Maximally-Disordered Distillable Quantum States

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Abstract

We explore classical to quantum transition of correlations by studying the quantum states located just outside of the classically-correlated-states-only neighborhood of the maximally mixed state (the largest separable ball (LSB)). We show that a natural candidate for such states raises the possibility of a layered transition, i.e., an annular region comprising only classical and the classical-like bound entangled states, followed by free or distillable entanglement. Surprisingly, we find the transition to be abrupt for bipartite systems: distillable states emerge arbitrarily close to the LSB. For multipartite systems, while the radius of the LSB remains unknown, we determine the radius of the largest undistillable ball. Our results also provide an upper bound on how noisy shared entangled states can be for executing quantum information processing protocols.

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Introduction: The states of a bipartite quantum system can either be separable or inseparable: In the former, the correlations between the subsystems are entirely classical (i.e., the correlations can be mimicked by a local description of the individual subsystems), and the states can be written as a convex combination of product states:

$$\rho^{AB} = \sum p_i \rho_i^A \otimes \rho_i^B,$$

where $A, B$ denote the subsystems and the positive probabilities $p_i$ add up to one. The density matrices $\rho_i$ act on the individual Hilbert spaces corresponding to the subsystems $A$ and $B$. If a state cannot be written in the form (1) it is said to be inseparable or entangled and can exhibit correlations that cannot be explained by any classical theory [1]. Recent advances have led to discoveries of two qualitatively different forms of entanglement, namely, bound entanglement [4, 9, 10, 11] and distillable entanglement [5, 6, 7, 8]. It is of fundamental importance how the different classes of quantum states are distributed in the Hilbert space. In particular, an understanding of the distribution of different states along the boundaries in the Hilbert space, where transitions from classically correlated states to quantum correlated states occur, has the potential to provide new insights into the very nature of quantum non-locality. A legitimate starting point would be to obtain a picture of such a distribution relative to the maximally mixed state: the state which is completely disordered (both locally and globally) and does not have any correlation between the subsystems. It is natural to expect that as one starts to move away from the maximally mixed state one would encounter more ordered states, as well as emergence of correlations between the subsystems. It has been shown that the neighbourhood of a maximally mixed state consists entirely of separable states [16, 17, 18], and recently the exact size of this separable neighbourhood for bipartite systems has been obtained [19].

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Thus, transitions from classical correlations to quantum can be observed just beyond the boundary of the LSB.

While determining the exact size of the separable neighbourhood constitutes a major progress, answers to several questions related to the characterization and distribution of the inseparable states beyond the LSB remain unknown, primarily because of the existence of two very different types of entanglement, viz., bound entanglement (BE) and distillable entanglement. The former class, though quantum correlated, is qualitatively very similar to classically correlated states because of their positivity under partial transposition (PPT) and usefulness that is no better than classically correlated states for reliable quantum information processing. The latter class, however, is negative under partial transposition (NPT) and shows distinct and demonstrable non-local features in quantum communication protocols.

We first ask how one may construct entangled states that are closest to the LSB. It was often thought that the entangled states closest to the maximally mixed state can be reached via mixing of an entangled state with the maximally mixed state, e.g., the pseudo-pure states used in room temperature NMR quantum computing. We will refer to this class of states as the Werner-type states. We show that while this is true for the simplest case of $2 \otimes 2$, for all other bipartite systems, there is a finite gap between the boundary of the LSB and the nearest inseparable state (which turns out to be distillable as well) that can be reached via such mixing.

Such a finite gap between the most-disordered distillable Werner-type states and the LSB opens up several interesting possibilities. For instance, we can think of a PPT neighbourhood of the maximally mixed state distinct from the LSB within which there is no NPT state. Moreover, there can as well be a largest undistillable neighbourhood distinct from both the largest separable and the largest PPT ones, with bound entangled states as the only inseparable states in the intermediate regime. We therefore ask at what distances and in what order, if any, relative to the boundary of the LSB can distillable and bound entangled states be found? For bipartite quantum systems, we show, using explicit constructions, that instead of the different types of inseparable states appearing in a layered fashion, both distillable and NPT n-copy undistillable states (conjectured to be NPT bound entangled) can be found arbitrarily close to the boundary of the LSB. The class of distillable states that lie arbitrarily close to the boundary of the LSB are therefore, the maximally-disordered distillable states. Such states are constructed via perturbations of the separable states on the surface of the LSB. A subset of such separable states on the surface are the maximally mixed states of $(D - 1)$ dimensional subspaces where $D$ is the dimension of the full Hilbert space.

