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On the re-aeration coefficient in channels of complex shape

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ABSTRACT: The recovery of rivers from spills of organic effluents is influenced by the diffusion of oxygen from the atmosphere, which is quantified by the re-aeration coefficient. Many tens of formulae for estimating this coefficient from simple hydraulic variables exist. As well as it being difficult to choose the most appropriate formula for any particular river reach, another issue exists for rivers of complex cross-sectional shape. In these cases there are significant transverse variations in water depth and flow velocity. Hence, the re-aeration coefficient must vary transversely also. The paper presents initial results from a theoretical analysis aimed at exposing the significance for estimated re-aeration coefficients of properly capturing the transverse heterogeneity of the physical processes. Three strategies for estimating the re-aeration coefficient for a channel of complex shape, consisting of a rectangular main channel surrounded by two symmetrical rectangular floodplains, were considered. Firstly, a simplistic approach in which the coefficient was evaluated only for the hydraulic conditions in the main channel, and expected to be dubious because it ignored transverse variations in the hydraulic conditions. Secondly, a naïve approach in which the coefficient was evaluated using cross-sectional average hydraulic conditions, and expected to be better than the simplistic approach because it attempted to recognize transverse variations in the hydraulic conditions. Thirdly, a robust approach in which the coefficient was evaluated as the cross-sectional average of three local values of the coefficient (one value for each flow zone, based on local hydraulic conditions), and expected to give the most reliable results because the transverse heterogeneity of the hydraulic conditions was properly captured. Using a typical empirical formula for the re-aeration coefficient and a modified flow resistance formula, general expressions for the re-aeration coefficient for each strategy were obtained in terms of the ratios of flood plain roughness to main channel roughness (γ), flood plain width to main channel width (β) and flood plain water depth to main channel water depth (η). Computations were undertaken for $1 < \gamma < 4$, $0.5 < \beta < 4$ and $0.05 < \eta < 0.4$. The results show that in comparison to the robust approach the simplistic approach overestimates the coefficient by up to 100%, with their ratio increasing with increasing γ and β, but gradually decreasing with increasing η. The results for the naïve approach are more complex. In comparison to the robust approach: when γ is low, it overestimates the coefficient (by up to 10%) and β has little effect, but when γ is high, it underestimates the coefficient (by up to 15%) and their ratio increases towards unity with increasing β; also their ratio gradually decreases with increasing η for all γ and β. In conclusion, although it may be tempting to evaluate the re-aeration coefficient from cross-sectional average hydraulic conditions, significant errors may be incurred.

1 INTRODUCTION

The re-aeration coefficient quantifies the rate at which oxygen diffuses into water bodies from the atmosphere. Hence it is a key parameter in (a) maintaining natural healthy fluvial ecosystems via stream metabolism and (b) determining the recovery of rivers from dissolved oxygen depletion episodes caused by pollution incidents. It is not surprising, therefore, that much effort has been devoted to developing ways of reliably predicting the coefficient's value over the wide range of flow types and magnitudes encountered in natural and manmade water courses.

Indeed over the last 50 years there have been many significant developments that have enhanced our understanding of the relevant physical processes and of the influence of hydraulic conditions. These have come from a range of theoretical, experimental and empirical studies. The ultimate aim of this work has been to predict the re-aeration coefficient for a river reach given sufficient information on the reach's hydraulic characteristics. Hence many tens of formulae have been published with the number of hydraulic parameters in them ranging from, typically, 2 to 5. The most common parameters are water depth, flow velocity and longitudinal channel slope: others used include Froude number and shear velocity, see for example Wilson & Macleod (1974), Rathbun (1977), Genereux & Hemond (1992), Moog & Jirka (1998), Melching & Flores (1999), Aristegi et al. (2009) and Raymond et al. (2012). Unfortunately, because the phenomenon is influenced by issues such as variable scales of process spatial heterogeneity and turbulent perturbations, which are unique to each reach, it is difficult to choose the most appropriate formula to use for any particular location.

