# Lectures Notes on
# Dynamic Optimizing Models of Money

**or**

# How I Learned to Stop Worrying and Love Monetary Economics

James M. Nason†

February 12, 2018

*Abstract*

These lectures notes are intended for use in first and second year macro and money theory courses for PhD students. First, the first section offers a short review of U.S. macro data and evidence about the response of aggregate output and prices to identified supply and monetary shocks. Next, a small scale new Keynesian model is presented to motivate the usefulness of including money in dynamic stochastic general equilibrium (DSGE) models. The remainder of the notes study DSGE models with money-in-the-utility-function (MIUF) or cash-in-advance (CIA) constraints. The notes emphasize the pros and cons of MIUF and CIA constraints in DSGE models.

† *e-mail*: jmnason@ncsu.edu, *voice*: (919) 513–2884, *address*: Department of Economics, North Carolina State University, Campus Box 8110, Raleigh, NC 27695–8110.

# I. Introduction to Monetary Business Cycle Models

Real business cycle (RBC) theory became the dominate modeling paradigm in macroeconomics by 1990. Within a few years, research showed RBC theory lacks a straightforward propagation mechanism that matches actual observations. Further, research on the recessions of the 1980s and 1990s suggested there was a substantial monetary component driving these downturns in the U.S. economy. By the mid-1990s, empirical evidence and advances in monetary business cycle models began a new push to explain business cycle fluctuations in much the same way as Keynesian IS-LM models of the 1960s. In these new Keynesian models, the key propagation mechanisms involved sticky prices and nominal wages.

A problem is that without some friction private agents have no incentive to hold fiat currency in stochastic dynamic general equilibrium (DSGE) models. For example, agents have no reason to place positive and finite value on fiat currency in a RBC model. Consider a canonical one-sector growth model. Since this model satisfies the first two welfare theorems (*i.e.*, has a complete set of contingent claims markets), fiat currency has zero value. The nominal aggregate price level is non-positive in this economy. Some technology or friction must be grafted onto RBC models to provide money with a strictly positive price level. Two approaches receives the most attention. One approach is money-in-the-utility function (MIUF), which is either explicitly or implicitly almost always part of new Keynesian (NK) DSGE models. Another approach uses the transactions technology of the cash-in-advance (CIA) constraint. Both approaches to modeling money in DSGE models have strengths and weaknesses.

Before studying DSGE models with MIUF or a CIA constraint, these notes aim to motivate including money in macro models. First, there is a brief survey of the data and evidence that nominal shocks matter for business cycle fluctuations. Next, a small scale NK model is used to discuss whether money is a necessary ingredient to construct monetary models.

### *I.A A Sketch of the Evidence*

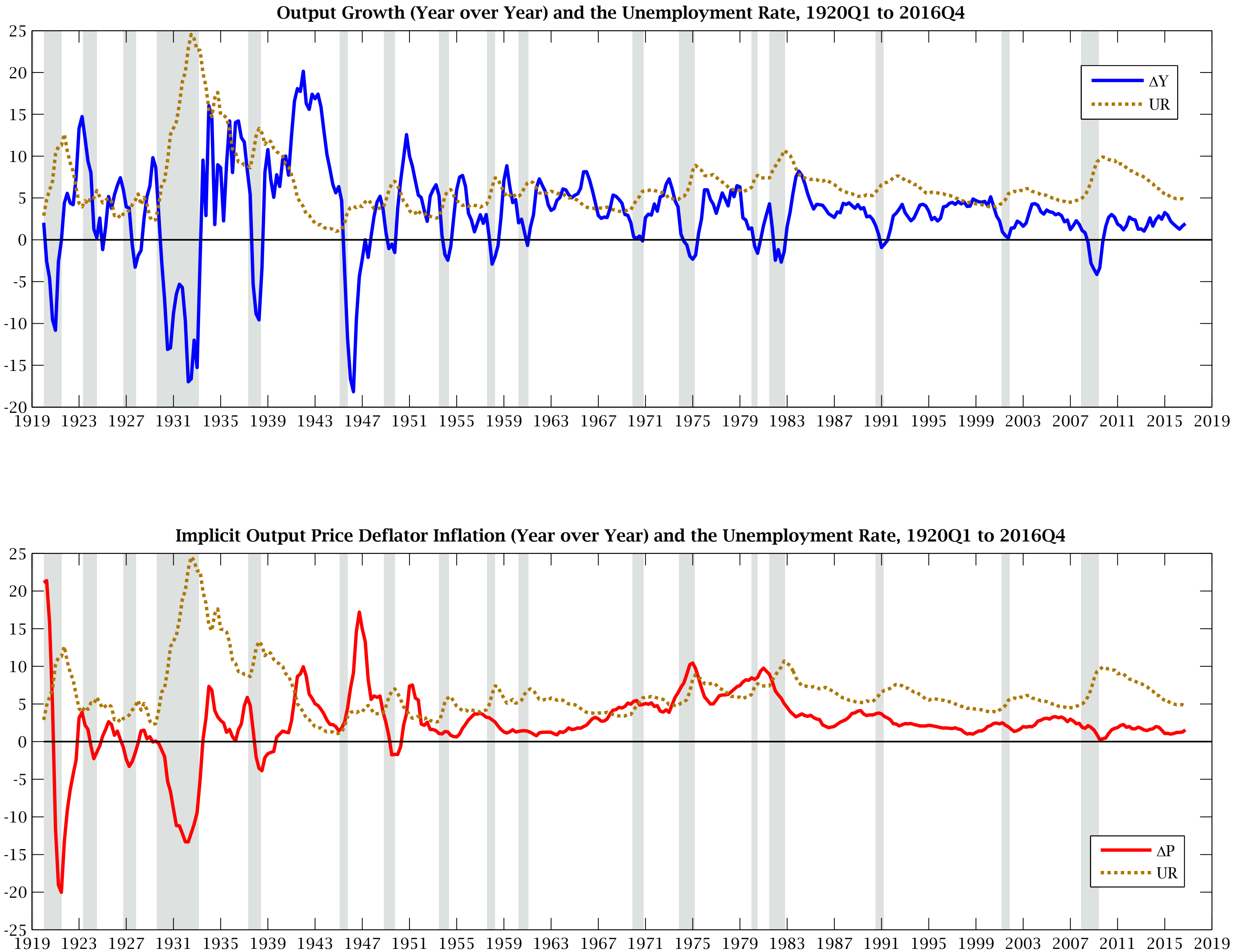
Much of modern macroeconomics is concerned with telling stories about output's response to identified productivity and monetary policy shocks. These stories are often embedded in monetary DSGE models. Macroeconomists evaluate the usefulness of these stories to describe the world by studying the fit of monetary DSGE models to the data.

Some of this data appears in figures 1 to 6. Figure 1 plots the real side of the U.S. economy and inflation from 1920$Q$4 to 2016$Q$4. Figure 1 plots output (real GDP) growth (year over year), inflation (growth in the output price deflator, year over year), and the unemployment rate. Short and long term private and government interest rates are displayed in figures 2, 3, and 4. The latter two figures depict term, risk, and liquidity spreads using these interest rates. Term, risk, and liquidity spreads are defined as the difference between the long yield and short rate, private long yield net of the government long rate, and gap between private and government short rates, respectively. Figures 5 and 6 show (year over year) growth rates of M1, M2, M3, and the monetary base. NBER dated recessions are the vertical gray bands in figures 1 to 6.

The top panel of figure 1 contains output growth (year over year) as the solid blue line. The dotted yellow line is the unemployment rate. The deepest troughs occur in 1932 and 1946 for output growth while its greater peak is in 1942. Output growth displays substantially less variation after 1948 compared with the interwar period. There is also a drop in output growth volatility between 1984 and 2007. The unemployment rate peaks in 1933 at close to 25 percent. Since the Great Depression, the highs are little more than 10 percent in the unemployment rate during the 1981–1982 and 2007–2009 recessions. After the last recession, nearly seven years are needed for the unemployment rate to drop in half. Output growth and the unemployment appear to move inversely around NBER dated recessions, which suggests visual support for Okun's rule. However, figure 1 offer no evidence that output growth is structurally causal prior to the unemployment rate.

The bottom panel of figure 1 presents inflation (year over year) and the unemployment rate (in levels). The solid (red) line plots inflation. The sharpest drop in inflation is seen in 1921 when the rate of deflation reaches $-20$ percent. During the Great Depression, there is sustained deflation that troughs at nearly $-18$ percent in 1932. The largest inflation spike is about 17 percent in 1946. The next peaks in inflation are about 10 percent in 1974 and 1981. Inflation is similar to output growth in that both are less volatile post-1948 and again post- 1984. However, these series are dissimilar in that inflation does not peak and trough with NBER dated recessions as does output growth. Also, inflation and the unemployment rate display Phillips curve-like comovement at times during the sample. However, this comovement is weak compared with the negative comovement observed for output growth and the unemployment rate.

**Figure 1: U.S. Real GDP Growth, Inflation, and the Unemployment Rate**



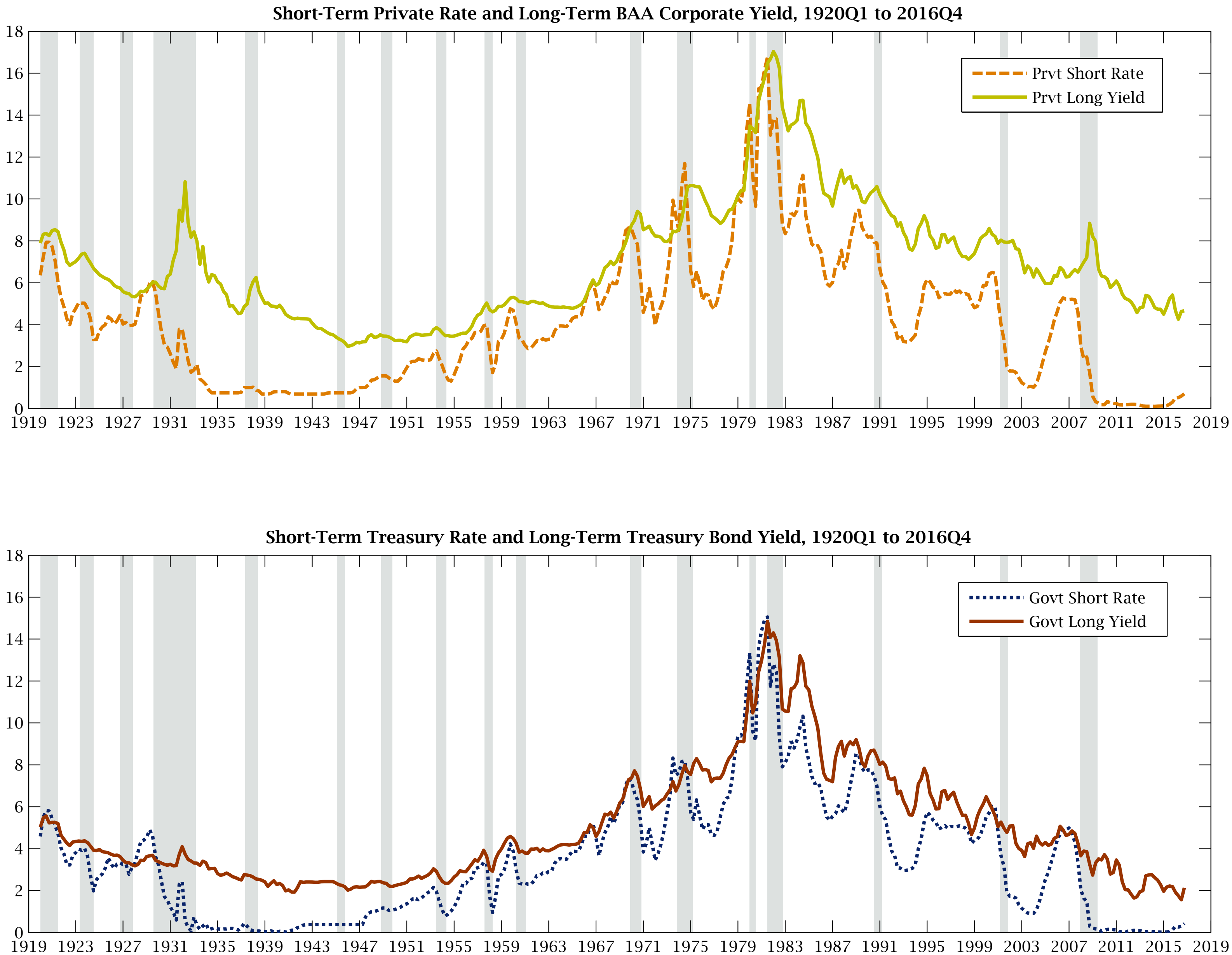

Note: The plots contain vertical gray bands that denote NBER dated recessions.

