Unemployment and the Stock Market when Households Lack Commitment*

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Abstract

A simple model attributes movements in stock prices, unemployment rates, and interest rate spreads to liquidity and productivity shocks. The environment is a Mortensen-Pissarides economy with a single twist: households receive uninsurable idiosyncratic preference shocks, and, due to limited commitment, self-insure by accumulating claims on firms’ profits. The model generates a strong aggregate demand channel that produces (potentially multiple) equilibria in which employment, the real interest rate, and the stock market are positively correlated. We calibrate the model to the U.S. economy and use it to interpret recent movements in the three variables. The model suggests a prominent role for liquidity shocks in unemployment rates and interest-rate spreads over 1981-2018, and reproduces much of the joint movement in the stock market and unemployment. We consider an experiment where a perfect storm of an increase in expenditure risk and aggregate demand coincides with an exogenous drop in the velocity of publicly-provided assets. Under both rational expectations and an adaptive learning rule, the economy converges to a self-fulfilling equilibrium with high unemployment and low stock-market capitalization. The results from this counterfactual highlight the fact that the economy is most fragile when aggregate demand is strong and the economy relies on privately-issued liquid assets.

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1 Introduction

Since the 2007-09 financial crisis attention has refocused to the interaction between financial and labor markets and the resulting implications for aggregate economic outcomes. In this paper, motivated by empirical evidence of a strong co-movement between the stock market and unemployment (see, for example, Farmer (2012) and Hall (2017)), we develop a simple model that attributes observed data on stock prices, unemployment rates, and real interest rates to exogenous liquidity and productivity shocks. The framework is a standard Mortensen and Pissarides (1994) model with a single change: a limited commitment problem in the goods market. Households encounter idiosyncratic spending shocks that, because of their limited ability to commit to repaying unsecured debt, they finance by pledging the value of some assets. Consumers’ liquid assets are shares in a mutual fund comprised of stocks and government bonds. This simple twist of an otherwise standard model imparts a key role to the stock market in generating booms and busts. Higher stock market valuations relax consumers’ liquidity constraints, thereby creating an aggregate demand channel that strengthens firms’ hiring incentives. The creation of new firms and jobs, likewise, enhances market capitalization and feeds back into stronger consumer demand. The key insight is that these strategic complementarities between the labor market and the stock market provide a potential explanation for the observed co-movement between stock prices and unemployment rates: a strong stock market does not just reflect but also promotes a robust labor market.

In our model, firm profits and government bonds play an insurance role analogous to capital in the Aiyagari (1994) model. In each period, markets open sequentially as in Berentsen, Menzio, and Wright (2011) and Branch, Petrosky-Nadeau, and Rocheteau (2016) with three separate stages. A frictional labor market opens first where firms with vacant positions and unemployed workers participate in a stochastic matching process. Consumption and production take place in the last two stages. In the second stage, buyers and firms trade consumption goods early in a competitive market. During that second stage early-consumption market, households face idiosyncratic spending shocks and, for some purchases, a limited ability to commit to repaying debt precludes financing these purchases with unsecured credit. Instead, some buyers are liquidity constrained and finance their purchases us-
ing the value of their mutual funds as collateral. Firms, likewise, can choose to speed up the production process, subject to a convex cost, in order to meet the demand from these early-consumption households.\footnote{The convexity of the cost function is crucial to generate the aggregate demand externality. This feature implies that firms price at marginal cost, but above average cost. Alternatively, imperfect competition can complement the aggregate demand channel in important ways. For instance, monopolistic competition generates variety effects. With more firms selling differentiated goods, consumers are better able to diversify their consumption basket, which augments the initial expenditure shock. See \textcite{BibbI2012} for a business cycle treatment of firm entry—which emphasizes the role of sunk entry costs—and \textcite{Silva2017} for a New Monetarist application.} Thus, assets play a dual role in this economy. When stock market valuations are high, consumer liquidity constraints are relaxed, which raises aggregate demand and strengthens the incentive for firms to create new openings. As firms create new jobs, enhancing firm value and stock market capitalization, household liquidity constraints are further relaxed. These complementarities between the labor and stock markets impart an important role for an endogenous real interest rate that carries a liquidity premium and creates an aggregate demand channel with implications for aggregate economic outcomes.

A novel finding is the possibility of multiple steady states: an equilibrium with a high employment rate and real interest rate that coexists with equilibria featuring low employment rates, low real interest rates, and low stock market valuations. The underlying strategic complementarity follows directly from the aggregate demand channel: high stock market capitalization reduces households’ liquidity constraints, increases aggregate demand, and thereby raises prices in the early-consumption market. Firms’ revenues increase and lead new firms to enter production, which further propagates the high stock market valuation. A lower need for liquidity, in turn, boosts real interest rates, which dampens firm value and entry. Conversely, the economy can be stuck at an equilibrium with low aggregate wealth, low employment, and low real interest rates where households are severely liquidity constrained.

A natural question is how our analysis differs from \textcite{Aiyagari1994} and other related Bewley-type models, beyond the obvious difference that consumers face random spending opportunities in our model and idiosyncratic income risk in standard incomplete markets models. If firms face a linear cost for speeding up production to the second stage, then the price for early-consumption goods – and, in turn, the value of firms – is independent of the spending shocks: the risk channel affects the labor market solely through the endogenous real interest rate that includes a premium
when consumers are liquidity constrained. As the frequency of spending shocks increase, i.e. a greater expenditure risk, the real interest rate decreases which has a modest effect on firms’ discounted future profits and, hence, job creation. In our model, the complementarity is strengthened via the aggregate demand channel so that if those spending shocks are large enough then stock market valuations are high and sufficient liquidity is provided, in equilibrium, so that households are no longer liquidity constrained. Thus, negative effects on the stock market both directly reduce job creation and liquidity in a mechanism distinct from Aiyagari (1994). This model accordingly captures the idea of stock market crashes compounding recessions expounded in Farmer (2012).

The ability of the model to generate multiple steady-state equilibria motivates the quantitative exercise explored later in the paper. We calibrate the model to match key long-run properties of the U.S. economy and then attribute short-run fluctuations to a sequence of unanticipated, permanent liquidity and productivity shocks. We identify the sequence of shocks by equating the model-implied paths for unemployment rates and interest-rate spreads with those observed in the data. Peering at the data through the lens of the model, fluctuations in unemployment and real interest rates can be attributed to time-variation in the expenditure risk from early consumption, financed in part by household asset holdings, and firms’ incentives to produce in the early-consumption market and service liquidity-constrained consumers. The periods in the data with low unemployment and large interest-rate spreads coincide with more frequent spending shocks and strong aggregate demand for early consumption. The model captures well movements in the stock market and can account for about 50% of the stock market capitalization to GDP ratio during peak episodes such as 2000 and 2016-19.

We then turn to our main counterfactual exercise to better understand how the aggregate demand channel can generate large increases in unemployment and stock market crashes. We consider a “perfect-storm” scenario by freezing the model economy in 2006, a period our calibration identifies as having a strong liquidity-constrained demand, and a large decrease in the velocity of government bonds, modeled as a reduction in the extent to which government bonds are pledgable to finance consumption. The latter proxies for a situation where an economy becomes more reliant on privately-issued liquidity. In this perfect-storm scenario, the model predicts the
existence of three steady-state equilibria, one of which is quantitatively consistent with the long-run unemployment rates in the U.S. The other two equilibria feature lower stock market values and higher unemployment. We then compute the impulse response to an expectations shock that dislodges the economy from its long-run equilibrium.

To compute the transition path following the expectations shock, one needs to take a stand on how households and firms form expectations. As a benchmark, we assume that agents hold rational expectations and compute the rational expectations equilibrium path following the shock. The middle steady-state is indeterminate and, therefore, there is a continuum of paths from the initial shock to this steady-state. We focus on the perfect foresight path, and find a large stock market and labor market response, with unemployment over-shooting its steady-state value before holding steady at a 12% unemployment rate. Because perfect foresight is a strong assumption, we also formulate an adaptive learning version of the model in the spirit of Evans and Honkapohja (2001). Under adaptive learning, agents generate forecasts about future real interest rates and firm values by taking a geometrically weighted average of past data. Depending on the details of how they average, learning will converge to one of the two steady-states with lower stock market values and high unemployment rates.

The results from this counterfactual highlight an important observation. According to the model, the economy is most fragile at times when spending shocks are most frequent/aggregate demand is strongest, and the economy is reliant on privately-issued liquid assets. A policy implication reinforces the prominent role that public provision of liquidity can play in avoiding recessions and financial crises. Further, the link between market capitalization, liquidity, and real interest rates, as well as the possibility of self-fulfilling collapses to low aggregate demand and high unemployment equilibria, provides a potential explanation of secular stagnation, i.e. Eggertsson, Mehrotra, and Summers (2016), and the excess volatility puzzle.

1.1 Related literature

The theory proposed in this paper is closely related to a class of incomplete market models where households hold assets with a precautionary savings motive to insure themselves against idiosyncratic shocks. Most closely related is Aiyagari (1994) where
households self-insure by acquiring claims to physical capital. In our model, the risk arises from idiosyncratic spending shocks, rather than income shocks, and assets are claims on aggregate firm values. Firms use labor acquired in a frictional labor market to produce consumption goods. As we show, the defining feature to our environment is the limited commitment problem. Spending shocks imply an aggregate demand channel that generates a positive feedback between employment and stock market valuations. We show that adding this aggregate demand channel causes stock market capitalization to rise more sharply with expenditure shocks than through the interest rate channel alone that is emphasized in incomplete markets models.\(^2\) Moreover, the interest rate follows a U-shaped pattern, initially decreasing from the higher liquidity premia and eventually rising as consumers are increasingly satiated, whereas it declines monotonically in a standard incomplete markets setup.

The current paper is also related to a burgeoning literature that incorporates incomplete markets into realistic business cycle models. For instance, Kaplan and Violante (2010) use a life-cycle version of a standard incomplete markets model to assess how much consumption insurance in data is derived from a precautionary motive under permanent and transitory earnings shocks. The HANK model in Kaplan, Moll, and Violante (2018) focuses on the precautionary savings motives of households subject to uninsurable risk and explores its implications for monetary policy in a model that captures empirical features of the wealth distribution.\(^3\) Kruse, Mukoyama, and Şahin (2010) endogenize income risk through labor market matching and assess the implications for optimal provision of unemployment insurance. Unlike most of the incomplete markets literature, we assume quasi-linear household preferences, which imply a degenerate wealth distribution. We maintain the quasi-linearity assumption for simplicity and tractability, and to clearly demonstrate the implications of the aggregate demand channel for the joint determination of employment, interest rates and the stock market. We acknowledge that doing so abstracts from important features of an economy with a non-degenerate wealth distribution.

This paper is also related to research on unemployment and financial frictions. Most closely related is a recent paper by Bethune and Rocheteau (2019). They use the

\(^2\)Therefore, the aggregate demand channel generated through expenditure risk can mitigate the unemployment volatility puzzle emphasized by Shimer (2005), and can arise from shocks to either liquidity or productivity.

\(^3\)For a recent survey of the HANK literature, see Kaplan and Violante (2018).
same multi-stage market structure in which households face a limited commitment problem subject to idiosyncratic expenditure shocks. In Bethune and Rocheteau (2019) the primary difference is the structure of the late stage market leads households to face employment and expenditure risk in a way so that the wealth distribution matters for equilibrium outcomes. Bethune, et al. extend the set of assets to include fiat money and focus on the long-run economic properties including the long-run Phillips curve relationship. We, in turn, focus on the possibility of multiple steady-states and study the dynamics along a transition path following liquidity shocks. The possibility of multiple steady-states in a frictional labor market model also arises in the environment of Kaplan and Menzio (2016), where consumers’ intensity of search for sellers in a goods market depends on their employment status.

Our model is also closely related to a literature that incorporates labor market and goods market frictions. For instance, Wasmer and Weil (2004) and Petrosky-Nadeau (2013) incorporate a frictional credit market used by investors to finance job posting costs. Bethune, Rocheteau, and Rupert (2015), like our model, incorporates a limited commitment problem in the goods market into a Mortensen-Pissarides framework. In our paper, a fraction of households facing idiosyncratic spending opportunities have access to unsecured credit. Bethune, Rocheteau, and Rupert (2015) assume that all consumers access unsecured credit, which affects the incentives of firms to post vacancies, but are only able to explain a small fraction of the increase in unemployment during the Great Recession. In Branch, Petrosky-Nadeau, and Rocheteau (2016), those households who are liquidity constrained can use their home equity as an asset to serve as collateral in financing early consumption opportunities. They also model a two-sector frictional labor market, one for consumption goods and one for construction, which makes the supply of housing endogenous. In Branch, et al. there is the possibility of multiple steady-states as well that arise because the matching probability in the frictional goods market depends on firm entry whereas in this paper the goods market is competitive and claims to the firms’ shares serves as the private liquidity.

We also come into contact with an extensive literature that studies the connections between the stock market and unemployment. Recently, Farmer (2012) presents evidence that stock market prices and unemployment are cointegrated. Hall (2017) noting the co-movement between stock prices and unemployment, integrates a com-
mon factor into a Mortensen-Pissarides model of unemployment. In Hall (2017) that common factor is fluctuations in the discount rate. Our model provides some micro-foundations to Hall’s analysis by endogenizing the real interest rate at which firms and households discount future payoffs. A key take-away from the micro-foundations introduced through the limited commitment friction is the strong complementarities between labor and goods markets that yield a multiplicity of equilibria. The quantitative exercises exploit these complementarities.