In rivers of complex shape an additional difficulty exists that has been largely overlooked, namely that the re-aeration coefficient may vary over the width of the channel because of transverse variations of the hydraulic conditions. For example, during high flows conditions in a river's main channel can be markedly different to those near to inundated banks and on floodplains. As far as the authors are aware this issue has not previously been studied, so that its effect on predicted values of the re-aeration coefficient is not known.

Therefore the aim of the paper is to quantify some of the likely consequences of this issue. This is achieved via a simple theoretical analysis of conditions in a river reach having a cross-section consisting of three zones, namely a rectangular main channel surrounded by two symmetrical rectangular floodplains. Three strategies for computing the re-aeration coefficient of the whole cross-section were considered: there were termed "simplistic", "naïve" and "robust". Using a typical empirical formula for the local re-aeration coefficient and a modified flow resistance formula, general expressions for the global re-aeration coefficient for each strategy were obtained in terms of the ratios of floodplain roughness to main channel roughness, flood plain width to main channel width and flood plain water depth to main channel water depth.

The following sections cover some relevant background, details of the computational strategies, a summary of results, a discussion of some relevant issues and conclusions.

2 BACKGROUND

Ever since the pioneering study on natural purification in rivers by Streeter & Phelps (1925) there has been interest in predicting the re-aeration coefficient in rivers, i.e. the rate at which oxygen in the atmosphere is absorbed by oxygen deficient water. The integrity of this work relies heavily on estimates of the coefficient obtained in the field using, typically, one of three basic techniques: the dissolved oxygen balance technique, the disturbed equilibrium technique and the tracer-gas technique (Rathbun 1977, Melching & Flores 1999, Jha et al. 2004). In the first, all dissolved oxygen inputs, outputs, sinks and sources in the reach of interest are measured and the coefficient is inferred by fitting a mathematical model of the dissolved oxygen balance. In the second the coefficient is inferred from similar sets of dissolved oxygen measurements taken under two different dissolved oxygen deficit conditions. In the third the coefficient is calculated from the observed losses to the atmosphere of an injected gas, taking into account the effects of dilution and dispersion occurring in the reach, as simultaneously measured using a soluble conservative tracer. Examples of less frequently used techniques are the night-time regression method and the time lag between noon and the peak of oxygen saturation method (Aristegi et al. 2009). Although Rathbun (1997) reports that the tracer-gas method is the most reliable, and it appears to be the preferred choice in much recent work, other methods continue to be advocated (Jain & Jha 2005, Aristegi et al. 2009).

Ever since measurements of the re-aeration coefficient have been available studies have been undertaken to derive empirical or semi-empirical equations enabling the coefficient to be estimated from commonly available hydraulic variables. The number of parameters appearing in these equations ranges from 2 to 5 (or more) with 3 being common. Probably the most frequent combination is water depth, flow velocity and longitudinal channel slope. Useful reviews of this work are given by, for example, Wilson & Macleod (1974), Rathbun (1977), Moog & Jirka (1998), Melching & Flores (1999), Aristegi et al. (2009) and Raymond et al. (2012). In summary, a large number of predictive equations have been recommended, but none has been found to be significantly superior to any other over a wide range of stream types and sizes. It seems that we need to be armed with a collection of likely equations and a way of deciding under which hydraulic conditions it is appropriate to use them. Unfortunately, the latter appears to be a weak link in our understanding. It has to be recognized also that the exercise is often compromised by the relatively large uncertainty in the values of some published data and, in some cases, by a lack of knowledge of the range of hydraulic conditions over which the equations were originally derived.

3 ANALYSIS

From the very large number of formulae for estimating the re-aeration coefficient for a channel, the following typical one was selected (Jha et al. 2001):

$$K = 5.8U^{0.5}H^{-0.25} \tag{1}$$

where K is the re-aeration coefficient, U is the flow velocity and H is the water depth. Its choice was based, primarily, on simplicity. The well-known Manning's open channel flow resistance equation is given by:

$$U = R^{0.667}S^{0.5}n^{-1} \tag{2}$$

where R is the hydraulic radius, S is the longitudinal channel slope and n is the Manning roughness coefficient of the channel. Using Equation 2 to eliminate U from Equation 1, and approximating the hydraulic radius by the water depth, gives:

$$K = 5.8S^{0.25}H^{0.083}n^{-0.5} \tag{3}$$

Applying Equation 3 locally to the main channel and the floodplains gives:

$$K_{mc} = 5.8S^{0.25}H_{mc}^{0.083}n_{mc}^{-0.5} \tag{4}$$

$$K_{fp} = 5.8S^{0.25}H_{fp}^{0.083}n_{fp}^{-0.5} \tag{5}$$

where subscripts *mc* and *fp* refer to the main channel and floodplains, respectively, and it is assumed that all parts of the channel cross-section have the same longitudinal slope.