Figure 2 reports private and government nominal short term interest rates and long-term yields from 1920$Q4$ to 2016$Q4$. The private (government) short term interest rate and long term yield appear in the top (bottom) panel of figure 2. The solid lines are private (light green) and government (brick) long term interest rates in the top and bottom panels of figure 2. In the top and bottom panels of figure 2, the private short and government rates are the tan and dashed (tan) and dotted (blue) lines.

The interest rate plots in figure 2 reveal that in the U.S. short and long rates fell from the beginning of the sample to 1948, except for spikes around the time of the Great Depression. From 1948 to the early 1980s interest rates increased. However, private and government short rates display cycles associated with NBER dated recessions. These fluctuations in short rates begin with the 1953–1954 recession. Short and long term rates peak around the 1981–1982 recession at more then 16 (14) percent for returns on private (government) securities. Subsequently, there is steady drop in private and government short and long rates to the end of the sample. Also, the cycles in private short and government rates become more pronounced post-1980. During the same period, business cycle-like behavior also appear in private long and government yields. Another important feature of the bottom panel of figure 2 is that there are two extended periods of near zero short term government rates during the sample. Although low from 2008 to the end of the sample, the short term government interest rate is matched by near zero returns from 1932 to 1948.

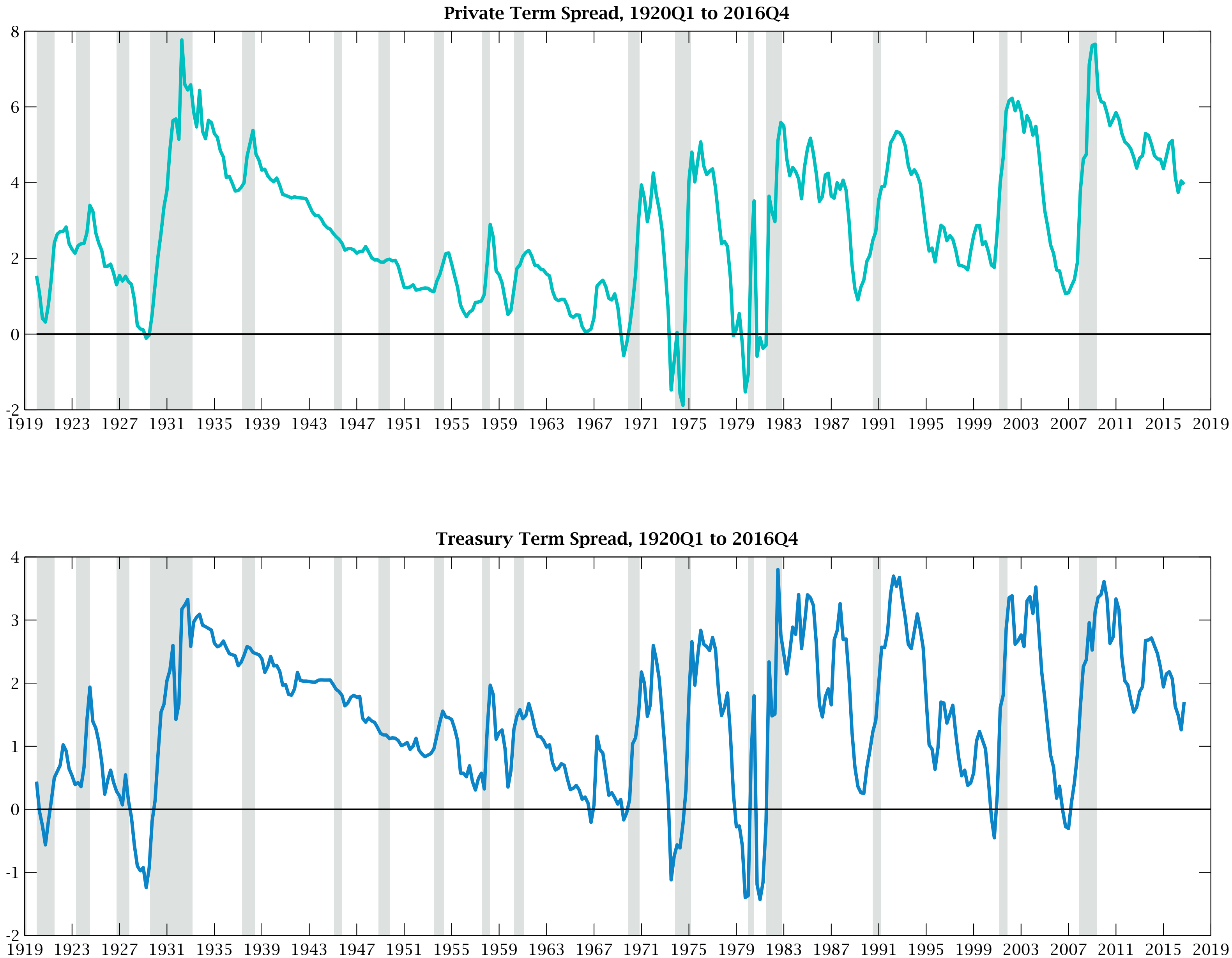
Term spreads are displayed in figure 3 from 1920$Q4$ to 2016$Q4$. The top and bottom panels of figure 3 plot solid lines, which are a private (teal) and a government (light blue) spreads of yields on long maturity securities minus returns on short term securities. The private and government term spreads are similar along several dimensions. The term spreads are low before a NBER dated business cycle peak, rise during the recession, and peak at NBER dated recession troughs, especially post-1954. Another feature common to both term spreads is a persistent decline from 1932 to 1953. A difference is the government term spread inverts (*i.e.*, the short rate is greater than the long rate) nine times in the sample, which is more than twice as many as found for the private term spread. However, the largest inversions (in absolute value) are found in the private term spread during the 1973–1975 recession and before the recession of 1980. At the other end, the largest private term spreads are seen in 1932 and 2009.

## Figure 2: Short-Term and Long-Term Private and Government Interest Rates



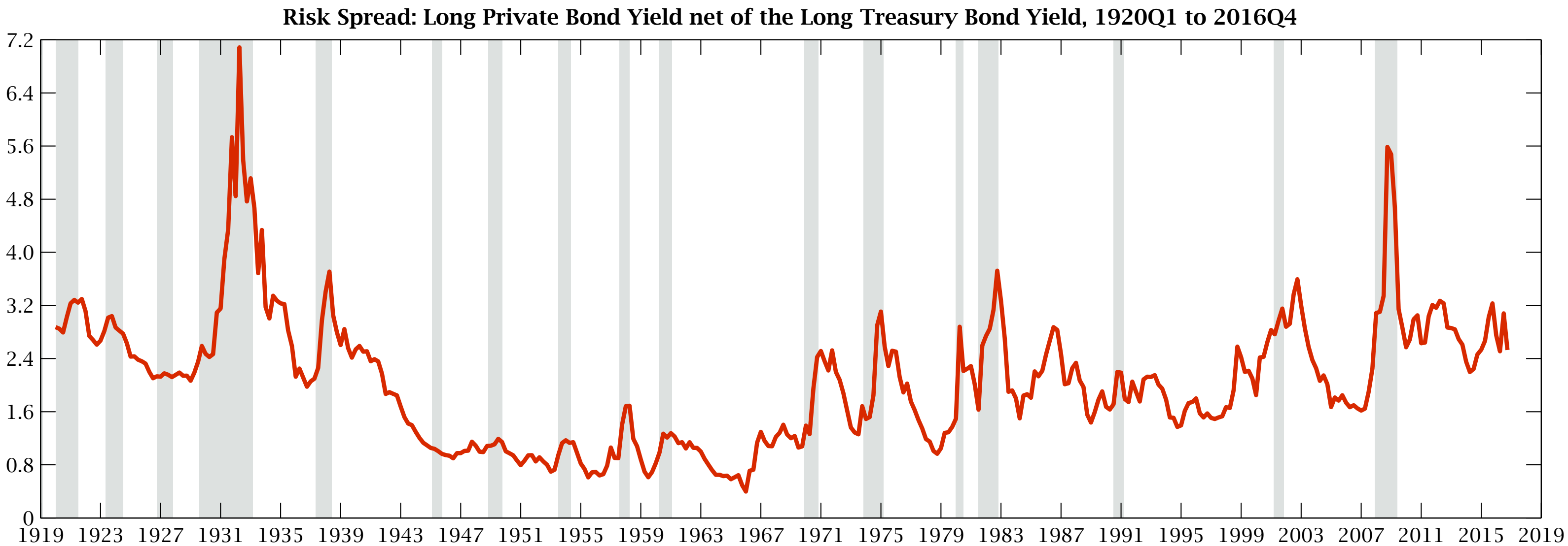

Note: The plots contain vertical gray bands that denote NBER dated recessions.

## Figure 3: Private and Government Term Spreads




Note: The plots contain vertical gray bands that denote NBER dated recessions.

## Figure 4: Risk and Liquidity Spreads





Note: The plots contain vertical gray bands that denote NBER dated recessions.

Figure 4 contain risk and liquidity spreads from $1920Q4$ to $2016Q4$. The risk spread is plotted as a solid (red) line in the top panel of figure 4. The most striking feature of the risk spread is that it is framed by spikes during the Great Depression and 2007–2009 recession. The former spike is greater than seven percent while the latter is about 5.6 percent. Otherwise, the risk spread is never greater than four percent during the sample. Examples are the 1937–1938 and 1981–1982 recession and in 2002. Nonetheless, the risk spread exhibits counter-cyclical business cycle comovement peaking during NBER recessions.

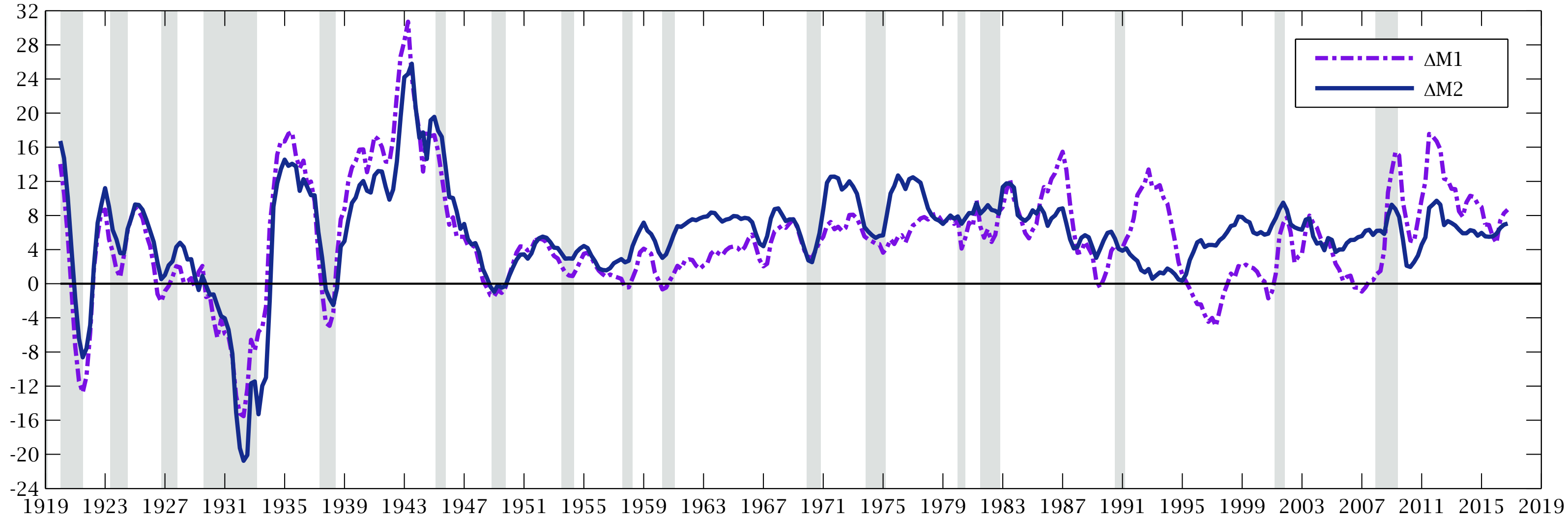
The liquidity spread is dominated by a spike during the 1973–1975 recession as shown in the bottom panel of figure 4. The next two largest peaks occur during the interwar period's recessions of 1920–1921 and 1929–1933 (*i.e.*, the Great Depression). Post-1975, the liquidity spread is greater during the double dip recessions of 1980 and 1981–1982, and the stock market event of 1987. There is a spike in the liquidity spread during the 2007-2009 recession, but it is smaller than the previously mentioned ones. Also, the liquidity spread peaks during NBER dated recessions similar to the risk spread.