The framework in our model is inspired by monetary theory, and in particular the class of New Monetarist models that incorporate unemployment and money. The timing structure of our model comes from Berentsen, Menzio, and Wright (2011). In Berentsen, Menzio, and Wright (2011) households have access to a single liquid asset, fiat money, and trade goods with firms in a decentralized goods market characterized by search frictions. In Section 5.4 of our paper we explicitly show that the set of steady-states in our framework is qualitatively different from the pure currency economy of Berentsen, Menzio, and Wright (2011). Several New Monetarist papers emphasize the dual role of assets as collateral. For instance, in Lagos (2010) consumption is financed with loans collateralized by Lucas trees (a real asset) and fluctuations in liquidity premia are shown to be important in explaining the equity premium puzzle. Similarly, Rocheteau and Wright (2013) the asset is again a Lucas tree, and with endogenous firm entry the model exhibits multiple steady-states and cycles reminiscent of recurring bubbles and crashes. Finally, Lagos and Rocheteau (2008) study the co-existence of money and capital when claims to capital can collateralize consumption.

Finally, our quantitative experiments include a treatment where rational expectations is replaced by an adaptive learning. There is an extensive literature that replaces rational expectations with an econometric learning rule (c.f. Marcet and Sargent (1989) and Evans and Honkapohja (2001)). This literature is based on the “cognitive consistency principle,” which states that rational expectations is an unreasonably strong assumption and so it is cognitively consistent to model agents who form expectations like a good economist who specifies a model and updates it based on recent data. Like this literature, we assume that households and firms’ forecasting model nests the rational expectations equilibrium. However, learning might converge

4The model under consideration here abstracts from features that may capture the equity premium. The households in our model pay a (potentially) substantial liquidity premium on stocks.
to a different steady-state equilibrium than one might predict if the agents all had rational expectations. In particular, we show the possibility of a crash equilibrium and excess volatility in stock prices and unemployment. In Branch, Petrosky-Nadeau, and Rocheteau (2016), learning generates the degree of home price appreciation observed in the data and, as a result, can explain the large increase in unemployment during the Great Recession.

2 Motivating Evidence

This article is motivated by the co-movement of stock market capitalization, interest rate spreads, and unemployment. We first document the negative relationship between the stock market capitalization and unemployment rate, and then turn to the role of interest rate spreads. Farmer (2012) provides evidence, from a vector error correction model, that variations in stock prices have out-of-sample predictive power for unemployment rates. The negative relationship is apparent in Figure 1, which is based on his calculations.

Second, given the stable relationship between unemployment and stock prices, to what extent do liquidity constraints and expenditure risk contribute to depressing employment and the stock market? To this end, it is important to quantify the major types of liquidity shortfalls facing U.S. households. Surveys report that U.S. households do face considerable expenditure risk. The Federal Reserve’s Report on the Economic Well-being of U.S. Households documents that 32% of respondents would not be able to cover expenses if they lose their main source of income for 3 months, even by borrowing. Moreover, during the pre-Affordable Care Act surveys, 30% or more of respondents experienced a major out-of-pocket medical expense in the 12 months prior to the survey, and as few as 22% during post-ACA surveys. These out of pocket expenses are large with a mean expense of $2,383. Moreover, 27% of respondents reported foregoing some type of medical care due to lack of affordability. These reports also find that 27% of households would have to borrow or sell something to pay for even modest unexpected expenses. For larger expenses, 20% report they would borrow or sell assets.

Third, to the extent that liquidity constraints are quantitatively important, they should affect firm values. In particular, we present estimates show that suggest the
Figure 1: Calculations based on Figure 5 in Farmer (2012), period 1960-2019. The variables are the logarithm of the Wilshire 5000, normalized by the nominal wage, and the logarithm of the logistic transformation of the unemployment rate (%).

real interest rate, along with hiring costs, is an important factor in firm values. We derive our regression equation by first noting that in Mortensen and Pissarides (1994), free entry of vacancy creation implies the following relationship:

$$J_t = \frac{(1 + \rho)\gamma}{q_t}$$

where $J$ is the value of the firm, $\gamma$ is the vacancy posting cost, $\rho$ is the rate of time preference, and $q$ is the vacancy rate. Equation (1) does not allow for any variable interest rates (say, through a liquidity premium) that would affect the equilibrium value of firms.

Now, let us generalize (1) in two ways. We can proxy for variable hiring costs by assuming the vacancy posting cost depends on the wage $W_t$, since it is generally a labor-intensive activity, and allow for interest rates to vary (which we interpret, as
our model implies, as reflecting time-varying liquidity premia).\(^5\) The formulation is then

\[ J_t = \frac{(1 + r_t)W_t \gamma}{q_t} \]

Now, divide through by the match productivity \(X_t\), and let \(J_{x,t} = J_t/X_t\) denote the firm value relative to output (or the stock market capitalization to GDP) and \(W_{x,t} = W_t/X_t\). Then apply logarithms:

\[ \log J_{x,t} = \log(1 + r_t) + \log(W_{x,t}) + \log \gamma - \log q_t \]

Finally, since risk-adjusted interest rates are very low, we can use the first-order approximation \(\log(1 + r_t) \approx r_t\), which can be decomposed as \(\rho - \text{spread}_t\) for rate of time preference \(\rho\), which serves as the natural interest rate. Thus,

\[ \log J_{x,t} \approx \rho + \log \gamma - \text{spread}_t + \log(W_{x,t}) - \log q_t, \]

which motivates the following regression:

\[ \log J_{x,t} = \lambda_0 + \lambda_1 \text{spread}_t + \lambda_2 W_{x,t} + \lambda_3 \log q_t + \varepsilon_t \quad (2) \]

A value of \(\lambda_1\) statistically significant from zero is evidence that the liquidity premium is associated with firm value after controlling for hiring costs. The stock market capitalization to GDP is increasing over time, however, with the rise of the publicly traded share of the economy. To correct for these issue, we estimate (2) in first differences (growth rates of the firm value, labor share, and the vacancy filling rates).\(^6\)

To estimate (2), we construct the job finding from unemployment flows as in Shimer (2005), which requires data on the total unemployed and those unemployed for less than 5 weeks. We also use data on vacancies, a flow variable, which we convert to a quarterly series and then divide by the number of unemployed to obtain the quarterly market tightness. We use the constant returns to scale property to obtain the vacancy filling probability: \(q(\theta) = f(\theta)/\theta\). This approach does not require us to impose a matching function. We calculate \(W_x\) as the aggregate nominal wage divided

\(^5\)Hagedorn and Manovskii (2008), for instance, decompose the costs of vacancy creation into labor and capital costs.

\(^6\)While endogeneity bias is a potential concern, in providing motivating evidence we report only OLS estimates.
by nominal output. The liquidity premium is constructed as in Krishnamurthy and Vissing-Jorgensen (2012). We calculate the difference between Moody’s Aaa-rated long-maturity corporate bond yields and the returns on long-term government bonds, where the latter are available until 1999. From 2000 onward, we use the yields on Treasuries with 20-year maturities. The data appendix describes each series.

Table 1 describes the regression results. A 1 percentage point change in the interest rate spread is associated with a percentage change reduction of the stock market capitalization of \(-10.33\%\). Or, more intuitively, a one standard deviation increase in the interest rate spread (\(\approx 30 \text{ bp}\)) is associated with a \(-3.09\%\) reduction in the stock-market capitalization to GDP ratio. The measure is statistically significant. Moreover, running the regression without the spread variable produces an adjusted R-squared of only 0.083 compared with 0.135 with the interest rate spread.

|            | coef | std err | t    | P>|t| | 0.025 | 0.975 |
|------------|------|---------|------|------|-------|-------|
| Intercept  | 0.0098 | 0.005  | 1.929 | 0.055 | -0.000 | 0.020 |
| q          | -0.1268 | 0.077  | -1.638 | 0.103 | -0.279 | 0.026 |
| spread     | -10.3305 | 4.822 | -2.142 | 0.034 | -19.845 | -0.816 |
| \(W_Y\)   | 0.1958 | 0.561  | 0.349 | 0.728 | -0.912 | 1.304 |

Table 1: Standard errors are heteroskedasticity and autocorrelation robust with a small-sample correction.

Our model generates the negative co-movement between stock prices and unemployment and the interest-rate spread. The motivating evidence suggests that idiosyncratic spending opportunities exist and liquidity premia affect firm values. These empirical regularities motivate the structure of our model.

3 Environment

The set of agents consists of a unit measure of households, composed of one buyer and one worker. Time is discrete and is indexed by \(t \in \mathbb{N}\). Each period of time is divided into three stages. The first stage stage is a frictional labor market where unemployed workers and vacant firms participate in a stochastic matching process.
Consumption and production take place in the last two stages. In the second stage, buyers and firms trade consumption goods early in a Walrasian market. In the last stage, buyers and firms have a late opportunity to trade goods and assets and wages are paid. We take the late-consumption good traded in the last stage as the numéraire.

\[ u_t = \beta \left[ \varepsilon(y_t) + x_{bt} + x_{wt} \right], \]

where \( \beta = (1 + \rho)^{-1} \in (0, 1) \) is a discount factor, \( y_t \in \mathbb{R}_+ \) is the buyer’s early consumption, \( x_{bt} \in \mathbb{R} \) is the buyer’s late consumption, and \( x_{wt} \geq 0 \) is the worker’s late consumption. If \( x_b < 0 \), then the buyer is self-employed and produces the numéraire good. Because of the linear preferences in terms of the numéraire good we can either treat the buyer and the worker as distinct agents, or as a joint entity with a consolidated budget constraint and impose conditions on primitives so that \( x_b \geq 0 \) holds.\(^7\)

\(^7\)The quasi-linearity in preferences keeps the model tractable and, in particular, implies that individual histories in the labor and goods markets are independent of asset holdings made in the third-stage; that is, the equilibrium wealth distribution is degenerate. Under more general preference specifications, households would self-insure against idiosyncratic employment and expenditure shocks, rendering analysis of the model more complex. This case is studied, in steady-state, by Bethune and Rocheteau (2019). While households in this model do not need to save for precautionary reasons in response to unemployment risk, they do have a precautionary demand for assets due to the spending shocks \( \epsilon_t \). This paper focuses on an aggregate demand channel, and its implications for complementarities between labor and goods markets, and abstracts from consequences of wealth heterogeneity. Analogously, Angeletos (2007) focuses on investment income risk in a tractable setting which abstracts from the role of labor income risk.
The utility function for early consumption, $\varepsilon v(y_t)$, is twice continuously differentiable, strictly increasing, and concave, with $v(0) = 0$, $v'(0) = \infty$, and $v'(\infty) = 0$. The multiplicative term, $\varepsilon$, is an idiosyncratic preference shock that is equal to $\varepsilon = 1$ with probability $\alpha$ and $\varepsilon = 0$ otherwise. These preference shocks correspond to liquidity shocks in the banking literature (e.g., Diamond and Dybvig (1983)) according to which some buyers have the desire for early consumption.

Each firm is a technology to produce $\bar{z}$ units of numeraire with one unit of indivisible labor (one worker) as the only input. Production takes time so that $\bar{z}$ is available in the last stage. The firm can speed up the production process and serve $y$ units of goods to early consumers at cost $c(y)$ in terms of numeraire, where $c' > 0$ and $c'' \leq 0$. Unless stated otherwise, we assume $c(0) = 0$, and $c'(0) = 0$. There is an upper bound $\bar{y}$ such that $\bar{z} = c(\bar{y})$. One can impose conditions on fundamentals that ensures $y \in (0, \bar{y})$, so that the constraint can be ignored. The output in the last stage is $\bar{z} - c(y)$. With probability $\lambda$, the buyer can access intra-period credit. In that case, repayment can be fully enforced. With probability $1 - \lambda$, a firm cannot monitor the buyer.

In order to hire a worker at time $t$, a firm must advertise a vacant position, which costs $k > 0$ units of the numéraire good at $t - 1$. The measure of matches between vacant jobs and unemployed households in period $t$ is given by $m(s_t, o_t)$, where $s_t$ is the measure of job seekers and $o_t$ is the measure of vacant firms (openings). The matching function, $m$, has constant returns to scale, and it is strictly increasing and strictly concave with respect to each of its arguments. Moreover, $m(0, o_t) = m(s_t, 0) = 0$ and $m(s_t, o_t) \leq \min(s_t, o_t)$. The exit probability out of unemployment for a worker is $e_t = m(s_t, o_t)/s_t = m(1, \theta_t)$ where $\theta_t \equiv o_t/s_t$ is referred to as labor market tightness. The vacancy filling probability for a firm is $q_t = m(s_t, o_t)/o_t = m(1/\theta_t, 1)$.

Employment (measured after the matching phase at the beginning of the second stage) is denoted $n_t$ and the economy-wide unemployment rate (measured after the matching phase) is $u_t$. Therefore, $u_t + n_t = 1$. An existing match is destroyed at the beginning of a period with probability $\delta$. A worker who loses her job in period $t$ becomes a job seeker in period $t + 1$. So, workers who lose their jobs must go through at least one period of unemployment, i.e. $s_{t+1} = u_t$. An employed worker in period $t$ receives a wage in terms of the numéraire good, $w_{1,t}$, paid in the last stage. An unemployed workers enjoys $w_0$, which represents unemployment benefits and the
value of leisure.

