have that $[C^H(\mathsf{c}(\hat{\mathbf{x}}))]_d \neq \varnothing$. Now note that $\tilde{w}^o \notin R^H(\mathsf{c}(\tilde{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}))$, as:

- $\tilde{w}^o \notin R^H(\mathsf{c}(\tilde{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \{y^1, \ldots, y^{n_{o+1}-2}\})$ for $o < O$ by the definition of $\tilde{w}^o$ and $(\tilde{\mathbf{x}}, \mathbf{z}, (y^1, \ldots, y^{n_{o+1}-2}))$ is weakly observable by Lemma A.1, and so Lemma A.2 implies that $\tilde{w}^o \notin R^H(\mathsf{c}(\tilde{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}))$.
- $\tilde{w}^O \notin R^H(\mathsf{c}(\tilde{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \{y^1, \ldots, y^{N-1}\})$ by the definition of $\tilde{w}^O$ and $(\tilde{\mathbf{x}}, \mathbf{z}, (y^1, \ldots, y^{N-1}))$ is weakly observable by Lemma A.1, and so Lemma A.2 implies that $\tilde{w}^O \notin R^H(\mathsf{c}(\tilde{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}))$.

Now, by combining the Case 2 assumption $\tilde{w}^o \in \mathsf{c}(\hat{\mathbf{x}})$, our preceding result that $\tilde{w}^o \notin R^H(\mathsf{c}(\tilde{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}))$, and Condition C4 applied to $\mathbf{z}$, we obtain that

$$[C^H(\mathsf{c}(\hat{\mathbf{x}}))]_{\mathsf{d}(y^{n_{o+1}})} \nsubseteq R(\mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z})) \tag{26a}$$

for $o < O$ and that

$$[C^H(\mathsf{c}(\hat{\mathbf{x}}))]_{\mathsf{d}(y^1)} \nsubseteq R(\mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z})) \tag{26b}$$

for $\mathsf{d}(\tilde{w}^O) = \mathsf{d}(y^1)$.[19]

[19]Note that $\mathsf{d}(\tilde{w}^o) \in \tilde{D} \smallsetminus \{\hat{d}\}$, since $\mathsf{d}(\tilde{w}^o) = \mathsf{d}(w^{o+1})$ is associated with the contract $w^{o+1} \in \mathsf{c}(\mathbf{y}) \smallsetminus \mathsf{c}(\hat{\mathbf{x}})$ for $o < O$ and since $\mathsf{d}(\tilde{w}^O) = \mathsf{d}(w^1)$ by the construction of $\tilde{w}^O$.

Next, note that, for $o < O$, we have that

$$\tilde{w}^o \notin R^H(\mathsf{c}(\tilde{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \{y^1, \ldots, y^n\}) \tag{27a}$$

for all $n \le n_{o+1} - 2$ by Lemma A.2 as both $\tilde{w}^o \notin R^H(\mathsf{c}(\tilde{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \{y^1, \ldots, y^{n_{o+1}-2}\})$ (by the definition of $\tilde{w}^o$) and $(\tilde{\mathbf{x}}, \mathbf{z}, (y^1, \ldots, y^n))$ is weakly observable by Lemma A.1. Similarly, for $o = O$,

$$\tilde{w}^O \notin R^H(\mathsf{c}(\tilde{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \{y^1, \ldots, y^n\}) \tag{27b}$$

for all $n \le N - 1$ by Lemma A.2 as both $\tilde{w}^O \notin R^H(\mathsf{c}(\tilde{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \{y^1, \ldots, y^{N-1}\})$ (by the definition of $\tilde{w}^O$) and $(\tilde{\mathbf{x}}, \mathbf{z}, (y^1, \ldots, y^{N-1}))$ is weakly observable by Lemma A.1.

We now show that

$$\mathsf{d}(y^{n_{o+1}}) = \mathsf{d}(\tilde{w}^o) \notin \mathsf{d}(\{w^2, \ldots, w^o\}) \tag{28a}$$

for $o < O$ and

$$\mathsf{d}(y^1) = \mathsf{d}(\tilde{w}^O) \notin \mathsf{d}(\{w^2, \ldots, w^O\}) \tag{28b}$$

via the following argument:

- If $o < O$, fix an $o' \in \{2, \ldots, o\}$. Recall that $y^{n_{o'}}$ is the most preferred contract in $X_{\mathsf{d}(y^{n_{o'}})} \smallsetminus R^H(\mathsf{c}(\tilde{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \{y^1, \ldots, y^{n_{o'}-1}\})$ with respect to $\succ_{\mathsf{d}(y^{n_{o'}})}$ by Condition C5. Since $o' \le o < o+1$, we have that $\tilde{w}^o \notin R^H(\mathsf{c}(\tilde{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \{y^1, \ldots, y^{n_{o'}-1}\})$ as $n_{o'} - 1 \le n_{o'+1} - 2$ by (27). Since $\mathsf{d}(\tilde{w}^o)$ prefers each contract in $[\mathsf{c}(\hat{\mathbf{x}})]_{\mathsf{d}(\tilde{w}^o)}$ to each contract in $[\mathsf{c}(\mathbf{x}) \smallsetminus \mathsf{c}(\hat{\mathbf{x}})]_{\mathsf{d}(\tilde{w}^o)}$ by Claim 6, $\mathsf{d}(\tilde{w}^o)$ has at least one contract (namely, $\tilde{w}^o$) in $X_{\mathsf{d}(\tilde{w}^o)} \smallsetminus R^H(\mathsf{c}(\tilde{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \{y^1, \ldots, y^{n_{o'}-1}\})$ that he prefers to any contract in $[\mathsf{c}(\mathbf{x}) \smallsetminus \mathsf{c}(\hat{\mathbf{x}})]_{\mathsf{d}(\tilde{w}^o)}$. Since $y^{n_{o'}} \in \mathsf{c}(\mathbf{x}) \smallsetminus \mathsf{c}(\hat{\mathbf{x}})$ (by the definition of $y^{n_{o'}}$), we must then have that $\mathsf{d}(y^{n_{o'}}) = \mathsf{d}(w^{o'}) \neq \mathsf{d}(\tilde{w}^o)$.
- If $o = O$, fix an $o' \in \{2, \ldots, O\}$. Recall that $y^{n_{o'}}$ is the most preferred contract in $X_{\mathsf{d}(y^{n_{o'}})} \smallsetminus R^H(\mathsf{c}(\tilde{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \{y^1, \ldots, y^{n_{o'}-1}\})$ with respect to $\succ_{\mathsf{d}(y^{n_{o'}})}$ by Condition C5. Since $o' \le O$, we have that $\tilde{w}^O \notin R^H(\mathsf{c}(\tilde{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \{y^1, \ldots, y^{n_{o'}-1}\})$ as $n_{o'} - 1 \le N - 1$ by (27).

