Imperfect Knowledge, Liquidity and Bubbles*

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Abstract

Insufficient liquidity can lead to substantial movements in asset prices. There is a single asset traded in a centralized market that facilitates exchange in decentralized trade. If the asset is in short supply the price includes a liquidity premium. Traders have imperfect knowledge about future asset prices and estimate, in real-time, an econometric forecasting model. A permanent decrease in the supply of assets, or an increase in collateral requirements, can lead to over-shooting of the price. When price includes a liquidity premium there can be recurrent bubbles and crashes. Liquidity and adaptive learning play key roles in fitting the empirical distribution of price-dividend ratios.

1 Introduction

It has long been recognized that financial assets have important roles beyond a store-of-value including the provision of liquidity services. Assets that can be considered safe are increasingly viewed to be in short supply: liquid assets are used as collateral in over-the-counter transactions and bilateral agreements while a rise in global demand by investors, governments and central banks, and changes to macro-prudential policies, are likely to exacerbate

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supply imbalances. When financial assets play a similar liquidity role as money, variations in the supply of assets can affect asset prices. For example, Krishnamurthy and Vissing-Jorgensen (2012) show that changes in the (relative) supply of treasury debt, corporate and agency bonds affect the price of these assets. Equities play a role, as well in secured lending markets, such as bilateral and trilateral repo markets.\footnote{Krishnamurthy, Nagel, and Orlov (2014) and Copeland, Martin, and Walker (2011) estimate 5-10\% of the collateral in the very large U.S. repo markets comes from equity.} Caballero, Farhi, and Gourinchas (2008) link global capital flows to a shortage in the supply of assets.

Search-and-matching models of asset pricing are useful environments for studying prices in economies with a shortage of liquid assets since they make explicit the liquidity properties of assets: assets can serve as collateral to facilitate bilateral, or over-the-counter, trade when limited enforcement or imperfect recognizability preclude unsecured credit arrangements as a means of payment. In search models, when the amount, or supply, of the asset is sufficiently low, the asset price will reflect its dual roles as a store of value and as a provider of liquidity services. The component of the asset price attributable to a liquidity premium, is sometimes referred to as a rational bubble.\footnote{There is a long history of interpreting fiat monetary equilibria as a rational bubble and extending that interpretation to assets, more broadly, since fiat money is an asset with a constant, zero payment forever. See, Tirole (1985).} Although search-based models have been useful in explaining certain empirical properties of asset prices, such as the risk-free rate and equity premium puzzles (see, Lagos (2010a)), to date, they have not been successful in generating other salient features of asset prices such as excess volatility and the rapid price appreciations and depreciations typically attributed to speculative bubbles.

This paper presents a search-based asset pricing model with imperfect knowledge and adaptive learning that explores the role that liquidity plays in generating asset price bubbles and crashes.\footnote{Models such as Lagos and Wright (2003) feature cycles that are interpretable as bubbles and crashes. Similar to their results in this paper, crashes feature welfare reducing collapses in consumption. A key distinction of the learning dynamics introduced here is the more gradual run-up of asset prices that can be an empirical feature in some asset prices.} The economic environment is an extension of Lagos (2010a) and Rocheteau and Wright (2011): there is a single asset, similar to a Lucas Tree, that pays an i.i.d. dividend and is traded in a centralized, competitive market. The supply of this asset is subject to occasional, small iid shocks that captures asset float and other exogenous factors that affect asset supply. Absent trading frictions, this asset would price at the discounted present value of the dividend flow. However, the economy also consists of a decentralized market where buyers and sellers are bilaterally matched and buyers submit to sellers a “buyer-takes-
all” offer. Unsecured credit is not available in these pairwise meetings because of limited enforcement. Instead, the “safe” asset can serve as collateral for secured credit – equivalently, exchanged *quid pro quo* for goods – giving rise to an endogenous liquidity role for financial assets. In a stationary (rational expectations) equilibrium, the asset price consists of two components: the expected present-value of future dividends and a liquidity premium.

In the model, the asset price is determined, in part, by the expected future price of the asset. The departure point of this paper is to replace rational expectations with price expectations formed from an adaptive learning rule as in Evans and Honkapohja (2001).4 The imperfect knowledge environment under consideration assumes that individuals understand a lot about the economic environment, but they do not know – or harbor some doubt about – the particular values of the dividend process, the asset supply process and other values/coefficients that determine asset prices. As a result, individuals draw inferences about the asset price process from recent data by adopting an econometric forecasting model that nests the rational expectations equilibrium. These agents are Bayesian and, because of uncertainty about their model, they place a prior on structural change in their econometric model. This imperfect knowledge framework implies that individuals forecast via an AR(1) econometric model whose parameters are updated in real time with a form of discounted least squares (“constant gain learning”). The priors for this Bayesian model are specified in such a manner that beliefs are, on average, close to rational expectations.

This paper identifies several channels through which liquidity and imperfect knowledge, or adaptive learning, interact to affect asset prices. First, although over time beliefs tend to converge toward rational expectations, the combination of constant gain learning and a positive liquidity premium can lead individuals to temporarily believe that asset prices follow a random walk without drift. Random walk beliefs arise for a very intuitive reason. Suppose there is a slight (temporary) upward drift to asset prices, perhaps arising from a tightening of financial frictions. Individuals’ econometric models will pick up that drift, leading to higher expectations about future asset prices that feed back onto higher asset prices. Thus, random-walk beliefs are nearly self-fulfilling and, consequently, such beliefs tend to persist for a substantial length of time. Furthermore, these beliefs generate excess volatility in asset prices, characterized by significant bursts and collapses in asset prices that are reminiscent of speculative bubbles and crashes. During a bubble episode, the asset becomes more liquid in over-the-counter markets as sellers are willing to part with more goods in exchange for the asset; a bubble leads to greater economic activity. Conversely, during a crash episode,

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4Baranowski (2012) is the first paper to study learning in a monetary search model.
the asset becomes less liquid and, as a result, there is less economic activity.

Second, a decline in the quantity of liquid assets will introduce just the type of drift in asset prices that can lead to random-walk beliefs and cause a substantial overshooting of the new stationary equilibrium price. Third, structural changes to the economic environment that increase the demand for collateral can also lead to a drift in asset prices that lead to random-walk beliefs and an over- or under-shooting of the equilibrium price. For example, a financial innovation – such as securitization – that increases sellers’ willingness to accept the asset as collateral will increase its liquidity properties. Or, substitution away from assets no longer perceived to be safe will increase the demand for the liquid asset as collateral. Alternatively, financial innovations that relax the collateral constraint have an ambiguous effect on price that depends, in part, on the strength of expectational feedback. It is shown that relaxing the borrowing constraint will lead to either over or under-shooting of asset prices.

It is tempting to conclude that the liquidity premium is just another fundamental factor, like dividends, driving asset prices. However, the paper demonstrates that assets in a search-based asset pricing model have both a direct and indirect effect with important theoretical implications. The direct effect is the liquidity premium that arises absent any learning dynamics. The indirect effect is that the liquidity role of the asset alters the way expectations about future prices and dividends affect the contemporaneous asset price. To assess the relative significance of the liquidity channel, Section 5 turns to an extension of the benchmark model that is calibrated to match key properties of U.S. financial data. It is shown that the model with liquidity captures regularities of the empirical distribution of S&P 500 price-dividend ratios particularly on the tails of the distribution.

1.1 Related Literature

This paper is the first to introduce econometric learning into a search model that emphasizes liquidity considerations and the “moneyness” of assets. As such, the paper contributes to a growing body of research.\footnote{See Nosal and Rocheteau (2011) for an extensive survey of search-based monetary and asset-pricing theory. Search-based models of asset pricing and liquidity include Duffie, Garleanu, and Pedersen (2005), Geromichalos, Licari, and Suarez-Lledo (2007), Lagos (2010a), Lagos and Rocheteau (2009), Weill (2008), Vayanos and Weill (2008), Vayanos and Wang (2012a), Lagos and Wright (2005), Lester, Postlewaite, and Wright (2012), Rocheteau and Wright (2011).} The first treatment of dynamic equilibria in the monetary model of Lagos-Wright is Lagos and Wright (2003). This model was extended to include free-entry
by Rocheteau and Wright (2005). The first Lagos-Wright model with both a real asset and money was Lagos and Rocheteau (2008). Lagos (2010a) and Lagos (2010b) was the first to incorporate liquidity considerations in standard asset pricing models by including equity into Lagos-Wright as a claim to a stochastic dividend stream. Rocheteau and Wright (2011) takes the Lagos-Wright model with endogenous firm entry and the Lagos (2010a) Lucas trees structure and examines the dynamic equilibria. In Weill (2007), Lagos, Rocheteau, and Weill (2011) the search friction is interpreted as liquidity shocks that arise with a fixed probability.

The framework employed here is also related to an extensive literature that employs adaptive learning in macroeconomics. Most closely related are papers that incorporate constant gain learning in studies of monetary policy and asset pricing: see, for example, Branch and Evans (2011); Sargent (1999); Orphanides and Williams (2005); Cho, Williams, and Sargent (2002); Williams (2004); Cho and Kasa (2008); McGough (2006).) Branch and Evans (2011), in particular, find that risk-averse agents in a mean-variance asset pricing model forecast both the risk (conditional variance) and return (conditional mean) of stock prices using a forecasting model whose parameters are updated using constant gain least squares then traders may also come to believe that stocks follow a random walk. These nearly self-fulfilling random walk beliefs lead to recurrent bubbles and crashes in stock prices. While there are similarities in the mechanism generating bubbles in Branch and Evans (2011) and in this paper, there are important differences. The present environment includes a liquidity services role for assets and this liquidity demand can play an important role in generating bubbles. While Branch and Evans (2011) emphasize variations in the perceived riskiness of assets.\(^6\) By focusing on how learning affects liquidity, the bubbles and crashes identified here have implications for consumption and economic activity.

The results in this paper relate to a recent, and growing, literature that estimates time series variation in liquidity premia.\(^7\) See, for example, Dick-Nielsen, Feldhutter, and Lando (2012) and Bao and Pan (2012) for evidence of excess volatility in estimated bond liquidity premia. In search-based models there are alternative explanations for a time-varying liquidity premium, these include multiple equilibria in Rocheteau and Wright (2011), He, Wright, and Zhu (2012) and endogenous credit cycles in Gu and Wright (2010).

Is it reasonable to assume that individuals might have imperfect knowledge about the

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\(^6\) Another related paper is Adam, Marcet, and Nicolini (2015) who demonstrate that an asset pricing model with adaptive learning can replicate some empirical regularities. Their framework and focus is distinct from the present paper that focuses on the liquidity role of the asset and providing some quantitative sense of its relative role. The results here complement previous research on learning and asset prices.

\(^7\) See Vayanos and Wang (2012b) for a survey of recent theoretical and empirical work on this topic.
asset price process? The answer is yes, for a variety of reasons. First, we adhere to the cognitive consistency principle, as articulated by Sargent (1993) and Evans and Honkapohja (2013), that economic agents should be assumed to behave like a good econometrician who forecasts future economic variables using time-series econometric methods. Second, it is plausible to assume that the economy is subject to occasional structural change—such as a decline in the supply, or an increase in the global demand for safe assets—that is only revealed after a sufficient quantity of data bears evidence of the change. Third, on average, their beliefs are close to the rational expectations equilibrium values and are determined endogenously with the state variables, thereby, preserving the cross-equation restrictions that are a salient feature of equilibrium models.