There is a fixed supply of one-period real government bonds $A^g$. Each bond issued in the third stage is a claim to one unit of the numeraire in the following period. In the second stage buyers are anonymous and cannot commit to repay their debt. There are perfectly competitive mutual funds which buy stocks and bonds and issue risk-free shares. We let $r_t$ denote the rate of return of such claims from the last stage of $t - 1$ to the last stage of $t$. These claims are perfectly diversified and hence free of idiosyncratic risk. Moreover, they can be authenticated and transferred at no cost. Household wealth, $a_{t+1}$, thus comprises shares in mutual funds that acquire existing firms or invest in new firms by creating vacant positions.

The objective is to develop the simplest environment in which private liquidity, public liquidity, and credit coexist. We focus on stock mutual funds, government bonds, and debt obligations as the assets for the following reasons. Stocks are a primitive given the fundamental role of firms in labor search models, and government bonds provide a policy instrument. Finally, probabilistic access to credit by consumers enables us to characterize the space between no-and-full commitment. Though the economy remains cashless, firms’ revenues and interest rates are endogenous and sensitive to the government supply of bonds. A cashless economy is reasonable in other respects. First, Hu and Rocheteau (2013) establishes that fiat money is not essential in environments with Lucas trees. Moreover, Lagos (2010) studies a similar economy in which Lucas trees serve as the media of exchange to explain the equity premium puzzle. Finally, this simple version of the model focuses attention on the feedback between a frictional labor market and a goods market in economies with a limited commitment problem. The appendix, though, sketches out an extension to include fiat money and transaction costs for liquidating bond and stock holdings.

In summary, the novelty relative to a standard Mortensen-Pissarides model is the second stage where households receive opportunities to consume early and are subject to limited commitment. Like a Bewley-Aiyagari model, as in Bewley et al. (1980) and Bewley (1983), households receive preference shocks but only a fraction can issue IOUs to finance consumption. Moreover, as in Aiyagari (1994), households can self-insure against idiosyncratic risk by accumulating capital, here in the form of stock ownership. Relative to an Aiyagari model, the environment features both employment and expenditure risk, but it is the latter that matters for the determination of the
real interest rate.

4 Equilibrium

In the following, we characterize an equilibrium by moving backward from agents’ choice of asset holdings in the last stage, to the determination of prices and quantities for early consumption/production, and finally the entry of firms and the determination of wages in the labor market.

4.1 Goods and asset markets

As previously indicated, the lifetime utility of a household is the sum of the lifetime utility of the buyer and the lifetime utility of the worker. Therefore, in the following, we treat separately the two agents composing the households. Let $W_t(\omega_t)$ denote the lifetime expected discounted utility of a buyer at the beginning of the last stage with $\omega_t$ units of wealth in terms of the numeraire. Wealth $\omega_t$ is composed of shares of mutual funds net of debt obligations and tax liabilities, and assets $a_{t+1}$ taken into the third subperiod consist solely of mutual funds since we only consider intra-period debt. Similarly, let $V_t(a_t)$ be the buyer’s value function at the beginning of the second stage, before preference shocks for early consumption are realized.

The buyer’s problem can be written recursively as

$$ W_t(\omega_t) = \max_{x_t, a_{t+1}} \left\{ x_t + \beta V_{t+1}(a_{t+1}) \right\} \quad \text{s.t.} \quad a_{t+1} = (1 + r_{t+1}) (\omega_t - x_t) \geq 0. \tag{3} $$

From (3), the buyer chooses its consumption, $x_t$, and asset holdings, $a_{t+1}$, in order to maximize its lifetime utility subject to a budget constraint. The budget constraint says that next-period wealth is equal to the current wealth net of consumption capitalized at the gross interest rate, $1 + r_{t+1}$. Or, combining (3) leads to

$$ W_t(\omega_t) = \omega_t + \max_{a_{t+1} \geq 0} \left\{ - \frac{a_{t+1}}{1 + r_{t+1}} + \beta V_{t+1}(a_{t+1}) \right\} \tag{4} $$

From (4), $W_t$ is linear in wealth and $a_{t+1}$ is independent of $\omega_t$. The Euler equation
for the buyer's problem is:

\[ 1 = (1 + r_t) \beta V'(a_t). \]

The disutility cost of accumulating one unit of wealth in the last stage is equal to one. This investment yields \(1 + r\) and is valued according to the buyer's discounted marginal utility of wealth in the early-consumption stage, \(\beta V'(a_t)\).

We now turn to the goods market for early consumers. The expected discounted utility of a buyer at the start of the early-consumption stage holding assets \(a_t\) is

\[
V_t(a_t) = \alpha \left[ (1 - \lambda) \max_{p_t y_t \leq a_t} \{v(y_t) + W(a_t - p_t y_t - \tau_t)\} \right] + \lambda \max_{y_t \geq 0} \{v(y_t) + W(a_t - p_t y_t - \tau_t)\} + (1 - \alpha) W_t(a_t - \tau_t).
\]

With probability \(\alpha\) the buyer wants to consume early. In that case, the buyer can finance expenditures using intra-period credit with probability \(\lambda\), or with assets when payment cannot be enforced. The constraint, \(p_t y_t \leq a_t\), captures the assumption that buyers cannot commit and, hence, delay settlement. As a result, spending cannot exceed wealth. With probability, \(1 - \alpha\), the buyer does not want to consume early. In general, the buyer enters the late-consumption stage with \(a_t - p_t y_t - \tau_t\) units of wealth, where \(\tau_t\) are lump-sum taxes. Using the linearity of \(W_t\),

\[
V_t(a_t) = \alpha \left[ (1 - \lambda) \max_{p_t y_t \leq a_t} \{v(y_t) - p_t y_t\} + \lambda \max_{y_t \geq 0} \{v(y_t) - p_t y_t\} \right] + a_t - \tau_t + W_t(0). \tag{5}
\]

Denote the optimal early consumption under perfect credit \(y^*_t\) and without credit \(\hat{y}_t\). These quantities satisfy \(y^*_t = v'^{-1}(p_t)\) and \(\hat{y}_t = \min\{y^*_t, a_t/p_t\}\). If the payment constraint does not bind, then the buyer equalizes marginal utility to price. Otherwise, early consumption equals the buyer’s wealth.

The expected revenue of a firm in terms of the numeraire in period \(t\) is:

\[ z_t = \bar{z} + \max_{y \in [0, y]} \{p_t y - c(y)\}. \]

Relative to the standard MP model, the novelty is the second term that represents the firm’s profits from selling early. Assuming an interior solution, the optimal supply of goods in the early market is

\[ y^*_t = c'^{-1}(p_t). \tag{6} \]
The price of early consumption is equal to the firm’s marginal cost from producing early. Market clearing in the early-consumption stage requires

\[ n_t y_t^* = \alpha [\lambda y_t^* + (1 - \lambda) \hat{y}_t] \]  \hspace{1cm} (7)

There is a measure \( n_t \) of active firms in the early market, each of which produces \( y_t^* \). Household consumption is the sum of purchases by individuals with and without access to credit.

Finally, the buyer’s choice of assets is obtained by substituting (5) into (4) and taking the first-order condition:

\[ \frac{\rho - r_t}{1 + r_t} = \alpha (1 - \lambda) \left[ \frac{v'(\hat{y}_t)}{c'(y_t^*)} - 1 \right] \]  \hspace{1cm} (8)

The left side of (8) represents the cost of holding the asset, which approximately equals the difference between the rate of time preference and the real interest rate. The right side represents the expected marginal benefit from holding liquid wealth. The expected marginal benefit is the percentage increase of marginal utility with respect to marginal cost multiplied by the probability of having a liquidity shock and not being able to access credit. If buyers are not constrained by their asset holdings in the early-consumption stage, then \( r_t = \rho \). Otherwise, \( r_t < \rho \).

### 4.2 Labor market

We now turn to the second agent in a household. The lifetime expected utility of an employed worker, measured in either the second or third stage, is

\[ U_{1,t} = w_{1,t} + (1 - \delta) \beta U_{1,t+1} + \delta \beta U_{0,t+1} \]

The employed worker receives a wage, \( w_{1,t} \), and keeps her job in the following period with probability \( 1 - \delta \). Similarly, the Bellman equation of an unemployed worker:

\[ U_{0,t} = w_{0,t} + (1 - e_t) \beta U_{0,t+1} + e_t \beta U_{1,t+1} \]
The unemployed worker enjoys $w_{0,t}$ and finds a job in the following period with probability $e_t$. Therefore, the utility of a household in the third stage composed of a buyer with $a$ units of wealth and a worker with employment state $e$ is $W(a) + Ue$.

Arbitrage between acquiring existing firms or creating new ones equates the rate of return on a mutual fund, $1 + r_{t+1}$, to that of opening a vacancy, $q_{t+1}J_{t+1}/k$, so that

$$(1 + r_{t+1})k = q_{t+1}J_{t+1}$$

The rate of return from investing in a new firm in the last stage of $t$ is the expected value of the firm in $t + 1$, $q_{t+1}J_{t+1}$, net of the initial investment, $k$, expressed as a function of this initial investment. The value of a firm solves

$$J_t = z_t - w_1 + (1 - \delta)\frac{J_{t+1}}{1 + r_{t+1}}$$

(9)

The value of a firm equals expected revenue net of the wage plus the expected discounted profits of the job multiplied by the survival probability $1 - \delta$. Market tightness is determined by the arbitrage condition, $(1 + r_{t+1})k = q_{t+1}J_{t+1}$, where $J_t$ is given by (9):

$$\frac{(1 + r_t)k}{q_t} = z_t - w_1 + (1 - \delta)\frac{k}{q_{t+1}}.$$ 

Throughout, we take $w_1$ as exogenously given.\footnote{We abstract from bargaining and wage-determination considerations in the labor market. A fixed wage could arise if the worker has no bargaining power, in which case $w_1 = w_0$, the reservation wage. Including a formal wage determination process, with $w_1 < z_t$, would not change the main qualitative results. Instead, the quantitative analysis calibrates the value for $w_1$. The Appendix presents an extension to where $w_1$ is determined via Nash bargaining.}

In the second stage, firms evolve according to

$$n_{t+1} = (1 - \delta)n_t + m(1, \theta_{t+1})(1 - n_t).$$

Among the $n_t$ existing firms in period $t$, a fraction $1 - \delta$ survive. The measure of new firms equals the measure of job seekers in $t + 1$, $u_t$, multiplied by the job finding probability $e_{t+1} = m(1, \theta_{t+1})$. The value of buyers’ assets in the second stage is the
market capitalization of firms plus the total value of government bonds.

\[ a_t = n_t J_t + A_t^q = \frac{n_t(1 + r_t)k}{q_t} + A_t^q. \]  

(10)

By market clearing, the total value of the stock market and government bonds equals the value of assets held by buyers when entering the early-consumption stage, \( a_t \). Equation (10) implies a positive relationship between stock market capitalization, employment, and interest rates. Combining (6) (7), and (10) allows us to express the price as a function of assets and employment:

\[ c^{t-1}(p_t) = \frac{\alpha}{n_t} \left[ \lambda v^{t-1}(p_t) + (1 - \lambda) \min \left\{ v^{t-1}(p_t), \frac{n_t J_t + A_t^q}{p_t} \right\} \right]. \]

We are now ready to define an equilibrium as a bounded sequence, \( \{J_t, \theta_t, n_t, p_t, r_t\}_{t=0}^{+\infty} \), that solves:

\[ J_t = \frac{(1 + r_t)k}{q(\theta_t)} = \bar{z} + \max_y \{p_t y - c(y)\} - w_1 + (1 - \delta) \frac{J_{t+1}}{1 + r_{t+1}} \]  

(11)

\[ c^{t-1}(p_t) = \frac{\alpha}{n_t} \left[ \lambda v^{t-1}(p_t) + (1 - \lambda) \min \left\{ v^{t-1}(p_t), \frac{n_t J_t + A_t^q}{p_t} \right\} \right] \]  

(12)

\[ \frac{\rho - r_t}{1 + r_t} = \alpha(1 - \lambda) \left[ \frac{v' \left( \frac{n_t J_t + A_t^q}{p_t} \right)}{p_t} - 1 \right]^+ \]  

(13)

\[ n_{t+1} = (1 - \delta) n_t + m(1, \theta_{t+1})(1 - n_t), \]  

(14)

for some given \( n_0 \). Equation (11) determines the value of a firm and market tightness taking the real interest rate and the early-consumption price as given. Equation (12) determines the early-consumption price by market clearing while (13) determines the real interest rate from the buyer’s demand for liquid wealth. Equation (14) is the law of motion of employment.

5 Deconstructing the model

For this section, set \( \lambda = 0 \), so that we isolate the role of liquid mutual funds and bonds. To better understand the components of the model, we deconstruct it by...
starting with the textbook Mortensen-Pissarides model and adding one new ingredient at a time. By doing so, we will relate our model to another canonical model of the macroeconomic literature, the Bewley-Aiyagari model of incomplete markets. For sake of illustration, we use a continuous-time version of the model that allows us to represent dynamics graphically through phase diagrams. The Appendix presents results on local uniqueness of rational expectations equilibria in the discrete-time version of the model.