Since $\mathsf{d}(\tilde{w}^O)$ prefers each contract in $[\mathsf{c}(\hat{\mathbf{x}})]_{\mathsf{d}(\tilde{w}^O)}$ to each contract in $[\mathsf{c}(\mathbf{x}) \smallsetminus \mathsf{c}(\hat{\mathbf{x}})]_{\mathsf{d}(\tilde{w}^O)}$ by Claim 6, $\mathsf{d}(\tilde{w}^O)$ has at least one contract (namely, $\tilde{w}^O$) in $X_{\mathsf{d}(\tilde{w}^O)} \smallsetminus R^H\left(\mathsf{c}(\tilde{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \{y^1, \ldots, y^{n_{o'}-1}\}\right)$ that he prefers to any contract in $[\mathsf{c}(\mathbf{x}) \smallsetminus \mathsf{c}(\hat{\mathbf{x}})]_{\mathsf{d}(\tilde{w}^O)}$. Since $y^{n_{o'}} \in \mathsf{c}(\mathbf{x}) \smallsetminus \mathsf{c}(\hat{\mathbf{x}})$ (by the definition of $y^{n_{o'}}$), we must then have that $\mathsf{d}(y^{n_{o'}}) = \mathsf{d}(w^{o'}) \neq \mathsf{d}(\tilde{w}^O)$.

By the inductive assumption that Attribute A1 holds for all $o' \in \{1, \ldots, o-1\}$, we have that $\mathsf{d}(\hat{w}^{o'}) = \mathsf{d}(w^{o'+1})$ and so, by (28), we have that $\mathsf{d}(y^{n_{o+1}}) \notin \mathsf{d}(\{\hat{w}^1, \ldots, \hat{w}^{o-1}\})$ for $o < O$ and $\mathsf{d}(y^1) \notin \mathsf{d}(\{\hat{w}^1, \ldots, \hat{w}^{O-1}\})$. By repeated application of the inductive assumption that Attribute A2 holds for all $o' \in \{1, \ldots, o-1\}$, we have that $R^H(\mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \{w^1, \ldots, w^{o-1}\}) \smallsetminus R^H(\mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z})) = \{\hat{w}^1, \ldots, \hat{w}^{o-1}\}$, and we thus obtain that $R^H(\mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \{w^1, \ldots, w^{o-1}\}) \smallsetminus R^H(\mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z})) \subseteq X \smallsetminus X_{\mathsf{d}(y^{n_{o+1}})}$ for $o < O$ and $R^H(\mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \{w^1, \ldots, w^{O-1}\}) \smallsetminus R^H(\mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z})) \subseteq X \smallsetminus X_{\mathsf{d}(y^1)}$. Given that we have already established by (26) that $[C^H(\mathsf{c}(\hat{\mathbf{x}}))]_{\mathsf{d}(y^{n_{o+1}})} \nsubseteq R(\mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}))$ for $o < O$ and that $[C^H(\mathsf{c}(\hat{\mathbf{x}}))]_{\mathsf{d}(y^1)} \nsubseteq R(\mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}))$, we obtain (25), i.e., that

$$[C^H(\mathsf{c}(\hat{\mathbf{x}}))]_{\mathsf{d}(y^{n_{o+1}})} \nsubseteq R^H\left(\mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \{w^1, \ldots, w^{o-1}\}\right)$$

for $o < O$ and that

$$[C^H(\mathsf{c}(\hat{\mathbf{x}}))]_{\mathsf{d}(y^1)} \nsubseteq R^H(\mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \{w^1, \ldots, w^{O-1}\}).$$

Third, we will show that

$$[C^H(\mathsf{c}(\hat{\mathbf{x}}))]_{\mathsf{d}(\tilde{w}^o)} \subseteq R(\mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \{w^1, \ldots, w^o\}). \tag{29}$$