2 A Search-based Asset Pricing Model with Imperfect Knowledge

2.1 Basic Environment

Each time period consists of two subperiods: in the first subperiod agents gather in a decentralized, or over-the-counter, market (DM), where buyers and sellers meet in bilateral matches and exchange specialized, non-storable goods $q$; in the second subperiod, agents interact in a centralized market (CM) where each agent is free to consume and produce a non-storable general good $x$ using a linear production technology and trade, at price $p_t$, in a financial asset (claims to a Lucas tree), which is the only storable good in this economy.\footnote{The economic environment is based on Lagos (2010a), who developed the first Lucas-tree asset pricing model with search frictions. One difference between Lagos (2010a) and the environment in this section, is there are two goods, one storable asset, and sellers have a production technology for the specialized good. The environment is also very close to Rocheteau and Wright (2011) but without free entry of firms.}

The general good serves as the numeraire good. Following Rocheteau and Wright (2005) agents are heterogeneous with a continuum of buyers, who consume but do not produce $q$, and sellers, who produce but do not consume $q$, each with measure one. Households have lifetime utility $\hat{E}_0 \sum_{t=0}^{\infty} \beta^t (U(x_t) + u(q_t) - l_t)$, where $l_t$ are labor hours used in production according to $x_t = l_t$. $\hat{E}$ is the (possibly) non-rational expectations operator (to be specified below). As is standard in these models, define $U'(x^*) = 1$ and let $U(x^*) = x^*$. For simplicity, assume that the specialized good is produced in the decentralized market according to the cost function $c(q) = q$. The efficient quantity of the decentralized good is, therefore,
Finally, assume that agents rank alternative bundles of the specialized good by \( u(q) = \frac{q^{1-\sigma}}{1-\sigma} \).

The financial asset pays, at the beginning of the CM, a known dividend \( y_t \) units of \( x \) to holders of the asset and has an exogenous supply \( A_t \). Household holdings of the financial asset, at the end of period \( t \), are denoted \( a_t \). Under consideration here is a specification of the model that isolates the liquidity effect. In subsequent analysis, a version of the model with stochastic dividend growth is formulated to demonstrate the relative significance of the asset’s liquidity role. There is a stochastic process for \( A_t \) meant to capture exogenous variation in the supply of assets. Changes in asset supply are implemented via lump-sum transfers to buyers at the beginning of the centralized market. The exogeneity of the asset supply process is made for technical convenience.

The decentralized market, or over-the-counter, market operates as follows. Buyers and sellers meet in bilateral matches where the probability of a buyer and seller meeting is given by the constant probability \( \alpha \). The parameter \( \alpha \) captures the search friction present in over-the-counter markets. Because of a lack of commitment, or imperfect credit enforcement, trade involves issuing debt backed by collateral in the form of claims to the asset, or equivalently, a *quid pro quo* transfer of shares in the asset. Following Kiyotaki-Moore, buyers are restricted to borrowing up to a fraction \( \rho \) of the expected value of their assets. In the event of default, the seller receives the collateral before dividends are paid in the centralized market.

### 2.2 Timing Assumptions

The timing of the model assumes that buyers make asset holding decisions during the centralized market and these assets can be used for trade in the following period. Thus, asset demand will depend critically on expectations about the future price of the asset. As an

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\(^9\)With search frictions, with positive probability there may not be trade and the utility function \( u(q) \) may not be defined at zero. The literature deals with this possibility typically by altering the utility function to be \( u(q) = (q + \varphi)^{1-\sigma} \frac{\varphi^{1-\sigma}}{1-\sigma} \) and setting \( \varphi \) arbitrarily close to zero.

\(^10\)This variation could also be considered “asset float” – i.e., IPO lock-up expiration, stock splits, repurchase agreements, etc. – changes in the supply of safe government debt, or an increase in global demand for safe assets that is external to the economy. It is possible to extend the model to include an exogenous asset supply. For example, in Li, Rocheteau, and Weill (2012) asset holders can create (“counterfeit”) their own assets which places an endogenous bound on the acceptability of assets. In He, Wright, and Zhu (2012), and Petrosky-Nadeau and Rocheteau (2013), the asset is housing and a housing sector endogenously builds new houses.
alternative to rational expectations, this paper assumes that individuals behave like econometricians who hold a (correctly) specified model of the economy, but they must recover the parameters in real time from data.

Of course, along a learning path, the decisions that individuals make will be informed, in part, by their forecasting model for price; as beliefs adjust, so will the decisions made by agents. The present environment assumes agents, at any date, hold a set of consistent subjective probability beliefs about all payoff-relevant variables that are beyond their control, such as prices, dividends and asset supply. Given well-specified subjective probability beliefs, the individual agents solve their optimization problem.\footnote{In general, finding solutions to the optimization problem of a learning agent can be challenging since household first-order conditions typically include expectations about the agent’s own future consumption and the evolution of their beliefs is a state variable to the household’s problem. Following Sargent (1999), the learning literature typically assumes households are anticipated utility maximizers who, when solving for their optimal plan, do not take into account how their beliefs might evolve in the future. The quasi-linear utility function, which eliminates wealth effects, greatly simplifies this issue.}

We assume that all individuals (buyers and sellers) have the same beliefs: they observe the same information, have the same learning rule, and its common knowledge within bilateral matches that they hold identical expectations. Finally, it is also assumed that beliefs and endogenous state variables are not determined simultaneously. In particular, we assume that the current asset price is not observable when agents form expectations. This breaks the simultaneity of beliefs and outcomes that is a feature of rational expectations models but is not consistent with agents who form expectations as out-of-sample forecasts given the available data. Define $\Omega_t = \left( \{p_j\}_{j=0}^{t-1}, \{A_j, y_j\}_{j=0}^t \right)$ as the information set available to agents when they make decisions at time $t$.

That agents make decisions given separate forecasts of prices and dividends is a natural and reasonable assumption in a model of adaptive learning. In an imperfect knowledge environment, it is not reasonable to expect that agents know the pricing function – which depends on discounted expected dividends in a rational expectations equilibrium – when it depends on the beliefs, preferences, etc. of all agents in the market. Moreover, because of the self-referential nature of asset pricing models, the pricing function under learning is not a time-invariant function of expected dividends. Thus, it is natural to imagine that outside of a rational expectations equilibrium agents try to learn about the pricing process. (See Adam and Marcet (2010) for extensive discussion.) We do, however, discipline beliefs to be, on average, close to rational expectations in a sense that we make formal below.

In the centralized market, individuals hold expectations about the next period’s price,
conditional on all of the previously realized state variables, before they observe the current price. Thus, we interpret the optimization problem as determining a demand schedule that the agents turn into the Walrasian auctioneer in the centralized market and the auctioneer sets the price to clear the market.\footnote{This assumption has a somewhat similar interpretation as the limit orders in Biais and Weill (2009). It is also reminiscent of the temporary general equilibrium theory of Grandmont (1977).} More specifically, the timing is as follows:

- At the beginning of period $t$, after observing the realized price $p_{t-1}$, buyers and sellers update their information set $\Omega_t$ to include $p_{t-1}$ and realizations of the exogenous shocks $y_t, A_t$.

- At the beginning of the decentralized and centralized markets in time $t$, buyers and sellers hold expectations $\hat{E}[p_t + y_t | \Omega_t]$ and $\hat{E}[p_{t+1} + y_{t+1} | \Omega_t]$, with $y_t, A_t$ are observable but $p_t \not\in \Omega_t$.

- At the beginning of the centralized market, buyers submit their asset demand schedule to the auctioneer based on $\hat{E}[p_{t+1} + y_{t+1} | \Omega_t]$. The auctioneer clears the market.

2.3 Model

To solve the model for an equilibrium asset price, we proceed sequentially beginning with the bargaining solution in the decentralized market. During the decentralized market, at the beginning of time $t$, a buyer makes a take-it-or-leave it offer in the form of a pair $(q_t, B_t)$ that specifies the exchange of $q$ units of the good in exchange for $B$ units of credit to be repaid in the centralized market.\footnote{The take-it-or-leave it offer is a special case of proportional bargaining, where the buyer captures the entire surplus from trade. Proportional bargaining has certain theoretical properties, such as surpluses that increase along with the bargaining set, that are more attractive than other forms of bargaining such as Nash bargaining. See Nosal and Rocheteau (2011) for details. The qualitative results do not hinge on the proportion of the surplus assigned to the buyer.} This offer solves

$$(q_t, B_t) = \arg \max_{q_t, B_t} \{u(q_t) - B_t\}$$

subject to the seller’s participation constraint

$$-q_t + B_t \geq 0$$

and the borrowing constraint

$$B_t \leq \rho \hat{E}[p_t + y_t | \Omega_t] a_{t-1} \geq 0$$
The buyer’s offer maximizes his surplus from an offer where the term $B_t$ is the expected consumption foregone in the centralized market after transferring $B$ units of the asset to the seller. Because $p_t$, the price of the asset in the centralized market in time $t$, is not contemporaneously observable, the bargaining terms between buyer and seller depend on their (possibly) non-rational expectations of the value of the asset. Simplifying leads to

$$q_t = \arg \max_q \{u(q_t) - q_t\}$$

subject to

$$B(q) \equiv q_t \leq \rho \hat{E} \left[ p_t + y_t \vert \Omega_t \right] a_{t-1}$$

The seller will extend credit so that the anticipated value of the debt in the centralized market, $\rho \hat{E} \left[ p_t + y_t \vert \Omega_t \right] a_{t-1}$, is greater than their cost of producing $q_t$.

The solution to this bargaining problem is

$$q_t = \begin{cases} q^* & \text{if } \rho \hat{E} \left[ p_t + y_t \vert \Omega_t \right] a_{t-1} \geq q^* \\ \rho \hat{E} \left[ p_t + y_t \vert \Omega_t \right] a_{t-1} & \text{else} \end{cases}$$

If buyers have sufficient holdings of the asset they purchase the efficient quantity $q^*$, otherwise they borrow the maximum amount and receive $\rho \hat{E} \left[ p_t + y_t \vert \Omega_t \right] a_{t-1}$ in return.

The value function for a buyer in the decentralized market, given $(q_t, B(q_t))$ is given by the expression

$$V_t(a_{t-1}) = \alpha [u(q_t) + W_t(a_{t-1}, B(q_t))] + (1 - \alpha)W_t(a_{t-1}, 0)$$

where $W_t$ is the value function for a buyer in the centralized market:

$$W_t(a, B) = \max_{x_t, l_t, a_t} U(x_t) - l_t + \beta \hat{E} \left[ V_{t+1}(a_t) \vert \Omega_t \right]$$

subject to the constraints

$$x_t + p_t a_t + B_t = l_t + (p_t + y_t) a_{t-1} + T_t$$

where $T_t$ are the lump-sum transfers that distributes, without loss of generality, the changes in asset supply to buyers.