Monetary aggregate growth rates are found in figures 5 and 6 from $1920Q4$ to $2016Q4$. The top and bottom panels of figure 5 report (year over year) growth rates of M1, M2, and M3. The plots of M1 and M2, and M3 are dot-dashed (purple) and solid (dark blue) lines in the top panel and in the bottom panel M3 is the solid (orange) line. Figure 6 repeats the plots of M1 and M2 growth and adds the monetary base growth rate (year over year).

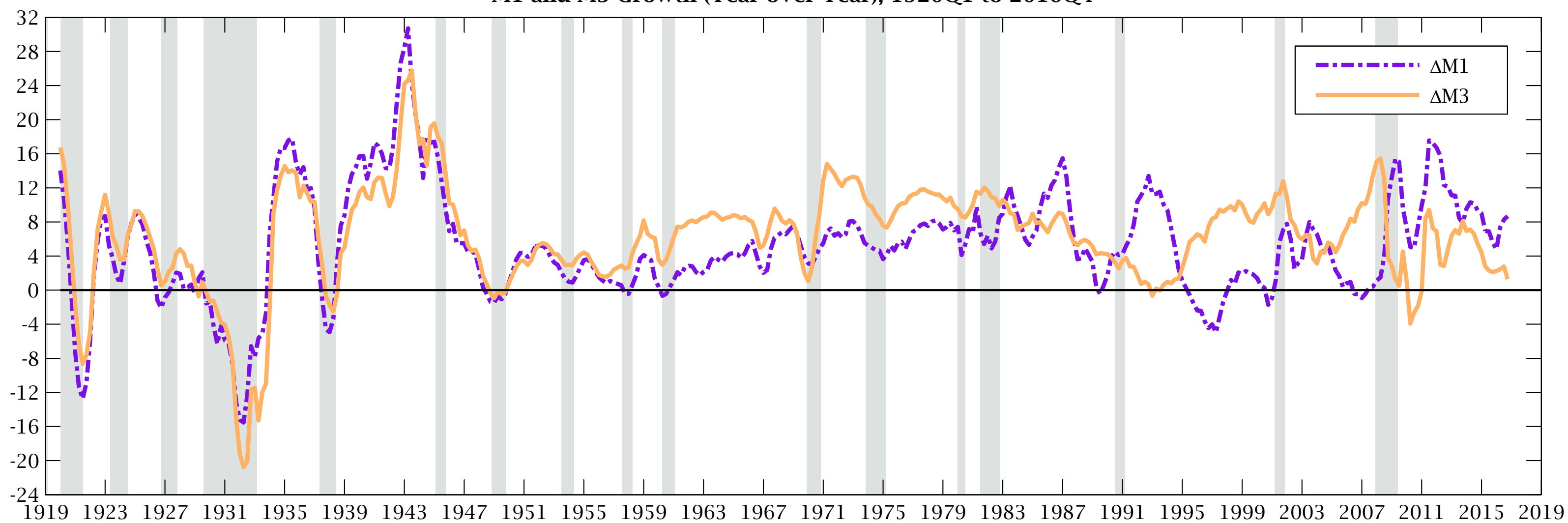
There is a drop in the volatility of the monetary growth rates post-1948 in figure 5. For example, M1 growth is about $-12$ percent in 1921, smaller than $-20$ percent in 1932, greater than 16 percent in 1935, and almost 32 percent in 1943. After 1948, the minimum is about negative four percent in 1997 and the maximum is 16 percent in 2011 for M1 growth. The growth paths of M2 and M3 growth display similar behavior during the sample. However, M3 growth has double digit growth for much of the 2000s. After the 2007–2009 recession, M3 growth turns negative. This makes M3 the only outside monetary aggregate to contract post-2007 while M1 and M2 growth are greater than 16 and eight percent in 2012. Nonetheless, M1, M2, and M3 fail to display consistent comovement with NBER dated recessions after 1948. The only exception is M2 growth is pro-cyclical during the 1950s, 1960s, and 1970s. This business cycle comovement disappears post-1983.

**FIGURE 5: GROWTH RATES OF M1, M2, AND M3**

M1 and M2 Growth (Year over Year), 1920Q1 to 2016Q4

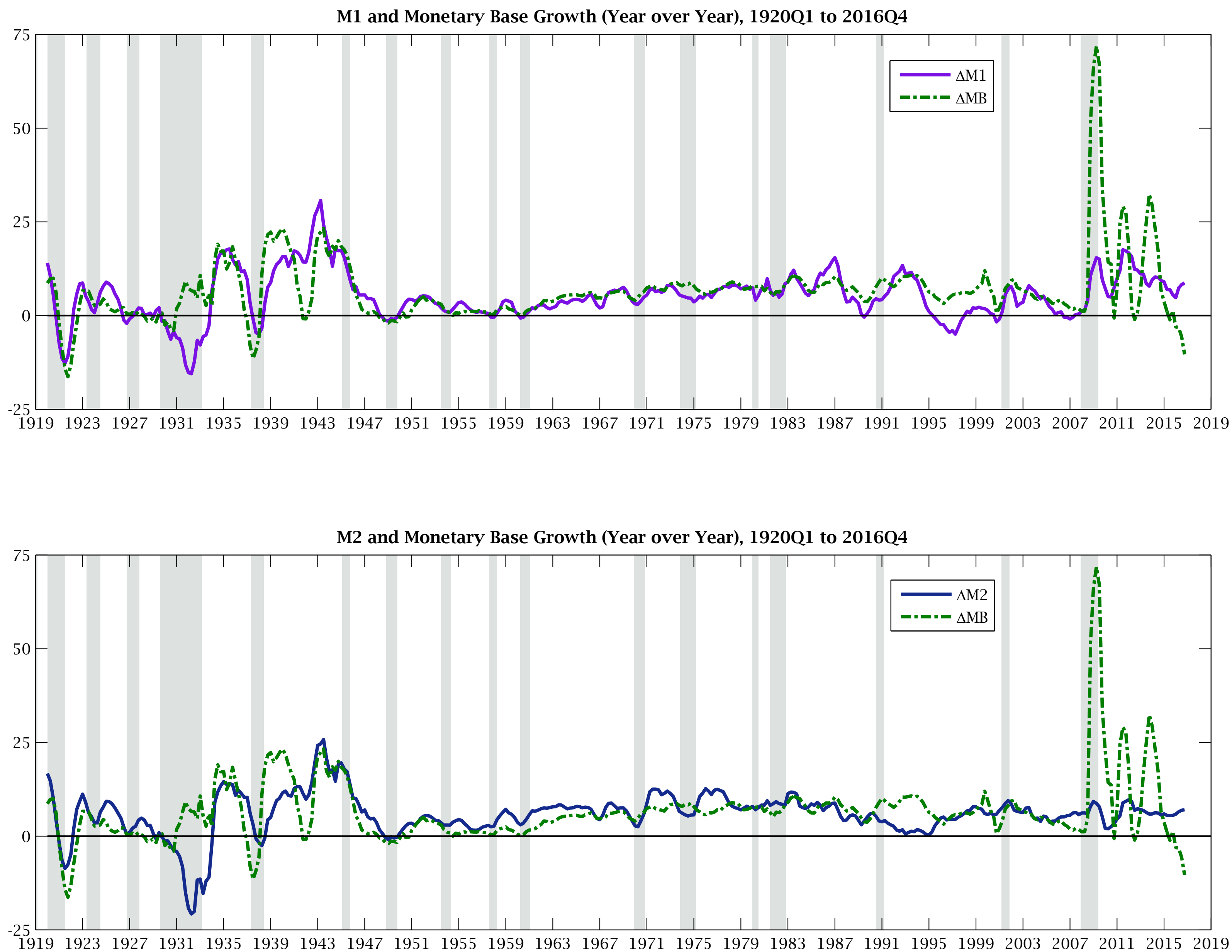



M1 and M3 Growth (Year over Year), 1920Q1 to 2016Q4




Note: The plots contain vertical gray bands that denote NBER dated recessions.

## Figure 6: Growth Rates of M1, M2, and the Monetary Base




Note: The plots contain vertical gray bands that denote NBER dated recessions.

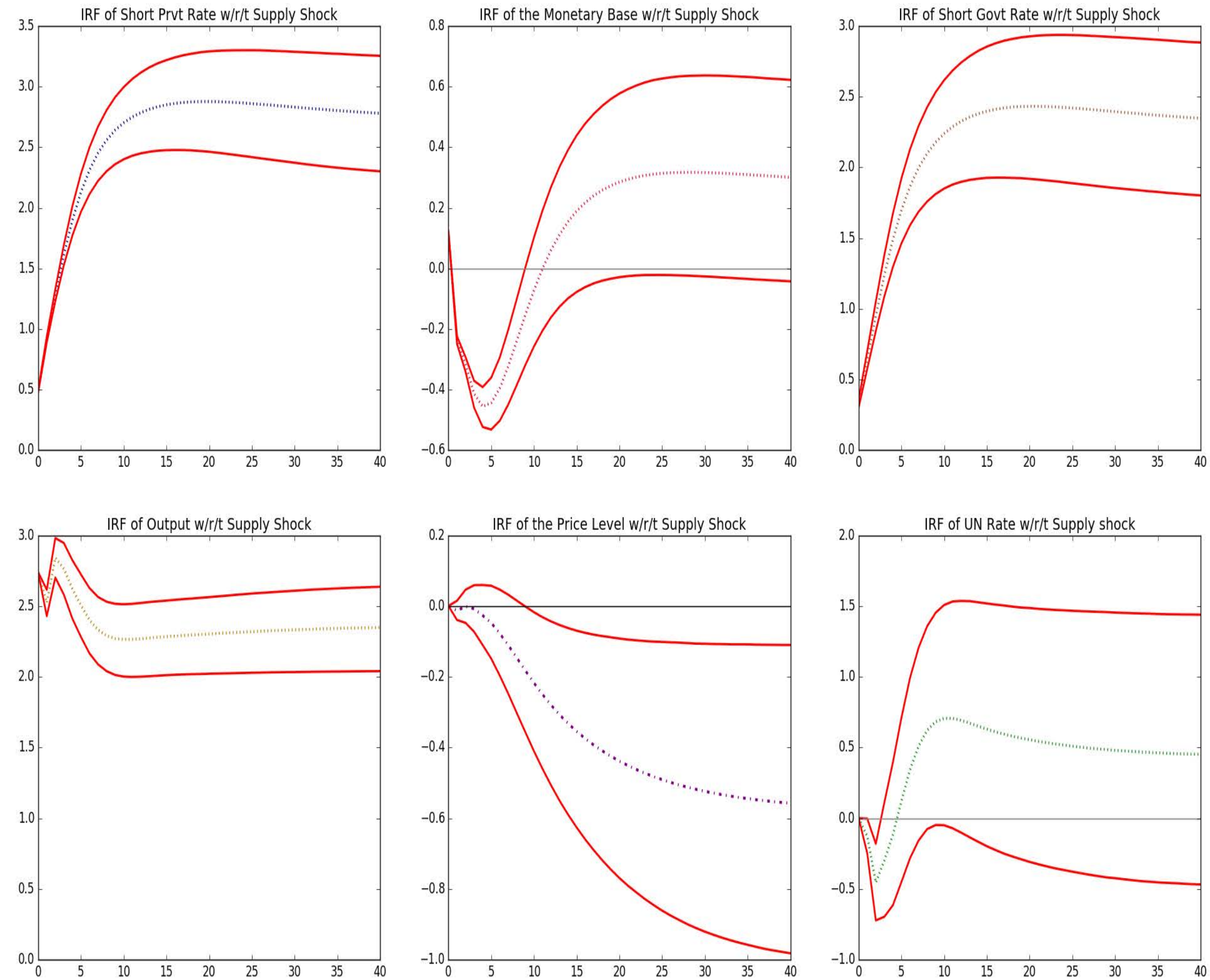
Figure 6 adds growth in the monetary base (year over year) to plots of of M1 and M2 growth. The (green) dot-dash plots of the top and bottom panels of figure 6 represent monetary base growth from 1920$Q4$ to 2016$Q4$. Similar to the monetary aggregate growth rates presented in figure 5, monetary base growth shows greater comovement with the U.S. business cycle during the interwar period than after. The reason is the monetary base was dominated by international flows of gold from 1920 to 1939. After 1948, movements in reserves are the most important source of changes in the monetary base. Still, the near 75 percent growth rate spike in the monetary base in 2009 is the most eye catching part of the top and bottom panels of figure 6. This growth in the monetary base drops to about 30 percent in 2012 and 2013. The monetary base displays similar growth rates only in the mid 1930s and from 1939 to 1944. An unresolved issue is that, although, growing post-2009 M1 and, to a lesser extent, M2, these inside monetary aggregates fail to match growth in the monetary base by several orders of magnitude.