Recall again that $\mathsf{d}(\tilde{w}^o) = \mathsf{d}(y^{n_{o+1}})$ for $o < O$ and $\mathsf{d}(\tilde{w}^O) = \mathsf{d}(y^1)$, so that we need to establish $[C^H(\mathsf{c}(\hat{\mathbf{x}}))]_{\mathsf{d}(y^{n_{o+1}})} \subseteq R(\mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \{w^1, \ldots, w^o\})$ for $o < O$ and $[C^H(\mathsf{c}(\hat{\mathbf{x}}))]_{\mathsf{d}(y^1)} \subseteq R(\mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \{w^1, \ldots, w^O\})$. As established by (24), we must have $[\mathsf{c}(\hat{\mathbf{x}})]_{\mathsf{d}(y^{n_{o+1}})} \subseteq R^H(\mathsf{c}(\tilde{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \{y^1, \ldots, y^{n_{o+1}-1}\})$ for $o < O$ and $[\mathsf{c}(\hat{\mathbf{x}})]_{\mathsf{d}(y^1)} \subseteq R^H(\mathsf{c}(\tilde{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \{y^1, \ldots, y^N\})$. Since $\tilde{\mathbf{x}}, \mathbf{z}, \mathbf{y}, \hat{\mathbf{x}}$ are all weakly observable and weakly compatible with $\hat{\succ}$, Lemma A.1 implies that $(\tilde{\mathbf{x}}, \mathbf{z}, \mathbf{y}, \hat{\mathbf{x}})$ is weakly observable. Thus, by

Lemma A.2, we have that both

$$R^H(\mathsf{c}(\tilde{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \{y^1, \ldots, y^{n_{o+1}-1}\}) \subseteq R^H(\mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\tilde{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \{y^1, \ldots, y^{n_{o+1}-1}\}),$$

implying $[\mathsf{c}(\hat{\mathbf{x}})]_{\mathsf{d}(y^{n_{o+1}})} \subseteq R^H(\mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\tilde{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \{y^1, \ldots, y^{n_{o+1}-1}\})$ for $o < O$, and

$$R^H(\mathsf{c}(\tilde{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \{y^1, \ldots, y^N\}) \subseteq R^H(\mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\tilde{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \{y^1, \ldots, y^N\}),$$

implying $[\mathsf{c}(\hat{\mathbf{x}})]_{\mathsf{d}(y^1)} \subseteq R^H(\mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\tilde{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \{y^1, \ldots, y^N\})$. Hence, for $o < O$, since

$$\mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\tilde{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \{y^1, \ldots, y^{n_{o+1}-1}\} = \mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \{w^1, \ldots, w^o\}$$

(by the construction of $\mathbf{w}$) we must also have

$$[\mathsf{c}(\hat{\mathbf{x}})]_{\mathsf{d}(y^{n_{o+1}})} \subseteq R^H(\mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \{w^1, \ldots, w^o\});$$

similarly, since

$$\mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\tilde{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \{y^1, \ldots, y^N\} = \mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \{w^1, \ldots, w^O\}$$

(by the construction of $\mathbf{w}$) we must have

$$[\mathsf{c}(\hat{\mathbf{x}})]_{\mathsf{d}(y^1)} \subseteq R^H(\mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \{w^1, \ldots, w^O\}).$$

In particular, we obtain that $[C^H(\mathsf{c}(\hat{\mathbf{x}}))]_{\mathsf{d}(y^{n_{o+1}})} \subseteq R^H(\mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \{w^1, \ldots, w^o\})$ for $o < O$ and $[C^H(\mathsf{c}(\hat{\mathbf{x}}))]_{\mathsf{d}(y^1)} \subseteq R^H(\mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \{w^1, \ldots, w^O\})$.

Now, for $o < O$, let $\hat{w}^o$ be the unique contract in $[C^H(\mathsf{c}(\hat{\mathbf{x}}))]_{\mathsf{d}(\tilde{w}^o)} = [C^H(\mathsf{c}(\hat{\mathbf{x}}))]_{\mathsf{d}(y^{n_{o+1}})}$, and let $\hat{w}^O$ be the unique contract in $[C^H(\mathsf{c}(\hat{\mathbf{x}}))]_{\mathsf{d}(\tilde{w}^O)} = [C^H(\mathsf{c}(\hat{\mathbf{x}}))]_{\mathsf{d}(y^1)}$. Thus, combining (25) and (29), we obtain that $\hat{w}^o \in R^H(\mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \{w^1, \ldots, w^o\}) \smallsetminus R^H(\mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \{w^1, \ldots, w^{o-1}\})$, so that Attribute A2 is satisfied. Moreover, by (24), we have that $\hat{w}^o \in R^H(\mathsf{c}(\tilde{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \{y^1, \ldots, y^{n_{o+1}-1}\})$ for $o < O$ as well as $\hat{w}^O \in R^H(\mathsf{c}(\tilde{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \{y^1, \ldots, y^N\})$, and so Attribute A3 is satisfied. Finally, Attribute A1 is satisfied as $\mathsf{d}(\tilde{w}^o) = \mathsf{d}(w^{o+1}) = \mathsf{d}(y^{n_{o+1}})$ for $o < O$ and $\mathsf{d}(\tilde{w}^O) = \mathsf{d}(w^1) = \mathsf{d}(y^1)$. This completes the proof of Case 2.

Substep 3: We will show that Attributes A1–A3 of the offer process $\mathbf{w}$ imply that the extended generalized pre-run rejection chain $(\mathbf{z}, \mathbf{y})$ at $\tilde{\mathbf{x}}$ under $\hat{\succ}$ satisfies Condition C3.