Combining these expressions, and making use of the quasi-linearity, leads to the following equation

$$W_t(a_{t-1}, B_t) = (p_t + y_t) a_{t-1} - B_t + T_t + \max_{x^*} \left[ U(x^*) - x^* \right]$$

$$+ \max_{a_t \geq 0} \left\{ -p_t a_t + \beta \hat{E} \left[ \alpha \left\{ u(q_{t+1}) - B(q_{t+1}) \right\} + (p_{t+1} + y_{t+1}) a_t \vert \Omega_{t+1} \right] \right\}$$
With these assumptions, asset demand $a_t$ is the solution to

$$\max_{a_t \geq 0} \left( \beta \hat{E}_t [p_{t+1} + y_{t+1}|\Omega_t] - p_t \right) a_t + \alpha \beta \left[ u(\hat{E} [q_{t+1}|\Omega_t]) - \hat{E} [B(q_{t+1})|\Omega_t] \right]$$

(1)

To derive this expression for asset demand, we impose that (i.) agents use point expectations, i.e. $\hat{E} [u(q_{t+1})|\Omega_t] = u(\hat{E} [q_{t+1}|\Omega_t])$, and, (ii.) expectations obey a law of iterated expectations, i.e. $\hat{E} \left[ \hat{E} (z|\Omega_{t+1}) |\Omega_t \right] = \hat{E} [z|\Omega_t]$ for any variable $z$. That agents use point expectations is a behavioral assumption that essentially holds that decisions only depend on the mean of their subjective beliefs.\(^{14}\) This assumption is made for technical convenience and is a standard restriction imposed in many rational expectations and adaptive learning models (see Evans and Honkapohja (2001)).\(^{15}\)

Notice that in (1), when $\alpha = 0$, i.e. there is no decentralized market, the buyer’s demand for the asset is equivalent to the risk-neutral Lucas asset pricing model. The first expression in (1) shows that a part of the demand for the asset depends on the expected return on the asset. The second expression is the liquidity demand for the asset and here it depends on the expected surplus from trading in the decentralized market.

There are three cases to consider:

1. When $\beta \hat{E} [p_{t+1} + y_{t+1}|\Omega_t] > p_t$, then households desire an infinite amount of the asset and the optimization problem does not have a solution.

2. When $\beta \hat{E} [p_{t+1} + y_{t+1}|\Omega_t] = p_t$ then households hold enough to purchase $q^* = 1$, $B_{t+1} = B(q^*) = 1$, and any $a_t \geq B_{t+1}$ is a solution to the optimization problem. In this case, there is no liquidity premium and the asset is priced as the discounted expected payment flow of the asset.

3. When $\beta \hat{E} [p_{t+1} + y_{t+1}|\Omega_t] < p_t$, then the household is liquidity constrained and $q_{t+1} = \rho \hat{E} [p_{t+1} + y_{t+1}|\Omega_{t+1}]$, and $a_t$ solves

\(^{14}\)Without this assumption, the optimal demand for $a_t$ will depend on the covariance between the asset’s return and the marginal return on wealth. One could include a learning model for “risk,” similar to Branch and Evans (2011) that will impact the quantitative results, but not affect the main qualitative channel through which liquidity affects asset prices under learning.

\(^{15}\)A closely related issue is the connection between the buyers’ solution to the dynamic programming formulation and the sequence problem. By assuming that asset demand solves (1), we are imposing that buyers satisfy a transversality condition. The adaptive learning literature often distinguishes between decisions that are based on satisfying the Euler equation or that also satisfy the anticipated lifetime budget constraint (“infinite horizon learning”). See Preston (2006) for an extensive discussion. It can be verified in the present environment, with quasi-linear utility and a linear production technology, that buyers asset demand will be the same regardless of Euler equation or infinite horizon learning.
The first-order condition from the buyer’s problem combined with a market clearing condition yields an expression for the equilibrium price.\footnote{The liquidity premium implies that there is a holding cost to the asset. Since sellers do not use the liquidity services of the asset, they will choose not to buy the asset in the competitive market.}

It remains to specify the processes for dividends and for the supply of the asset. Since we are interpreting the asset as a “safe asset” it is natural to assume that \( y_t = y \) is known with certainty (or subject to small iid shocks). Below, the case when dividends follow a geometric random walk with drift is considered. Assume also that the supply of the asset is given by the process \( \log A_t = \log A - \frac{1}{2} \log \hat{\varepsilon}_t \) where \( A > 0 \) and \( E \hat{\varepsilon}_t = 1 \) with a small compact support. The stochastic process for the supply of shares is meant to proxy for exogenous changes in asset supply.\footnote{Asset float has been shown to be an important factor in asset pricing (see Cochrane (2005), Baker and Wurgler (2000)).} Because the stochastic component of the transfers are unpredictable, buyers will not anticipate receiving transfers in period \( t + 1 \) when deciding on their asset demand at time \( t \).

With these assumptions in hand, it is straightforward to solve for the following equilibrium price

\[
    p_t = \beta \hat{E}_t (p_{t+1} + y_{t+1}) [1 + \mathcal{L}_t] \tag{2}
\]

where

\[
    \mathcal{L}_t = \begin{cases} 
    \alpha \rho \left\{ \left[ \rho \hat{E}_t (p_{t+1} + y_{t+1}) A_t \right]^{-\sigma} - 1 \right\} & \text{if } A_t < \frac{q^*}{\rho \hat{E}_t (p_{t+1} + y_{t+1})} \\
    0 & \text{else}
    \end{cases}
\]

where we now make use of the simplifying notation \( \hat{E}_t z = \hat{E} [z | \Omega_t] \). Recall, also that \( \hat{E}_t y_{t+1} = y \). This law of motion for the equilibrium price can be written compactly as

\[
    p_t = G(\hat{E}_t p_{t+1}, A_t) \tag{4}
\]

### 2.4 Rational Expectations Equilibria

A \textit{rational expectations equilibrium} is a sequence \( \{p_t\} \) that is a (bounded) solution to (2). In the analysis below, the model will be parameterized so that it is locally determinate and it is natural to focus on solutions to (2) that take the form of a noisy steady-state.
When $\alpha > 0$, therefore, and the supply of the asset $A$ is sufficiently low so that there is a liquidity premium, the asset trades at a price above its discounted cash flow, and these equilibria are sometimes called rational bubbles. Because the liquidity premium arises out of a fundamental property of the asset – that is, its ability to facilitate bilateral exchange – we refer to this as the fundamental price.

**Definition 1** The “fundamental,” or stationary, equilibrium price is the steady-state $\bar{p} = G(\bar{p}, A)$.

**Remark.** Of course, when $A$ is sufficiently high (or $\alpha = 0$) then there is no liquidity premium and $\bar{p} = \beta y/(1 - \beta)$, which is the expected present value of the dividend flow.

**Definition 2** A noisy steady-state rational expectations equilibrium is a function $p(A_t)$ defined so that $p(A_t) = G(\hat{p}, A_t)$ with $\hat{p}$ such that $\hat{p} = EG(\hat{p}, A_t)$, where the expectation is taken with respect to the distribution of $A_t$.

The following result is a direct application of a theorem in Evans and Honkapohja (1995).

**Proposition 3 (Evans and Honkapohja (1995))** Consider a family of distribution functions for $A_t$, indexed by $\alpha$, with $F_\alpha(-\alpha) = 0, F_\alpha(\alpha) = 1$ and $F_\alpha$ (weakly) converges as $\alpha \to 0$ to $F_0(A) = 1$. Define $\hat{p}(\alpha) = EG(\hat{p}(\alpha), A_t(\alpha))$ and $\bar{p} = G(\bar{p}, A)$ is the fundamental steady-state. Assume the model is parameterized so that $\partial G(\bar{p}, A)/\partial \bar{p} \neq 0, \partial^2 G(\bar{p}, A)/\partial \bar{p}^2 \neq 0$. Then there exists a noisy steady-state $p(A_t) = G(\hat{p}(\alpha), A_t)$ with $\hat{p}(\alpha)$ arbitrarily close to $\bar{p}$, for sufficiently small $\alpha$.

In a (noisy steady-state) rational expectations equilibrium, asset prices are small iid deviations from the fundamental, or stationary, equilibrium price.

### 3 Liquidity and Beliefs

This section presents results on the nature of beliefs that arise in equilibrium and along a typical learning path.

#### 3.1 Learning

Because the economic law of motion is a non-linear expectational difference equation, it is not reasonable to expect that agents will know the complete underlying economic structure
and be able to form rational expectations. In response, many modelers assume that agents behave like a good Bayesian who holds priors about the perceived model of the economy and updates those priors as new data become available. This adaptive learning approach assumes that agents hold a correctly specified model with unknown parameters and use a reasonable estimator to update their parameter estimates.$^{18}$

In practice, however, econometricians often misspecify their models. In particular, even though the actual data generating process may be non-linear, econometricians and professional forecasters typically estimate linear models such as vector autoregressive models. This section takes this approach seriously by imposing that agents form their expectations via a linear AR(1) model of the asset price. Although this forecasting model is misspecified, in a stochastic consistent expectations equilibrium (SCE) agents’ forecasting model is optimal within the class of misspecified models, i.e. the optimal linear projection, so that, within the context of their perceived model, they are unable to detect their misspecification. The projection parameters and the equilibrium stochastic process for asset prices are jointly determined so that a SCE preserves many of the cross-equation restrictions that are a salient feature of rational expectations models. The Appendix defines and presents additional details on the properties of the SCE. It is important to note that, although the AR(1) model may be misspecified for some AR coefficients, the forecasting model nests the (unique) noisy steady-state rational expectations equilibrium.$^{19}$ In fact, the stable SCE will coincide with the rational expectations equilibrium.

Specifically, assume that agents form expectations from the forecasting model

$$p_t = c_{t-1} + b_{t-1}p_{t-1} + \varepsilon_t$$

Thus,

$$\hat{E}_t p_{t+1} = c_{t-1} (1 + b_{t-1}) + b_{t-1}^2 p_{t-1}$$

Plugging (5) into (2) leads to the actual law of motion, given by

$$p_t = G(y + c_{t-1} (1 + b_{t-1}) + b_{t-1}^2 p_{t-1}; A_t)$$

where $G(\hat{E}_t (p_{t+1} + y_{t+1}); A_t)$ is given by the expression (2).

$^{18}$See Evans and Honkapohja (2001) for extensive treatment of adaptive learning and expectational stability.

$^{19}$In other models, a different forecast rule may be natural. For instance, in some settings it might be reasonable for agents to forecast price growth. Here the asset price is a function of expected future price and those expectations enter non-linearly. Thus, there is a natural dependence on expectations of price.
The forecasting model and beliefs in (5), formalize the nature of individuals’ imperfect knowledge. They have an imperfect understanding of the process that determines the market asset price, \( p_t \), and they specify a linear econometric forecasting model that nests the noisy rational expectations equilibrium price (i.e. where \( c = \hat{p}, b = 0 \)). Given these beliefs, they determine their optimal asset demand and bargain over terms of trade with sellers (who share the same beliefs) in the over-the-counter market.