5.1 A Mortensen-Pissarides economy

The Mortensen-Pissarides economy with a single good and frictionless goods market can be obtained by shutting down the idiosyncratic preference shocks, $\alpha = 0$, so that there is no early consumption. In this case, $z_t = \bar{z}$ and $r_t = \rho$ since stocks and bonds play no liquidity/insurance role. Hence, a change in $A^g$ has no effect on interest rates or output. An equilibrium can be reduced to a pair, $(J_t, n_t)$, which solves

$$
(\rho + \delta)J = \bar{z} - w_1 + \dot{J} \\
\dot{n} = m[1, \theta(J)](1 - n) - \delta n,
$$

where $\theta(J)$ is the solution to $J = k/q(\theta)$. It is easy to check that there is a unique steady state and, for any initial condition $n_0$, a unique equilibrium corresponding to the saddle path leads to the steady state. Along this equilibrium, $J$ is constant and equal to the discounted sum of the profits, $(\bar{z} - w_1)/(\rho + \delta)$, where the effective discount rate is the sum of the rate of time preference and the depreciation rate. Similarly, market tightness is constant. Graphically, in the left panel of Figure 3, the $J$-isocline is horizontal. The $n$-isocline is upward-sloping since a higher market value of firms induces a higher market tightness, and higher employment at the steady state.

---

See related versions by Rocheteau and Rodriguez-Lopez (2014) or Rocheteau, Weill, and Wong (2018). The continuous-time version is such that buyers receive opportunities to consume the DM good at Poisson rate $\alpha > 0$. When these preference shocks occur, buyers consume discrete amount of the goods. Buyers do not have access to the technology to produce the numeraire good when preference shocks occur, but they can produce in-between shocks. Firms general an output flow $\bar{z}$ that they can transform into the DM good at the cost $c(y)$. In contrast to the discrete-time economy, the matching flow, $m(s, o)$, can be any non-negative real number.
5.2 Mortensen-Pissarides with early consumption and perfect credit

We reintroduce preference shocks for early consumption by setting $\alpha > 0$ but keep the goods markets frictionless by assuming that buyers have access to perfect credit in the early-consumption stage, i.e., $\lambda = 1$. In that case an equilibrium is a list, $\{J_t, p_t, y_t^*, n_t\}$, that solves

$$\begin{align*}
(\rho + \delta) J &= \bar{z} + \max_y \{py - c(y)\} - w_1 + \dot{J} \\
\nu'(\frac{ny^*}{\alpha}) &= p = c'(y^*) \\
\dot{n} &= m [1, \theta(J)](1 - n) - \delta n
\end{align*}$$

From (16), assuming $c'' > 0$, each firm’s early-supply of goods decreases with $n$. As a result, the price of early consumption is a decreasing function of $n$. It implies that the firm’s total revenue on the right side of (15) is $z = z(n)$ with $z' < 0$. As there are no liquidity constraints, a change in government bonds $A^g$ has no effect on equilibrium.

The dynamic system, (15)-(17), can be reduced to two ODEs and two unknowns, $J$ and $n$. In the right panel of Figure 3, the $J$-isocline is decreasing in $n$, since higher $n$ means lower early-consumption prices and lower profits. As before there is a unique steady state and a unique equilibrium starting from any initial condition $n_0$. Along the saddle path trajectory, the value of firms is negatively correlated with $n$. A positive productivity shock that raises $\bar{z}$ shifts the $J$-isocline upward. So the value of firms and market tightness overshoot their steady-state values. As employment increases, $p_t$ decreases which brings $J_t$ and $\theta_t$ back to their steady states.

5.3 Mortensen-Pissarides with limited commitment

Households accumulate wealth to self-insure against the idiosyncratic risk of early consumption. To mimic the one-good economy of the Aiyagari model, we impose a linear cost function, $c(y) = y$, so that $p = 1$ and firms are indifferent between producing early or late. As a result, the marginal product of capital, as captured by $\bar{z} - w_1$, does not depend on market capitalization. An equilibrium can now be reduced to a triple, $(J, r, n)$,
\( \dot{J} = 0 \)

\( \dot{n} = 0 \)

(a) MP model.

\( \dot{J} = 0 \)

\( \dot{n} = 0 \)

(b) 2-good MP model with convex cost.

Figure 3: Phase diagrams: MP models.

\[
(r + \delta) J = \bar{z} - w_1 + \dot{J} \\
\rho - r = \alpha [v' (nJ + A^g) - 1]^+ \\
\dot{n} = m [1, \theta(J)] (1 - n) - \delta n.
\]

The novelty is Equation (18) that endogenizes the real interest rate. From (18) one can express \( r \) as an increasing function of \( nJ + A^g \) and reduce the system to two ODEs and two unknowns, \((J, n)\). The \( J \)-isocline, such that \( \dot{J} = 0 \), is given by \([r(nJ + A^g) + \delta] J = \bar{z} - w_1\). There is a negative relationship between \( J \) and \( n \). Intuitively, as the measure of firms increases, market capitalization increases for given \( J \). As households have more liquidity to finance demand shocks, \( r \) rises, which reduces the value of each firm. Thus, an increase in \( A^g \) lowers the \( J \)-isocline: raising real interest rates \( r \), reducing firm value \( J \), and hence depressing employment \( n \) via a reduced incentive to hire. Let \( \bar{M} \) denote the market capitalization above which \( r = \rho \).

For all \( nJ > \bar{M} \), \( J \) is constant and equal to \((\bar{z} - w_1)/(\rho + \delta)\). Let \( M \) denote the market capitalization such that \( r = -\delta \), i.e., \( v'(M + A^g) = 1 + (\rho + \delta)/\alpha \). As \( nJ \) approaches \( M \), \( J \) tends to \( +\infty \) and \( n \) tends to 0. The \( n \)-isocline gives a positive relationship between \( n \) and \( J \). So there is a unique steady state. Moreover, for given \( n_0 \) the equilibrium is unique. Along this equilibrium \( J \) decreases over time and \( r \) increases if \( n_0 \) is less than the steady state.

A positive productivity shock moves the \( J \)-isocline upward. If the initial steady state
is such that households are liquidity constrained, then $J$ overshoots its steady-state value. As $n$ increases, market capitalization increases as well and the real interest rate decreases, which brings the value of firms back to their steady state.

5.4 The general case

We now combine all the ingredients: (i) households are subject to idiosyncratic preference shocks for early consumption, $\alpha > 0$; (ii) they can access credit with probability $\lambda$; (iii) and the cost of early production, $c(y)$, is strictly convex. The early-consumption price, $p = c'(y')$, now depends on households’ liquid wealth, thereby providing another channel through which liquid wealth affects firms’ revenue.

An equilibrium is now a list, $(J, r, p, n)$, that solves

$$(r + \delta) J = \bar{z} + \max_y \{py - c(y)\} - w_1 + \dot{J}$$

$$c^{-1}(p) = \frac{\alpha}{n_t} \left[ \lambda v^{-1}(p) + (1 - \lambda) \min \left\{ v^{-1}(p), \frac{nJ + A^g}{p} \right\} \right]$$

$$\rho - r = \alpha (1 - \lambda) \left[ \frac{v'(nJ + A^g)}{p} - 1 \right]^+$$

$$\dot{n} = m [1, \theta(J)] (1 - n) - \delta n.$$
We can reduce these equations to a pair of ordinary differential equations by defining a sequence of functions. Households lacking credit are unconstrained if and only if $py^* \leq nJ + A^g$, in which case we have $y^* = (\alpha/n)y^*$. Noting that $y^* = v'^{-1}(c'(y^*)) = (y^*)^{-\sigma/\gamma}$, we have that $y^* = (\alpha/n)(y^*)^{-\sigma/\gamma}$, or

$$y^* = \left(\frac{\alpha}{n}\right)^{-\sigma/(\gamma+\sigma)}$$

Otherwise, $py^* > nJ + A^g$, or

$$\left(\frac{\alpha}{n}\right)^{\sigma(\gamma-1)/(\gamma+\sigma)} > nJ + A^g$$

Let $y^*(n)$ is the solution to $v'(y^*) = c'(\alpha y^*/n)$ and $\hat{y}(n, J, A^g)$ solve

$$\hat{y} = \frac{\alpha}{n} \left[ \lambda v'^{-1}(c'(\hat{y})) + (1 - \lambda) \frac{nJ + A^g}{c'(\hat{y})} \right]$$

It is easy to check that $y^*$ is a decreasing function of $n$ with $\lim_{n \to 0} y^* = +\infty$ and $\lim_{n \to +\infty} y^* = 0$ and $\hat{y}$ is an increasing function of $J$ and $A^g$ and a decreasing function of $n$. Thus, the buyer’s liquidity constraint is more likely to bind if $n$ is low and $J$ is large. Let $y^*(n, J, A^g) = \min\{\hat{y}(n, J, A^g), (\alpha/n)y^*(n)\}$.

We define the price, in turn, as $p(n, J, A^g) = c'[y^*(n, J, A^g)]$. The price is weakly decreasing in $n$ and weakly increasing in $J$ and $A^g$ (an aggregate demand effect). The total revenue of a firm is

$$z(n, J, A^g) = \bar{z} + p(n, J, A^g)y^*(n, J, A^g) - c[y^*(n, J, A^g)].$$

Revenue is weakly decreasing in $n$ and weakly increasing in $J$ and $A^g$. The real interest can also be expressed as a function of $n$ and $J$ as follows:

$$r(n, J, A^g) = \rho - \alpha(1 - \lambda) \left[ v^\gamma \frac{nJ + A^g}{p(n, J, A^g)} - 1 \right]^+,$$

where $y^b$ is an increasing function of $n, J,$ and $A^g$. So, $r$ is a weakly increasing function of $n, J,$ and $A^g$. 

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Using the functions $z(n, J, A^g)$ and $r(n, J, A^g)$ we reduce the dynamic system to two autonomous, nonlinear ODEs:

\[
\dot{J} = \left[ r(\tilde{n}, \tilde{J}, A^g) + \delta \right] J + w_1 - z(\tilde{n}, \tilde{J}, A^g) \equiv f(J, n)
\]

\[
\dot{n} = m \left[ 1, \theta(J) \right] (1 - n) - \delta n \equiv g(J, n).
\]

The right side of the $J$-ODE is monotone increasing in $n$ but can be non-monotone in $J$. As a result, the $J$-isocline can also be non-monotone, with important consequences for the multiplicity of steady states and dynamics. An increase in government bonds $A^g$ raises both interest rates and revenue, thus having an ambiguous effect on the $J$-nullcline.

![Figure 5: Multiple equilibria: full model.](image)

In Figure 5 we provide a numerical example for the following parameter values: $m(s, o) = s^\xi o^{1-\xi}$ with $(1 - \xi)/\xi = 0.2$, $c(y) = y^{1.9}/1.9$, $v(y) = y^{0.5}/0.5$, $\tilde{z} - w_1 = -0.5$, $\rho = 0.1$, $\alpha = \delta = 1$, and $A^g = 0$. Note that for this parametrization $\tilde{z} - w_1 < 0$, i.e., if the early-consumption opportunities are shut down, then firms make negative
profits. This numerical example exhibits multiple active steady states. There is an equilibrium with high employment, high value for firms, and high interest rates and a different equilibrium with low employment rate, low valuation of firms, and low interest rate. The logic for this multiplicity is as follows. At the high equilibrium market capitalization is high, which relaxes the liquidity constraint faced by households in the early-consumption stage. As a result, the aggregate demand for early consumption is high, which pushes $p$ up, raises firms’ revenue, and generates entry. The real interest is high because wealth is abundant, which reduces the liquidity/insurance premium of stocks. The high $p$ and high $r$ have opposite effects on $J$, but in our example the former dominates.

At the opposite, the economy can be stuck in an equilibrium with low aggregate wealth, low employment, and low real interest rates. In this equilibrium, households are severely liquidity constrained, which reduces aggregate early-consumption and depresses the price $p$. It also reduces the real interest rate. Firms’ profits are lower due to the lower $p$, which reduces entry and employment. We think of this type of equilibrium as capturing the notion of secular stagnation.

5.5 The aggregate demand channel: comparison of general case with Bewley-Aiyagari

In the Bewley-Aiyagari version of the model, spending shocks impact the goods and labor market through the endogenous real interest rate: if spending shocks occur more frequently, and consumers are liquidity constrained, then the real interest rate is lower and firm values are higher (less discounting of expected future profits). In the general version of the model in addition to this real interest rate channel there is the additional aggregate demand channel as spending shocks impact firm revenues in early consumption markets. This section further compares the steady-state properties of these two cases.

Figure 6 plots stock market capitalization ($nJ$) and real interest rates as a function of the frequency of spending shocks $\alpha$ in both the Bewley-Aiyagari version of the model (dashed line) and the full model with the aggregate demand channel. To generate this figure, we set $\bar{z} - w_1 = 0.055$, $\sigma = 0.9$, $\gamma = \xi = k = 0.5$, and $\delta = 0.25$. The Bewley-Aiyagari version arises when $\sigma = 0$. The top panel plots the steady-
state market capitalization as a function of $\alpha$. In both versions of the model, greater expenditure risk increases market capitalization, through more firm-entry ($n$) and firm value ($J$). However, in the full model the market value of firms increases substantially more as $\alpha \to 1$. Notice that the full model generates a kink at the point that households are no longer liquidity constrained. For $\alpha$ greater than the kink point, stock market capitalization increases linearly with $\alpha$.

The bottom panel of Figure 6 plots the associated real interest rate. When $\alpha = 0$, there is no expenditure risk and $r = \rho$ in both versions of the model. For small $\alpha > 0$, households are liquidity constrained and further increases in $\alpha$ tighten those liquidity constraints and reduce the real interest rate. This pattern continues for all values of $\alpha$ in the Bewley-Aiyagari version. In the full version of the model, the aggregate demand
channel causes stock market capitalization to increase substantially, which loosens consumers’ liquidity constraints and the real interest rate increases. Eventually, at the kink point in the top panel, consumers are no longer liquidity constrained and \( r = \rho \).