We also provide constructions for maximally disordered distillable states in multipartite quantum systems, where the exact size of the LSB is still unknown. However, using a result obtained in Ref. we are able to show that all quantum states that are on or inside the ball of radius $R_{LSB}$ ($D$ is now the total dimension of the multipartite system) are PPT and therefore represents the largest undistillable ball in the multipartite case. Proceeding in a similar way as in the bipartite case we construct multipartite distillable states that are arbitrarily close to this PPT ball. Interestingly this shows that given a composite quantum system, the radius of the largest undistillable neighbourhood is independent of all possible partitions and depends only on the total dimension. Remarkably, the preceding result coupled with the conjecture that the the LSB for multipartite systems is smaller than that for bipartite systems, lead to a new conjecture that the transition for general multipartite systems is indeed layered.

**Minimum distance of the Werner-type entangled states from the maximally mixed state:** In this paper, the distance between any two density matrices, $\rho$ and $\sigma$, will be given by the Hilbert-Schmidt distance defined as: $\delta = \|\rho - \sigma\| \equiv \sqrt{\text{tr}\left((\rho - \sigma)^2\right)}$. We begin by considering the class of bipartite mixed

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1 We also note that the existence of BE state that are NPT has been conjectured by several groups, but a rigorous proof is still pending.
states, \( \rho_x = x \rho + \frac{(1-x)}{D} I \), where \( x > 0 \) and \( \rho \) is any density matrix in \( d \otimes d \). The total dimension of the composite Hilbert space is denoted by \( D = d^2 \). Obviously if \( \rho_x \) is entangled for some values of \( x \), then \( \rho \) must be an entangled state. Entanglement properties of such states are of considerable importance because the generic quantum states used in NMR quantum computing, the so called pseudo-pure states \( [21] \), and the famous Werner states \( [21] \) are of similar form. In what follows we answer the question, are the states \( \rho_x \), when inseparable, the closest inseparable states to the maximally mixed state? In Ref \( [19] \) it was shown that the states \( \rho_x \) are always separable when \( x \leq \frac{1}{D+2} \). Moreover, \( \rho_x \) becomes inseparable when \( x \) is taken to be a maximally entangled state of Schmidt rank two and \( x > \frac{1}{D+2} \). The proof of inseparability follows from the fact that when \( x \) is chosen to be a maximally entangled state of Schmidt rank 2, then \( \rho_x \) becomes a full rank mixed state for \( x > \frac{2}{D+2} \) \( [19] \).

We first show that \( \rho_x \) indeed becomes distillable when \( x > \frac{2}{D+2} \), and \( \rho \) is one of the four Bell states, \( \{ |\Phi^\pm \rangle, |\Psi^\pm \rangle \} \) defined in the standard basis as \( |\Phi^\pm \rangle = \frac{1}{\sqrt{2}} (|00 \rangle \pm |11 \rangle) \) and \( |\Psi^\pm \rangle = \frac{1}{\sqrt{2}} (|01 \rangle \pm |10 \rangle) \). Let \( \rho = |\Phi^+ \rangle \langle \Phi^+ | \). Then \( \langle \Psi^- | (x \rho^{PT} + \frac{(1-x)}{D} I) |\Psi^- \rangle < 0 \) when \( x > \frac{2}{D+2} \). Since \( |\Psi^- \rangle \) is a Schmidt rank two state, the state \( \rho_x \) is therefore distillable.

Let us now note that when \( x \leq \frac{1}{D+2} \), the distance of the state \( \rho_x \) from the maximally mixed state is always less than or equal to \( R_{LSB} \) and the equality is achieved when \( \rho \) is any pure state and \( x = \frac{1}{D+2} \). The intermediate regime \( \frac{1}{D+2} < x \leq \frac{2}{D+2} \), therefore corresponds to the states that are outside the LSB and the inseparability/separability property of these states remains to be determined.