Plots of these data are useful for thinking about business cycle comovement. Nevertheless, studying business cycle comovement this way is limited. Looking only at data gives no information about structural relationships. Models need assumptions and restrictions to address questions about the responses, say, of output to productivity and monetary policy shocks.

Figures 7 and 8 provide a start to answer these questions using some of the data reported in figures 1 to 6. The (incomplete) answers are provided by structural impulse response functions (IRFs). The IRFs are identified either by a supply shock or by a monetary policy shock. Figure 7 (8) contains the IRFs with respect to a supply (monetary policy) shock of the private short term rate, monetary base, short government interest rate, aggregate output, the aggregate price level, and the unemployment rate from impact to a horizon of 40 quarters.

The IRFs are calculated using a second-order VAR estimated on a sample starting in 1960$Q4$ and ending in 2006$Q4$. The supply (monetary policy) shock is identified as the orthogonalized forecast innovation of output growth (the short term government interest rate) by imposing the ordering described in the previous paragraph on a Cholesky decomposition of the covariance matrix of the reduced form VAR residuals. Figures 7 and 8 depict the structural IRFs as dotted lines. The solid (red) lines are one standard deviation confidence bands of the IRFs. The confidence bands are the 16th and 84th percentiles of 50,000 replications of the structural VAR generated using Monte Carlo integration methods.

## Figure 7: Responses to an Identified Supply Shock, 1960$Q4$ to 2006$Q4$



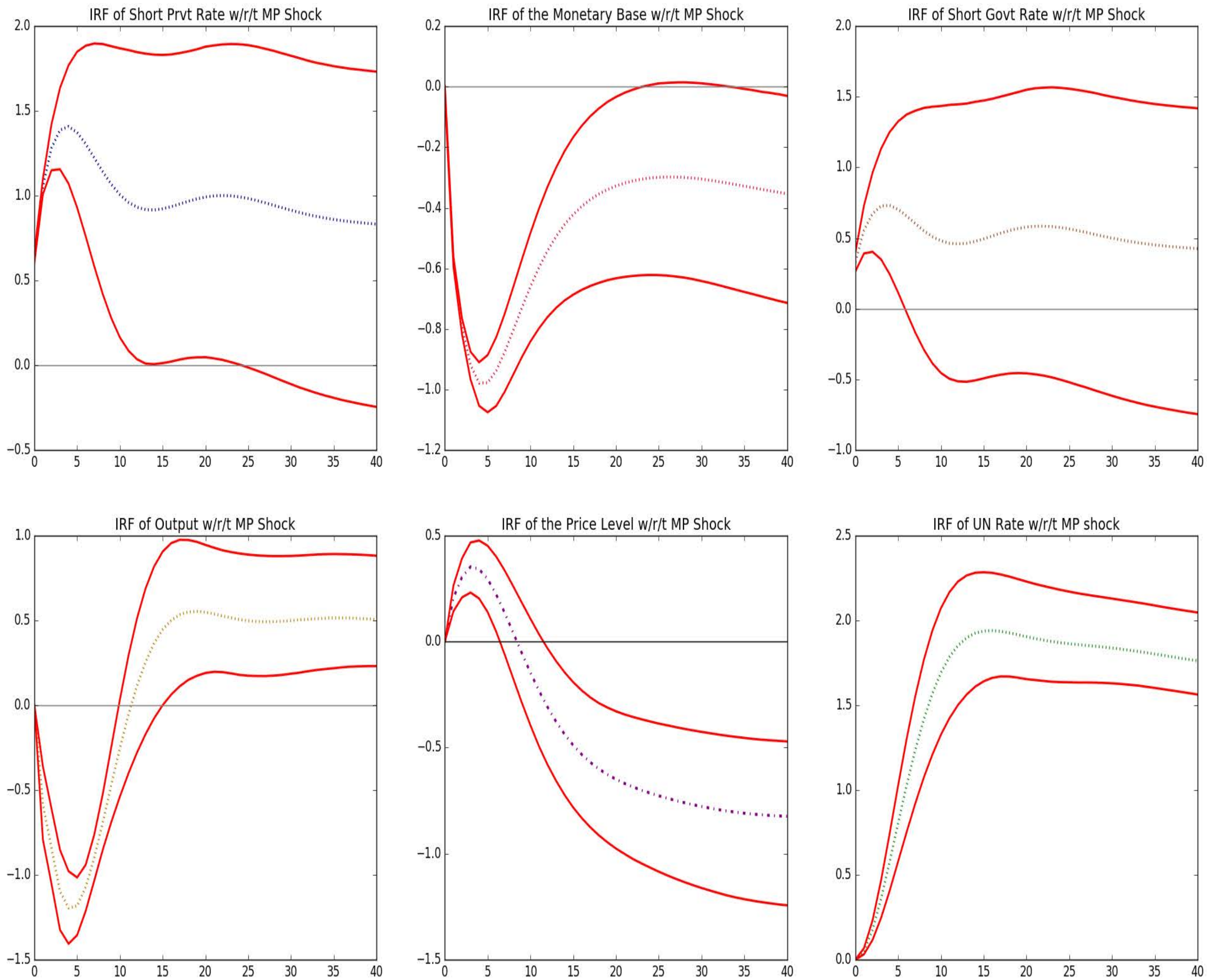

Note: The IRFs are the dotted (blue) lines from impact to a horizon of 40 quarters. The solid (red) lines are one-standard deviation confidence bands.

Dynamic responses to an identified supply shock appear in figure 7. The top left and right panels of figure 7 show interest rates rise from impact to about a three year horizon before leveling off in. The confidence bands of these IRFs do not include zero at each horizon from impact to ten years. The price level falls from about the one year to the ten horizon in the bottom middle panel of figure 7. Note the confidence bands of the IRF of the price level cease to contain zero (quarter by quarter) after the three year horizon. The supply shock also results in the monetary base having an inverted hump shape IRF in the top middle panel of figure 7. However, the confidence bands of the IRF cover zero at every quarter in the IRF horizon after three years. The bottom right panel of figure 7 shows the unemployment rate has a V-shaped response in response to the supply shock. This IRF falls for the first two quarters, subsequently rises to about the three year horizon before leveling off. However, after about one year, zero is covered by the confidence bands of the IRF of the unemployment rate quarter by quarter.

The bottom left panel of figure 7 displays the IRF of output with respect to the supply shock. Output is higher at impact, rises for two quarters, peaks at less than one year, before falling for the next four quarters, and then levels to the end of the remainder IRF horizon. The confidence bands of these IRFs are far from zero from impact to the 40 quarter horizon. The identified supply shock produces a small humped shaped dynamic response in output. This IRF better resembles a mix of the transition path of output in the short run to a transitory (*i.e.* demand) shock and in the long run to a permanent (*i.e.*, productivity) shock produced by a RBC model; see Cogley and Nason (1995).

In summary, the supply shock produces the expected responses in the private short term rate, monetary base, government short term rate, aggregate output, the aggregate price level, and the unemployment rate. Output is permanently higher. Unemployment is lower in the short run, but rises in the medium and long run. A possible explanation of the unemployment rate's response is that it reflects an increase in the real reservation wage as labor supply schedules shift up and to the left. The price level is lower from one to ten years, which combined with higher nominal rates signals higher real rates. Higher real rates suggest the demand for credit has increased, which is consistent with a decline in the monetary base. Banks reduce reserves in response to increased loan demand, which is consistent with higher real rates. Nonetheless, the response of the monetary base to the supply shock indicates it returns to steady state.

**Figure 8: Responses to an Identified Monetary Policy Shock, 1960$Q4$ to 2006$Q4$**




Note: The IRFs are the dotted (blue) lines from impact to a horizon of 40 quarters. The solid (red) lines are one-standard deviation confidence bands.

Figure 8 displays the impact of an identified monetary policy shock on the same variables. An unanticipated contractionary monetary policy shock yields a drop in real activity, (eventually) lower prices, higher nominal interest rates, and less inside money. These responses are consistent with the beliefs many economists have about the effects a monetary policy shock has on real and nominal activity.

Real activity consists of output and the unemployment rate for the VAR used to compute the IRFs reported in figure 8. The bottom left panel of figure 8 shows output has an inverted-hump shaped response IRF that troughs at a one-year horizon before rising to zero at the three-year horizon. Beyond three years, a monetary policy shock results in a greater level of output, which is not a standard result. The unemployment rate rises from impact to about a four-year horizon before falling slightly to the 10-year horizon as seen in the bottom right panel of figure 8. Note also the confidence bands of the IRF of output and the unemployment covers zero for only a few or no quarters of the 40 quarter horizon.

The price level rises at impact, peaks at quarter three, before falling and turning negative at the two-year horizon as shown in middle of the bottom row of panels in figure 8. This hump shaped IRF represents the "price puzzle." The price level (or inflation) is higher in reaction to a contractionary monetary policy shock. A good introduction to the price puzzle is Hanson (2004). Nonetheless, the monetary policy shock produces a lower price level from the three- to ten-year horizon as expected in response to the monetary policy shock. Except when the price level's IRF crosses the x-axis around the three-year horizon, a reasonable inference is the responses of the price level to the monetary shock is different from zero quarter by quarter.

The monetary policy shock appears to lower activity in credit markets. For example, the monetary base has an inverted-hump shaped IRF for the monetary base in the middle panel of the top row of figure 8 while the top left panel of figure 8 displays a hump shaped IRF for the private short term rate. These responses suggest a decline in credit market activity because an unexpected increase in the government short term rate reduces the monetary base. The drop in outside money is often taken as an incentive for banks to raise the cost of loans. The significance of a lower monetary base and an increase in the private appears to be limited quarter by quarter to the IRFs at impact to the three- or four-year horizon, according to the confidence bands reported in the top left and bottom middle panels of figure 8.

### I.B Motivating Money with a New Keynesian Model that Lacks It

New Keynesian (NK) models most often assign no role to money. Instead, the focus is on the responses of aggregate variables and inflation to an identified (and unanticipated) monetary policy shock; see Woodford (2003). Monetary policy is defined by an interest rate or Taylor rule

$$(1 - \rho_R \mathbf{L})R_t \;=\; \pi_t \;+\; (1 - \rho_R)\left[\kappa_\pi(\pi_t \;-\; \overline{\pi}) \;+\; \kappa_y \tilde{y}_t\right] + \varepsilon_{R,t}, \quad \varepsilon_{R,t} \sim \mathcal{N}\left(0,\, \sigma^2_{\varepsilon_R}\right), \qquad (1)$$

in new Keynesian models, where $\rho_R \in (-1,\, 1)$, $\kappa_\pi,\ \kappa_y > 0$, $\overline{\pi}$ is the inflation target of the monetary authority and $\tilde{y}_t$ is the output gap. The monetary policy shock is $\varepsilon_{R,t}$, which represents the source of the non-systematic variation in the (nominla) policy rate $R_t$. Movements in the final targets generate systematic variation in monetary policy. The final targets are deviations of inflation, $\pi_t$, from its target, $\overline{\pi}$, and movements in the output gap, $\tilde{y}_t$. The monetary authority changes its intermediate target, the policy rate $R_t$, to equate $\pi_t$ to $\overline{\pi}$ and set $\tilde{y}_t = 0$. The interest rate rule (1) guides the monetary authority in its quest to achieve these goals.