The strong strategic complementarities between labor markets and goods markets renders the creation of private liquid assets elastic with respect to spending shocks. Figure 6 highlights the unique implications relative to a Bewley-Aiyagari model. These insights are important for interpreting the quantitative experiments that follow in the remainder of the paper.

### 5.6 Comparison to a pure-currency economy

In the following we show that the set of steady states arising from our model differs qualitatively from the one of a pure currency economy where fiat money is the only means of payment (Shi (1998) or Berentsen, Menzio, and Wright (2011)).\(^\text{10}\) Let \( \pi \) denote the growth of the money supply. A steady-state equilibrium of a pure currency economy is a list, \((J, p, y^s, n)\), that solves:

\[
\begin{align*}
(\rho + \delta) J &= \bar{z} + \max_y \{py - c(y)\} - w_1 \\
p &= c'(y^s) \\
\rho + \pi &= \alpha \left[ \frac{\nu'(ny^s/\alpha)}{c'(y^s)} - 1 \right] \\
\delta n &= m \left[ 1, \theta(J) \right] (1 - n).
\end{align*}
\]

The third equation pins down \( y^s \) as a decreasing function of \( n \). From the second equation, \( p \) is a decreasing function of \( n \). From the first equation, \( J \) is a decreasing function of \( n \). From the Beveridge curve \( n \) is increasing with \( J \). So the steady state of the pure currency economy is unique. Our economy differs from the pure currency economy in two ways. First, in our model the real interest rate is endogenous and depends on the measure of firms and their valuation. As discussed above, the logic for the determination of the real interest rate is similar to the one in the Aiyagari model.

\(^\text{10}\)The comparison here is not directly to Berentsen, Menzio, and Wright (2011), which includes a frictional goods market where the probability of matching is affected by matching in the labor market.
Second, the price of early consumption depends on market capitalization through a limited commitment problem. This is this second channel that has the potential of generating multiple steady states in our model.

6 Quantitative analysis

The frequency is monthly and the time range is 1948-2018. The matching function is taken from den Haan, Ramey, and Watson (2000): 
\[ m(s, o) = so/(s^\xi + o^\xi)^{1/\xi}. \]
The cost of early production is 
\[ c(y) = y^{1+\sigma}/(1 + \sigma), \]
and the utility function is 
\[ v(y) = y^{1-\gamma}/(1 - \gamma). \]
We choose \( \sigma = 0.2 \), which represents a 20% markup of price to average cost. The construction of empirical separation rates \( s_t \) and job finding rates \( e_t \) uses unemployment data as in Shimer (2005). We set \( \delta \) and \( e \) according to their respective means. The parameter \( \xi \) to target the mean job finding rate. The implied rate of employment from the Beveridge curve is 
\[ n = e/(s + e). \]

We assume a liquidity premium or convenience yield of liquid assets of 75 basis points, which is close to the spread estimated by Krishnamurthy and Vissing-Jorgensen (2012) for Treasury securities.\(^{11}\) This spread is also similar to the baseline results of several incomplete market models, e.g., Aiyagari (1994) and Angeletos (2007). We also assume an annual risk-free interest rate of 4%, which implies a monthly target for \( r_t \). We choose \( \rho \) to attain a risk-free rate of 4%, \( \gamma \) to target the interest rate spread, and \( A^\theta \) to match the semi-elasticity of the spread with respect to the debt-to-GDP ratio of \(-0.746\) calculated in Krishnamurthy and Vissing-Jorgensen (2012), and \( k \) to satisfy the arbitrage condition 
\[ J = (1 + r)k/q. \]

The calibration of \( \alpha \) and \( w_1 \) depends on evidence on health expenditure shocks from the Federal Reserve Report on the Economic Well-being of U.S. Households in 2015. As previously mentioned, the share in the sample reporting a major health expense ranges from 22% – 30% and so we take 26% as a midrange value along with a mean expense of $2,383. Hence, \( \alpha \) is chosen to match health expenditures shocks of 26% annually, and \( w_1 \) is chosen to target the fraction of the wage spent on unexpected health costs. Finally, we choose \( \lambda \) to represent the fraction of households with revolving credit sufficient to replace income, as reported in Braxton, Herkenhoff, and

\(^{11}\)They consider the percentage spread between Moody’s Aaa-rated long-maturity corporate bonds yields and yields on long-term maturity Treasury bonds.
Phillips (2019). The appendix provides explicit details of how the calibration procedure was implemented. Table 2 summarizes the parameters, values, and respective targets, while Figure 7 plots the steady-state relationship.

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<tr>
<th>Parameter</th>
<th>Values</th>
<th>Calibration Strategy</th>
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<td>$\overline{z}$</td>
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</tr>
<tr>
<td>$\alpha$</td>
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</tr>
<tr>
<td>$\sigma$</td>
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<td>ratio of price to average cost</td>
</tr>
<tr>
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<td>consistency with market tightness</td>
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<tr>
<td>$\xi$</td>
<td>1.326</td>
<td>consistency of tightness with job finding probability</td>
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<tr>
<td>$A^g$</td>
<td>0.225</td>
<td>Semi-elasticity of interest rate spread</td>
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<tr>
<td>$\lambda$</td>
<td>0.430</td>
<td>Fraction of households with revolving credit sufficient to replace income</td>
</tr>
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</table>

Table 2: Parameterization

6.1 Inspecting data through the lens of the model

The calibrated parameter values in Table 2 pin down a unique equilibrium in the model that effectively capture several key long-run empirical moments of the U.S. aggregate economy. However, the exercise is silent as to whether the model can capture time-series properties of financial and labor markets. We turn now to a quantitative exercise that equates model implied paths for the unemployment rate and the interest-rate spread with their empirical counterparts. The exercise is to find a sequence of unanticipated, permanent shocks to $(\alpha_t, \overline{z}_t)$ for which the model’s rational expectations equilibrium paths for $n_t$ and $\rho - r_t$ reproduce the same series in the data over the period 1981-2018. We choose this particular sample period to cover the low inflation era, given that we restrict ourselves to a real economy. We use this quantitative exercise to achieve two goals. First, we use the model as a lens through which we can interpret recent labor market and financial market experiences. While the exercise mechanically reproduces employment and interest-rate spread data, we can independently assess the model’s quantitative predictions for
Figure 7: Steady-state at calibrated parameter values

stock market capitalization, relative to aggregate output. Second, the shock estimates for $\alpha_t$ provide the basis for the counterfactual exercise to follow.

Figure 8 plot the results. The bottom two panels, the interest-rate spread (e.) and unemployment rate (f.), are the targets for calibrating $(\alpha_t, \bar{z}_t)$, whose estimates are plotted in panels (a.) and (c.). Panel (d.) plots the resulting model-implied paths for firm revenues from early consumption (relative to GDP $z_t$). Clearly, to match data on unemployment rates and interest-rate spreads, the model calls for substantial movements in the extent of the limited commitment problems, i.e. the frequency of spending shocks $\alpha$, as well as the firm’s revenue from producing for the late-consumption market, i.e. $\bar{z} - w_1$. To interpret the magnitudes, recall that $\alpha (1 - \lambda)$ of buyers will face an early-consumption preference shock and will not have access to credit to finance their purchase. So when $\alpha \approx 0.20$ in 2006 and with a $\lambda = 0.43$, 11.4% of buyers will use their mutual fund (liquid assets) to finance early consumption purchases. Similarly, when $\bar{z} - w_1 < 0$, as it is most of the time from the mid-90’s on, firms would not choose to enter the market, or post vacancies, if
it were not for the greater revenue they could earn by choosing to produce for the early-consumption market. The interpretation then of the time-series in panel (c.) is the movements in $\bar{\varepsilon} - w_1$ measure the extent to which firms have an incentive to produce in the more profitable early-consumption market.

The results in Figure 8 indicate that the model interprets movements in the unemployment rate and interest-rate spreads as the result of fluctuations in expenditure risk and the limited commitment problem that manifests in the expansion and contraction of the early-consumption market—that is, time variation in the aggregate demand channel. In the data, there are three large episodes that feature spikes in the interest-rate spread and low unemployment rates. Through the lens of the model, these events arise as heightened expenditure risk leads to large increases in the demand for early-consumption goods, a fraction of which is financed through liquid assets, and increasing incentives for firms to enter the early-consumption market.

While we are mechanically matching unemployment rates and interest-rate spreads, we can assess the model’s performance by comparing the rational expectations equilibrium path for the stock market capitalization to GDP ratio in the model to the data. The outcome of this comparison is presented in Figure 8.b. The solid line is the rational expectations equilibrium path for stock market capitalization to GDP, where GDP is measured as firm production $z_t$, which includes production in both consumption markets. The dashed line is computed from the data, measuring stock market capitalization from the balance-sheet of non-financial companies. The model, with time-varying $(\alpha_t, \bar{\varepsilon}_t)$, does a good job capturing the overall timing and trends in market capitalization. It predicts a large increase in market-capitalization to GDP during the dot-com boom, the mid 2000’s, and the post-financial crisis recovery period. The model does not generate the degree of volatility observed in the data. For example, the model explains about half of the large increase in market capitalization in 2000 and in 2019. Given the simplicity of the model, and well-known limitations of rational expectations models of asset-pricing, the results in Figure 8 are a surprising success for the model.

Similarly, the model does a good job capturing the joint movements of unemployment and the stock market identified by Farmer (2012), and others, and illustrated in Figure 1. Figure 8f plots the market capitalization (left axis) and the unemployment

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12This is the “Buffet” measure of market capitalization. FRED code: “NCBEILQ027S”.

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Figure 8: Results from calibration exercise. Values for $\alpha_t$ and $\bar{z}_t$ chosen so that model implied paths for the interest rate spread (e.) and unemployment (f) match the data. Panel (b) compares model implies stock market capitalization to the data. Panel (d.) plots the resulting relationship between the equilibrium outcomes in the stock and labor markets.

rate (right axis) over the sample period. Clearly, this dynamic version of the model captures the negative relationship between the stock market and unemployment remarkably similar to Figure 1. In the counterfactuals that follow we look deeper into
the model implications of the time-variation in aggregate demand suggested by Figure 8.

Finally, note that there is an important interaction between the nonlinearities implied by labor market search frictions and the aggregate demand externality. As the unemployment rate falls very low, in the range of 4% in 2000 and 2019, the congestion externalities in matching make any further reductions in unemployment very difficult. As a result, it takes large increases in spending, as evidenced by the spike in the fraction of consumers with liquidity shocks, to promote more hiring given the plummeting vacancy filling rates. The period 2017-2019 is associated with a reduction in the exogenous component of revenue, the increase in the probability of a liquidity shock is even more pronounced.

6.2 Counterfactual: a perfect storm

Section 5 demonstrated that when the strategic complementarities in the model are strong, then there can exist multiple steady-states including a low employment/low market capitalization equilibrium. The strategic complementarities are strongest when $\alpha$ is high, $\bar{z} - w_1 < 0$, and the exogenous liquidity in the form of government bond supply $A^g$ is low. In this section, we take the values for $(\alpha, \bar{z}) = (0.21, -0.06)$ taken from the quantitative exercise in Figure 8 for the period covered by 2006, and we decrease the value of $A^g$ by 30%. The decrease in $A^g$ is formally equivalent to introducing a pledgability constraint on public bonds that can serve as collateral and the shock is that consumers can only pledge 70% of $A^g$. We think of the latter as an exogenous decrease in the velocity of government bonds. This particular value is guided by data on government bond velocity. For instance, Figure 9 plots the government bond velocity in the U.S. over time and shows that post-2000 until 2006 there was a 30% decrease in the velocity. \(^{13}\)

The “perfect storm” counter-factual then asks what would happen if, at the same time there is an increase in aggregate demand for early-consumption goods, there is a large decrease in the velocity of publicly provided liquidity and a shock to expectations about future stock market returns. We model this counterfactual as an unanticipated

\(^{13}\)We measure the government bond velocity as the ratio of nominal GDP to the nominal value of “safe” government bonds, using the methodology in Gorton, Lewellen, and Metrick (2012). A similar magnitude is seen if velocity is defined as nominal consumption to the safe bond supply.
permanent shock to \((\alpha, \bar{z}, A^g)\) and examine the impulse response to an expectations shock.\(^{14}\) In this counter-factual, there are now three steady-state equilibria, with differing levels of employment rates and market capitalization: see Figure 10. While not shown, the high and low steady-states are determinate and the middle steady-state is indeterminate.\(^{15}\)

### 6.2.1 Modeling expectations

When computing the transition paths following the unanticipated shocks, one must take a stand on how households and firms form expectations. As a benchmark, we assume that expectations are formed rationally and, given the shocks, solve for the perfect foresight path. However, rational expectations, and even more so for perfect foresight, is a strong assumption that requires the agents in the economy to perfectly understand the dynamic evolution and behavior, including beliefs and preferences, of all other agents. As an alternative, we formulate an adaptive learning version of the model, following in the footsteps of Marcet and Sargent (1989) and Evans and Honkapohja (2001), among others. The adaptive learning literature assumes that

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\(^{14}\)Although we technically decrease the lump-sum financed bonds \(A^g\), there are alternative interpretations. We could instead imagine that various frictions may limit the number of these bonds that can be used to finance consumption purchases and our liquidity shock would arise from a further limiting of them. We find decreasing \(A^g\), though, to be a convenient formalization of an exogenous decrease in liquidity that, in order to maintain the same level of economic activity, requires an expansion of endogenous privately-generated liquidity in the form of claims on firms.