Note first that repeated application of Attribute A2 implies that

$$\{\hat{w}^1, \ldots, \hat{w}^O\} = R^H(\mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \{w^1, \ldots, w^O\}) \smallsetminus R^H(\mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}))$$

by Lemma A.2 as $(\hat{\mathbf{x}}, \mathbf{z}, \mathbf{w})$ is weakly observable by (17). Moreover, by repeated application of Attribute A3 we have that

$$\{\hat{w}^1, \ldots, \hat{w}^O\} \subseteq R^H(\mathsf{c}(\tilde{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \{y^1, \ldots, y^N\})$$

by Lemma A.2 as $(\tilde{\mathbf{x}}, \mathbf{z}, \mathbf{w})$ is weakly observable by Lemma B.1 as $(\mathbf{z}, \mathbf{y})$ is a generalized pre-run rejection chain at $\tilde{\mathbf{x}}$ under $\hat{\succ}$. Combining these last two observations yields that

$$R^H(\mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \{w^1, \ldots, w^O\}) \smallsetminus R^H(\mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z})) \subseteq R^H(\mathsf{c}(\tilde{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \{y^1, \ldots, y^N\}). \tag{30}$$

By the assumption that $\mathbf{z}$ satisfies Condition C3, we have $R^H(\mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z})) \smallsetminus R^H(\mathsf{c}(\hat{\mathbf{x}})) \subseteq R^H(\mathsf{c}(\tilde{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}))$ and so, as $(\tilde{\mathbf{x}}, \mathbf{z}, \mathbf{y})$ is weakly observable,[20] we have by Lemma A.2 that $R^H(\mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z})) \smallsetminus R^H(\mathsf{c}(\hat{\mathbf{x}})) \subseteq R^H(\mathsf{c}(\tilde{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \{y^1, \ldots, y^N\})$. Combining this with (30), we then obtain that

$$R^H(\mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \{w^1, \ldots, w^O\}) \smallsetminus R^H(\mathsf{c}(\hat{\mathbf{x}})) \subseteq R^H(\mathsf{c}(\tilde{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \{y^1, \ldots, y^N\}).$$

By (17), we have that $\mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \{w^1, \ldots, w^O\} = \mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \{y^1, \ldots, y^N\}$; therefore, $R^H(\mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \{y^1, \ldots, y^N\}) \smallsetminus R^H(\mathsf{c}(\hat{\mathbf{x}})) \subseteq R^H(\mathsf{c}(\tilde{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \{y^1, \ldots, y^N\})$, i.e., $(\mathbf{z}, \mathbf{y})$ satisfies Condition C3.

Substep 4: We will show Attributes A1–A3 of the offer process $\mathbf{w}$ imply that the extended generalized pre-run rejection chain $(\mathbf{z}, \mathbf{y})$ at $\tilde{\mathbf{x}}$ under $\hat{\succ}$ satisfies

[20] Recall that $(\tilde{\mathbf{x}}, \mathbf{z}, \mathbf{y})$ is weakly observable by Lemma B.1 as $(\mathbf{z}, \mathbf{y})$ is a generalized pre-run rejection chain at $\tilde{\mathbf{x}}$ under $\hat{\succ}$.

Condition C4.

Consider a doctor d. There are two cases to consider:

Case 1: $d \in \mathsf{d}(\{w^2, \ldots, w^O\}) \cap \tilde{D}$**.** By Condition C5 applied to $\mathbf{z}$ and the construction of the generalized pre-run rejection chain $\mathbf{y}$, there exists an n such that w^o was d's most preferred contract in $X_d \smallsetminus R^H(\mathsf{c}(\tilde{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \{y^1, \ldots, y^n\})$. Since $w^o \in \mathsf{c}(\mathbf{x}) \smallsetminus \mathsf{c}(\hat{\mathbf{x}})$ by the construction of $\mathbf{w}$, Claim 6 implies that $[\mathsf{c}(\hat{\mathbf{x}})]_d \subseteq R^H(\mathsf{c}(\tilde{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \{y^1, \ldots, y^n\})$. As $(\tilde{\mathbf{x}}, \mathbf{z}, \mathbf{y})$ is weakly observable,[21] Lemma A.2 implies that $[\mathsf{c}(\hat{\mathbf{x}})]_d \subseteq R^H(\mathsf{c}(\tilde{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \mathsf{c}(\mathbf{y}))$ so Condition C4 is satisfied for d.

Case 2: $d \in (D \smallsetminus \mathsf{d}(\{w^2, \ldots, w^O\})) \cap \tilde{D}$**.** Assume that

$$[\mathsf{c}(\hat{\mathbf{x}})]_d \nsubseteq R^H(\mathsf{c}(\tilde{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \mathsf{c}(\mathbf{y})) \tag{31}$$

as otherwise $(\mathbf{z}, \mathbf{y})$ immediately satisfies Condition C4 for d.

We argue first that $[C^H(\mathsf{c}(\hat{\mathbf{x}}))]_d \nsubseteq R^H(\mathsf{c}(\tilde{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \mathsf{c}(\mathbf{y}))$: As $(\tilde{\mathbf{x}}, \mathbf{z}, \mathbf{y})$ is weakly observable by Lemma B.1, Lemma A.2 combined with (31) implies $[\mathsf{c}(\hat{\mathbf{x}})]_d \nsubseteq R^H(\mathsf{c}(\tilde{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}))$; Condition C4 applied to $\mathbf{z}$ then yields $[C^H(\mathsf{c}(\hat{\mathbf{x}}))]_d \nsubseteq R^H(\mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}))$. Since $(\tilde{\mathbf{x}}, \mathbf{z}, \hat{\mathbf{x}})$ is weakly observable by Lemma A.1 and $\mathsf{c}(\tilde{\mathbf{x}}) \subseteq \mathsf{c}(\hat{\mathbf{x}})$, Lemma A.2 implies $[C^H(\mathsf{c}(\hat{\mathbf{x}}))]_d \nsubseteq R^H(\mathsf{c}(\tilde{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}))$. Given that $C^H(\mathsf{c}(\tilde{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}))$ and $C^H(\mathsf{c}(\hat{\mathbf{x}}))$ are feasible, we must have that either $[C^H(\mathsf{c}(\tilde{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}))]_d = [C^H(\mathsf{c}(\hat{\mathbf{x}}))]_d$ or $[C^H(\mathsf{c}(\hat{\mathbf{x}}))]_d \nsubseteq \mathsf{c}(\tilde{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z})$.