Money has no role in a monetary policy regime defined by the interest rate rule (1). The question is why. The answer is that the monetary authority supplies sufficient fiat currency to accommodate its demand at the policy rate, $R_t$. Still, this begs the question of the determination of equilibrium in the money market.

The monetary authority supplies money to ensure its policy rate clears the money market. Since money demand is flat with respect to $R_t$, understanding the incentives that give households, workers, and firms reasons to hold money is useful. One place to locate these incentives is the opportunity cost of holding fiat currency. The opportunity cost is the real rate. This turns the question into the determination of real rates in NK models.

A small scale NK model is a vehicle to study this question. Along with the interest rate rule (1), the dynamic IS schedule

$$\tilde{y}_t \;=\; \gamma_f\, \mathbf{E}_t \tilde{y}_{t+1} \;+\; \gamma_b \tilde{y}_{t-1} \;-\; \phi(R_t \;-\; \mathbf{E}_t \pi_{t+1}) \;+\; \varepsilon_{y,t}, \quad \varepsilon_{y,t} \sim \mathcal{N}\left(0,\, \sigma^2_{\varepsilon_y}\right), \qquad (2)$$

and the hybrid new Keynesian Phillips curve (NKPC)

$$\pi_t \;=\; \delta_f\, \mathbf{E}_t \pi_{t+1} \;+\; \delta_b \pi_{t-1} \;-\; \lambda \tilde{y}_t \;+\; \varepsilon_{\pi,t}, \quad \varepsilon_{\pi,t} \sim \mathcal{N}\left(0,\, \sigma^2_{\varepsilon_\pi}\right), \qquad (3)$$

complete the small scale NK model. Long run monetary neutrality holds if $\gamma_b = 1 - \gamma_f$. Assuming $\delta_b = 1 - \delta_f$ produces a vertical hybrid NKPC in the long run. Also, $\varepsilon_{y,t}$ is often interpreted as an aggregate demand shock while $\varepsilon_{\pi,t}$ is described as a markup shock.

There are five unknowns in the small scale NK model of equations (1), (2), and (3). The unknowns are $\tilde{y}_t,\ \pi_t,\ R_t,\ \mathbf{E}_t\tilde{y}_{t+1}$, and $\mathbf{E}_t\pi_{t+1}$. The problem is to solve for the "state" variables of the model. The state variables are the one-step ahead expectations of the output gap and inflation. The goal is to compute these expectations. One way to do this is to marginalize the small scale NK model with respect to $\tilde{y}_t$ and $\pi_t$. The idea is to construct reduced form equations for these variables using the Taylor rule (1), dynamic IS equation (2), and hybrid-NKPC (3); see Nason and Smith (2008).

The problem can also be solved by adding a Fisher equation to the small scale NK model. An unadorned version of the Fisher equation is $r_t^E = R_t - \mathbf{E}_t\pi_{t+1}$, where $r_t^E$ is the (expected) real rate. An interpretation of the Fisher equation is that it represents the demand side of the money market because $r_t^E$ is the opportunity cost of holding money. The opportunity cost of money is the income foregone by not holding assets with a greater (real) return. Changes in $r_t^E$ alter the demand for interest bearing assets and money as agents adjust their portfolios.

Lets show this by substituting for $\mathbf{E}_t\pi_{t+1}$ in the dynamic IS schedule (2) and the hybrid NKPC (3). The dynamic IS schedule (2) becomes

$$\tilde{y}_t \;=\; \gamma_f\,\mathbf{E}_t\tilde{y}_{t+1} \,+\, \gamma_b\tilde{y}_{t-1} \,-\, \phi r_t^E \,+\, \varepsilon_{y,t}, \tag{4}$$

which describes the equilibrium in the real economy under the Fisher equation. Solving the dynamic IS schedule (4) forward gives

$$\tilde{y}_t \;=\; \vartheta_1\tilde{y}_{t-1} \,+\, \vartheta_2\sum_{j=0}^{\infty}\vartheta_3^j\,\mathbf{E}_t r_{t+j} \,+\, \varepsilon_{y,t}, \tag{5}$$

where the $E$ superscript on the real rate is dropped to avoide redundant notation and $\vartheta_i$, $i = 1$, 2, and 3, are functions of the forward and backward roots (or eignevalues) of the second-order stochastic difference equation, which in turn are nonlinear functions of $\gamma_f,\ \gamma_b$, and $\phi$. The solution is a AR(1) in the output gap, which is backward looking in its own lag, and forward looking in the expected present discounted value of current and future real rates; see Sargent (1987, pp. 191–192) and Hansen and Sargent (2013, pp. 95–103). The solved dynamic IS schedule (5) appears independent of nominal variables.

The Fisher equation (1) and the solved dynamic IS (5) make the nominal side of the small scale NK model interest rate determined. Inflation become interest rate determined by substituting these equations into the hybrid NKPC (3)

$$\pi_t = \delta_f R_t + \delta_b \pi_{t-1} - \lambda\vartheta_1 \tilde{y}_{t-1} - (\delta_f + \lambda\vartheta_2) r_t^E - \lambda\vartheta_2 \sum_{j=1}^{\infty} \vartheta_3^j \mathbf{E}_t r_{t+j} + \varepsilon_{\pi,t} - \lambda\varepsilon_{y,t}, \quad (6)$$

The revised hybrid-NKPC (6) determines $\pi_t$ with pre-determined variables, $\pi_{t-1}$ and $\tilde{y}_{t-1}$, the current nominal and (expected) real rates, and the expected future path of the real rate. Next, consider replacing $\pi_t$ and $\tilde{y}_t$ in the interest rate rule (1) using the revised hybrid-NKPC (6) and the solved dynamic IS (5). The result is the policy rate, $R_t$, becomes a linear function of its own lag, pre-determined variables $\pi_{t-1}$, and $\tilde{y}_{t-1}$, $r_t^E$, and its future path. Since the interest rate rule (1) is the device the monetary authority employs to control $R_t$, the monetary authority must convince households, workers, and firms that their expectations about current and future real rates will be met or validated by monetary policy.

Monetary policy seen this way is, at least in part, about matching anticipated real returns on interest earning assets with private sector expectations. Monetary policy has to satisfy these expectations to generate a determinate (*i.e.* stable) monetary equilibrium in which $P_t \in (0, \infty)$. However, adding several interest bearing assets to macro models, and NK models in particular, adds complexity that makes solving these model difficult. In an earlier tradition, money served to summarize the impact of inflation expectations on portfolio decisions in a direct and concise way in monetary models.

This suggests asking

**Exercise Warming Up:** These are questions about the small scale NK model.

$(i)$ Solve the second-order difference equation (2) by pencil and paper. Show your entire work. If you need to place restriction on $\gamma_f$, $\gamma_b$, and $\phi$, list these. Interpret your solution.

$(ii)$ Does your answer to $(i)$ affect your view of how inflation is determined in equilibrium.

$(iii)$ What is the impact of your answers to $(i)$ and $(ii)$ on monetary policy under the interest rate rule (1)? Does solving the NK model of equations (1), (2), and (3) suggest that monetary policy in this economy is about $r_t^E$ rather than $R_t$? Why? Does this alter your views about the usefulness of the NK model? Explain

## II. Money-in-the-Utility Function

Most NK models, either explicitly or implicitly rely on MIUF to motivate the existence of nominal prices and wages. Quite literally, MIUF places real balances in the period utility of the representative agent. Real balances equals nominal cash balances or money, $M_t$, divided by the aggregate price level, $P_t$. Assume a perfectly inelastic labor supply, which makes period utility of the agent $\mathcal{U}(c_t, m_t)$, which is strictly concave, continuously differentiable on the positive real line, and satisfies the Inada conditions (*i.e.*, $\lim_{c\longrightarrow 0}\mathcal{U}_c(c, \cdot) = \infty$, $\lim_{c\longrightarrow\infty}\mathcal{U}_c(c, \cdot) = 0$, $\lim_{m\longrightarrow 0}\mathcal{U}_m(\cdot, m) = \infty$, and $\lim_{m\longrightarrow\infty}\mathcal{U}_m(\cdot, m) = 0$), where $c_t$ denotes consumption, $m_t = M_t/P_t$ are real balances the household owns. The Inada conditions guarantee positive demands for consumption and real balances. Also, assume $\mathcal{U}_m(\cdot, m) \leq 0$, for all $m > \underline{m}$. The inequality states that at high levels of real balances, $m$, the marginal utility of real balances becomes non-positive. This is important for the steady state equilibrium. For example, the MIUF specification $\mathcal{U}(c, m) = \ln c + \psi \ln m,\ \psi > 0$, violates the inequality.

### *II.A Deflation, Inflation, and Monetary Steady State Equilibria*

The household's model is completed with the budget constraint

$$w_t + (1 + r_t)b_{t-1} + \frac{M_{t-1}}{P_t} = c_t + b_t + \frac{M_t}{P_t} - \frac{X_t}{P_t},$$

where $w_t$ is exogenous income, $b_t$ is the net amount of unit discount bonds the household has outstanding at the end of date $t$, $r_t$ is the real return on the bonds, and $X_t$ is a nominal lump-sum transfer (or tax) from the government. The budget constraint of the government or monetary authority is $X_t = M_t - M_{t-1}$. For the moment, assume a perfect foresight equilibrium (*i.e.*, there is no uncertainty, say, from unanticipated income shocks). The dynamic Lagrange of the household is

$$\mathcal{L}_t = \sum_{j=0}^{\infty} \beta^{t+j}\mathcal{U}(c_{t+j}, m_{t+j}) + \sum_{j=0}^{\infty} \lambda_{t+j}\Bigg[ w_{t+j} + (1 + r_{t+j})b_{t-1+j} + \frac{M_{t-1+j}}{P_{t+j}} - c_{t+j} - b_{t+j} - \frac{M_{t+j}}{P_{t+j}} - \frac{X_{t+j}}{P_{t+j}} \Bigg],$$

where $\lambda_t$ is the shadow price of another unit of income. The control variable is $c_t$ and the state variables are $b_t$ and $m_t$. The first order necessary conditions (FONCs) are

$$\beta^t \mathcal{U}_{1,t} - \lambda_t = 0,$$

$$-\lambda_t + \lambda_{t+1}(1 + r_{t+1}) = 0,$$

and

$$\beta^t \frac{\mathcal{U}_{M,t}}{P_t} - \frac{\lambda_t}{P_t} + \frac{\lambda_{t+1}}{P_{t+1}} = 0.$$

Combine the first two FONCs to obtain the Euler equation for state variable $b_t$

$$\frac{1}{1 + r_{t+1}} = \beta \frac{\mathcal{U}_{c,t+1}}{\mathcal{U}_{c,t}}. \tag{7}$$

The Euler equation for $m_t$ is

$$\frac{\mathcal{U}_{M,t}}{\mathcal{U}_{c,t}} = 1 - \beta \frac{\mathcal{U}_{c,t+1}}{1 + \pi_{t+1}}, \tag{8}$$

which substitutes the first FONC into the third FONC for $\lambda_t$ and $\lambda_{t+1}$, where $1 + \pi_t = P_t/P_{t-1}$.

The optimality conditions (7) and (8) contain restrictions on consumption, real balances, the real rate, and inflation that must be satisfied for any equilibrium path of the economy. The real price of the bond equals the rate at which the household is willing to move consumption intertemporally, according to the Euler equation (8). The rate at which the household is willing to trade real balances for consumption intratemporally equals one minus the real return to $M_t$ valued in utils of consumption, which describes the optimality condition (7).

The demand for money is found by eliminating $\beta \mathcal{U}_{c,t+1}$ using the optimality conditions (7) and (8). The result is

$$\frac{\mathcal{U}_{M,t}}{\mathcal{U}_{c,t}} = 1 - \frac{1}{(1 + r_{t+1})(1 + \pi_{t+1})}.$$

Since the nominal rate $R_t \approx r_t + \pi_t$, given $r_t \times \pi_t \approx 0$, the household's money demand becomes

$$\frac{\mathcal{U}_{M,t}}{\mathcal{U}_{c,t}} = \frac{R_{t+1}}{1 + R_{t+1}}.$$

The left hand side of the equality captures the marginal rate of substitution between money and consumption. Along any candidate equilibrium path, this marginal rate of substitution must equal the nominal return on the unit discount bond discounted by $1 + R_t$. The household is willing to move cash from date $t$ to date $t + 1$ up to the point where the net benefit in utility equals the foregone discounted nominal interest. Cash (and real) balances generate an opportunity cost for the household. The ratio on the right hand side of the equality is this cost.