We now define three ways of estimating the global re-aeration coefficient for the channel. Firstly a simplistic approach, which uses information from only the main channel; secondly a naïve approach which uses cross-sectional average values of water depth (H_{csa}) and channel resistance (n_{csa}); thirdly a robust approach, which uses local values of the re-aeration coefficient from the main channel and the floodplains. Thus the simplistic approach uses Equation 4, whilst the naïve approach uses the following equations to calculate H_{csa} and n_{csa}, and uses them in Equation 3 in lieu of H and n.

$$H_{csa} = \frac{H_{mc} \times B_{mc} + 2 \times H_{fp} \times B_{fp}}{B_{mc} + 2 \times B_{fp}} \tag{6}$$

$$n_{csa} = \frac{n_{mc} \times B_{mr} + 2 \times n_{fp} \times B_{fp}}{B_{mc} + 2 \times B_{fp}} \tag{7}$$

Finally, the robust approach uses Equations 4 and 5 in the following formula to calculate the cross-sectional average value of the re-aeration coefficient (K_{csa}):

$$K_{csa} = \frac{K_{mc} \times B_{mc} + 2 \times K_{fp} \times B_{fp}}{B_{mc} + 2 \times B_{fp}} \tag{8}$$

Note that the cross-sectional averaging accomplished in Equations 6, 7 and 8 uses the width of the flow zones, which reflects the fact that re-aeration takes place through the water surface.

After some straightforward manipulation the following general equations were derived for the three approaches in terms of the ratios of flood plain roughness to main channel roughness (γ), flood plain width to main channel width (β) and flood plain water depth to main channel water depth (η):

$$K_s = 5.8S^{0.25}H_{mc}^{0.083}n_{mc}^{-0.5} \tag{9}$$

$$K_n = \frac{5.8S^{0.25}H_{mc}^{0.083}(1+2\beta\eta)^{0.083}n_{mc}^{-0.5}(1+2\beta\gamma)^{-0.5}}{(1+2\beta)^{-0.417}} \tag{10}$$

$$K_r = \frac{5.8S^{0.25}H_{mc}^{0.083}\left(1+2\beta\eta^{0.083}\gamma^{-0.5}\right)}{1+2\beta} \tag{11}$$

where the subscripts *s*, *n* and *r* refer to the simplistic, naïve and robust approaches.

Expressing the former two re-aeration coefficients as a ratio of the latter one enables the consequences of using them to be studied. Thus:

$$\frac{K_s}{K_r} = \frac{1+2\beta}{1+2\beta\eta^{0.083}\gamma^{-0.5}} \tag{12}$$

$$\frac{K_n}{K_r} = \frac{(1+2\beta\eta)^{0.083}(1+2\beta\gamma)^{-0.5}}{\left(1+2\beta\eta^{0.083}\gamma^{-0.5}\right)(1+2\beta)^{-1.417}} \tag{13}$$

4 RESULTS

Equations 12 and 13 were computed for $1 < \gamma < 4$, $0.5 < \beta < 4$ and $0.05 < \eta < 0.4$. Some typical results from Equation 12 are shown in Figures 1–3.