- If $[C^H(\mathsf{c}(\tilde{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}))]_d = [C^H(\mathsf{c}(\hat{\mathbf{x}}))]_d$, note that, by Lemma 3, we have that $[C^H(\mathsf{c}(\hat{\mathbf{x}}))]_d$ contains the worst contract in $\mathsf{c}(\hat{\mathbf{x}})$ with respect to $\succ_d$. Hence, if $[C^H(\mathsf{c}(\tilde{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}))]_d = [C^H(\mathsf{c}(\hat{\mathbf{x}}))]_d$, then Condition C5 applied to $\mathbf{z}$ implies $[\mathsf{c}(\hat{\mathbf{x}}) \smallsetminus C^H(\mathsf{c}(\hat{\mathbf{x}}))]_d \subseteq R^H(\mathsf{c}(\tilde{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}))$. As $(\tilde{\mathbf{x}}, \mathbf{z}, \mathbf{y})$ is weakly observable by Lemma B.1, Lemma A.2 implies $[\mathsf{c}(\hat{\mathbf{x}}) \smallsetminus C^H(\mathsf{c}(\hat{\mathbf{x}}))]_d \subseteq R^H(\mathsf{c}(\tilde{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \mathsf{c}(\mathbf{y}))$. Combining this with (31) yields $[C^H(\mathsf{c}(\hat{\mathbf{x}}))]_d \nsubseteq R^H(\mathsf{c}(\tilde{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \mathsf{c}(\mathbf{y}))$.
- If $[C^H(\mathsf{c}(\hat{\mathbf{x}}))]_d \nsubseteq \mathsf{c}(\tilde{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z})$, then the construction of $\mathbf{y}$ implies that $[C^H(\mathsf{c}(\hat{\mathbf{x}}))]_d \subseteq \mathsf{c}(\mathbf{y})$ only if $[\mathsf{c}(\hat{\mathbf{x}}) \smallsetminus C^H(\mathsf{c}(\hat{\mathbf{x}}))]_d \subseteq R^H(\mathsf{c}(\tilde{\mathbf{x}}) \cup$

[21] Recall that $(\tilde{\mathbf{x}}, \mathbf{z}, \mathbf{y})$ is weakly observable by Lemma B.1 as $(\mathbf{z}, \mathbf{y})$ is a generalized pre-run rejection chain at $\tilde{\mathbf{x}}$ under $\hat{\succ}$.

$\mathsf{c}(\mathbf{z}) \cup \mathsf{c}(\mathbf{y}))$.[22] But if $[\mathsf{c}(\hat{\mathbf{x}}) \smallsetminus C^H(\mathsf{c}(\hat{\mathbf{x}}))]_d \subseteq R^H(\mathsf{c}(\tilde{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \mathsf{c}(\mathbf{y}))$, then (31) again implies $[C^H(\mathsf{c}(\hat{\mathbf{x}}))]_d \subseteq C^H(\mathsf{c}(\tilde{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \mathsf{c}(\mathbf{y}))$.

Having established that $[\mathsf{c}(\hat{\mathbf{x}})]_d \nsubseteq R^H(\mathsf{c}(\tilde{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \mathsf{c}(\mathbf{y}))$ implies $[C^H(\mathsf{c}(\hat{\mathbf{x}}))]_d \nsubseteq R^H(\mathsf{c}(\tilde{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \mathsf{c}(\mathbf{y}))$, note that Condition C3 applied to $(\mathbf{z}, \mathbf{y})$ implies $[C^H(\mathsf{c}(\hat{\mathbf{x}}))]_d \nsubseteq R^H(\mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \mathsf{c}(\mathbf{y}))$.

This completes the proof of Claim 3. □

As explained in the discussion after the statement of Claim 3, Claim 3 implies Claim 2. This completes the proof of Claim 2. □

□

C Proofs for Section 4

Proposition 6. *If the choice function of every hospital is observably substitutable across doctors then for any preference profile $\succ$ and any two orderings $\vdash, \vdash'$, the set of all contracts available to hospitals at the end of the the cumulative offer process for $\vdash$ coincides with the set of all contracts available to hospitals at the end of the cumulative offer process for $\vdash'$.*

Proof. Fix a preference profile $\succ$. Let $\vdash$ be one ordering and $\mathbf{x} = (x^1, \ldots, x^M)$ be the corresponding complete offer process, and let $\vdash'$ be another ordering and $\mathbf{y} = (y^1, \ldots, y^N)$ be the corresponding complete offer process.