A central issue for monetary theory is that a steady state exists in which real balances have (strictly) positive and finite value, $m^* \in (0, \infty)$. Brock (1974, 1975) and Obstfeld and Rogoff (1983, 1986) develop the relevant results. Assume additive separability between consumption and real balances in period utility

$$\mathcal{U}(c_t, m_t) = u(c_t) + v(m_t).$$

The optimality condition (8) becomes

$$\frac{u'(c_t) - v'(m_t)}{P_t} = \beta \frac{u'(c_{t+1})}{P_{t+1}},$$

Let the growth rate of money be $1 + \mu = M_{t+1}/M_t$. Apply this to the previous expression and assume that the real side of the economy is at a steady state to produce

$$\left[u'(c^*) - v'(m_t)\right] m_t = \left[\frac{\beta u'(c^*)}{1 + \mu}\right] m_{t+1}.$$

This is a nonlinear difference equation in real balances. Lets rewrite it as

$$\mathcal{A}(m_t) = \mathcal{B}(m_{t+1}). \tag{9}$$

Without loss of generality, assume $\lim_{m \longrightarrow 0} v_m(m) m = 0$. This assumption yields at least two steady state equilibria. These exists one steady state equilibrium at $m = 0$. The other generates $m^* > 0$, which implies $P^* > 0$. The former steady state equilibrium forces both sides of the difference equation (9) to be zero. This explains the need for the assumption that $m$ plunges to zero faster than the marginal utility of real balances goes to infinity.

The equilibrium with positive steady state real balances is defined by the equality of (9). Figure 2.1 of Walsh (2017, p. 52) is a visual display of this result. The function $\mathcal{B}(m_{t+1})$ is the ray from the origin. The function $\mathcal{A}(m_t)$ initially falls as $m$ increases because $u'(c^*) < v'(m)$

for small $m$. As $m$ increases, the marginal utility of real balances falls and $\mathcal{A}(m_t)$ achieves its minimum and then begins to rise. Define $m^*$ as the steady state level of real balances at which the functions $\mathcal{A}(m_t)$ and $\mathcal{B}(m_{t+1})$ intersect. If $m > m^*$, the transversality condition

$$\lim_{j\longrightarrow\infty} \beta^j u'(c^*)m_{t+j} = 0$$

is violated. This rules out implosive paths for the aggregate price level, $P_{t+j} \longrightarrow 0$, as $j \longrightarrow \infty$. However, there is no reason to rule out a hyperinflation. Let $m$ decrease from $m^*$ to zero. First, $\mathcal{A}(m_t)$ must cross the $m$ axis to the right of the origin as $m$ becomes small. This forces $m = \overline{m}$ and $\mathcal{A}(\overline{m}) = 0$. Hence, an initial level of $m$ between $\overline{m}$ and $m^*$ moves toward $\overline{m}$ and then jumps to the origin with a steady state of $m = 0$.

This is a speculative hyperinflation because inflation is not driven by fundamentals. The price level is growing faster than the stock of nominal balances, but not because there is rapid money growth, $0 < \mu$. Instead, the hyperinflation is speculative. Households anticipate that $P_t$ is going to infinity faster than the government can print cash. Households have an incentive not to hold money because the purchasing power of money, $1/P_t$, is expected to fall to zero.

There are at least three ways to kill off hyperinflations. One way is with the assumption $\lim_{m\longrightarrow 0} \mathcal{A}(m) < 0$. Although this rules out paths that converge to $m = 0$, the problem is that $m < 0$ becomes the unique steady state solution. This is clearly impossible. A second way is to rule out hyperinflation is with the restriction on the real balance component of utility

$$\lim_{m\longrightarrow 0} v_m(m)m = \underline{m} > 0.$$

The implication is $\lim_{m\longrightarrow 0} \mathcal{A}(m) < 0$. This is not the worse of it. According to Obstfeld and Rogoff (1986), $\lim_{m\longrightarrow 0} v(m) = -\infty$. The restriction is the utility derived from real balances plunges to negative infinity as real balances collapse to zero. The household can never be compensated for the drop in real balances with a finite increase in the consumption good. The definition $\mathcal{A}(m) = \left[u'(c^*) - v'(m)\right]m$ is the source of this result. The restriction on $v(m)$ implies that $c$ must rise, driving its marginal utility to zero as $m$ rises. This is an incredible restriction on preferences, which questions the reasonableness of the MIUF approach.

A third way to rule out hyperinflation is for the government to fractional back fiat currency. Assume households know the government will always and everywhere trade some of the consumption good for a unit of fiat currency. Even if the government will give the household no more than a tiny $\epsilon$ of the consumption good per unit of fiat currency, a lower bound is placed on the value of real balances (and a upper bound on the aggregate price level); see Obstfeld and Rogoff (1983, pp. 683–684). Since the government is ready, willing, and able to back fiat current, households expect to obtain a strictly positive amount of consumption for fiat currency always and everywhere. Hence, the purchasing power of money is strictly positive. However, the fractional backing of fiat currency suggests that monetary economics cannot be separated from issues of the fiscal finance of governments.

### *II.B The Fisher Equation and MIUF*

The MIUF approach has its costs. Care needs to be taken when real balances are placed in the household's utility function. The restrictions on the MIUF necessary to annihilate hyperinflationary equilibria is a leading example. However, MIUF is widespread because of its ease at placing a positive value on fiat currency in equilibrium

A widely used MIUF specification is

$$\mathcal{U}\left(c_t,\ \ell_t,\ \frac{M_t}{P_t}\right) = \frac{\left[c_t^{\psi}\left(\frac{M_t}{P_t}\right)^{1-\psi}\right]^{1-\alpha}}{1-\alpha}\left[\frac{\ell_t^{1-\nu}}{1-\nu}\right],$$

where $\ell$ is household leisure, $0 < \psi < 1$, and $0 < \alpha,\ \nu$. Remember that if any of the curvature parameters equal one, the household has log utility. Note also that MIUF imposes an externality on the household. Observe that $P_t$ enters household preferences through real balances. Household welfare is affected by the actions of other households and the monetary authority through the determination of $P_t$.

The rest of the economy consists of the household's budget constraint, technology, and perfectly competitive money, bonds, and goods markets. The budget constraint is

$$y_t + (1-\delta)k_t + (1+r_t)b_t + \frac{M_t}{P_t} = c_t + k_{t+1} + b_{t+1} + \frac{M_{t+1}}{P_t} - \frac{X_t}{P_t}, \qquad (10)$$

where $y_t$ and $k_t$ are output and the capital stock, the depreciation rate $\delta \in (0, 1)$. Note the change in timing of nominal balances compared with the previous monetary model. The household also owns a constant returns to scale (CRS) technology

$$y_t = k_t^{\theta}\left[z_t n_t\right]^{(1-\theta)}, \quad \theta \in (0, 1), \tag{11}$$

where $n_t$ is labor input, $\ell_t = 1 - n_t$, and $z_t$ is labor augmenting total factor productivity (TFP).

The key to the propagation mechanism of the MIUF model is the interaction of the money demand function and expected inflation effect. Lets see why. The optimality conditions with respect to $b_{t+1}$ and $M_{t+1}$ are

$$\left[c_t^{\psi}\left(\frac{M_t}{P_t}\right)^{1-\psi}\right]^{1-\alpha}\frac{\ell_t^{1-\nu}}{c_t} = \beta\mathbf{E}_t\left\{\left[c_{t+1}^{\psi}\left(\frac{M_{t+1}}{P_{t+1}}\right)^{1-\psi}\right]^{1-\alpha}\frac{\ell_{t+1}^{1-\nu}}{c_{t+1}}\left(1 + r_{t+1}\right)\right\}, \tag{12}$$

and

$$\left[c_t^{\psi}\left(\frac{M_t}{P_t}\right)^{1-\psi}\right]^{1-\alpha}\frac{\ell_t^{1-\nu}}{c_t}$$

$$= \beta\mathbf{E}_t\left\{\left[c_{t+1}^{\psi}\left(\frac{M_{t+1}}{P_{t+1}}\right)^{1-\psi}\right]^{1-\alpha}\frac{\ell_{t+1}^{1-\nu}}{c_{t+1}}\frac{P_t}{P_{t+1}}\left[1 + \frac{1-\psi}{\psi}\left(\frac{P_{t+1}c_{t+1}}{M_{t+1}}\right)\right]\right\}, \tag{13}$$

respectively. Bond market optimality has the usual interpretation. The household buys an additional bond to move consumption intertemporally up to the point the loss in current utility equals the discounted expected gain in utility from consuming the return to the bond.

Optimality in the money market requires the household to give up some marginal utility of consumption today to gain another unit of nominal balances up to point where the discounted expected benefit of the unit of nominal balances, which is the unit of nominal balances plus the consumption velocity of money weighted by the share of real balances to consumption in utility valued at the additional marginal utility of consumption. MIUF is the motivation for the household to value fiat currency.

The optimality conditions (12) and (13) yield the money market arbitrage condition

$$\mathbf{E}_t\left\{\left[c_{t+1}^{\psi}\left(\frac{M_{t+1}}{P_{t+1}}\right)^{1-\psi}\right]^{1-\alpha}\frac{\ell_{t+1}^{1-\nu}}{c_{t+1}}\frac{1}{1+\pi_{t+1}}\left[\frac{1-\psi}{\psi}\left(\frac{P_{t+1}c_{t+1}}{M_{t+1}}\right) - \left(r_{t+1} + \pi_{t+1}\right)\right]\right\} \approx 0, \tag{14}$$

where $r_{t+1}\pi_{t+1} \approx 0$. The arbitrage forces the marginal utility of an extra unit of nominal balances to equal the nominal return on $b_{t+1}$, $r_{t+1} + \pi_{t+1}$ . Hence, equilibrium in the money market is driven by two factors. First, the household values real balances for its contribution to its welfare net of the real return to cash. The other factor is the opportunity cost of holding money, which is the real return to the bond.

The term in the brackets of the arbitrage condition (14) suggests a money demand function for the MIUF model. This term implies, $\ln M_t - \ln P_t = \ln c_t - (r_t + \pi_t)$ + expectation error. Money demand displays liquidity preference (*i.e.*, changes in the nominal rate alter the demand for money), a positive consumption elasticity of money demand, and an error term that is a function of the expected marginal utility of real balances and nominal rate.

The bond optimality condition (12) restricts the Fisher equation of this model. Rewrite this Euler equation as

$$1 = \mathbf{E}_t \left\{ \Gamma_{t+1} \left( \frac{M_{t+1}}{M_t} \right)^{(1-\psi)(1-\alpha)} \left(1 + \pi_{t+1}\right)^{(1-\psi)(1-\alpha)-1} \left(1 + r_{t+1}\right) \left(1 + \pi_{t+1}\right) \right\},$$

where $\Gamma_{t+1} \equiv \beta \left( \frac{c_{t+1}}{c_t} \right)^{\psi(1-\alpha)-1} \left( \frac{\ell_{t+1}}{\ell_t} \right)^{1-\nu}$, which folds the growth rate of leisure into the SDF. The previous Euler equation becomes

$$1 = \mathbf{E}_t \left\{ \Gamma_{t+1} \left(1 + \mu_{t+1}\right)^{(1-\psi)(1-\alpha)} \left(1 + \pi_{t+1}\right)^{(1-\psi)(1-\alpha)-1} \left(1 + R_{t+1}\right) \right\},$$

where $R_t = (1 + r_t)(1 + \pi_t)$ and denoting the growth rate of money as $1 + \mu_{t+1} = M_{t+1}/M_t$. Identify the inverse of $\Gamma_t$ with the stochastic risk-free (real) rate, $r_{F,t+1}$ and log linearize the bond market optimality condition to produce Fisher's equation

$$\mathbf{E}_t \left\{ R_{t+1} \right\} = \mathbf{E}_t \left\{ r_{F,t+1} \right\} + \mathbf{E}_t \left\{ \pi_{t+1} \right\} + (1 - \psi)(1 - \alpha) \mathbf{E}_t \left\{ \pi_{t+1} - \mu_{t+1} \right\}. \qquad (15)$$

Fisher's equation holds exactly in this model when utility is log, $\alpha = 1$, or if expected inflation equals expected money growth. Otherwise, an increase in expected inflation generates positive or negative changes in the expected nominal bond return, given the preference parameters $\alpha$ and $\psi$. Hence, the monetary propagation mechanism of the MIUF approach is either the expected inflation effect or the liquidity effect. Greater curvature in the utility function (*i.e.*, a larger $\alpha$ gives greater risk aversion) is required for the liquidity effect to dominate. Otherwise, the expected inflation effect drives fluctuations in the nominal return to the unit discount bond. Changes in the intertemporal opportunity cost of money matter more for movements in the expected nominal rate. In either case, movements in $\mathbf{E}_t\pi_{t+1}$ generate variation in the nominal rate independent of real factors (*i.e.*, consumption growth and leisure). This produces fluctuations in consumption because of the impact on the Euler equation of capital.