The results show that in comparison to the robust approach the simplistic approach always overestimates the re-aeration coefficient. Over the ranges of the parameters considered the error increases with increasing γ and β, but decreases with increasing η. The error ranges from about 4% ($\beta = 0.5$, $\gamma = 1$, $\eta = 0.4$) to about 119% ($\beta = 4$, $\gamma = 4$, $\eta = 0.05$). These results make physical sense because if K_s is used to estimate the re-aeration coefficient the contributions from the shallower and more

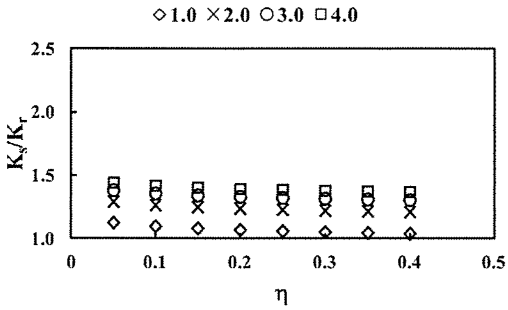

Figure 1. Variation of the ratio of K_s to K_r for four values of γ for the case of $\beta = 0.5$.

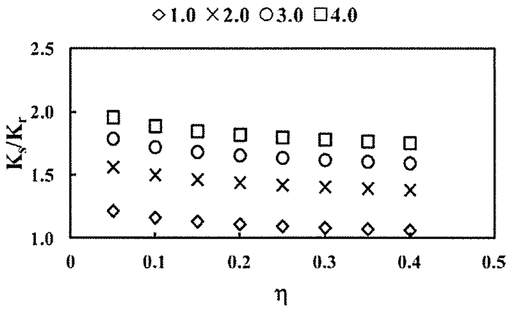

Figure 2. Variation of the ratio of K_s to K_r for four values of γ for the case of $\beta = 2$.

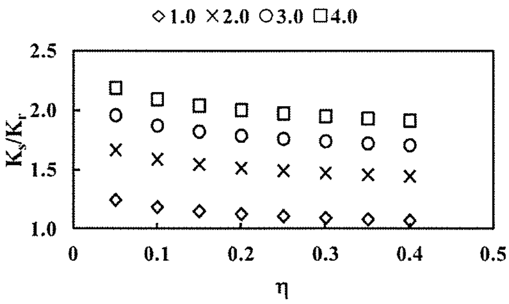

Figure 3. Variation of the ratio of K_s to K_r for four values of γ for the case of $\beta = 4$.

slowly moving water on the floodplains (compared to the main channel) are ignored. When β and γ are large and η is small, the rate of re-aeration on the floodplains is much smaller in comparison to the value in the main channel than when β and γ are small and η is large, so that the former conditions create larger errors than the latter conditions. This happens due to the magnitude and sign of the exponents in Equation 1. Note that when $\eta = 0$ the main channel is at its bank full state and there is no water on the floodplains, when $\eta = 0.5$ the water depth on the floodplains is half the water depth in the main channel and $\eta = 1$ represents a theoretical upper limit when the floodplain and main channel water depths are equal.

Corresponding results from Equation 13 are shown in Figures 4–6.

These results are more complex and show that in comparison to the robust approach the naïve

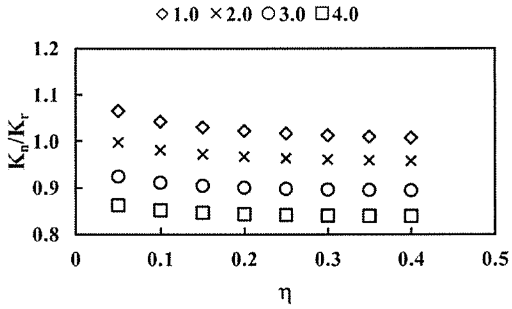

Figure 4. Variation of the ratio of K_n to K_r for four values of γ for the case of $\beta = 0.5$.

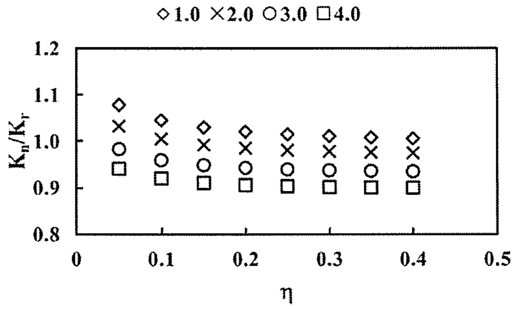

Figure 5. Variation of the ratio of K_n to K_r for four values of γ for the case of $\beta = 2$.