We show first that $\mathsf{c}(\mathbf{x}) \smallsetminus \mathsf{c}(\mathbf{y}) = \varnothing$. Suppose by way of contradiction that $\mathsf{c}(\mathbf{x}) \smallsetminus \mathsf{c}(\mathbf{y}) \neq \varnothing$ and let m be the smallest integer such that $x^m \notin \mathsf{c}(\mathbf{y})$. Let $\mathbf{x}' = (x^1, \ldots, x^{m-1})$. Three facts follow immediately:

1. $\mathsf{d}(x^m) \notin \mathsf{d}(C^H(\mathsf{c}(\mathbf{x}')))$, as $\mathbf{x}$ is an observable offer process.
2. $\mathsf{d}(x^m) \in \mathsf{d}(C^H(\mathsf{c}(\mathbf{y})))$, as $x^m \succ_{\mathsf{d}(x^m)} \emptyset$, $x^m \notin \mathsf{c}(\mathbf{y})$, and $\mathbf{y}$ is a complete offer process.

[22] To see this, note first that the generalized pre-run rejection chain constructed in Step 2 has the property that, for all $n \geq 2$, y^n is the most preferred contract for $\mathsf{d}(y^n)$ in $X_{\mathsf{d}(y^n)} \smallsetminus R^H(\mathsf{c}(\tilde{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \{y^1, \ldots, y^{n-1}\})$. Next, note that $y^1 \notin [C^H(\mathsf{c}(\hat{\mathbf{x}}))]_d$ since $y^1 \in \mathsf{c}(\hat{\mathbf{x}}') \smallsetminus \mathsf{c}(\hat{\mathbf{x}})$. Hence, given that $[C^H(\mathsf{c}(\hat{\mathbf{x}}))]_d \nsubseteq \mathsf{c}(\tilde{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z})$ and that $[C^H(\mathsf{c}(\hat{\mathbf{x}}))]_d$ contains the worst contract in $\mathsf{c}(\hat{\mathbf{x}})$ with respect to $\succ_d$ (Lemma 3), $[C^H(\mathsf{c}(\hat{\mathbf{x}}))]_d \subseteq \mathsf{c}(\mathbf{y})$ only if $[\mathsf{c}(\hat{\mathbf{x}}) \smallsetminus C^H(\mathsf{c}(\hat{\mathbf{x}}))]_d \subseteq R^H(\mathsf{c}(\tilde{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \mathsf{c}(\mathbf{y}))$.

3. $\mathsf{d}(x^m) \notin \mathsf{d}(\mathsf{c}(\mathbf{y}) \smallsetminus \mathsf{c}(\mathbf{x}'))$, as $\mathsf{c}(\mathbf{y}) \cap X_{\mathsf{d}(x^m)} \subseteq \mathsf{c}(\mathbf{x}')$ since $x^m \notin \mathsf{c}(\mathbf{y})$, each $x \in X_{\mathsf{d}(x^m)}$ such that $x \succ_{\mathsf{d}(x^m)} x^m$ is in $\mathsf{c}(\mathbf{x}')$, and $\mathbf{y}$ is a complete offer process.

Now, since $\mathbf{x}'$ and $\mathbf{y}$ are both compatible with respect to the same preference profile $\succ$, we can apply Lemma A.1 to infer that $(\mathbf{x}', \mathbf{y})$ is weakly observable. Since C^h is observably substitutable across doctors for all $h \in H$, we must have that, if $\mathsf{d}(x^m) \notin C^H(\mathsf{c}(\mathbf{x}'))$ and $\mathsf{d}(x^m) \notin \mathsf{d}(\mathsf{c}(\mathbf{y}) \smallsetminus \mathsf{c}(\mathbf{x}'))$, then $\mathsf{d}(x^m) \notin C^H(\mathsf{c}(\mathbf{x}') \cup \mathsf{c}(\mathbf{y})) = C^H(\mathsf{c}(\mathbf{y}))$, where the last equality follows from the fact that $\mathsf{c}(\mathbf{x}') \subseteq \mathsf{c}(\mathbf{y})$ by construction. But this statement and the three facts we showed previously can not be true simultaneously; thus, we have a contradiction.

The proof that $\mathsf{c}(\mathbf{y}) \smallsetminus \mathsf{c}(\mathbf{x}) = \varnothing$ is analogous.

□

Theorem 5. *If the choice function of every hospital is observably substitutable across doctors, then the cumulative offer mechanism is stable.*

Proof. Fix a profile of choice functions $C = (C^h)_{h \in H}$ that is observably substitutable across doctors, a preference profile $\succ = (\succ_d)_{d \in D}$ for the doctors, and an ordering $\vdash$ of the elements of X. For any $t \geq 1$, let y^t denote the (unique) contract that is offered in Step t of the cumulative offer process with respect to $\vdash$ and $\succ$ and set $A^t \equiv \{y^1, \ldots, y^t\}$.

We first show by induction on t that $C^H(A^t)$ is a feasible outcome. For $t = 0$, there is nothing to show. So suppose the statement is true up to some $t \geq 0$ and consider Step $t+1$. Let $h^{t+1} \equiv \mathsf{h}(y^{t+1})$. Note that for any $h \neq h^{t+1}$, we have that $A^t_h = A^{t+1}_h$ and $C^h(A^t) = C^h(A^{t+1})$. Now consider an arbitrary contract $x \in C^{h^{t+1}}(A^{t+1}) \smallsetminus \{y^{t+1}\}$. Note that if $x \in R^{h^{t+1}}(A^t)$, observable substitutability across doctors implies $\mathsf{d}(x) \in \mathsf{d}(C^{h^{t+1}}(A^t))$. Hence, $x \in C^{h^{t+1}}(A^{t+1}) \smallsetminus \{y^{t+1}\}$ and the inductive assumption imply that $\mathsf{d}(x) \notin \mathsf{d}(C^h(A^t)) = \mathsf{d}(C^h(A^{t+1}))$, for all $h \neq h^{t+1}$. This shows that $C^H(A^{t+1})$ is a feasible outcome.