\(^{15}\)In the counterfactual experiment, we hold the steady-state employment rate in the high steady-state fixed at its calibrated value. This necessitates changing the parameter values for \(\gamma, \delta\), at the expense of no longer matching the moments used earlier when calibrating these parameters.
rational expectations is too strong of an assumption and replaces it with a “cognitive consistency principle” that states that agents should be modeled like a good economist who formulates and adapts a well-specified forecasting rule. To discipline that rule, expectations are typically assumed to come from a model that nests the rational expectations equilibrium.

In the present environment, without aggregate stochastic shocks, the limit point of a learning process should be the rational expectations steady-state equilibria. Thus, we adopt a “steady-state” learning rule, as follows. Without imposing rational ex-
pectations, the equations governing a temporary equilibrium are:

\[ J_t = z + \sigma \frac{1}{1+\sigma} \max \left\{ \alpha J_t, \frac{\alpha}{n_t} \gamma^{(1+\sigma)/(\gamma+\sigma)} \right\} + \frac{(1-\delta)J_{t+1}^e}{1+r_{t+1}^e} \]

\[ 1 + r_t = \frac{1 + \rho}{1 + \alpha \left[ \frac{1}{\sigma^{(1-\gamma)/(1+\sigma)n_t}} \sigma^{(1+\gamma)/(1+\sigma)} - 1 \right]^\gamma} \]

\[ n_t = (1-\delta)n_{t-1} + \left[ \frac{J_t}{k(1+r_t)} \right]^{(1-\xi)/\xi} (1-n_{t-1}) \]

Thus, agents in the economy need to generate one-step ahead forecasts of \( J_{t+1}^e \), future profitability of a firm, and the real interest rate \( r_{t+1}^e \). Steady-state learning dictates that those forecasts should come from a geometrically weighted average of past-data, written recursively as

\[ J_{t+1}^e = J_t^e + g \left( J_t - J_t^e \right) \]
\[ r_{t+1}^e = r_t^e + g \left( r_t - r_t^e \right) \]

The parameter \( 0 < g < 1 \) is called the constant gain coefficient. Notice the timing: like rational expectations, expectations and outcomes at time \( t \) are determined simultaneously. The dynamic stability of a rational expectations equilibrium depends, in a complicated way, on whether the particular steady-state of interest is determinate or indeterminate and on the size of the gain coefficient \( g \). If \( g = 1 \) then the economy will jump right to a steady-state and for \( g < 1 \) there is slow adaption to a steady-state that is stable under this learning rule. In most cases, the determinate steady-states are also stable under learning. However, it is well-known since Van Zandt and Lettau (2003) that for gain coefficients that are large enough that an indeterminate steady-state can be stable under this learning rule. The key detail yielding this result is that expectations and outcomes are determined simultaneously.

Most learning models, however, break the simultaneity of expectations and outcomes by adopting a timing convention that expectations are formed before new data are realized, that depend in a self-referential way on those expectations. Under this
alternative “$t - 1$” timing, the steady-state learning rules would be

\[
J_{t+1}^e = J_t^e + g (J_{t-1}^e - J_t^e)
\]

\[
r_{t+1}^e = r_t^e + g (r_{t-1}^e - r_t^e)
\]

With these alternative learning rules, firms make entry decisions given the previous period’s assessment of firm value $J_{t-1}$ and households decide on asset-holdings before $r_t$ is realized. In our counterfactual analysis, we take the contemporaneous timing as our benchmark and illustrate robustness to the “$t-1$” dating.

6.2.2 Results

The counterfactual experiment imagines that the economy has been in the high employment equilibrium. Then, at $t = 1$ there is an increase in aggregate demand, parameterized by $\alpha = 0.21$, an increased incentive for firms to produce the early consumption good, $\bar{z} - w_1 = -0.06$, and an exogenous decrease in liquid bonds $A^g = 0.15$. That generates the set of steady-state equilibria in Figure 10. Simultaneously, there is a transitory expectations shock with $J^e, r^e$ decreasing outside of the local basin of attraction of the high employment steady-state. We choose a learning gain coefficient $g = 0.92$ as this produces locally stable learning dynamics for all 3 steady-state equilibria. Figure 11 plots the results.

In this perfect-storm counterfactual the economy converges to the middle steady-state, exhibiting a lower employment rate and lower stock market capitalization. Figure 11 demonstrates that the economy ends up in this steady-state under both rational expectations and the benchmark learning rule. Under rational expectations, the expectations shock produces an immediate and large decrease in the stock market, slightly overshooting the intermediate steady-state. The interest-rate spread increases, more than doubling its original value, over-shooting the new equilibrium which is about double the original spread. The combination of lower firm values and a higher real interest rate, leads to substantially lower employment rates that bottom out with an unemployment rate of 13% before stabilizing at the steady-state value of 11% unemployment. Under learning, the economy exhibits a similar trajectory though the learning dynamics feature oscillations near the equilibrium which helps arrest some of the decline in employment.
Figure 11: Perfect storm counterfactual: increase in demand/supply for early consumption good, a decrease in exogenous liquidity, and a pessimistic expectations shock. Stock market capitalization and interest-rate spreads are computed relative to the calibrated steady-state. Expectations are formed by rational expectations (dashed) or learning (solid) with gain $g = 0.92$.

### 6.3 Robustness to learning-rule assumptions

We briefly illustrate the implications from different treatments of the learning rule. In particular, we consider a slightly smaller gain coefficient than used in Figure 11, and the dynamics under the “t-1”-dating for both large and small expectations shocks.

The middle, indeterminate equilibrium is stable under learning for gain parameters $g$ that are large enough. The dynamics in Figure 11 exhibit small and dampening oscillations around the equilibrium. That suggests that for some gain parameters smaller than $g = 0.92$, the steady-state is unstable and feature bounded periodic dynamics near the steady-state. Figure 12 confirms this intuition. The figure was generated
under the same “perfect-storm” conditions as Figure 11, except now \( g = 0.90 \). The previous perfect foresight dynamics are presented for comparison. Now, the learning dynamics drop near the middle steady-state but, rather than converging to the equilibrium, the learning dynamics converge to a two-cycle around the equilibrium. Thus, in this case the counterfactual predicts lower average employment rates and market capitalization, but employment periodically increases and decrease by about 5% and there is substantial volatility in stock market capitalization and the real interest rate.

With the “t-1”-timing convention, the indeterminate steady-state is unstable under learning. That means that the counterfactual experiment converges either to the low equilibrium, i.e. the “secular stagnation” equilibrium with permanently lower employment and stock market capitalization, or return to its original equilibrium.
Figure 13: Counterfactual: large expectations shock and a crash. Stock market capitalization and interest-rate spreads are computed relative to the calibrated steady-state. Expectations are formed by rational expectations (dashed) or learning (solid).

Figures 13 and 14 illustrate the possibilities. In Figure 13 there is a large expectations shock, equivalent to expecting the stock market to drop by 30%. Again, there is a large and abrupt stock market crash that eventually converges to an equilibrium that is about 18% of its previous value. Similarly, the interest rate spread increases substantially and the employment rate ends up at about 60%. The figure compares the results to what would happen under rational expectations, which converges to the middle steady-state with substantially higher employment rates. Thus, with this alternative learning rule and a large expectations shock the economy converges to a crash equilibrium.

With a smaller expectations shock, Figure 14 shows that the economy will con-
verge back to the original equilibrium. There is again a large drop in stock market capitalization, but with the less pessimistic shock to expectations, the drop is not as pronounced, never enters the basin of attraction for the crash equilibrium, and promptly returns on a path to the original equilibrium. The smaller drop in the stock market and resulting recovery implies that the employment cost from this counterfactual is relatively small. In this scenario, unemployment increases from 6.5% to 7.5% before recovering. There is also a substantial but quickly diminishing impact on real interest rates.

Figure 14: Counterfactual: small expectations shock and $t-1$-dating in the learning rule. Stock market capitalization and interest-rate spreads are computed relative to the calibrated steady-state. Expectations are formed by rational expectations (dashed) or learning (solid).
7 Conclusion

We have studied the effects of changes in household liquidity constraints on the labor and stock markets. We generalized the Mortensen-Pissarides model along a single dimension: idiosyncratic expenditure risk introduces a limited commitment problem. For some consumption shocks, households can finance their purchases with unsecured debt but in others they must use their liquid assets, in the form of a mutual fund composed of stocks and government bonds, as collateral for intraperiod loans. This single twist of an otherwise standard model introduces strong complementarities into the economy. When stock market valuations are high, then household liquidity constraints are relaxed which, in turn, strengthens firms incentives for job creation. Similarly, when firms create more jobs that increases the stock market values and further increases consumption demand. A novel finding is that these complementarities can produce multiple steady-state equilibria with high employment/high stock market capitalization co-existing with low employment/low stock market values.

We calibrated the model to the long-run properties of the U.S. economy and exploited the multiple steady-states in the model in a counterfactual exercise. Our quantitative analysis suggests that variations in the unemployment rate and interest-rate spreads arise because of fluctuations in the extent of expenditure risk; that is, in the frequency of consumption opportunities that need to be financed with short-term debt backed by the value of households’ liquid asset holdings. Our counterfactual froze the economy at one such point and then considered a perfect storm of increased consumption risk and a decrease in the velocity of government bonds. This scenario coincides with an economy that has low unemployment, high stock market capitalization, and high real interest rates but that is also dependent on the private-provision of liquid assets. We show in this case that multiple steady-states exist, making the economy fragile and susceptible to self-fulfilling collapses in employment and stock market values. Under a variety of assumptions about how expectations are formed (e.g. rational expectations and learning), we show that a fragile economy collapses to a secular stagnation equilibrium with high unemployment, low stock prices, and low real interest rates.
References


Supplemental Appendix

A  Data appendix

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Table 3: Data sources used in motivating evidence and model calibration.

B  Details on Equilibrium Determinacy

Section 5 presented results on the set of equilibria in a continuous time approximation of the model. While the continuous time formulation enhances tractability, in this Appendix we provide additional details on the set of rational expectations equilibria in the discrete-time version of the model.

Mortensen-Pissarides Economy

Here we shut-down the early consumption preference shocks: \( \alpha = 0 \). An equilibrium is a pair \((J_t, n_t)\) that is non-explosive solution to the following pair of non-linear
difference equations:

\[ J_{t+1} = -\frac{1 + \rho}{1 - \delta} z + \frac{1 + \rho}{1 - \delta} J_t \]

\[ n_{t+1} = (1 - \delta) n_t + \left( \frac{J_{t+1}}{(1 + \rho)k} \right)^{\frac{1 - \xi}{\xi}} (1 - n_t) \]

There is a unique steady-state \( \bar{J} = \frac{1 + \rho}{\delta + \rho} z \) and \( \bar{n} = 1/(1 + \delta ((\delta + \rho)k/z)^{(1-\xi)/\xi}) \).

Moreover, the eigenvalues of the Jacobian matrix, evaluated at the unique steady-state, are

\[ \lambda_1 = \frac{1 + \rho}{1 - \delta} \]

\[ \lambda_2 = 1 - \delta - \left( \frac{z}{(\delta + \rho)k} \right)^{(1-\xi)/\xi} \]

We can summarize the results as follows.

**Proposition 1** Let \( \alpha = 0 \). There exists a unique steady-state. For \( z/k < (2 - \delta)^{\xi/(1-\xi)}(\delta + \rho) \), there exists a unique perfect foresight equilibrium path to the steady-state for any given \( n_0 \). Else, there is a degenerate equilibrium with a unique perfect foresight path for \( n_0 = \bar{n} \), and no equilibrium otherwise.

**Perfect Credit**

Now we consider the case of a DMP model with two goods, and perfect credit for the early-consumption good. In this case, \( r_t = \rho \), and equilibrium is a (non-explosive) solution to the difference equations

\[ J_{t+1} = -\frac{1 + \rho}{1 - \delta} z + \frac{1 + \rho}{1 - \delta} J_t - \frac{\sigma(1 + \rho)}{(1 + \sigma)(1 - \delta)} \left( \frac{n_t}{\alpha} \right)^{\frac{\gamma(1 + \sigma)}{\sigma + \gamma}} \]

\[ n_{t+1} = (1 - \delta) n_t + \left( \frac{J_{t+1}}{(1 + \rho)k} \right)^{(1-\xi)/\xi} (1 - n_t) \]
A steady-state is a pair \((\bar{J}, \bar{n})\) that solves
\[
\bar{J} = \frac{1 + \rho}{1 - \delta} z + \frac{\sigma(1 + \rho)}{(1 + \sigma)(1 - \delta)} \left( \frac{\alpha}{\bar{n}} \right)^{-\gamma(1 + \sigma) \over \sigma + \gamma} \\
\bar{J} = (1 + \rho)k \left( \frac{\delta \bar{n}}{1 - \bar{n}} \right)^{\xi/(1 - \xi)}
\]

The first equation is monotonically decreasing in \(n\), with \(\bar{J} \to \infty\) as \(\bar{n} \to 0\), while the second equation monotonically increases with \(n\) and features \(\bar{J} \to \infty\) as \(\bar{n} \to 1\), while also \(\bar{J} = 0\) when \(\bar{n} = 0\). Thus, there exists a unique steady-state. Similarly computing the eigenvalues of the Jacobian, we can establish the following result.