Agents update their parameter estimates recursively in real-time using discounted least-squares, i.e. “constant gain learning.” Let \( \theta' = (c, b), X' = (1, p) \). Agents update parameter estimates according to the following recursive algorithm

\[
\begin{align*}
\theta_t &= \theta_{t-1} + \gamma S_{t-1} X_{t-1} (p_t - \theta' X_{t-1}) \quad (7) \\
S_t &= S_{t-1} + \gamma (X_t X'_t - S_{t-1}) \quad (8)
\end{align*}
\]

The equations in (7)-(8) are the updating equations for recursive least squares where the data are discounted by a constant “gain” \( \gamma \). Here \( S_t \) is an estimate of \( E X_t X'_t \), the second moment matrix of the regressors. Least-squares updating arises when the constant gain \( \gamma \) is replaced by a decreasing sequence \( \gamma_t = t^{-1} \). Sargent and Williams (2005) demonstrate that constant gain learning equations (7)-(8) arise from an approximate Bayesian learning process in which the prior on parameter drift, a common assumption in applied econometric work, is proportional to the ratio of observation noise variance to the covariance of the regressors, with the speed of drift controlled by the constant gain \( \gamma \). An alternative interpretation of (7)-(8) is that agents use least squares modified to discount past data due to a concern about (possible) structural change of an unknown form. In the stochastic simulations below, we set \( \gamma = 0.10 \) which equates to an effective sample size of approximately 80 years.\(^ {20} \)

The central interest in this paper is asset price dynamics under constant gain learning. There are, of course, related issues such as the existence, expectational stability (“E-stability”), and stability under learning of a SCE. These issues are discussed, in some detail, in the Appendix where numerical analysis also shows that the noisy rational expectations equilibrium is stable under learning.

\(^ {20} \)Empirical estimates of constant gains typically fall in the range of \( 0.02 - 0.10 \), see Branch and Evans (2006). Bubbles and crashes arise more often in stochastic simulations for larger values of \( \gamma \) but also introduce a greater amount of volatility.
Figure 1: Asset price dynamics. Plots typical stochastic paths for asset prices under rational expectations (left) and constant gain learning (right).

3.2 A Special Case: $\sigma = 1$

The main point of this paper is to study how learning impacts, and is impacted by, the liquidity properties of assets and the implications for asset prices. Adaptive learning leads to qualitatively distinct dynamics. For example, Figure 1 plots two stochastic simulations of the model, given by equation (4), that compares typical asset price dynamics under learning to rational expectations.\textsuperscript{21} As is evident, on average, both formulations feature asset prices that are small iid deviations from steady-state. The learning model, though, features an "escape" with rapid price appreciation followed by an abrupt collapse. The learning dynamics are the result of two forces, the "escape dynamics" that push the system away from steady-state in response to stochastic shocks, and the "mean dynamics" that govern the transition path back to the steady-state. The remainder of this section and Section 4 present analytic and numerical analysis that explains the underlying mechanisms driving the learning dynamics evident in Figure 1 and analyzes how the liquidity features of assets affect those mechanisms.

The remainder of this section details the mean dynamics and illustrates how the liquidity properties of the asset can affect the qualitative nature of learning dynamics. The following section focuses on mechanisms that can trigger an "escape" from a neighborhood of the rational expectations equilibrium. There are two central ingredients to the results that come below: the asset is in a sufficiently low supply (i.e. illiquid) that is imperfectly known by agents, and a prior belief of possible structural change.

\textsuperscript{21}The model is parameterized according to Table 1, that follows, except $A = .15, \gamma = .10$. 

15
3.2.1 Asymptotic Learning Dynamics

In the case that \( \sigma = 1 \) the form of the asset pricing equation is simplified and additional analytic results are available on the global learning dynamics. In this case, the law of motion for asset prices is

\[
p_t = (1 - \alpha)\beta\hat{E}_t(p_{t+1} + y_{t+1}) + \alpha\beta A^{-1}(1 + \varepsilon_t) \tag{9}
\]

\[
= (1 - \alpha)\beta [y + c(1 + b)] + \alpha\beta A^{-1} + (1 - \alpha)\beta b^2 p_{t-1} + \alpha\beta A^{-1} \varepsilon_t \tag{10}
\]

\[
\equiv T(a, b)'X_{t-1} + \alpha\beta A^{-1} \varepsilon_t \tag{11}
\]

where \( \varepsilon_t \) is white noise with variance \( \sigma^2 \). Without loss of generality, it is assumed that \( \rho = 1 \) when deriving equation (9). For fixed belief parameters \((c, b)\), the law of motion in (9) is conditionally linear and the analysis in Branch and Evans (2011) can be used to gain analytic insight into the nature of real-time learning dynamics.

For small gains \( \gamma \), it is possible to obtain results on the asymptotics of \( \theta_t^\prime = (c_t, b_t) \) by studying a continuous time approximation to the constant gain least-squares recursive algorithm. More specifically, Evans and Honkapohja (2001) demonstrate that asymptotically the dynamics are governed by the “mean dynamics” ordinary differential equation (ODE)

\[
\frac{d\theta}{d\tau} = S^{-1}M(\theta)(T(\theta) - \theta) \tag{12}
\]

\[
\frac{dS}{d\tau} = M(\theta) - S \tag{13}
\]

where \( \tau = \gamma t \), \( M(\theta) \) is the unconditional covariance matrix of the regressors holding \( \theta \) fixed. That the mean dynamics ODE governs learning dynamics is intuitive since under (12) the parameters \((c, b)\) are adjusted in the direction of the asymptotic moments implied by the actual law of motion generating the data given \((c, b)\), with weighting terms that depend on estimates of the covariance matrix. It is straightforward to see that the fundamental equilibrium is a locally stable rest point since \((1 - \alpha)\beta < 1\).

Under decreasing gain learning, \( \gamma \) is replaced with \( 1/t \) and it can be shown that in the limit as \( t \to \infty \) the learning dynamics converge with probability one to the fundamental equilibrium. We next summarize the analytical results for constant gain learning by directly applying the results in Branch and Evans (2011).

The first result establishes that for a sufficiently small constant gain the perceived coefficients \( \theta_t \) will be an approximately normal random variable with a mean equal to its fundamental equilibrium value and a variance that depends on both the constant gain and other parameters of the model including the illiquidity parameter \( A \). The second result
shows that from a given initial condition \((\theta_0, S_0)\) the solution to the “mean dynamics” of the
ODE (12)-(13) give the expected transition path to the fundamental equilibrium values. All
proofs are contained in the Appendix.

**Proposition 4** The belief parameters \(\theta_t\) are approximately distributed as
\(\theta_t \sim N(\bar{\theta}, \gamma V)\) for small \(\gamma > 0\) and large \(t\), where \(\bar{\theta} = (\bar{p}, 0)\)' and for appropriately defined \(V\).

**Proposition 5** Define \(\phi_t = (\theta_t, \text{vec}(S_t))'\). For any \(\phi_0\) within a suitable neighborhood of the
fundamental equilibrium, define \(\hat{\phi}(\tau, \phi_0)\) as the solution to the differential equation (12)-(13),
with initial condition \(\phi_0\). Fix \(T > 0\). The mean dynamics of (7)-(8) satisfy
\(E\phi_t \approx \hat{\phi}(\gamma t, \phi_0)\) for \(\gamma\) sufficiently small and \(0 \leq t < T/\gamma\).

There are important conclusions to draw from these propositions. First, the fundamental
equilibrium provides a natural benchmark in the sense that the coefficients for the forecast
rule under learning are centered on these equilibrium values. Second, for \(\gamma \to 0\), the learning
dynamics are arbitrarily close to their fundamental equilibrium values with high probability.
Third, for any initial condition, and finite period of time, the expected learning path to the
fundamental equilibrium will be the solution path to the mean dynamics equations (12)-(13).
One can think of constant gain learning, which respond more strongly to recent shocks, as
re-initializing the mean dynamics. Moreover, these initial conditions can be drawn from the
asymptotic distribution for \(\theta\) computed in Proposition 4.

### 3.2.2 Learning Dynamics and Random-Walk Beliefs

This section turns to demonstrating a key theoretical possibility in the mean dynamics.
We present these results using the baseline parameterization from Table 1. The value of \(\beta\)
accords with a 2% real interest rate, while \(\alpha\) was chosen to match the velocity of collateral
estimate in Singh (2011). All values of \(\sigma < 2\) yield a determinate model. In the baseline
parameterization \(\sigma = 1.75\), though the subsequent analysis considers alternative values for \(\sigma\).
The values of \(A\) and \(y\) were chosen so that there was a liquidity premium in the fundamental
price, and the value of \(\sigma_\pi^2\) ensures that the supply shocks are “small”. We choose \(\rho = 1\),
though consider alternative values below. In this case, the mean price, i.e. the fundamental
equilibrium, is \(\bar{p} \approx 2.3\). With these parameter values, the model is illustrative.

Figure 2 plots representative mean dynamics where initial values for \(c, b > 0\) are drawn
from the principal axis of the asymptotic distribution in Proposition 5. To generate this
figure the parameter values are chosen according to Table 1 except \(\sigma = 1\). The initial values
Table 1: Baseline parameterization

<table>
<thead>
<tr>
<th>$\beta$</th>
<th>$\sigma$</th>
<th>$\alpha$</th>
<th>$A$</th>
<th>$y$</th>
<th>$\sigma^2$</th>
<th>$\rho$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.98</td>
<td>1.75</td>
<td>0.10</td>
<td>0.40</td>
<td>0.01</td>
<td>0.004</td>
<td>1</td>
</tr>
</tbody>
</table>

Figure 2: Mean Dynamics (baseline parameterization).

are $c = 3.73$ and $b = .4$ corresponding to an increase in the perceived mean and serial correlation of the asset price. The top panel plots the perceived value for the intercept, $c$, while the bottom panel plots the perceived lag coefficient $b$.

Figure 2 illustrates two aspects of the expected learning path. First, the fundamental equilibrium is a stable rest point of the mean dynamics implying that learning paths converge to the fundamental equilibrium. Second, the transition path to the fundamental equilibrium is interesting. At first the path for $(c, b)$ moves back toward the fundamental equilibrium then abruptly reverses course with $c \approx 0$ and $b \approx 1$ for a finite period of time, before converging to the fundamental equilibrium values. The mean dynamics, therefore, show that the expected learning path has agents perceiving, for stretches of time, that the asset price follows a random walk without drift.

Random-walk beliefs play a key role in learning dynamics. First, the additional serial correlation introduced through beliefs has important implications for the dynamic nature
of asset prices. This serial correlation can be nearly self-confirming as it introduces serial correlation that otherwise would not exist and the data can be slow to reveal that misspecification.\footnote{See Sargent (1999) for further discussion.} Second, random-walk beliefs can arise through learning as temporary deviations from the fundamental equilibrium. Third, as the next section will demonstrate, random walk beliefs can lead to speculative bubbles and crashes. In essence, agents come to believe that recent innovations in asset prices are permanent shifts and not mean-reverting fluctuations. They trade on these beliefs and bid asset prices up.

### 3.2.3 Implications of Asset Shortages

This section examines the comparative effects of asset shortages via the parameter $A$. Of course, if $A$ is sufficiently large then there is no liquidity premium. Smaller values of $A$ then yield a larger liquidity premium as there is not sufficient quantities of the asset to secure the efficient quantity of trade in the decentralized market. The onset of random-walk beliefs depends on a complicated interaction of the entire set of structural parameters. It is natural, though, to focus on the role asset supply plays in the onset of random-walk beliefs.

Beliefs and the liquidity premium are jointly determined in equilibrium outcomes. The size of the liquidity premium affects the likelihood of observing random walk beliefs, and hence, bubbles and crashes through two primary channels. First, smaller values of $A$ increase the mean asset price and increase asset volatility, which in turn affects the asymptotic distribution for the learning coefficients $(c_t, b_t)$. Second, smaller values of $A$ affect the T-map which in turn affects the learning dynamics through the least-squares updating equations. It turns out that smaller values of $A$ make the onset of random-walk beliefs more likely.