We now show that if \( x \leq \frac{2}{D+2} \), then states of the form \( \rho_x = x \rho + \frac{(1-x)}{D} I \) are always separable for all \( \rho \). It was proved in Ref \( [17] \) that a full rank mixed state is separable if the minimum eigenvalue is greater or equal to \( \frac{1}{D+2} \). We show that if \( x \leq \frac{2}{D+2} \) then for any \( \rho \) the minimum eigenvalue of \( \rho_x \) is always greater or equal to \( \frac{1}{D+2} \) from which our conclusion immediately follows. Let us first suppose that \( \rho \) is not a full rank state. Then the minimum eigenvalue of \( \rho_x \) is \( \frac{(1-x)}{D} \geq \frac{1}{D+2} \) if \( x \leq \frac{2}{D+2} \). Now assume that \( \rho \) is a full rank state and the minimum eigenvalue of \( \rho \) is \( \lambda \). Then the minimum eigenvalue of \( \rho_x \) is \( \lambda x + \frac{(1-x)}{D} \). Then \( \lambda x + \frac{(1-x)}{D} \geq \frac{1}{D+2} \), which implies that \( x \leq \frac{2}{D+2} \). Since \( D \lambda \leq 1 \), we have \( \frac{2}{D+2} \leq \frac{2}{D+2} \).

Now note that the distance of the state \( \rho_x \) from the maximally mixed state is given by \( x \sqrt{Tr (\rho^2) - \frac{1}{D}} \). Thus, the inseparable states (as shown earlier, such states are also distillable) nearest to the maximally mixed state that can be reached via perturbation can be expressed as a mixture of a pure entangled state and the maximally mixed state and therefore at a distance \( R = \frac{2}{D+2} \sqrt{\frac{(D-1)}{D}} \). We note that \( R \geq R_{LSB} \) and the equality is achieved only when \( D = 4 \), corresponding to the systems in \( 2 \otimes 2 \). From the ratio \( \frac{R}{R_{LSB}} = \frac{D}{D+2} \) implies that as the dimension of the Hilbert space increases, the entangled states that can be reached via perturbation start to move away farther from the boundary of the LSB, and the ratio becomes as large as 2.

**Constructions of distillable states arbitrarily close to the boundary of the LSB:** We first note that the radius of the largest separable ball is, in fact, the distance between the maximally mixed state and the maximally mixed state in any \( (D-1) \) dimensional subspace, i.e., \( R_{LSB} = \| \frac{1}{D} I - I_{D-1} \|_1 \), where \( I_{D-1} = \frac{1}{D-1} (I - |\phi \rangle \langle \phi |) \), for some pure state \( |\phi \rangle \). This turns out to be a useful observation. We now focus our attention on a class of operators that are the partial transposition of \( \frac{1}{D-1} I_{D-1} \):

\[
\sigma_\phi(k) = \frac{1}{D-1} (I - |\phi \rangle \langle \phi |)^{PT},
\]

where \( |\phi \rangle \) is a pure state of Schmidt rank \( k, k = 1, \cdots, d \), and is of the form \( \sum_{i=0}^{k-1} \eta_i |ii \rangle \) where \( \eta_i \)'s are real and positive and \( \sum_{i=0}^{k-1} |\eta_i|^2 = 1 \). The superscript \( PT \) denotes partial transposition. One can now check

\[2\] A bipartite quantum state \( \rho \) is said to be distillable iff there exists an integer \( n \) and a Schmidt rank two state \( |\psi \rangle \) such that \( \langle \psi | (\rho^{PT})^\otimes n |\psi \rangle < 0 \) \( [10] \).
that the operator $\sigma_\varphi(k)$ is indeed a density matrix. Firstly, it has trace one, since the trace of its partial transpose is one and trace is invariant under partial transposition. Now note that the eigendecomposition of the operator $|\varphi\rangle \langle \varphi|^{PT}$ is given by

$$
(|\varphi\rangle \langle \varphi|)^{PT}_{ij} = \sum_{i=0}^{k-1} (1 - |\eta_i|^2) |ii\rangle \langle ii| + \sum_{i,j=0, i<j}^{k-1} \eta_i \eta_j \left| \psi_{ij}^+ \right\rangle \left\langle \psi_{ij}^+ \right| - \sum_{i,j=0, i<j}^{k-1} \eta_i \eta_j \left| \psi_{ij}^- \right\rangle \left\langle \psi_{ij}^- \right| ,
$$

(3)

where $\left| \psi_{ij}^\pm \right\rangle = \frac{1}{\sqrt{2}}(|ij\rangle \pm |ji\rangle)$. Note that the eigenvectors of $\left| \varphi\rangle \langle \varphi|\right)^{PT}$ span $k^2$ dimensional subspace. Now substituting (3) in (2) one can see that the operator is Hermitian and positive semidefinite. Therefore it is a density matrix.