This suggest questions about the impact of TFP and money growth on the model. Consider

**Exercise MIUF:** Let the money growth shock and TFP follow

$$\mu_{t+1} = (1 - \rho_\mu)\mu^* + \rho_\mu\mu_t + \eta_{t+1}, \quad \eta_{t+1} \sim \mathcal{N}\left(0, \sigma_\eta^2\right), \tag{16}$$

and

$$\ln z_{t+1} = \gamma + \ln z_t + \varepsilon_{t+1}, \quad \varepsilon_{t+1} \sim \mathcal{N}\left(0, \sigma_\varepsilon^2\right), \tag{17}$$

where $\rho_\mu \in (-1, 1)$ and $\gamma > 0$. Define the stationary component of the aggregate price level to be $\widetilde{P}_t = P_t A_t / M_t$.

$(i)$ In MIUF models, does an increase in expected inflation induce the household to demand more real balances or less? Explain.

$(ii)$ Stochastically detrend the MIUF model just presented and compute the steady state of its endogenous variables. Describe the steady state.

$(iii)$ An economy exhibits superneutrality when changes in the money growth rate have no impact on the levels of real variables. What restrictions on the utility function parameters are necessary for this MIUF model to possess supernuetrality at the steady state? [Hint: Examine optimality in the labor market.] Does it matter that leisure is non-additively separable from consumption and real balances in the utility function? Explain.

$(iv)$ Discuss the impact of the restrictions on the preference parameters suggested by part $(ii)$ of this question on the transition dynamics to the steady state (you do not have to linearize the model to answer the question, but it might help). Do anticipated or unanticipated movements in money growth matter for real variables along the transition path? That is, does this economy posses expectational neutrality? Explain.

$(v)$ Discuss the role MIUF has in consumption. [Hint: Is the cost of obtaining consumption lower when the household has more real balances?]

## III. Shopping Time Technology and the CIA Constraint

MIUF is a straightforward way to introduce fiat currency into a DSGE model. Since real balances provide utility, the household is given a reason to hold fiat currency. In some cases, the aggregate price level has strictly positive and finite value in equilibrium. Still this begs the question of the source(s) of the transactions services that makes real balances valuable for the household. The answer to the question is that MIUF is an indirect utility function. The task is to study the primitives that underly this indirect utility function.

Suppose the household has to use its real balances and some of its time endowment to shop. The maintained assumptions are that it takes time to shop and the costs of shopping are inversely related to real balances. These suggest the shopping time function

$$c_t = \mathcal{C}\left(n_{c,t}, \frac{M_t}{P_t}\right), \quad \mathcal{C}_x \geq 0, \quad \mathcal{C}_{x,x} \leq 0, \quad x = n_{c,t}, \frac{M_t}{P_t},$$

where $n_{c,t}$ is the part of the household time endowment spent shopping and $1 = \ell_t + n_t + n_{c,t}$. No restriction is placed on the cross derivative of the shopping time technology, $\mathcal{C}_{n,m}$, but assume that $\mathcal{C}(\cdot, \cdot)$ satisfies the requirements of the inverse function theorem, $n_{c,t} = \mathcal{N}\left(c_t, \frac{M_t}{P_t}\right)$. Next, assume the household's direct utility function is $\mathcal{W}(c_t, 1 - n_t - n_{c,t})$. Substitute for $n_{c,t}$ in $\mathcal{W}(\cdot, 1 - n_t - n_{c,t})$ to construct the indirect utility function

$$\mathcal{W}\left(c_t, 1 - n_t - \mathcal{N}\left(c_t, \frac{M_t}{P_t}\right)\right) = \mathcal{U}\left(c_t, \frac{M_t}{P_t}, n_t\right).$$

The difference between $\mathcal{W}(\cdot, \cdot)$ and $\mathcal{U}(\cdot, \cdot, \cdot)$ is the indirect utility function contains a market price and a nominal stock of wealth, $M_t$.

Restrictions on the transactions technology translate into restrictions on the MIUF. For example, when $\mathcal{U}_{c,M}$ is positive, additional real balances yield higher marginal utility of consumption. Hence, consumption and real balances are complements. This restriction on the indirect utility function gives MIUF the interpretation that real balances produce transactions services. An implication is an increase in expected inflation lowers real economic activity in response to a higher rate of money growth.

On the other hand, $\mathcal{U}_{c,M} \leq 0$ implies the household is willing to substitute real balances for consumption (or the converse). By making consumption and real balances substitutes, real balances serve as an asset in utility because higher real balances are associated with lower consumption. When money growth generates higher expected inflation, greater real economic activity results under the assumption $\mathcal{U}_{c,M} \le 0$.

The direct utility function determines the sign of the cross derivative, $\mathcal{U}_{c,M}$. Differentiation with respect to $c$ followed by differentiation with respect to $m$ yields

$$\mathcal{U}_{c,M} \quad = \quad \Big[W_{\ell,\ell}\mathcal{N}_c \; - \; \mathcal{W}_{c,\ell}\Big]\mathcal{N}_M \; - \; \mathcal{W}_{\ell}\mathcal{N}_{c,M}.$$

Theory provides no restrictions to sign $\mathcal{U}_{c,M}$. The lack of information to sign the cross derivatives $\mathcal{W}_{c,\ell}$ and $\mathcal{N}_{c,M}$ suggests MIUF models have free parameters that allow a range of responses to money growth shocks. The problem is there are no a priori restrictions that aid in choosing among these responses. Further, the assumption the transactions technology $\mathcal{C}(\cdot, \cdot)$ depends only on real balances and not other assets begs the question of why only fiat currency is the medium of exchange, especially when innovations are generated by financial markets.

A clear understanding of the relationship between the transaction technology and MIUF models is gained by studying

**Exercise Shopping Time:** Assume lifetime expected discounted household utility is

$$\mathbf{E}_t \left\{ \sum_{j=0}^{\infty} \beta^j W \left( c_{t+j}, \; 1 \; - \; n_{t+j} \; - \; \mathcal{N}\left( c_{t+j}, \; \frac{M_{t+j}}{P_{t+j}} \right) \right) \right\}.$$

The budget constraint of the household is (10) with the technology (11). All the usual parameter restrictions apply and the stochastic processes of money growth (16) and labor augmenting technical change (17) are given in **Exercise MIUF**.

$(i)$ What restriction (or restrictions) on $\mathcal{N}(\cdot, \cdot)$ necessary to prevent the nominal return on bonds, $R_t$, from being zero? Interpret the restriction(s).

$(ii)$ Compare and contrast your answer to part $(i)$ with the equilibrium condition (14). [Hint: Focus on the restriction(s) required for money to have positive and finite value in the MIUF model's equilibrium.]

$(iii)$ Provide restrictions on utility to guarantee superneutrality in the steady state of the transactions services model of real balances.

### *III.A The CIA Constraint*

Explaining the existence of fiat currency is a deep and fundamental issue in monetary economics. The question is which frictions (*i.e.*, market incompleteness) in an economy give agents incentives to hold fiat currency. The problem is equivalent to asking why fiat currency has strictly positive and finite value when it is dominated in (nominal) rate of return by other assets. Economists have been studying the primitives responsible for giving fiat currency finite value for a long time. The new monetarist research program seeks to answer these questions. Williamson and Wright (2010a, b) and Logos, Rocheteau, and Wright (2017) survey this research. Also, see Walsh (2017, section 3.4) for an introduction to this class of monetary models.

One restriction that gives fiat currency positive and finite value is the CIA constriant. The CIA constraint appears ad hoc or not a deep primitive. However, Camera and Chien (2016) argue the differences between a monetary model with a CIA constraint and the conical new monetarist model are more similar than not. Under a CIA constraint, the household has to own cash before purchasing goods and services. In this case, barter is not possible.

An inequality that captures this notion is the simple CIA constraint

$$c_t \ \leq \ \frac{M_t}{P_t}. \qquad (18)$$

Consumption is constrained to be no more than market value of real balances. The household can use no other real or nominal resources to purchase the consumption good during date $t$ under the CIA constraint (18). There are variations of the CIA constraint that add real wages, and subtract bank deposits or government bonds from the right hand side of the inequality or add investment to the left hand side; see Nason and Cogley (1994) and Belongia and Ireland (2014). However, the central point remains that only the real balances the household owns at the end of date $t-1$ and carry into date $t$ are available to purchase the consumption good.

Lets solve this constrained optimization problem using dynamic programing methods and Bellman's equation. Bellman's equation is

$$\mathcal{J}\left(k_t,\, b_t,\, \frac{M_t}{P_t},\, z_t,\, \mu_t\right) \ =$$

$$\mathbf{Max}_{(c_t, n_t, k_{t+1}, b_{t+1}, M_{t+1})} \left[\mathcal{V}(c_t,\, 1-n_t) \ + \ \beta\mathbf{E}_t\left\{\mathcal{J}\left(k_{t+1},\, b_{t+1},\, \frac{M_{t+1}}{P_{t+1}},\, z_{t+1},\, \mu_{t+1}\right)\right\}\right], \qquad (19)$$

subject to the budget constraint (10), the production technology (11), and the CIA constraint (18) given the state variables of the capital stock, the stock of nominal balances taking the real return to the unit discount, the aggregate price level, and the stochastic processes of money growth (16) and labor augmenting technical change (17) parametrically, where $\mathcal{J}\left(k_t,\, b_t,\, \frac{M_t}{P_t},\, z_t,\, \mu_t\right)$ is the value function of the household, and the period utility function of the household, $\mathcal{V}(c_t,\; 1 - n_t)$, has standard restrictions. Denote the multipliers of the budget constraint (10) and the CIA constraint (18) by $\lambda_{1,t}$ and $\lambda_{2,t}$, respectively. The latter shadow price represents the transactions services of real balances. The FONCs of the dynamic program (19) are

$$\mathcal{V}_{c,t} \;-\; \lambda_{1,t} \;-\; \lambda_{2,t} \;=\; 0, \tag{20}$$

$$-\,\mathcal{V}_{n,t} \;+\; \lambda_{1,t}(1 \;-\; \theta)\frac{y_t}{n_t} \;=\; 0, \tag{21}$$

$$-\,\lambda_{1,t} \;+\; \beta \mathbf{E}_t\left\{\frac{\partial \mathcal{J}_{t+1}}{\partial k_{t+1}}\right\} \;=\; 0, \tag{22}$$

$$-\,\lambda_{1,t} \;+\; \beta \mathbf{E}_t\left\{\frac{\partial \mathcal{J}_{t+1}}{\partial b_{t+1}}\right\} \;=\; 0, \tag{23}$$

and

$$-\,\frac{\lambda_{1,t}}{P_t} \;+\; \beta \mathbf{E}_t\left\{\frac{\partial \mathcal{J}_{t+1}}{\partial M_{t+1}}\right\} \;=\; 0. \tag{24}$$

The impact of the CIA constraint (18) is the marginal utility of consumption is the sum of the shadow price of a unit of real income, $\lambda_{1,t}$, plus the shadow price of a unit of real balances, $\lambda_{1,t}$. Rather than a direct utility effect, the CIA approach places a value in the marginal utility of consumption on holding real balances because it is the transactions medium.