To more closely examine the effect of changes in asset supply, Figure 3 plots the 95% confidence ellipses for different values of $A$ in the range $[0.1, 0.7]$ and the baseline parameterization with $\sigma = 1$. Each confidence ellipse is centered at $(\bar{p}, 0)$. The arrow indicates the direction in which $A$ increases. There are three features of note in Figure 3. For each ellipse, the principal axis is pointed in the direction of a random walk without drift. One can expect many trajectories moving along that direction. Second, when the asset is in shorter supply, the liquidity premium is higher and the mean value for the asset price is higher. Third, as $A$ increases the confidence ellipses become more tightly concentrated around the fundamentals equilibrium. In particular, for $A$ sufficiently large there are no liquidity effects and the confidence ellipse collapses to a point at $(\bar{p}, 0)$. When the asset is in short supply, then the learning dynamics converge to a distribution with considerable variation in the
belief coefficients \((c_t, b_t)\).

To examine the effect that \(A\) has on the learning dynamics, Figure 4 plots the mean dynamics for various values of \(A\) in the range \([0.05, 0.20]\), holding all of the other parameter values the same as in Table 1. The figure is generated by choosing initial values from the respective principal axes of the 95% confidence ellipses. In each case, the initial value for \(b\) is fixed at 0.60, and then \(c\) is chosen on the principal axis. For each value of \(A\), the learning dynamics converge to the fundamental equilibrium, though, the transitional dynamics can be different. The thick line corresponds to \(A = 0.05\), where the liquidity premium is strongest. As asset supply becomes larger – or, a smaller liquidity premium – the dynamics are drawn towards a random walk but gradually peak further away from \(c = 0, b = 1\). One way to interpret constant gain learning is it re-initializes the mean dynamics which provide the subsequent expected learning path. Since the confidence ellipses are more tightly concentrated for large values of \(A\), even for larger gains, it is less likely that beliefs would land at the value of \(c = 0.60\) that triggers the random walk beliefs in Figure 4. Thus, if random-walk beliefs do not arise in the mean dynamics, which approximates learning dynamics for small gains, they are unlikely to appear with larger constant gains.
Figure 4: Mean dynamics for various values of $A$ (initial conditions drawn from the corresponding 95% confidence ellipse).

4 Liquidity, Bubbles and Crashes

This section demonstrates that changes in the liquidity properties of an asset in an imperfect knowledge environment have theoretical implications for asset prices.

4.1 Real-time Asset Price Dynamics

This subsection uses stochastic simulations of the real-time learning dynamics to demonstrate that bubbles and crashes – i.e., substantial deviations from the fundamental equilibrium values – can arise as an endogenous response to the asset supply shocks. For example, Figure 5 plots a real-time simulation of price dynamics. As in Figure 1, the figure is computed with the parameter values in Table 1 except with $A = 0.15$ and a constant gain of $\gamma = 0.10$. To generate this figure the model is initialized at the stationary rational expectations equilibrium, expectations are generated according to (5) with parameters updated via constant gain least-squares, and price is determined by (4). The left panels plot the asset price and the

\footnote{To prevent explosive dynamics, we impose the following restrictions on the learning dynamics: (1.) agents only update their estimates of $b$ provided that it lies below $1/\beta$; (2.) agents forecast a non-negative return on the asset, i.e. $\hat{E}_t p_{t+1} + y \geq 0$.}
quantity traded in the decentralized market. The right panels plot the estimated coefficients \((c_t, b_t)\). This is the expanded version of Figure 1.

Under constant gain learning, the economy hovers near its stationary rational expectations equilibrium price most of the time with small iid deviations. Since these learning dynamics are near the noisy rational expectations equilibrium, the figure illustrates that, on average, beliefs are close to their rational expectations values. At about period 200, there is an abrupt qualitative change in the dynamics with a bubble in the asset price. This bubble features a price that increases over 3 times above its fundamental value. The bubble persists only for a finite length of time before returning to a neighborhood of the stationary equilibrium. The pattern of beliefs correspond with what was observed in Figure 2 and in Proposition 5, in that for finite stretches of time agents believe that inflation follows a random walk. In simulations, these large deviations from rational expectations are recurrent.

Figure 5 provides an intuitive story for the existence of asset price bubbles. Because of imperfect knowledge about the economy, individuals learn about the price process via an econometric forecasting rule that remains robust to structural change and model misspecification by weighting recent data more heavily than past data. Occasional economic shocks can give the asset price process an upward drift that is captured by agents’ econometric model as serial correlation. Eventually this serial correlation becomes self-fulfilling and, for
a finite stretch of time, agents come to believe that asset prices follow a random-walk. With random-walk beliefs, buyers will interpret recent price deviations as permanent shifts in the price process and will demand more of the asset. Eventually, though, as the asset price becomes sufficiently high, the liquidity premium vanishes and the learning dynamics return to a neighborhood of the rational expectations equilibrium. Importantly, Figure 5 shows that the collapse in the bubble can be costly as there is a sharp decline in \( q_t \), the quantity traded in the decentralized market. This figure is a key result of this paper.  

Figure 6 demonstrates that the first deviation away from the equilibrium price can be a crash. In this case, the logic is symmetric to why bubbles arise. A sequence of positive asset supply shocks place a downward drift on price leading agents’ econometric model to pick up this trend with higher estimated values of \( b_t \) and lower values of \( a_t \). In turn, asset price expectations decrease which leads to a lower demand for the asset and a smaller quantity of over-the-counter trade. A downward price spiral arises as there is a further downward drift in price, higher estimated values of \( b_t \) until the coefficients arrive at a random walk model which, as argued above, is nearly self-confirming. These beliefs only persist for a finite period of time and eventually the stability of the stationary equilibrium takes over and price dynamics return to their stationary equilibrium value.

Although the dynamics presented in figures 5-6 are the result of a single simulation, the dynamics are a generic feature of the model under certain parameterizations. The analytic results show that, for small gains, that the price dynamics will be small iid deviations from the fundamental price. For the right sequence of shocks, an “escape dynamic” occurs that alters the qualitative nature of the dynamics. That this escape coincides with the perceived process for price following a random walk is predicted by the confidence ellipsoids and the mean dynamic paths. One could formally analyze the “most likely unlikely” sequence of

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24 Many bubble episodes, in practice, feature more gradual run-ups in price that are followed by an abrupt crash. To generate more bubbles with these features likely requires abandoning technically convenient modeling assumptions – such as quasi-linear preferences, linear production technologies, etc. We leave this issue to be explored in future research.

25 It is possible, using the techniques in Cho, Williams, and Sargent (2002), to examine which of the “escape paths” are most likely to drive the economy away from the stationary equilibrium and lead to random-walk beliefs by identifying the “most likely unlikely” sequence of shocks that move asset price a given distance away from the stationary equilibrium value. In principle, one can compute these escape paths analytically in special cases, but more typically it is necessary to resort to simulations.

26 Notice in Figure 6 that, at the end of the simulation shown, the values for \( c, b \) have not returned to their equilibrium values though the forecasting model implied mean is consistent with the equilibrium price. If the simulation were to continue the agents would eventually learn that \( b = 0 \) and \( c \approx 8 \).
shocks that trigger these escape dynamics – essentially moving the perceived coefficients onto the mean dynamic paths that feature random-walk beliefs – using the large deviation theory advanced by Williams (2004). Although such an analysis is beyond the scope of the present paper, we note that random walk beliefs are intuitive following a sequence of shocks that impart the appropriate amount of drift into the price process. That the econometric model detects this drift as serial correlation is expected, as noted by Sargent (1999), since random-walk models approximate well low frequency drift.

### 4.2 A Change in Asset Supply

An imbalance in the supply of assets can be interpreted, within the context of the model, as a decrease in the (mean) asset share supply $A$. Since $A$ is the per-capita supply of the asset, values of $A < 1$ imply a demand imbalance that manifests as a liquidity premium in the stationary equilibrium price.

This subsection considers the following experiment of a decrease in the (mean) asset supply. The model parameters are chosen according to the baseline parameterization with an initial asset share supply $A = 0.40$. The model is initialized in a stationary equilibrium and individuals’ beliefs are set to their rational expectations equilibrium values. Then $A$ is
lowered permanently to $A = 0.2$, a change that reflects a decrease in the supply of assets. Although the economy is initially in a rational expectations equilibrium, the agents in the economy have imperfect information about the change in the supply of assets and the greater liquidity premium that will eventually arise. Figure 7 plots the resulting belief and price dynamics. At time 0, the asset supply $A$ decreases, raising the liquidity premium and the asset price without a corresponding increase in price expectations (which are determined by an adaptive learning rule). Initially, the asset price is below the new stationary equilibrium price. The increase in the asset price is tracked by agents’ econometric model as an increase in the persistence of prices, reflected in an increase in the estimated value of $b_t$. As the mean dynamics predict (for the special case), eventually agents’ beliefs are that prices follow a random walk. At this point, there is a burst in prices as the asset price increases to nearly double its new long-run value before converging to the new equilibrium price. The price dynamics are consistent with the observation that a shortage in assets will lead to transitional dynamics that feature temporary asset bubbles.

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27Figures 7 and 8 are generated as the average time-path across 10,000 stochastic simulations of length 1,000.
28These results are similar to McGough (2006) who examines changes to the natural rate of unemployment in the model of policymaker learning developed in Sargent (1999).
The dynamics in Figure 7 are economically intuitive. The decrease in the mean-value of $A$, is an unanticipated shock. The agents in the model observe the new value for $A_t$, however, they do not know that it is permanently lower. With fixed initial beliefs, initially the price of the asset increases but not to its new fundamental value. Having observed higher prices, agents re-estimate their model and increase their expectations of price, which leads to additional trade in the decentralized market. The drop in $A$ increases price initially which is then reinforced through agents' beliefs as they observe higher prices and expect that their liquidity constraints will be further eased. This becomes self-reinforcing and price overshoots its fundamental value. Because the fundamental equilibrium is stable under learning, eventually beliefs and price converge to the new fundamental price.

4.3 Implications of Changes in Collateral Demand

An imbalance in the demand for assets can, alternatively, arise from a greater demand for the use of assets as collateral in over-the-counter markets. Recently, policymakers such as the IMF, and other market observers, have identified several structural developments and policy changes that have the potential to increase the global demand for safe assets. Among these changes, are new financial regulations that require an increasing number of over-the-counter transactions to be cleared through central clearinghouses. Additionally, many central banks use assets as collateral in repurchase agreements. This subsection investigates the potential impact of increased demand for collateral in over-the-counter transactions.

The model, of course, consists of bilateral trade and does not readily feature a third party clearing house. However, the model lends itself to the following interpretation: as the probability $\alpha$ of a match between buyer and seller increases, buyers will find themselves in more frequent need of collateral to secure trade in over-the-counter transactions. Alternatively, changes in perceived riskiness of other assets can increase demand for collateral in the liquid asset. Hence, an increase in $\alpha$ can be interpreted as a structural increase in the demand for collateral.

Figure 8 present an analogous experiment to Figure 7 where initially $\alpha = 0.01$, signifying an illiquid market with a low probability of over-the-counter trade and then $\alpha$ is increased to $\alpha = 0.2$. The figure demonstrates, for similar reasoning as a change in supply, that changes in the demand for collateral leads to an over-shooting of the new equilibrium price.