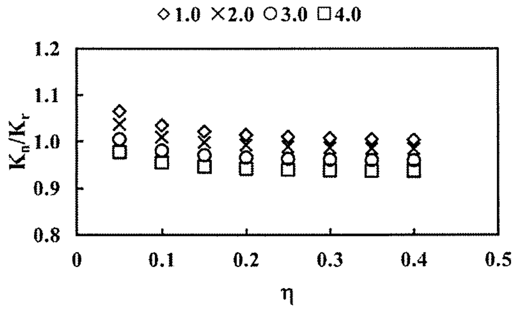

Figure 6. Variation of the ratio of K_n to K_r for four values of γ for the case of $\beta = 4$.

approach sometimes overestimates the re-aeration coefficient and sometimes underestimates it. Over the ranges of the parameters considered the error ranges from about −16% (β = 0.5, γ = 4, η = 0.4) to about +8% (β = 2, γ = 1, η = 0.05). Generally, low γ promotes overestimation, and under these conditions β has little effect, whilst high γ promotes underestimation, and under these conditions β has a noticeable effect (increasing β reducing the underestimation). Whether the coefficient is overestimated or underestimated, the ratio K_n/K_r decreases with increasing η. Although these trends are difficult to interpret physically, it is reasonable that the errors in K_n are smaller than the errors in K_s because K_n takes some (if not a completely satisfactory) notice of the hydraulic conditions on the floodplains.

5 DISCUSSION

The motivation behind the analysis reported above was that although progress has been made in recent years in improving the computation of channel conveyance (McGahey et al. 2008) and, to a lesser extent mass transport (Manson & Wallis 2004), in one-dimensional computation river models, by taking a proper account of the transverse heterogeneity of the relevant physical processes, the same cannot be said for water quality modeling. The results presented here show that if cross-sectional average values of flow velocity and water depth are used to calculate the re-aeration coefficient (the naïve approach considered above, and often the only one available in software packages) errors of the order of 10–20% are likely for the scenarios considered.

An important question is, how significant are these errors? It could be argued that they are not very significant at all because (a) most of the published equations for predicting the re-aeration coefficient are not very reliable (see, for example, Wilson & Macleod 1974, Melching & Flores 1999, Aristegi et al. 2009) and (b) much larger differences would probably be encountered if several different published equations for predicting the re-aeration coefficient were used (see, for example, Genereux & Hemond 1992, Aristegi et al. 2009). Similarly, the results of some typical Streeter-Phelps (1925) computations suggests that percentage errors in the timing of the critical dissolved oxygen state and in its concentration are no larger than percentage errors introduced into the re-aeration coefficient itself. Nevertheless, although the errors associated with the naïve approach may seem tolerable (also bearing in mind the inherently more uncertain nature of water quality modeling compared to flow and mass transport modeling), the errors may be larger in channels of different shape and if other empirical equations for the re-aeration coefficient were used. Of course, on the other hand, the errors may be smaller. It is clear that more work on predicting re-aeration coefficients in channels of complex shape is needed.

Although the methodology described herein could be used to investigate the issue for a wider range of scenarios, some significant improvements to it could be made before extending the work. For example, a fully two-dimensional hydraulic analysis based on the SKM approach (Shiono & Knight 1991, Knight 2013) would enable a more accurate cross-sectional average estimate of the re-aeration coefficient to be made.

6 CONCLUSIONS

A simple theoretical analysis has highlighted the scale of errors that can be made when estimating re-aeration coefficients in channels of complex shape. In particular, errors due to the imperfect treatment of the transverse heterogeneity of the relevant hydraulic conditions have been considered. As would be expected, basing the estimation of the coefficient only on conditions in the main channel can lead to large errors (of the order of 100% compared to a properly evaluated cross-sectional average value). Using cross-sectional average values of the hydraulic conditions gives smaller errors, but significant errors (10–20%) may still be incurred. There is a clear practical significance of this issue for dissolved oxygen modeling scenarios, such as stream metabolism and pollution incident studies, but the scale of the problem will remain unclear until wider ranges of channel shapes and empirical formulae for the re-aeration coefficient have been included in the analysis.

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