When $k = 1$, $|\varphi\rangle$ is a product state. Hence, $\sigma_\varphi(1)$ is the maximally mixed state of a $(D - 1)$ dimensional subspace and is on the surface of the LSB. The same cannot be said when $k \geq 2$ because $|\varphi\rangle$ is now a pure entangled state. In fact we next show that the states, $\sigma_\varphi(k)$ are of full rank and also lie on the surface of the LSB for all $k \geq 2$. They are therefore separable.

Substituting (3) in (2) one can obtain the spectral decomposition of $\sigma_\varphi(k)$,

$$
\rho_\psi,\varphi(\epsilon, k) = \epsilon |\psi\rangle \langle \psi| + (1 - \epsilon) \sigma_\varphi(k) ,
$$

(5)

where $|\psi\rangle$ is a pure entangled state to be fixed later and $0 < \epsilon < 1$. We next prove that for the cases, $k=1,2$ there exist states, $|\psi\rangle$, such that $\rho_\psi,\varphi(\epsilon, k)$ are distillable for some finite range of $\epsilon$. In the rest of this discussion, we will ignore the subscript $\varphi$ in the cases when it is apparent from the context.

**k=1:** Without loss of generality, we can write $\rho_\psi(\epsilon, 1) = \epsilon |\psi\rangle \langle \psi| + \frac{1-\epsilon}{D-1} \mathbf{I}_{D-1}$, where $\frac{1-\epsilon}{D-1} \mathbf{I}_{D-1} = \frac{1-\epsilon}{D-1} (\mathbf{I} - |00\rangle \langle 00|)$. Let $|\psi\rangle$ be a pure entangled state of Schmidt rank two: $|\psi\rangle = \frac{1}{\sqrt{2}} (|10\rangle - |11\rangle - |00\rangle)$). To prove that $\rho_\psi(\epsilon, 1)$ is distillable, we need to show that there exists a Schmidt rank two state $|\chi\rangle$ such that $\langle \chi| \rho_\psi^{PT} |\chi\rangle < 0$. Construct the following state

$$
|\chi\rangle = \alpha |00\rangle + \frac{\beta}{2} (|10\rangle - |11\rangle - |00\rangle + |01\rangle) ,
$$

(6)

where $\alpha = |\alpha| e^{i\vartheta_1}$ and $\beta = |\beta| e^{i\vartheta_2}$ are complex quantities. By showing that the reduced density matrices are of rank 2, one can easily verify that the state $|\chi\rangle$ is indeed of Schmidt rank two. We choose $\vartheta_1, \vartheta_2$ such that $\vartheta_1 - \vartheta_2 = \pi$. With this constraint, the normalization condition for $|\chi\rangle$ reads as $|\alpha|^2 + |\beta|^2 + |\alpha||\beta| = 1$. 


One can now easily check that: \( \langle \chi | \rho^T_\psi (\epsilon, 1) | \chi \rangle = -\frac{\epsilon}{2} |\alpha| |\beta| + \frac{3(1-\epsilon)^2}{4(D-1)} |\beta|^2 \), where we have used \( \theta_1 - \theta_2 = \pi \).

Let \( |\alpha| = y \). Requiring \( \langle \chi | \rho^T_\psi (\epsilon, 1) | \chi \rangle < 0 \) for distillability, one obtains

\[
\epsilon > \frac{1}{1 + yA},
\]

where \( A = \frac{4(D-1)}{9} \) is a constant.

\( k=2 \): Let \( \rho_\psi (\epsilon, 2) = \epsilon |\psi\rangle \langle \psi| + (1-\epsilon) \sigma_\varphi (2) \) where \( \sigma_\varphi (2) = \frac{1}{D-1} (I - |\Phi^+\rangle \langle \Phi^+|)^{PT} \). That is, we have chosen \( |\varphi\rangle \) to be the the Bell state \( |\Phi^+\rangle \). We now choose \( |\psi\rangle \) as the singlet state. Therefore, the state under consideration is: \( \epsilon |\Psi^-\rangle \langle \Psi^-| + \frac{1-\epsilon}{D-1} (I - |\Phi^+\rangle \langle \Phi^+|)^{PT} \). By noting that \( |\Phi^+\rangle \) is the eigenvector corresponding to the negative eigenvalue of \( (|\Psi^-\rangle \langle \Psi^-|)^{PT} \) it follows, \( \langle \Phi^+| \rho^{PT} (\epsilon, 2) |\Phi^+\rangle = -\frac{\epsilon}{2} \). Hence, the state \( \rho_{\psi, \varphi} (\epsilon, 2) \) is distillable when \( \epsilon > 0 \).