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29 See Iorgova, Al-Hassan, Chikada, Fandl, Morsy, Pihlman, Schmieder, Severo, and Sun (2012).
4.4 Implications of Changes in Collateral Requirements

The analysis so far sets the parameter $\rho = 1$ in the borrowing constraint. This section considers the effect of financial innovations that loosen the borrowing constraint, i.e. an increase in $\rho$. It is well-known that increasing the maximum amount that an agent can borrow can have competing effects on the liquidity premium (see He, Wright, and Zhu (2012)). As $\rho$ increases the demand for the safe asset increases because now it can facilitate more trade in the decentralized market. However, larger values of $\rho$ also relax the borrowing constraint and so reduces the liquidity premium. Which effect dominates depends on the strength of the liquidity effect.

To demonstrate the potential impact of looser borrowing restrictions, this section considers two different cases. The first where a change in $\rho$ from a small value of $\rho = 0.10$ to a larger value of $\rho = 0.25$ leads to a higher fundamental asset price. This case will only arise when the liquidity effect is not particularly strong, i.e. for $\sigma < 1$. As an illustrative example, set $\sigma = 0.6$. The second case is when a change in $\rho$ from the relatively large value of $\rho = 0.8$ to $\rho = 1.0$ leads to a smaller fundamental asset price. This comparative static effect arises, for example, in the baseline parameterization. Figure 9 plots the results.

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$30$Changes in financial innovations in a learning model play an important role in explaining the joint dynamics of housing prices and unemployment in Branch, Petrosky-Nadeau, and Rocheteau (2015).
In both panels, an increase in $\rho$ that allows buyers to borrow against a greater fraction of their assets leads to a change in the fundamental price. The left panel is for the case where the fundamental price increases while the second has a decrease in the fundamental price. In the left panel the asset price overshoots its new equilibrium price. In the right panel, the opposite occurs.

5 Quantitative Results

In the baseline version of the model under learning with iid (or, constant) dividends, the more scarce liquid-assets are the more likely it is that destabilizing learning dynamics will arise. However, it is also well-known that in other models with serially correlated dividends, or stationary dividend growth, that the learning dynamics can generate large swings in asset prices. The search model is a natural laboratory to study the relative roles of the liquidity and store-of-value channels of the asset in generating excess volatility and large price-movements.\footnote{Here we focus on fitting the distribution of price-dividend ratios, in particular the tails of the distribution. The ability of adaptive learning models to fit other empirical features of equity prices have been demonstrated by Branch and Evans (2009) and Adam, Marcet, and Nicolini (2015).}
5.1 A Quantitative Model

It is standard to model the stochastic dividend growth process as iid so that dividends follow a geometric random-walk with drift. To incorporate non-stationary dividends into the search-based model this section works with an extension of the benchmark model by adapting Lagos (2010a) to the present environment. The model is extended to include three different non-storable consumption goods. A “general” good that can be produced and consumed in the centralized market. A numeraire good that can not be produced, whose supply is set equal to dividends, but can be consumed by asset-holders in the centralized market. Finally, there is the specialized good produced by sellers and consumed by buyers in the decentralized market. The source of exogenous variation in the economy is the state of aggregate productivity that is assumed to follow a geometric random-walk with drift. Aggregate productivity affects the production/output of all goods in the economy. Finally, we continue to assume that the supply of the (storable) liquid asset is iid and we further maintain that it is correlated with exogenous dividend growth. This latter assumption captures the empirical fact that firms pay out to shareholders through both dividends and share repurchases (e.g. Grullon and Michaely (2002)). The Appendix provides a detailed specification of the model.

The search-based asset pricing model with non-stationary dividend growth and iid share supply can be written in the following state-space form:

\[
\frac{p_t}{y_t} = \nu_t \left( \frac{y_t}{y_{t-1}} \right)^{\sigma-1} - 1 \quad (14)
\]

\[
\nu_t = \left\{ 1 + \beta(1 - \alpha) \tilde{E}_{t+1} + \alpha \beta \left( \tilde{E}_{t+1} \right)^{1-\sigma} \bar{A}^{-\sigma} \right\} \left( \frac{y_t}{y_{t-1}} \right)^{1-\sigma} \quad (15)
\]

\[
\frac{y_t}{y_{t-1}} = g \varepsilon_t \quad (16)
\]

where \( g > 1, \varepsilon_t \sim N(1, \sigma^2) \) and \( \bar{A} \) is the mean value for the supply of the liquid asset. This version of the model is written in terms of the price-dividend ratio which is stationary and whose value is determined by (14) where \( \nu_t = (p_t/y_t + 1)(y_t/y_{t-1})^{1-\sigma} \). The auxiliary variable is determined by the expectational difference equation (16). It is this latter object that agents must forecast using their adaptive learning rule as in the previous sections. Finally, equation (16) specifies the stochastic process for dividends.

This section calibrates the model (14)-(16) in order to give a sense of the liquidity role of the asset in stock price dynamics. Table 2 details the calibrated parameter values.

The basic time unit is a month. The value of \( \beta \) corresponds approximately to a 3% (annualized) real interest rate. The remaining parameter values were chosen to match U.S.
Table 2: Calibration.

<table>
<thead>
<tr>
<th>(\beta)</th>
<th>(g)</th>
<th>(\sigma^2)</th>
<th>(\sigma)</th>
<th>(\alpha)</th>
<th>(A)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.9930</td>
<td>1.0011</td>
<td>0.000233</td>
<td>9</td>
<td>0.0000001</td>
<td>([.005,.006])</td>
</tr>
</tbody>
</table>

S&P 500 real stock price and real dividends.\(^{32}\) The values for \(g\) and \(\sigma^2\) match the moments \(E \ln(y_t/y_{t-1}) = .00096\) and \(\sigma_{\ln y} = .0011\). The exercise conducted in the next subsection compares the distribution of stock prices in an economy with a liquidity role for stocks to one without. Thus, we are interested in the period 1994-2007 that is generally thought of as the asset shortage period, and omitting the years of the financial crisis.\(^{33}\) Over this period the mean price-dividend ratio in the data is 62.8. There is no good data on the liquidity role, as defined in the model, of stocks. While there is evidence of the role stocks play as collateral in repo transactions, and retirement account loans, precise estimates of the liquidity effect, as defined in the model, are not available. However, Krishnamurthy and Vissing-Jorgensen (2012) and Krishnamurthy (2002) provide estimates for the liquidity premium in U.S. Treasury prices. Their estimates for the (annualized) liquidity premium range from 35bp to 140bp. Treasuries, of course, play a more prominent role than equities as a source of collateral in secured debt. Using their estimates provide a reasonable upper-bound benchmark for the liquidity role that stocks can play in generating empirically plausible price-dividend ratios. Moreover, as will be seen below even with this calibration the direct effect of the liquidity role on stock prices is often a very small fraction of the price-dividend ratio in the model. The values for \(\sigma, \alpha, A\) were chosen to match the mean price-dividend ratio and the range of liquidity premia reported in Krishnamurthy and Vissing-Jorgensen (2012).

5.2 Results

This subsection reports results from monte carlo simulations of the model (14)-(16) calibrated according to Table 2. In the experiment, we set the constant gain in the adaptive learning

\(^{32}\)The data used for calibration were obtained from Robert Shiller’s Irrational Exuberance website: http://aida.wss.yale.edu/ shiller/data.htm.

\(^{33}\)The model is not inconsistent with the financial crisis. For example, if the financial crisis decreased counter-parties willingness to accept equity as collateral, the model could generate a crash via the learning dynamics with an exogenous decrease in the acceptability, or demand, for collateral.
rule to $\gamma = 0.10$. The model is simulated for 20,000 periods, counting the first 10,000 as a burn-in period to ensure that the learning dynamics have converged to its long-run stationary distribution, and then the distribution for equilibrium price-dividend ratios ($p/y$) is calculated. The simulations are repeated 1000 times and the results are averaged across simulations.

To see the relative impact of the liquidity role, Figure 10 plots the histograms from two separate monte-carlo experiments. The first is for the calibrated model as in Table 2. The second is adopting the calibrated parameters in Table 2 except setting $\alpha = 0$ to shut down the liquidity role of the asset. The two model economies are identical except for the additional role for the stock as collateral in over-the-counter transactions. The figure plots the histograms from these simulations. The dotted line is the dividends-only model and the solid line includes the liquidity premium. The dashed line plots the histogram for (monthly) S&P 500 price-dividend ratios over the 1994.1-2007.12 period. In the data, there are two peaks to the histograms with a price-dividend ratio of just under 60 and another peak just above 80. The data have observed values ranging from about 30 to about 100. Recall that under rational expectations, the price-dividend ratio will be iid deviations around the mean p-d ratio of 62.8. Thus, the distribution will have a single peak. The model with liquidity and learning, however, broadly captures these features. There are peaks to the distribution at 50 and just under 80, with a slight peak at 60. The range of outcomes from the calibrated model are in line with those observed over the data. Though, the data predict bubbles with greater frequency and the model predicts crashes occur more often than in the data. The model without asset shortages (dashed line) also has two peaks, though, at lower price-dividend ratios. Importantly, the model without the asset shortages is not able to capture the tail behavior in the data as well as the model with the liquidity premium.

As suggested in the earlier analysis, the right tail behavior is from learning dynamics that deviate from a small neighborhood of the rational expectations equilibrium. This logic follows from the fact that the rational expectations equilibrium features iid p-d ratios and learning dynamics, on average, are distributed normally around the rational expectations equilibrium. To give an example of a typical learning path that generates p-d ratios in the right tail of Figure 10, Figure 11 plots one typical bubble path. The figure plots a 200 period segment from a long simulation: most of the simulation features p-d ratios that are distributed around the mean p-d ratio of 62.8 but feature occasional, recurrent bubbles and crashes. While not shown, the distribution of p-d ratios in the RE version of the model is tightly and normally distributed around the mean p-d ratio. In this particular episode, price
Figure 10: Distribution of price-dividend ratios in model and the data. The figure plots histograms from monte carlo simulations in the model with and without a liquidity role. The dashed line is the histogram from data on the S&P 500 price-dividend ratio over the period 1994.1-2007.12.

builds slowly up to a peak price-dividend ratio of nearly 140. Preceding this bubble are belief coefficients that are drawn toward a random-walk forecasting model for the price-dividend ratio. There is an abrupt collapse in the price-dividend ratio as the market returns to its stable equilibrium value.

The economic intuition for bubble paths in the model with liquidity is as follows. The equilibrium price-dividend ratio is determined by expectations and iid deviations to asset supply and dividend growth. The price-dividend ratio reflects the dual roles of the asset in generating a dividend cash flow and in facilitating trade. The right combination of dividend growth and asset supply shocks can create a drift in asset prices that is then captured as serial correlation by the agents’ AR(1) econometric forecasting model. Not all sequences of shocks lead to random-walk beliefs, only those that push the learning dynamics sufficiently far away from the rational expectations equilibrium, that is, what Sargent (1999) calls the “most likely unlikely sequence of shocks.” Intuitively, the mechanism for generating bubbles is that the “most likely unlikely sequence of shocks” leads agents’ to forecast the price-dividend ratio as a random walk. Thus, recent innovations are viewed as permanent shifts in the long-run
price-dividend ratio. This leads sellers in the decentralized market to loosen their liquidity constraint on buyers, which increases the demand for the asset by buyers, leading to further increases in the asset price. However, the bubble does not persist forever as there are two forces working against it. First, the liquidity premium shrinks along the bubble path until eventually the efficient level of trade takes place in the decentralized market and there is no liquidity premium at all. Second, the mean dynamics feature random-walk beliefs only for a finite stretch of time and eventually the learning dynamics converge back to the rational expectations equilibrium.