**Proposition 2** In the DMP with 2 goods and perfect credit, there is a unique steady-state. Moreover, for \(k\) sufficiently large, there exists a unique (non-explosive) perfect foresight path, for any given \(n_0\), that converges to the steady state. For \(k\) sufficiently small, the steady-state is a source and there exists a degenerate equilibrium with \(n_0 = \bar{n}\).

**Bewley-Aiyagari Economy**

Now we consider the Bewley-Aiyagari version of the economy: there’s two goods, a limited commitment problem for the early-consumption good, and a linear cost function for the early-consumption good. These assumptions imply that the real interest rate is endogenous, compared to the standard DMP model or the model with perfect credit. However, \(py - c(y) = 0\), which is interpreted as the marginal product of a job (asset) is independent of the market value of the liquid assets. With these assumptions, an equilibrium is a (non-explosive) solution to the pair of equations
\[
J_t = z + (1 - \delta) \frac{J_{t+1}}{1 + r_{t+1}} \\
1 + r_t = \frac{1 + \rho}{1 + \alpha \left[ (n_t J_t)^{-\gamma} - 1 \right]^+} \\
n_{t+1} = (1 - \delta) n_t + \left[ \frac{J_{t+1} (1 - \alpha + \alpha (n_{t+1} J_{t+1})^{-\gamma})}{k (1 + \rho)} \right]^{(1 - \xi) / \xi} (1 - n_t)
\]
The first two equations, in turn, can be re-written as

$$J_t = z + \frac{(1-\delta)(1-\alpha)}{1+\rho} J_{t+1} + \frac{(1-\delta)\alpha}{(1+\rho)} n_{t+1}^{-\gamma} J_{t+1}^{-\gamma}$$

As before, the steady-state can be calculated as a pair $(\bar{J}, \bar{n})$ that solve the equations

$$J \left[ 1 - \frac{(1-\delta)(1-\alpha)}{1+\rho} - \frac{(1-\delta)\alpha}{(1+\rho)} \bar{n}^{-\gamma} J^{-\gamma} \right] = z$$

$$\left( \bar{J}/k \right)^{(1-\xi)/\xi} \left[ \frac{1 - \alpha + \alpha \left( \bar{n} \bar{J} \right)^{-\gamma}}{1+\rho} \right]^{(1-\xi)/\xi} = \frac{\delta \bar{n}}{1 - \bar{n}}$$

As before, the first equation implies $\bar{J}$ is decreasing in $n$, with $\bar{J} \to \infty$ as $\bar{n} \to 0$. The second equation implies that $\bar{J}$ is increasing in $n$ with $\bar{J} \to \infty$ as $\bar{n} \to 1$ and $\bar{J} = 0$ at $\bar{n} = 0$. Again, it follows that there exists a unique steady-state.

Expressions for the eigenvalues of the Jacobian are complicated and analytic results are not, in general, available. However, for the following special case we can provide a uniqueness result.

**Proposition 3** In the Bewley-Aiyagari version of the model, there exists a unique steady-state. Furthermore, for $\xi$ sufficiently large, there exists a unique (non-explosive) perfect foresight equilibrium, for any given $n_0$, that converges to the steady-state.

**The general case**

The equilibrium path is found as the solution to the following equations:

$$J_t = z + \frac{\sigma}{1+\sigma} \max \left\{ \alpha J_t, \left( \frac{\alpha}{n_t} \right)^{\gamma(1+\sigma)/(\gamma+\sigma)} \right\} + \frac{(1-\delta)J_{t+1}}{1 + r_{t+1}}$$

$$1 + r_t = \frac{1 + \rho}{1 + \frac{\alpha}{\alpha^{(1-\gamma)/(1+\sigma)} n_t^{(\sigma+\gamma)/(1+\sigma)} - 1} +}$$

$$n_{t+1} = (1-\delta)n_t + \left[ \frac{J_{t+1} \left( 1 + \alpha \left[ \frac{1}{\alpha^{(1-\gamma)/(1+\sigma)} n_t^{(\sigma+\gamma)/(1+\sigma)} - 1} \right] k (1+\rho) \right) \right]^{(1-\xi)/\xi} (1-n_t)$$
Too see this, note that in a constrained equilibrium,

\[ py_s - c(y_s) = \frac{\sigma}{1 + \sigma} y_s^{1+\sigma} \]

Since \( nJ = py_s \alpha \), \( \alpha J = y_s^{1+\sigma} \), the surplus is \( py_s - c(y_s) = \frac{\sigma}{1+\sigma} \alpha J \). In the interest rate equation,

\[
v'(nJ/p)/p = (nJ)^{-\gamma} p^{\gamma-1} = (nJ)^{-\gamma} y_s^{\sigma(\gamma-1)} = (nJ)^{-\gamma} (\alpha J)^{\gamma(\gamma-1)/(1+\sigma)} = \frac{1}{\alpha^\sigma (1-\gamma)/(1+\sigma) n^{\gamma} J^{(\sigma+\gamma)/(1+\sigma)}}
\]

Define \( \bar{n} = J(\bar{J}) \) as the implicit steady-state function defined from the firm’s profit recursion assuming the liquidity constraint binds. It can be shown that:

1. \( \bar{n} \to 0 \) as \( \bar{J} \to 0 \);
2. \( \bar{n} \to 0 \) as \( \bar{J} \to \infty \);
3. if \( z < 0 \) then \( \bar{n} > 0 \) for \( 0 < \bar{J} < \infty \).

These properties imply that the function \( J(\bar{J}) \) is non-monotonic. However, the second equation, as before features \( \bar{n} \) increasing with \( \bar{J} \). The non-monotonicity of the firm’s steady-state profit equation raises the possibility of multiple steady-state equilibria. Moreover, it is apparent that the slope of the profit functions are complicated expressions which raise the possibility of bifurcations not only in the number of steady-states but also the stability of steady-states.

C Equilibrium with Nash-bargained wage

In this Appendix, we consider an extension of the analysis in Section 5.4 by allowing for the wage to be determined endogenously via Nash bargaining between firms and workers. We also briefly discuss the effects of a Nash-bargained wage on the J-isocline (the n-isocline is unaffected).
In general, the bargained wage solves

\[ w = (1 - \alpha_L)w_0 + \alpha_L (z + k\theta) \]

where \( \alpha_L \) is the bargaining power. In the version of the model with just stocks, we can rewrite the wage equation as a function of \( n \) and \( J \) by using market clearing, the pricing relationship, and free entry:

\[ w(n, J) = (1 - \alpha_L)w_0 + \alpha_L \left[ \bar{z} + \frac{\sigma + \sigma nJ}{1 + \sigma} + k \left( \frac{J}{k} \right)^{1/\xi} \right] \]

Note that the wage depends positively on \( n \) and \( J \). From the second term in the bracket, higher \( nJ \) means greater stock market capitalization, which boosts the early-consumption price and productivity. Higher \( J \) also boosts hiring (and \( \theta \)) through the free entry condition. This effect raises wages since workers can find alternative jobs more easily should wage negotiations fail. The latter effect is standard in Mortensen and Pissarides (1994) but the former effect is novel due to the aggregate demand externality.

We consider the equilibrium with no credit or bonds \((\lambda = 0, A^\theta = 0)\) in which the wage is determined according to Nash bargaining. The set of equations governing the equilibrium path is:

\[
(r + \delta) J = \bar{z} + \max_y \{py - c(y)\} - w_1 + \dot{J} \\
w_1 = (1 - \alpha_L)w_0 + \alpha_L \left[ \bar{z} + \max_y \{py - c(y)\} + k\theta \right] \\
c^{-1}(p) = \frac{\alpha J}{p} \\
\rho - r = \alpha(1 - \lambda) \left[ v'(\frac{nJ}{p}) \frac{nJ}{p} - 1 \right]^+ \\
\dot{n} = m [1, \theta(J)] (1 - n) - \delta n
\]

We can reduce this system of equations by substituting the wage equation into the
firm’s Bellman:

$$(r + \delta)J = (1 - \alpha_L)(\bar{\varepsilon} + \max_y \{py - c(y)\} - w_0) + \alpha_Lk\theta + \dot{J}$$

The value of the firm now depends on the weighted average of the (endogenous) match surplus and hiring costs through market tightness. The market tightness depends implicitly on firm value $J$ through the free entry condition.

The endogenous wage affects the curvature of the $J$-nullcline. Suppose the levels of $n$ and $J$ are such that profits $z - w_1$ are the same in both cases. Then, as $J$ rises, the wage rises, taking $n$ as fixed. Thus, profits fall faster with the endogenous wage, which reduces job creation and hence the $n$ associated for a given $J$. However, for lower $J$, the endogenous wage falls below the exogenous wage $w_1$, so the $J$ nullcline can rise higher. Appendix (C) describes the equilibrium under Nash bargaining of wages.

D Calibration Details

The calibration targets are the replacement ratio of the unemployed of 0.4, the interest rate spread of 75 basis points, an annual frequency of liquidity shocks of 0.26, the job finding and separation rates, an annual interest rate of 4%, elasticity of marginal cost $\sigma = 0.2$, expenditure shocks relative to the wage $\epsilon$, semi-elasticity of the interest rate spread with respect to debt-to-GDP equal to $-0.746$, and the mean proportion of households with at least one unsecured credit card between 2000 and 2007 from the Survey of Consumer Finances, which was 74.7%.

In order to determine the targets $\epsilon$ and $\alpha$, we examine evidence on health expenditure shocks from the Federal Reserve Report on the Economic Well-Being of U.S Households in 2015. 26% of households in the sample experienced a major out-of-pocket health expense and the mean of that expense was $2,383.\textsuperscript{17} To obtain the wage figure in the data, we take hourly wages for 2015, multiply them by mean weekly hours in 2015, and multiply them by 52 to obtain the annual wage. However, as the model frequency is monthly, we multiply the percentage of the wage spent by 12.\textsuperscript{18}

\textsuperscript{17}This figure excludes just one extreme outlier with an out-of-pocket expense of 1 million dollars.

\textsuperscript{18}The FRED codes are CES0500000003 and AWHAETP for average hourly wages and and weekly wages.
We obtain the job finding rates $e$ and separation rates $s$ using worker flows as in Shimer (2005). Next-period unemployment satisfy $U_{t+1} = U_t(1 - e_t) + U^s_{t+1}$, given newly unemployed $U^s_{t+1}$. We rearrange to isolate $e_t$. Given that job losers have on average half a month to find a new job before being recorded as unemployed, the newly unemployed satisfies $U^s_{t+1} = s_t(1 - U_t)(1 - 1/2)e_t$. Rearranging determines $s_t$.\textsuperscript{19} We find the series’ means, $\bar{e} = 41.5\%$ and $\bar{s} = 3.10\%$, and also back out the corresponding employment target: $n = e_t/(s_t + e_t) = 92.8\%$.

We start with an initial guess of government bonds relative to GDP $x = A^g/(nz)$ as well as labor income to GDP $w_1/z$. Using the latter, we back out the firm value relative to GDP from the steady-state Bellman equation:

$$J/z = (1 + r)/(r + \delta)(1 - w_1/z)$$

and the productivity level $z$:

$$z = \bar{z} + py_s - c(y_s)$$

$$= \bar{z} + \frac{\sigma}{1 + \sigma}py_s$$

$$= \bar{z} + \frac{\sigma}{1 + \sigma}\frac{\alpha py_b}{n}$$

$$= \bar{z} + \frac{\sigma}{1 + \sigma}\frac{\alpha}{n}ew_1$$

Dividing through by $z$ and rearranging yields

$$z = \bar{z}\left(\frac{1}{1 - \frac{\sigma}{1 + \sigma}\frac{\alpha ew_1/z}{n}}\right)$$

Upon obtaining $J/z$ and $z$, we back out $J, w_1$, and $A^g$. Note that, as $py_s = \frac{2}{n}\epsilon w_1$, we can find the price as

$$p = \left(\frac{\alpha \epsilon w_1}{n}\right)^{\sigma/(1+\sigma)}$$

In a liquidity-constrained steady state equilibrium, $py_b = \lambda py^* + (1 - \lambda)(nJ + A^g) =$ earnings, respectively.

\textsuperscript{19}We use series on aggregate unemployment (FRED code UNEMPLOY), aggregate employment (CE160V) the the aggregate number employed for less than five weeks (UEMPLT5).
\( w \epsilon, \) where \( \epsilon \) is the fraction of the wage devoted to health expenditure. Dividing through by \( nz \) and applying \( y^* = p^{-1/\gamma} \) yields

\[
\lambda p^{(\gamma - 1)/\gamma} / (nz) + (1 - \lambda)[J/z + A^g/(nz)] = (w_1/z)(\epsilon/n)
\]

Using this expression \( \lambda p^{(\gamma - 1)/\gamma} / (nz) \) and the expression for the price from above, we find

\[
p = \left[ (\alpha/n)(\lambda p^{(\gamma - 1)/\gamma} + (1 - \lambda)(nJ + A^g)) \right]^{\sigma/(\sigma+1)}
\]

Given \( p \), we solve for \( \gamma \) using (13):

\[
\frac{\rho - r}{1 + r} - \alpha(1 - \lambda) \left[ \left( \frac{(nJ + A^g)}{p} \right)^{-\gamma}/p - 1 \right]
\]

and recover the quantities \( y^* = p^{1/\sigma} \) and \( y^b = ny^*/\alpha \).