The preceding results show that one can have distillable states arbitrarily close to the surface of the LSB as the states \( \sigma_\varphi (k) \) are on the surface of the LSB. Recently, new classes of NPT states that are \( n \)-copy undistillable for any \( n \geq 1 \) have been obtained in Ref.\[22\]. One such class corresponds to the states that are of the form \( |\Psi\rangle \), where \( k \geq 3 \). Hence, we also have \( n \)-copy undistillable states on the immediate boundary of the LSB since \( \sigma_\varphi (k) \) lies on the LSB for all \( k \geq 1 \).

**Constructions for multipartite systems:** Some of our results and also that of Ref.\[19\] can be directly applied to multiparty systems but with some caution. A multipartite state can be separable in all bipartite cuts, but can still be inseparable. Such true bound entangled states do exist.\[11\]. However, positivity/negativity of a multipartite state under partial transposition under all possible bipartite partitions do indeed provide useful information regarding distillability. For instance, a multipartite state cannot be distillable, and at the same time, PPT/separable across all distillable entanglement across at least one bipartite cut, then the state is distillable. We now show that the boundary of the ball of radius \( \frac{1}{\sqrt{D(D-1)}} \) around the maximally mixed state contains distillable multipartite states, where \( D \) is now the total dimension of the multipartite system. The construction is very similar to that in the case of bipartite systems. We provide it for the \( 2 \otimes 2 \otimes 2 \) case, but it can be trivially generalized for higher dimensions. Consider the following class of states

\[
\hat{\rho} (\epsilon, k)_{123} = \epsilon |\psi\rangle_{123} \langle \psi| + (1-\epsilon) \tilde{\sigma}_{123},
\]

where

\[
|\psi\rangle_{123} = \frac{1}{\sqrt{6}} (|100\rangle - |111\rangle - 2 |011\rangle)_{123}
\]

and \( \tilde{\sigma} = \frac{1}{7} (I - |000\rangle \langle 000|)_{123} \). To show that the state \( |\psi\rangle \) is inseparable/distillable, it is sufficient to show that the state is inseparable across at least one bipartite cut. Let us consider the \( 1 : (2, 3) \) cut, and denote the states of the qubits 2 and 3 as follows: \( |00\rangle_{23} = |0\rangle_{23} , |11\rangle_{23} = |1\rangle_{23} , |01\rangle_{23} = |2\rangle_{23} , |10\rangle_{23} = |3\rangle_{23} \). Substituting these in the above two equations, we get exactly the same distillable states that we have used in the bipartite case.

We close our discussion of the multipartite case with the following claim: *For a multipartite quantum system of total dimension \( D \), the exact radius of the largest PPT ball around the maximally mixed state is given by \( \frac{1}{\sqrt{D(D-1)}} \). It is also the largest undistillable ball.* First, using the result of Ref.\[19\], we observe that for all possible bipartite cuts, the largest separable ball has the radius \( \frac{1}{\sqrt{D(D-1)}} \). Next, if a state is NPT, then it has to be NPT in one of the bipartite cuts and therefore, it lies outside the ball of radius \( \frac{1}{\sqrt{D(D-1)}} \). Hence,
the radius of the largest PPT ball is at least $\frac{1}{\sqrt{D(D-1)}}$. However, we have just shown that on the boundary of such a ball there exists distillable states. Hence, the largest PPT ball and the largest undistillable ball are the same for the multipartite case, with a radius of $\frac{1}{\sqrt{D(D-1)}}$.

**Discussions:** We have shown that the inseparable states nearest to the maximally mixed state can never be reached via mixing of an entangled state with the maximally mixed state other than in $2 \otimes 2$. We have also addressed the issue of distribution of different types of inseparable states on the boundary of the LSB for the bipartite case and have shown via explicit constructions that distillable states exist arbitrarily close to the surface of the LSB. Hence, these states are the *maximally-disordered* distillable states. Thus, for the bipartite case, our results show that the LSB, the largest PPT ball, and the largest undistillable balls all have the same radius. In the case of multipartite systems, immediate extensions of the results for the bipartite case imply that the radius of the largest PPT ball and the largest undistillable ball are the same. This shows that for any composite quantum system, the radius of the largest undistillable ball around the maximally mixed state is independent of all possible partitions and depends only on the total dimension. However, it is likely that there is a gap between the LSB and the largest undistillable/PPT ball for the multipartite case. Whether such a layering exists in the multipartite case, as opposed to the bipartite case, remains an open problem.

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