A natural next question is what fraction of the stock price can be attributed directly to a liquidity premium. The model was calibrated, in part, so that the non-stochastic steady-state price in the model with asset shortages was 35 bp higher than the model without asset shortages. However, along a dynamic learning path there is not an obvious counter-factual for computing a liquidity premium. Changing the liquidity properties of the asset has a direct effect on the environment, parameterized by $\alpha, \sigma, \bar{A}$, and an indirect effect by altering the expectational feedback and the learning dynamics. However, it is possible to calculate the fraction of the price-dividend ratio, along a dynamic equilibrium path, that is directly attributable to the liquidity role by calculating $\alpha \beta (E_t \nu_{t+1})^{1-\sigma} \bar{A}^{-\sigma} / \left( \frac{p_t}{y_t} \right)$. 

Figure 11: Bubble: an example of a stochastic simulation in the model with stochastic dividend growth that features a bubble.
Figure 12 plots the histogram of the liquidity direct effect measured in annualized percentage points. The distribution peaks at the calibrated (annualized) liquidity premium value of 35 bps. However, along a given learning path there can be substantial variation in the liquidity value of the asset. At times, the fraction of the price directly attributable to its liquidity role can occasionally rise to as much as 1-2% and seldomly up to about 10%. Often times, liquidity directly accounts for a much smaller fraction of the overall price. Thus, quantitatively the liquidity value of the asset can be significant, even though on average it is quite small.

The liquidity role of the asset imparts a direct and indirect effect on price. The direct effect is the fraction of the equilibrium asset price directly attributable to its liquidity role. In the calibrated version of the model this effect is small. Importantly, though the indirect effect changes the qualitative nature of the learning dynamics – the liquidity channel alters the impact of expectations on asset prices as demonstrated in the previous sections – in such a way that the distribution of price-dividend ratios features more action in the tails. Thus, a scarce liquid asset will have a greater likelihood of observing bubbles and crashes in the data.
6 Conclusion

This paper proposes a search-based asset pricing model with imperfect knowledge and adaptive learning as a means for generating bubbles and crashes in asset prices. The results presented here demonstrate that bubbles and crashes can arise in response to changes in the supply of assets, changes in the over-the-counter liquidity of the asset, and as an endogenous response to transitory, economic shocks. There were two key features to the analysis: first, because of search frictions in over-the-counter market, and a shortage of assets that can serve as collateral, the financial asset’s price includes a liquidity premium; second, imperfect knowledge about the future asset price leads individuals to formulate, and estimate in real-time, an econometric forecasting model. The adaptive learning process of revising the estimates of the econometric model can lead individuals to temporarily believe that price follows a random walk without drift, which can be nearly self-confirming. Random walk beliefs lead asset prices to deviate from their fundamental price as agents interpret recent price innovations as permanent, leading to a positive feedback loop that results in rapid price appreciation and increased trade in over-the-counter markets. Thus, learning can relax the liquidity constraints along a (nearly) self-confirming bubble path. As that liquidity premium vanishes, the learning dynamics return price to its fundamental value, however, the route back to equilibrium features a crash in asset prices and an abrupt decline in over-the-counter trade.

The liquidity role of the asset alters the manner in which expectations enter the model. The greater the shortage in the safe asset, the more likely bubbles/crashes are to arise. Quantitatively, the liquidity role is important for matching the empirical distribution of price-dividend ratios. Bubble-like asset price dynamics can arise in models without liquidity effects, however, this paper presents new results on how learning can interact with liquidity effects and produce bubbles or crashes.

Appendix

Details on Stochastic Consistent Expectations Equilibria

This subsection presents insights on the nature of beliefs in an SCE. Branch and McGough (2005) characterize an equilibrium where agents hold linear beliefs, as in (6), and the state variable follows a non-linear reduced-form, as in (2), such that the belief parameters $c, b$ are linearly consistent with the associated equilibrium dynamics. To state a precise definition of a stochastic expectations equilibrium (SCEE), the following adapts Branch and McGough (2005) to the present environment. Define the following notation: for any initial distribution $\lambda_0$ on a compact set, with the initial condition $p_0$ chosen with respect to this distribution,
and for \( t \geq 1 \), let \( \lambda_t(\lambda_0) \) be the unconditional distribution of \( p_t \), and let \( \Lambda_t(\lambda_0) \) be the unconditional joint distribution of \((p_t, p_{t-1})\), as determined by (6).

**Definition 6** The triple \((\{p_t\}, c, b)\) is a stochastic consistent expectations equilibrium (SCE) provided the following hold:

1. \( p_t \) is generated by (6);

2. there exists a unique distribution \( \Lambda \) so that for initial distribution \( \lambda_0 \), the distribution \( \lambda_t(\lambda_0) \) converges weakly to \( \lambda \);

3. for any \( \lambda_0 \), \( \lim_{t \to \infty} E_{\lambda_t(\lambda_0)}(p_t) = c \) and \( \lim_{t \to \infty} \text{corr}_{\lambda_t(\lambda_0)}(p_t, p_{t-1}) = b \).

An SCE occurs when there is a unique distribution to which \( p_t \) converges weakly, for any initial condition, and the asymptotic mean and autocorrelation coincide with the beliefs of agents. It is in this sense that agents are unable to detect their misspecification within the context of their model as a check of regression residuals would not reveal any first order autocorrelation that would lead the forecaster to reject the econometric model. There is a close connection between an SCE and a restricted perceptions equilibrium (RPE) as discussed in Evans and Honkapohja (2001) and Branch (2006). In an RPE, the belief coefficients are restricted to be the best linear projection of the stochastic process onto the restricted set of regressors. The connection to the SCE is that a first-order SCE (as defined above) is an RPE. However, a higher-order SCE is not necessarily an RPE.

Notice that if \( \bar{p} = G(\bar{p}; A) \) is a steady-state of the model (2) then in an SCE \( c = \bar{p} \). In an SCE, the mean asset price will coincide with the mean price under rational expectations. Thus, showing existence of an SCE is straightforward: if \( \bar{p} \) is a fixed point of \( G \) then the pair \((\bar{p}, 0)\) characterizes an SCE. Branch and McGough refer to an SCE with zero autocorrelation as a ‘trivial SCE,’ though in the present context it accords with the (locally unique) rational expectations equilibrium. Showing existence of non-trivial SCE is challenging and many of the sufficient conditions in Branch and McGough (2005) are violated in the present environment. In a linear model (which arises here when \( \sigma = 1 \), Hommes and Zhu (2012) show that a non-trivial SCE do not exist for iid stochastic shocks but do exist if the shocks are serially correlated. Moreover, Branch and McGough (2005) showed that non-trivial SCE, when \( \bar{p} \neq 0 \), will be unstable under learning.

To illustrate the possible types of equilibria it is useful to define the map \( T : \mathbb{R} \times [0, 1] \rightarrow \mathbb{Z} \).
Figure 13: Stochastic Consistent Expectations Equilibria. Left panel is for the baseline parameterization. Right panel sets $\beta = 0.99, A = 0.1, \sigma = 0.05$.

$\mathbb{R}^2$ as follows

\[
T_c(c, b) = \lim_{t \to \infty} E p_t \\
T_b(c, b) = \lim_{t \to \infty} corr(p_t, p_{t-1})
\]

and $T(c, b)' = (T_c(c, b), T_b(c, b))$. The map $T$ can be interpreted as follows: given fixed beliefs $(c, b)$, the actual law of motion is given by (6) and the corresponding asymptotic mean and first-order autocorrelation is given by $T(c, b)$. If $\tilde{p}$ is a fixed point of (2) then an SCE is a pair $(\tilde{p}, \tilde{b})$ where $\tilde{b} = T_b(\tilde{p}, \tilde{b})$ is a fixed point of the map $T$. Unsurprisingly, a noisy steady-state $(\tilde{c}, \tilde{b}) = (\tilde{p}, 0)$ is a fixed point of the T-map. Figure 13 plots two different examples of SCE. In each plot of $T_b$ the parameters are set to the baseline parameterization in Table 1. Where the line $T(b)$ crosses the 45° line is a SCE. There is a SCE at $b = 0$, corresponding to the fundamental equilibrium and, depending on the value of $\sigma$, there can exist an SCE at $b = 1$. Note that for small values of $\sigma$ there exists an equilibrium with self-fulfilling serial correlation. That serially correlated beliefs can be nearly self-confirming is a key insight into the learning dynamics.

The learning literature, e.g. Evans and Honkapohja (2001), has shown that the T-map can be used to compute a stability condition, known as E-stability, which often governs
whether or not equilibrium parameters are locally stable under learning and that the differential equation, used to define E-stability, also provides information on the global dynamics under learning. The mathematical theorems underlying the E-stability principle rely on the stochastic approximation approach, and those theorems could be applied to the present non-linear environment. However, the form of the T-map is sufficiently complicated that general results are not available. It is possible to numerically solve for the E-stability dynamics and present available analytic results for the special case $\sigma = 1$.

The E-stability principle states that locally stable rest points of the ordinary differential equation

$$\frac{d(c, b)'}{d\tau} = (T(c, b) - (c, b))'$$

will be attainable under least squares and closely related learning algorithms.\(^{34}\). That the E-stability principle governs stability under learning is intuitive since under (17) the parameters $(c, b)$ are adjusted in the direction of the asymptotic moments implied by the actual law of motion generating the data given $(c, b)$. Local stability of (17) then answers the question of whether, under these E-stability dynamics, a small displacement of $(c, b)$ from a SCE would return to the equilibrium.

A numerical investigation revealed that only the trivial SCE, i.e. the fundamental equilibrium, is E-stable as anticipated by the results in Branch and McGough (2005), who showed that non-trivial SCE under unstable under learning. Figure 14 demonstrates the E-stability dynamics for the baseline parameterization in Table 1.

Figure 14 plots the resting points of the E-stability ODE and the associated vector field where the arrows indicate the direction of adjustment in (17). The figure shows that beliefs with $b = 1$ are unstable under the E-stability dynamics. In contrast, the fundamental equilibrium with $c = \bar{p}$ and $b = 0$ is a sink under learning. Figure 14 also demonstrates that the path to the fundamental equilibrium may include non-linear paths, providing a brief glimpse at the global learning dynamics presented in the main text.

**Proof to Propositions 4-5.**

Propositions 4 and 5 provide asymptotic approximations to the learning algorithm

$$\theta_t = \theta_{t-1} + \gamma S_{t-1} X_{t-1} (\pi_t - \theta'_{t-1} X_{t-1})'$$

$$S_t = S_{t-1} + \gamma (X_t X'_t - S_{t-1})$$

It is possible to re-write the equations for real-time learning in the form

$$\phi_t = \phi'_{t-1} + \gamma H(\phi'_{t-1}, \bar{X}_t)$$

\(^{34}\)Here $\tau$ denotes “notional” time
where $\bar{X}_t = (1, p_t, p_{t-1}, r_t)'$. The superscript $\gamma$ highlights the dependence of the parameter estimates on $\gamma$. The stochastic approximation approach is to compare the solutions to the continuous time ODE and the discrete time algorithm, thus, define the corresponding continuous time sequence for $\phi_t^\gamma$ as $\phi_t^\gamma = \phi_t^{\gamma}$ if $\tau_t^\gamma \leq t < \tau_{t+1}^\gamma$ where $\tau_t^\gamma = \gamma t$.