Next, we will show that $A \equiv C^H(A^T)$ is stable. By construction, A is individually rational for hospitals. Moreover, each doctor only proposes acceptable contracts. To see that A is unblocked, consider an arbitrary set of contracts $Z \subseteq X \smallsetminus A$ such that $Z \succ_d A$ for all $d \in \mathsf{d}(Z)$. As every doctor proposes during the cumulative offer process every contract preferable to their assigned contract, we must have $Z \subseteq A^T \smallsetminus A$. Since $A = C^H(A^T)$ and $Z \subseteq X \smallsetminus A$, irrelevance of rejected contracts implies $A = C^H(A \cup Z)$.[23] Hence, Z is not a blocking set of A.

$\square$

Theorem 6. *If* $|H| > 1$ *and that the choice function of some hospital is not observably substitutable across doctors, then there exist unit-demand choice functions for the other hospitals such that no cumulative offer mechanism is stable.*

Proof. Let $h \in H$ be an arbitrary hospital and assume that C^h is not observably substitutable across doctors. Let $\mathbf{x} = (x^1, \ldots, x^M) \in X_h$ be an observable offer process for which there exists a contract $x \in \mathsf{c}(\mathbf{x})$ such that $x \in R^h(\{x^1, \ldots, x^{M-1}\}) \smallsetminus R^h(\{x^1, \ldots, x^M\})$ even though $\mathsf{d}(x) \notin \mathsf{d}(C^h(\{x^1, \ldots, x^{M-1}\}))$. Assume without loss of generality that $\mathbf{x}$ is *minimal* in the sense that, for all observable offer processes $\mathbf{y} = (y^1, \ldots, y^N)$ such that $\mathsf{c}(\mathbf{y}) \subsetneq \mathsf{c}(\mathbf{x})$, $y \in R^h(\{y^1, \ldots, y^{N-1}\}) \smallsetminus R^h(\{y^1, \ldots, y^N\})$ implies $\mathsf{d}(y) \in \mathsf{d}(C^h(\{y^1, \ldots, y^{M-1}\}))$.

Let $\bar{x}$ be a contract between $\mathsf{d}(x)$ and a hospital $\bar{h} \neq h$ and $\bar{x}^M$ be a contract between $\mathsf{d}(x^M)$ and $\bar{h}$.

For the doctors, we define $\succ$ by setting

1. for all m, m' such that $m < m'$ and $\mathsf{d}(x^m) = \mathsf{d}(x^{m'})$, $x^m \succ_{\mathsf{d}(x^m)} x^{m'} \succ_{\mathsf{d}(x^m)} \emptyset$,
2. $\bar{x} \succ_{\mathsf{d}(x)} \emptyset$ and, for all $m \in \{1, \ldots, M-1\}$ such that $\mathsf{d}(x^m) = \mathsf{d}(x)$, $x^m \succ_{\mathsf{d}(x)} \bar{x}$, and
3. $\bar{x}^M \succ_{\mathsf{d}(x^M)} x^M$ and, for all $m \in \{1, \ldots, M-1\}$ such that $\mathsf{d}(x^m) = \mathsf{d}(x^M)$, $x^m \succ_{\mathsf{d}(x^M)} \bar{x}^M$.

For $\bar{h}$, we set

$$C^{\bar{h}}(Y) = \begin{cases} \{\bar{x}\} & \bar{x} \in Y \\ \{\bar{x}^M\} & \bar{x} \notin Y \text{ and } \bar{x}^M \in Y \\ \varnothing & \text{otherwise.} \end{cases}$$

We show first that for any ordering $\vdash$, the set of contracts proposed in the cumulative offer process with respect to $\succ$ and $\vdash$ must be $\mathsf{c}(\mathbf{x}) \cup \{\bar{x}, \bar{x}^M\}$. This will be sufficient to prove Theorem 6 since $C^H(\mathsf{c}(\mathbf{x}) \cup \{\bar{x}, \bar{x}^M\}) = C^h(\{x^1, \ldots, x^M\}) \cup \{\bar{x}\}$

[23] Example 5 in Appendix C.1 shows that the irrelevance of rejected contracts condition is necessary to guarantee the existence of stable outcomes even when the choice functions of hospitals are observably substitutable and observably size monotonic.

and $\mathsf{d}(\bar{x}) = \mathsf{d}(x) \in \mathsf{d}(C^h(\{x^1, \ldots, x^M\}))$, so that the outcome of any cumulative offer process for $\succ$, $C^H(\mathsf{c}(\mathbf{x}) \cup \{\bar{x}, \bar{x}^M\})$, is not even feasible.

For the remainder, fix an arbitrary ordering $\vdash$ of the set of contracts and let $\mathbf{y}$ be the sequence of contracts that is produced by the cumulative offer process with respect to $\succ$ and $\vdash$. Note that we must have $\mathsf{c}(\mathbf{y}) \subseteq \mathsf{c}(\mathbf{x}) \cup \{\bar{x}, \bar{x}^M\}$ since doctors only rank contracts in the latter set as acceptable. Now suppose first that there is an m such that $x^m \notin \mathsf{c}(\mathbf{y})$. Without loss of generality, assume that $\{x^1, \ldots, x^{m-1}\} \subseteq \mathsf{c}(\mathbf{y})$. By the rules of cumulative offer processes, $\mathbf{y}$ must be a complete offer process with respect to $\succ$. Since $x^m \notin \mathsf{c}(\mathbf{y})$ and $x^m \succ_{\mathsf{d}(x^m)} \emptyset$, we must have $\mathsf{d}(x^m) \in \mathsf{d}(C^H(\mathsf{c}(\mathbf{y})))$. We will distinguish two cases:

1. $\mathsf{d}(x^m) \in \mathsf{d}(C^h(\mathsf{c}(\mathbf{y})))$

   Since $x^m \notin \mathsf{c}(\mathbf{y})$, the minimality of $\mathbf{x}$ implies that, for all observable offer processes $\tilde{\mathbf{y}} = (\tilde{y}^1, \ldots, \tilde{y}^O)$ such that $\mathsf{c}(\tilde{\mathbf{y}}) \subseteq \mathsf{c}(\mathbf{y})$, $\tilde{y} \in R^h(\{\tilde{y}^1, \ldots, \tilde{y}^{O-1}\}) \setminus R^h(\{\tilde{y}^1, \ldots, \tilde{y}^O\})$ only if $\mathsf{d}(\tilde{y}) \in \mathsf{d}(C^h(\{\tilde{y}^1, \ldots, \tilde{y}^{O-1}\}))$. Hence, Lemma A.1 implies that $((x^1, \ldots, x^{m-1}), \mathbf{y})$ is weakly observable.[24] Now given that $x^m \notin \mathsf{c}(\mathbf{y})$, the compatibility of $\mathbf{y}$ with $\succ$ implies that $\{x^m, \ldots, x^M\}_{\mathsf{d}(x^m)} \cap \mathsf{c}(\mathbf{y}) = \varnothing$. Since $\mathbf{x}$ is observable, we must have $\mathsf{d}(x^m) \notin \mathsf{d}(C^h(\{x^1, \ldots, x^{m-1}\}))$. But given that $\{x^1, \ldots, x^{m-1}\} \subseteq \mathsf{c}(\mathbf{y})$ and $\mathsf{d}(x^m) \in \mathsf{d}(C^h(\mathsf{c}(\mathbf{y})))$, there must exist an $n \leq N$ and a contract $y \in \{x^1, \ldots, x^{m-1}\}_{\mathsf{d}(x^m)}$ such that $y \in R^h(\{x^1, \ldots, x^{m-1}\} \cup \{y^1, \ldots, y^{n-1}\}) \setminus R^h(\{x^1, \ldots, x^{m-1}\} \cup \{y^1, \ldots, y^n\})$ and $\mathsf{d}(y) \notin C^h(\{x^1, \ldots, x^{m-1}\} \cup \{y^1, \ldots, y^{n-1}\})$. This contradiction shows that $\mathsf{d}(x^m) \in \mathsf{d}(C^h(\mathsf{c}(\mathbf{y})))$ is impossible.

2. $\mathsf{d}(x^m) \in \mathsf{d}(C^{\bar{h}}(\mathsf{c}(\mathbf{y})))$

   By construction of $C^{\bar{h}}$ and $\succ$, we must have $\mathsf{d}(x^m) \in \{\mathsf{d}(\bar{x}), \mathsf{d}(\bar{x}^M)\}$. It is easy to see that $\mathbf{y}$ can only be a complete offer process with respect to $\succ$ when $m = M$ and $\bar{x}^M \in C^{\bar{h}}(\mathsf{c}(\mathbf{y}))$. But if $m = M$, we have that $\mathsf{c}(\mathbf{y})_h = \{x^1, \ldots, x^{M-1}\}$ and hence $\mathsf{d}(x) \notin \mathsf{d}(C^h(\mathsf{c}(\mathbf{y})))$. Since $\mathbf{y}$ is a complete offer process with respect to $\succ$, we must then have that $\bar{x} \in \mathsf{c}(\mathbf{y})$ and hence, $\bar{x}^M \notin C^{\bar{h}}(\mathsf{c}(\mathbf{y}))$. This contradiction shows that $\mathsf{d}(x^m) \in \mathsf{d}(C^{\bar{h}}(\mathsf{c}(\mathbf{y})))$ is impossible.

[24] That we are able to use Lemma A.1 follows since the cumulative offer process with respect to $\succ' \equiv \succ^{\mathsf{c}(\mathbf{y})}$ and $\vdash$ must also produce the offer process $\mathbf{y}$. Hence, we can restrict attention to an economy in which only contracts in $\mathsf{c}(\mathbf{y})$ are available. Since $\mathsf{c}(\mathbf{y}) \subsetneq \mathsf{c}(\mathbf{x})$, the minimality of $\mathbf{x}$ implies that the choice function of h is observably substitutable across doctors in this associated economy.

Now given that $\{x^1, \ldots, x^M\} \subseteq \mathsf{c}(\mathbf{y})$, the compatibility of $\mathbf{y}$ with $\succ$ implies that $\bar{x}^M \in \mathsf{c}(\mathbf{y})$ as $\bar{x}^M \succ_{\mathsf{d}(x^M)} x^M$. But then $\mathbf{y}$ is observable only if there is an n such that $\bar{x}^M \in R^{\bar{h}}(\{y^1, \ldots, y^n\})$. Since the last statement is only possible when $\bar{x} \in \{y^1, \ldots, y^n\} \subseteq \mathsf{c}(\mathbf{y})$, we must have $\mathsf{c}(\mathbf{y}) = \{x^1, \ldots, x^N\} \cup \{\bar{x}, \bar{x}^M\}$.

$\square$

References

Hatfield, J. W., S. D. Kominers, and A. Westkamp (2017). Stability, strategy-proofness, and cumulative offer mechanisms. Mimeo, University of Texas at Austin.