Constructing the envelop conditions of the CIA constraint model is an intermediate step between the FONCs and optimality conditions. The envelop conditions are

$$\frac{\partial \mathcal{J}_t}{\partial k_t} \;=\; \lambda_{1,t}\left[\theta\frac{y_t}{k_t} \;+\; (1 \;-\; \delta)\right], \tag{25}$$

$$\frac{\partial \mathcal{J}_t}{\partial b_t} = \lambda_{1,t}(1 + r_t), \tag{26}$$

and

$$\frac{\partial \mathcal{J}_t}{\partial M_t} = \frac{\lambda_{1,t} + \lambda_{2,t}}{P_t}. \tag{27}$$

The current value to the household's program of a unit of real balances is the sum of the shadow prices of the consumption good and real balances valued at the purchasing power of money.

An implication of the CIA constraint is the household treats fiat currency in the same way it does any financial asset. Push the envelope condition (27) ahead one period and combine it with the FONC (24) of money to produce the stochastic difference equation

$$\frac{\lambda_{1,t}}{P_t} = \beta \mathbf{E}_t \left\{ \frac{\lambda_{1,t+1} + \lambda_{2,t+1}}{P_{t+1}} \right\}. \tag{28}$$

This is familiar because it generates the solution

$$\frac{\lambda_{1,t}}{P_t} = \mathbf{E}_t \left\{ \sum_{j=1}^{\infty} \beta^j \frac{\lambda_{2,t+j}}{P_{t+j}} \right\}. \tag{29}$$

This is the familiar present discounted value asset pricing equation. The transversality condition, $\lim_{j \longrightarrow \infty} \beta^j \mathbf{E}_t\{\lambda_{1,t+j}/P_{t+j}\} = 0$, is invoked to construct the infinite sum of the present value relation (29).

The present value relation (29) states that the purchasing power of the shadow price of a unit of the consumption today is positive only if the purchasing power of the future stream of the value of the transactions services of real balances is positive. The CIA constraint (18) is expected to bind strictly only when $0 < \lambda_{2,t+j}$, for all $j \geq 0$. The present discounted value relation (29) and the CIA constraint (18) together insure the aggregate price level $P_t$ (or the purchasing power of money) is positive and finite. Nonetheless, if and only if the CIA constraint (18) holds with equality is $P_t \in (0, \infty)$ in equilibrium.

The FONCs connect the left hand side of the present discounted value relation (29) to the marginal utility of consumption, $\mathcal{V}_{c,t}$. Add $\lambda_{2,t}/P_t$ to both sides of the equality of the present discounted value relation (29) and use the FONC (20) of consumption to find

$$\mathcal{V}_{c,t} \;=\; \mathbf{E}_t \left\{ \sum_{j=0}^{\infty} \beta^j \frac{P_t}{P_{t+j}} \lambda_{2,t+j} \right\}.$$

Next, multiple and divide the right hand side of the previous equation by $P_{t+j-1}$ and remember the definition of inflation to show

$$\mathcal{V}_{c,t} \;=\; \mathbf{E}_t \left\{ \sum_{j=0}^{\infty} \Xi_{t+j} \lambda_{2,t+j} \right\}, \quad \Xi_{t+j} \;\equiv\; \beta^j \left( \prod_{i=0}^{j} \frac{1}{1 \,+\, \pi_{t+i}} \right), \quad \pi_t \;\equiv\; 0. \tag{30}$$

The present value relation (30) identifies $\mathcal{V}_{c,t}$ with the future path of the value of transactions services of real balances. For the marginal utility of consumption to be positive, the future expected discounted value of the flow of transactions services of real balances must be positive. This flow of transactions services of real balances is discounted at $\Xi_{t+j}$, which depends on the expected future real return to cash, $1/(1 + \pi_{t+j})$, or the inverse of the inflation rate. The discount rate $\Xi_{t+j}$ is uncertain. Higher future expected inflation rates lower the current marginal utility of consumption because it encourages the household to move consumption forward in time. The expected increase in future inflation acts like a tax on the future value of the transactions services of real balances. The expected inflation tax lowers the stock of future real balances the household anticipates it owns in the future and with it the ability of the household to transact in the future.

This suggests

**Exercise CIA Constraint 1:** Construct the MIUF version of the present value relation (30). Compare and contrast the two present value relations and their impact on the equilibrium determination of the aggregate price level.

The CIA approach to money imposes an important restriction on the optimality conditions of the economy that distinguishes it from the MIUF approach. The FONCs (20) of consumption and (24) of money and the envelope condition (27) of money yield

$$\lambda_{1,t} \quad = \quad \beta\, \mathbf{E}_t \left\{ \frac{\mathcal{V}_{c,t+1}}{1 + \pi_{t+1}} \right\}. \tag{31}$$

This is the forward looking stochastic discount factor (SDF) of the household. Rather than the current marginal utility of consumption equaling $\lambda_{1,t}$, the current shadow price of another unit of real resources is the discounted expected real return to cash between dates $t$ and $t+1$ valued at the date $t+1$ marginal utility of consumption. The CIA constraint (18) limits the household to use an extra unit of cash it obtains today to purchase an extra unit of the consumption good tomorrow. Since the purchasing power of money can change between dates $t$ and $t+1$, the SDF depends on the real return to cash (*i.e.*, the inverse of $\pi_{t+1}$), which is a random variable.

The SDF affects the optimality conditions of the economy. The FONC (21) of labor becomes

$$\mathcal{V}_{n,t} \; = \; \Gamma_t(1 \, - \, \theta)\frac{Y_t}{n_t}, \tag{32}$$

where the SDF is defined as $\Gamma_t \equiv \lambda_{1,t}$. Unlike the MIUF model, the CIA approach introduces nominal factors into labor market optimality. The marginal rate of substitution between labor and consumption involves explicit intertemporal factors because of the CIA constraint (18. The cash the household garners from current labor income is not available to buy consumption until date $t + 1$. The Euler equations for capital and the unit discount bond are found by pushing the envelop conditions (25) and (26) forward one period and substituting the result into the FONCs (22) and (23) to produce

$$\Gamma_t \; = \; \beta\,\mathbf{E}_t\left\{\Gamma_{t+1}\left[\theta\frac{y_{t+1}}{k_{t+1}} \; + \; (1 \, - \, \delta)\right]\right\}, \tag{33}$$

and

$$\Gamma_t \; = \; \beta\,\mathbf{E}_t\left\{\Gamma_{t+1}(1 \, + \, r_{t+1})\right\}, \tag{34}$$

respectively.

Inspection of the optimality conditions (31), (32), (33), and (34) reveals that superneutrality fails in this economy. The addition of the CIA constraint (18) introduces a welfare cost on the household. It cannot transact in the goods market without cash whenever the CIA constraint binds. Moreover, the household faces an externality, given the CIA constraint (18) binds. The equilibrium aggregate price level and the stock of real balances are determined by other agents, who are other households and the monetary authority.

The monetary propagation mechanism of the CIA model is more highly restricted than it is in the MIUF model. The expected inflation effect drives the monetary business cycle of the CIA model. Assume that utility is separable in consumption and leisure and is a power function in utility. In this case, the Euler equation (34) of bonds is equivalent to

$$\mathbf{E}_t\left\{\frac{c_{t+1}^{-\alpha}}{1+\pi_{t+1}}\right\} = \beta\,\mathbf{E}_t\left\{\frac{c_{t+2}^{-\alpha}}{1+\pi_{t+2}}\left(\frac{1+R_{t+1}}{1+\pi_{t+1}}\right)\right\}. \tag{35}$$

Remember the Euler equation balances the utility cost of postponing consumption to buy a bond against the utility benefits of future consumption obtained from the income generated by the bond under the CIA constraint (18).

Next, lets review a result about log linearizing Euler equations. Consider the Euler equation

$$1 = \beta\,\mathbf{E}_t\left\{\left(\frac{c_{t+1}}{c_t}\right)^{-\alpha}(1+q_{t+1})\right\},$$

where $q_{t+1}$ is the return on an asset. Assume consumption growth and asset returns are jointly log normally distributed and homoskedastic. The motivation for the assumption is to link risk aversion and fluctuations in consumption growth to time-variation in expected returns in the intertemporal capital asset pricing model (ICAPM). The ICAPM predicts that consumption growth is the factor that drives time-variation in asset returns. For example, this link (or the lack thereof) is the source of the risk-free rate puzzle (*i.e.*, short rates are low and lack volatility relative to the predictions of the ICAPM). Log linearizing the Euler equation and applying the property of log normality and homoskedasticity results in

$$0 = \ln\beta - \alpha\,\mathbf{E}_t\{g_{c,t+1}\} + \mathbf{E}_t\{q_{t+1}\} + \frac{1}{2}\left[\alpha^2\sigma_{g_c}^2 + \sigma_q^2\right] - \alpha\,\mathrm{Cov}(g_{c,t+1}, q_{t+1}),$$

where $\ln(1+q_t) \approx q_t$, $g_{c,t} = \ln(c_t/c_{t-1})$, and $\ln\mathbf{E}_t\{x_{t+1}\} = \mathbf{E}_t\{\ln x_{t+1}\} + \frac{1}{2}\mathrm{Var}(\ln x_{t+1})$ because of log normality and homoskedasticity. Since the risk-free rate is the zero-risk asset and uncorrelated with $g_{c,t+1}$, the linearized Euler equation for the risk-free rate is

$$\mathbf{E}_t\{r_{F,t+1}\} = -\ln\beta + \alpha\,\mathbf{E}_t\{g_{c,t+1}\} - \frac{\alpha^2}{2}\sigma_{g_c}^2. \tag{36}$$

This is the expected risk-free return generating equation. It predicts consumption growth is the lone factor producing interest rate fluctuations. A one percent change in consumption growth is expected to produce an $\alpha$ percent change in $r_{F,t+1}$.

These results lead to the log linearized version of the Euler equation (35)

$$\mathbf{E}_t\{R_{t+1}\} \;=\; \alpha\,\mathbf{E}_t\{g_{c,t+1}\} \;+\; \mathbf{E}_t\{\pi_{t+2}\},$$

where constants are ignored. If the expected risk-free rate is uncorrelated with expected consumption growth (see the generating equation (36) of $\mathbf{E}_t\{r_{F,t+1}\}$), the Fisher's equation of the CIA model depends on expectations of inflation at date $t+2$. Substitute for $\mathbf{E}_t\{g_{c,t+1}\}$ using equation (36) and add and subtract $\mathbf{E}_t\{\pi_{t+1}\}$ in the log linearized Euler equation to show

$$\mathbf{E}_t\{R_{t+1}\} \;=\; \mathbf{E}_t\{r_{F,t+1}\} \;+\; \mathbf{E}_t\{\pi_{t+1}\} \;+\; \mathbf{E}_t\{g_{\pi,t+2}\}, \tag{37}$$

where $g_{\pi,t+2} = \pi_{t+2} - \pi_{t+1}$ and constants are ignored. In the CIA model, the expected inflation effect is reinforced by potential persistence in inflation growth. Unlike the Fisher equation (15) of the MIUF model, there is no chance the liquidity effect operates in the CIA model.

This is the good moment to ask

**Exercise CIA Constraint 2:** Let the period utility of the household be

$$\mathcal{V}(c_t,\; 1 \,-\, n_t) \;=\; \frac{c_t^{1-\alpha}}{1 \,-\, \alpha}\left(\frac{\ell_t^{1-\nu}}{1 \,-\, \nu}\right),$$

with all the usual restrictions. The budget constraint of the household is (10), technology given by the CRS production function (11), and the CIA constraint is equation (18). Once again, all the usual parameter restrictions apply and the stochastic processes of money growth (16) and labor augmenting TFP (17) are given in exercise MIUF. To detrend $\Gamma_t$, work inside the expectation operator of the stochastic discount factor (31) to see that $\tilde{\Gamma}_t = \Gamma_t\, z_t^{-\alpha}$.

$(i)$ Construct the optimality conditions of the economy. Interpret these equations.

$(ii)$ Show superneutrality fails to hold in the steady state equilibrium of this model. Without re-solving the model, discuss the impact of utility non-separable in consumption and leisure on the steady state and superneutrality in this CIA model. In the MIUF economy, log utility permits superneutrality result. Does this result hold in this CIA model? Explain.

$(iii)$ Show the inflation tax lowers the real return to capital in this economy. Discuss the restrictions on utility, technology, and the technology and money growth shocks that raise the cost of the inflation tax in this CIA model.

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