This Appendix sketches the proof to the propositions by making use of Propositions 7.8 and 7.9 of Evans and Honkapohja, and using arguments in Chapter 14 of Evans and Honkapohja and Branch and Evans (2011). The “mean dynamics” are the solution to the ODE

$$\frac{d\phi}{d\tau} = h(\phi)$$

where $h(\phi) = E[H(\phi, \bar{X}_t)]$ is given by the expression in (12)-(13).

Let $\check{\phi}(\tau, \phi_0)$ be the solution to the mean dynamics differential equation $\dot{\phi} = h(\phi)$ from an initial condition $\phi_0$. Define $U^\gamma(\tau) = \gamma^{-1/2} \left( \phi^\gamma(\tau) - \check{\phi}(\tau, \phi_0) \right)$. For small $\gamma$ the probability distribution of $U^\gamma(\tau)$ converges to the probability distribution of the solution $U(t)$ to the differential equation

$$dU(\tau) = D_{\phi}h(\check{\phi}(\tau, \phi_0))U(\tau)d\tau + R^{1/2}(\check{\phi}(\tau, \phi_0))dW(\tau)$$
The results below establish that \( EU(\tau) = 0 \) so that, as \( \gamma \to 0, E\phi^\gamma(\tau) = \bar{\phi}(\tau, \phi_0) \) and \( \lim_{\tau \to \infty} \bar{\phi}(\tau, \phi_0) = \phi^*. \)

The technical conditions required for Proposition 4 and 5 can be verified by using the arguments in Branch and Evans (2011), and so they are omitted here.

Proposition 5 uses the following result from Evans and Honkapohja (2001):

**Proposition 7 (EH(2001))** Consider the normalized random variables \( U^\gamma(\tau) = \gamma^{-1/2} \left( \phi^\gamma(\tau) - \bar{\phi}(\tau, \phi_0) \right) \)

As \( \gamma \to 0 \), the process \( U^\gamma(\tau), 0 \leq \tau \leq T \), converges weakly to the solution \( U(\tau) \) of the stochastic differential equation

\[
dU(\tau) = D_\phi h(\bar{\phi}(\tau, \phi_0))U(\tau)d\tau + R^{1/2}(\bar{\phi}(\tau, \phi_0))dW(\tau)
\]

with initial condition \( U(0) = 0 \), where \( W(\tau) \) is a standard vector Wiener process, and \( R \) is a covariance matrix whose \( i, j \)th elements are

\[
R^{ij}(\phi) = \sum_{k=-\infty}^{\infty} \text{Cov} \left[ H^i(\phi, \bar{X}_k^\phi), H^j(\phi, \bar{X}_0^\phi) \right]
\]

Moreover, the solution to the stochastic differential equation has the following properties

\[
EU(\tau) = 0
\]

\[
\frac{d\text{Var}(U(\tau))}{d\tau} = D_\phi h(\bar{\phi}(\tau, \phi_0))V_\phi(\tau) + V_\phi D_\phi h(\bar{\phi}(\tau, \phi_0))' + R(\bar{\phi}(\tau, \phi_0))
\]

where \( V_\phi = \text{Var}(U(\tau)) \). This result indicates that, for finite periods of time, the learning dynamics weakly converge to the solution of the ODE \( \hat{\theta} = h(\theta) \), thus establishing Proposition 5.

Proposition 4 arises from the following result in Evans and Honkapohja:

**Proposition 8 (EH(2001))** Consider the normalized random variables \( U^{\gamma_k}(\tau) = \gamma_k^{-1/2} (\phi^{\gamma_k}(\tau) - \phi^*) \).

For any sequences \( \tau_k \to \infty, \gamma_k \to 0 \), the sequence of random variables \( (U^{\gamma_k}(\tau_k))_{k \geq 0} \) converges in distribution to a normal random variable with zero mean and covariance matrix

\[
V = \int_0^\infty e^{sB} R(\theta^*) e^{sB'} ds,
\]

where \( B = D_\phi h(\phi^*) \).

It follows then that \( \theta_t \sim N(\theta^*, \gamma V) \) for small \( \gamma \) and large \( t \). Using arguments in Evans and Honkapohja (2001), Chapter 14.4, \( V \) is the solution to the matrix Riccati equation

\[
D_\theta h(\phi^*)V + V (D_\theta h(\phi^*))' = -R(\phi^*)
\]
where \( R = E \mathcal{H}(\phi^*, \bar{X}) \mathcal{H}(\phi^*, \bar{X})' \).

**Details on the model with stochastic dividend growth.**

In this Appendix we provide the details for the extension of the basic model to include stationary dividend growth and iid asset supply. This version of the model is an extension of Lagos (2010a) to include a dividend specification so that dividend growth follows a geometric random walk and asset supply is subject to iid disturbances. These features lead to an empirically plausible model of stock prices.

The Lagos (2010a) model is an extension of the Lagos-Wright model to include two assets, claims to a tree with stochastic dividends and a risk-free one period bond. This model is then calibrated and shown to provide a good fit to the equity premium observed in the data. This Appendix adapts the Lagos (2010a) framework in order to extend the baseline model in this paper to include a stochastic dividend process that leads to dividend growth following a geometric random walk, a specification preferred in applied work. The benchmark model is specified to include three different goods: “general goods” that are produced and consumed in the CM; a numeraire good, which is exogenous and whose supply is derived from dividends; and, a specialized good produced and sold in the DM. The general goods are produced according to a linear production function \( z_t n_t \) where \( n_t \) is the net amount of labor rented at real-wage \( w \) and \( z \) is the state of aggregate technology. The process for \( z_t \) is assumed to be a geometric random-walk, i.e. \( z_{t+1}/z_t = g \varepsilon_{t+1}, \varepsilon_{t+1} \sim N(1, \sigma^2) \). This is the sole source of randomness in the economy. Consumption \( c_t \) of the general good yields flow utility \( \nu(c_t) = c_t^{1 - \sigma} / (1 - \sigma) \). Consumption of the numeraire good is denoted \( x_t \) and the total output of the numeraire is determined by dividends so that in equilibrium \( x_t = y_t \).

With trend growth in aggregate productivity we assume that the efficient production of the specialized good in the DM also has a stochastic trend by assuming a cost to production \( c(q_t; \mu_t) = \mu_t^{-1} q_t \) and it is imposed that \( \mu_t = z_t^\sigma \). The supply of the asset, \( A_t \), is assumed to be iid and correlated with dividend growth, in particular, \( A_t = \bar{A} (y_{t+1} / y_t)^{1-\sigma}, \bar{A} > 0 \). This specification captures the empirical feature of asset float that firms payout to shareholders through both dividends and share repurchases. The timing of the process for asset supply also captures that \( A_t \) is the supply of the asset that is used for trade in the following period.

Given the environment described above, it follows that the buyer’s problem is

\[
\max_{\tilde{E}_0} \sum_{t=0}^{\infty} \beta^t \{ U(x_t) + \nu(c_t) + u(q_t) - \omega_t l_t \}
\]
subject to
\[ x_t + w_t n_t + p_t a_t = w_t l_t + (p_t + y_t) a_{t-1} + T_t \]
\[ c_t = z_t n_t \]

where \(\omega_t\) parameterizes the disutility from labor. For balanced growth considerations it is assumed that \(\omega_t = z_t^{1-\sigma}\). The CM value function can be expressed as
\[
W(a_{t-1}, z_t) = \frac{\omega_t}{W_t} (p_t + y_t) a_{t-1} + \frac{\omega_t}{W_t} T_t + \max_{x, n} \left[ U(x_t) + \nu(z_t n_t) - \frac{\omega_t}{W_t} (x_t + w_t n_t) \right]
\]
\[ + \max_a \left[ -\frac{\omega_t}{W_t} p_t a_t + \beta E_t V(a_t, z_{t+1}) \right] \]

When a buyer and seller are matched in the DM, the buyer makes a take-it-or-leave-it offer. The solution to the bargaining problem is
\[
(q_t, d) = \arg \max_{q, d} \left[ u(q_t) - \mu_t^{-1} q_t \right]
\]
subject to
\[-\mu_t^{-1} q_t + \frac{\omega_t}{W_t} (p_t + y_t) d \geq 0\]

As before, the solution to this bargaining problem depends on whether buyers hold enough of the asset to purchase the efficient quantity:
\[
q_t = \begin{cases} 
q_t^* \equiv \mu_t^{1/\sigma} & \text{if } q_t^* \leq \mu_t \frac{\omega_t}{W_t} (p_t + y_t) a_{t-1} \\
\mu_t \frac{\omega_t}{W_t} (p_t + y_t) a_{t-1} & \text{else}
\end{cases}
\]

Given the solution to the bargaining problem, the DM value function is
\[
V(a_{t-1}, z_t) = \alpha \left[ u(q_t) + W(a_{t-1} - d, z_t) \right] + (1 - \alpha) W(a_{t-1}, z_t)
\]
\[
= \alpha \left[ u(q_t) - \frac{\omega_t}{W_t} (p_t + y_t) d \right] + \frac{\omega_t}{W_t} (p_t + y_t) a_{t-1} + W(0, z_t)
\]

In the CM value function consumption of the general and numeraire good is determined from the solution to the following static problem
\[
\max_{x, n} U(x_t) + \nu(z_t n_t) - \frac{\omega_t}{W_t} (x_t + w_t n_t)
\]

The solutions to the static problem, combined with the equilibrium condition that \(x_t = y_t\) and the balanced growth restriction of constant labor \(n_t\), imply that \(\frac{\omega_t}{W_t} = y_t^{-\sigma}\) and \(\omega_t = z_t^{1-\sigma}\). It follows then that the asset demand, for a liquidity constrained investor is,
\[
\max_a \left\{ -y_t^{-\sigma} p_t a_t + \beta E_t \left[ \alpha u(q_{t+1}) + (1 - \alpha)(p_{t+1} + y_{t+1}) y_{t+1}^{-\sigma} a_t \right] \right\}
\]

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Given the definition of $q_t$, the F.O.C. for asset holdings is

$$0 = -y_t^{-\sigma} p_t + \beta (1 - \alpha) \hat{E}_t (p_{t+1} + y_{t+1}) y_{t+1}^{-\sigma} + \alpha \beta \left[ \hat{E}_t (p_{t+1} + y_{t+1}) \right]^{1-\sigma} A_t^{-\sigma}$$

Solving for the price-dividend ratio leads to the expression

$$\frac{p_t}{y_t} = \beta (1 - \alpha) \hat{E}_t \left( \frac{p_{t+1}}{y_{t+1}} + 1 \right) \left( \frac{y_{t+1}}{y_t} \right)^{1-\sigma} + \alpha \beta \left[ \hat{E}_t \left( \frac{p_{t+1}}{y_{t+1}} + 1 \right) y_{t+1} \right]^{1-\sigma} A_t^{-\sigma} y_t^{\sigma-1}$$

After algebraic manipulations, the asset-pricing equations in the text follow immediately.

References