The loss is given by the market clearing differential \( \mathcal{L}_1 = py^* - (\alpha/n)(\lambda p^{(\gamma - 1)/\gamma}) + (1 - \lambda)(nJ + A^g) \). We find the zero of \( \mathcal{L}_1 \) with respect to \( w_1/z \). We obtain the vacancy posting cost \( k \) from \( k = Jq/(1 + r) \).

The final step is to determine the supply of government bonds \( A^g \). The loss function \( \mathcal{L}_2 \) takes \( x = A^g/(nz) \) as an argument and uses \( \mathcal{L}_1 \) to back out the remaining parameters. Given the full set of parameters, \( \delta, \gamma, z, w_1, w_0, \rho, \alpha, \sigma, k, \xi, A^g, \lambda \), we compute the loss as the difference in the semi-elasticity of the spread with respect to debt-to-GDP between the model and the data:

\[
\frac{\partial 100[\rho - r(A^g/(nz))]}{\partial \log(A^g/(nz))}
\]

We compute the model semi-elasticity using numerical differentiation and repeated application of the chain rule. The analogue in the data \(-0.746\), the empirical semi-elasticity computed by Krishnamurthy and Vissing-Jorgensen (2012) and reported in Table 1 of the article.\textsuperscript{20}

We solve for \( x \) by finding the root of \( \mathcal{L}_2 \). Given \( x \), we find the remaining parameters according to the steps listed above.

\textsuperscript{20} Their spread measure is the yield difference between Moody’s Aaa-rated long-maturity corporate bonds and Treasury bonds.
Additional details on the derivation of employment law of motion

We finally simplify the job finding probability \( e = m(1, \theta) \). First, combining the equilibrium expression for market tightness, \( J = (1 + r)k/q(\theta) \), with the job finding probability under the Den Haan Ramey Watson function, \( q = 1/(1 + \theta)^{1/\xi} \), we obtain

\[
\theta(n, J) = \left[ \left( \frac{J}{(1 + r(n, J))k} \right)^{\xi} - 1 \right]^{1/\xi}
\]

The job finding probability in terms of \( n, J \) is

\[
e(n, J) = \frac{\theta(n, J)}{(1 + \theta(n, J)^{1/\xi})}
\]

which can be written explicitly as

\[
e(n, J) = \left[ \left( \frac{J}{(1 + r(n, J))k} \right)^{\xi} - 1 \right]^{1/\xi}
\]

\[
\frac{J}{(1 + r(n, J))k}
\]

E  A monetary economy

In the main formulation of the model, households who are liquidity constrained have access a single asset, the mutual fund composed of stocks and government bonds. We focus on stock mutual funds, government bonds, and debt obligations as the assets for the following reasons. Stocks are a primitive given fundamental role of firms, and government bonds provide a policy instrument. Finally, probabilistic access to credit by consumers enables us to characterize the space between no-and-full commitment. Though the economy remains cashless, firms’ revenues and interest rates are endogenous and sensitive to the government supply of bonds.

A cashless economy is reasonable in other respects. First, Hu and Rocheteau (2013) establishes that fiat money is not essential in environments with Lucas trees. Moreover, Lagos (2010) studies a similar economy in which Lucas trees serve as the media of exchange to explain the equity premium puzzle. Second, the baseline economy endogenizes the real interest rate and firms’ revenue and links them to market
capitalization. The model nests Mortensen and Pissarides (1994) by shutting down the idiosyncratic preference shocks, and Bewley-Aiyagari by making firms indifferent between early and late production.

However, it is straightforward to incorporate fiat money into the model without disrupting its fundamental insights. In this Appendix we sketch out an extension where, depending on the size of the consumption shocks, households will choose to use fiat money or liquidate their other assets.

**Sketch of monetary extension**

We now add fiat money and government bonds to our economy. Fiat money grows at the gross growth rate, \( \mu_t = \frac{M_t}{M_{t-1}} \), through lump-sum transfers to buyers. There is a fixed supply, \( A^g \), of one-period real government bonds, where each bond pays off one unit of numeraire. The preference shock in the early-consumption period is an iid draw from the cumulative distribution \( F(\varepsilon) \). Whereas consumers can use fiat money for early consumption at no cost, there is a fixed cost \( \kappa \) of using liquid bonds and stocks. Moreover, households can liquidate all of their bond holdings but only a fraction fraction \( \psi \) of stocks on demand.

The problem of a buyer holding \( \omega_t \) units of wealth is:

\[
W_t(\omega_t) = \max_{x_t, \ell_{t+1}, a_{t+1}} \left\{ x_t + \beta V_{t+1}(a_{t+1}^s, a_{t+1}^g, a_{t+1}^m) \right\} \\
\text{s.t.} \quad \frac{a_{t+1}^s}{1 + r_{t+1}^s} + \frac{a_{t+1}^g}{1 + r_{t+1}^g} + (1 + \pi_{t+1}) a_{t+1}^m = \omega_t - x_t.
\]

The buyer’s portfolio in the second stage is now composed of three types of assets: real balances, \( a^m \), bonds, \( a^g \), and stocks, \( a^s \). Since they have different liquidity properties, assets offer generally different rates of return: \( 1/(1 + \pi_{t+1}) \) for fiat money, \( 1 + r_{t+1}^g \) for bonds, and \( 1 + r_{t+1}^s \) for stocks. Substituting \( x \) from (20) into (19),

\[
W_t(\omega_t) = \omega_t + \max_{x_t, \ell_{t+1}, a_{t+1}} \left\{ -\frac{a_{t+1}^s}{1 + r_{t+1}^s} - \frac{a_{t+1}^g}{1 + r_{t+1}^g} - (1 + \pi_{t+1}) a_{t+1}^m + \beta V_{t+1}(a_{t+1}^s, a_{t+1}^g, a_{t+1}^m) \right\}.
\]
The buyer’s value function in the AM is

\[ V_t(a^s_t, a^g_t, a^m_t) = \alpha \int \max_{\chi_t \in \{0, 1\}} \{ \varepsilon v(y_t) - \chi_t \kappa - p_t y_t \} dF(\varepsilon) + W_t(a^s_t + a^g_t + a^m_t) \]

s.t. \[ p_t y_t \leq a^m_t + \chi_t (a^g_t + \psi a^s_t) \]

where \( \chi_t = 1 \) if the buyer chooses to liquidate stocks and bonds and \( \chi_t = 0 \) otherwise. The buyer wants to consume early with probability \( \alpha \), in which case his marginal utility of consumption is determined by a draw from \( F(\varepsilon) \). The buyer’s surplus is reduced by the fixed cost \( \kappa \) if the buyer chooses to liquidate some bonds and stocks.

There is a threshold, \( \tilde{\varepsilon}_t \), such that:

\[ \max_{p_t y_t \leq a^m_t} \{ \tilde{\varepsilon}_t v(y_t) - p_t y_t \} = \max_{p_t y_t \leq a^m_t + a^g_t + \psi a^s_t} \{ \tilde{\varepsilon}_t v(y_t) - p_t y_t \} - \kappa. \]

For all \( \varepsilon \leq \tilde{\varepsilon}_t \) the buyer uses cash as means of payment whereas for all \( \varepsilon > \tilde{\varepsilon}_t \) the buyer uses both cash and stocks. The first-order conditions of the buyer’s portfolio problem are:

\[ i_t = \alpha \int_0^{+\infty} \left\{ \frac{\varepsilon v'((a^m_t + \chi_t (a^g_t + \psi a^s_t))/p_t)}{p_t} - 1 \right\}^+ dF(\varepsilon) \tag{21} \]

\[ \frac{\rho - r^g_t}{1 + r^g_t} = \alpha \int_{\tilde{\varepsilon}_t}^{+\infty} \left\{ \frac{\varepsilon v'[(a^m_t + a^g_t + \psi a^s_t)/p_t]}{p_t} - 1 \right\}^+ dF(\varepsilon) \tag{22} \]

\[ \frac{\rho - r^s_t}{1 + r^s_{t+1}} = \alpha v \int_{\tilde{\varepsilon}_t}^{+\infty} \left\{ \frac{\varepsilon v'[(a^m_t + a^g_t + \psi a^s_t)/p_t]}{p_t} - 1 \right\}^+ dF(\varepsilon) \tag{23} \]

where \( \{x\}^+ = \max\{x, 0\} \). The choice of real balances as given by (21) equalizes the expected marginal value of money in all trades to the cost from holding money relative to stocks. Let \( i^g \) denote the nominal interest rate on government bonds, and \( i^s \) the nominal interest rate on stocks. We have

\[ \frac{\rho - r^g_t}{1 + r^g_t} = \frac{i_t - i^g_t}{1 + i^g_t} \quad \text{and} \quad \frac{\rho - r^s_t}{1 + r^s_{t+1}} = \frac{i_t - i^s_t}{1 + i^s_t}. \]

So, (22) and (23) define the interest rate differential between government bonds and illiquid bonds and, stocks and illiquid bonds.

\[ i^s_t = \frac{(1 - \psi) i_t}{1 + \psi i_t} \]
The clearing of the AM goods market implies

\[ np_t c^{-1}(p_t) = \alpha \int_0^{+\infty} \min \left\{ p_t v^{-1} \left( \frac{p_t}{\bar{\varepsilon}} \right), \left( a_t^g + \psi a_t^s \right) \mathbb{1}_{\{\varepsilon \geq \bar{\varepsilon}_t\}} + a_t^m \right\} dF(\varepsilon), \]

where \( a_t^s = n_t J_t \) and \( a_t^g = A^g \). The left hand side is the aggregate supply of assets arising from the \( n \) firms. The right hand side is aggregate demand. Buyers with a preference shock less than \( \bar{\varepsilon}_t \) spend their real balances while buyers with a preference shock larger than \( \bar{\varepsilon}_t \) spend their real balances and some of their bonds and stocks. The rest of the model is similar to that of previous sections.

Suppose first that \( c(y) = y \) and \( y^s < \bar{\varepsilon} \). In this case, \( p_t = 1 \) and firms are indifferent between selling to early buyers or late buyers. Moreover, assume \( A^g = 0 \) and \( v = 1 \). Consider equilibria in which stocks do not pay a liquidity premium: \( r_t = \rho \). From (22) \( y = y^*_\varepsilon \) for all \( \varepsilon \geq \bar{\varepsilon}_t \) where \( \varepsilon v'(y^*_\varepsilon) = 1 \). The threshold for \( \varepsilon \) above which buyers liquidate stocks solves

\[ \kappa = [\bar{\varepsilon} v(y^*_\varepsilon) - y^*_\varepsilon] - [\bar{\varepsilon} v(a^m_\varepsilon) - a^m_\varepsilon]. \]

The threshold \( \bar{\varepsilon} \) increases with \( \ell \) and \( \kappa \). Real balances are determined by (21),

\[ i = \alpha \int_0^{\bar{\varepsilon}(a^m_\varepsilon)} [\varepsilon v'(a^m_\varepsilon) - 1] dF(\varepsilon). \]

Because \( r = \rho \) and \( p = 1 \) market capitalization, \( nJ \), is determined independently of \( \ell \). The condition for this equilibrium to occur is \( nJ \geq y^*_\varepsilon - \ell \). The buyer’s total wealth is large enough to finance the early consumption for the largest value of \( \varepsilon \). If this condition does not hold, then \( r \) falls below \( \rho \) and stocks pay a liquidity premium.

Suppose next that buyers receive a high preference shock, \( \varepsilon_H \), with probability \( \alpha_H \), a low preference shock, \( \varepsilon_L < \varepsilon_H \), with probability \( \alpha_L \), and no preference shock with complementary probability \( 1 - \varepsilon_H - \varepsilon_L \). Moreover, we consider equilibria where \( nJ + \ell < y^*_\varepsilon_H \) and we assume that \( \kappa \) is neither too low nor too large so that \( \varepsilon_L < \bar{\varepsilon} < \varepsilon_H \). Liquidity and interest rates are determined by:

\[ i = \alpha_L \{\varepsilon_L v'(a^m_\varepsilon) - 1\} + \alpha_H \{\varepsilon_H v'(\ell + nJ) - 1\} \]

\[ \frac{\rho - r}{1 + r} = \alpha_H \{\varepsilon_H v'(a^m_\varepsilon + nJ) - 1\}. \]
If \( i \) is not too large, \( a^m \geq y_{\varepsilon L}^* \), and aggregate liquidity is determined by \( i = \alpha_H \{ \varepsilon_H v'(a^m + nJ) - 1 \} \).

In this case the nominal interest rate on stocks is zero and the real interest rate is determined by \( r = -\pi/(1 + \pi) \). This equilibrium corresponds to a liquidity trap. If \( i \) is sufficiently large so that \( \ell < y_{\varepsilon L}^* \), then the nominal interest rate on stocks is positive and is affected by monetary policy.

Finally, if \( A^g > 0 \) and \( \psi < 1 \), then there exists liquidity trap equilibria where \( i^g = 0 \) and \( i_t^* = (1 - \psi)i_t/(1 + \psi i_t) > 0 \). The value of money solves:

\[
\frac{i - i^s}{v(1 + i^s)} = i = \alpha_H \{ \varepsilon v'(a^m + A^g + \psi nJ) - 1 \}.
\]

Note, however, that such liquidity trap equilibria do not exist if the distribution is continuous because it would require \( \tilde{\varepsilon} v'(a^m/p) = 1 \), which is inconsistent with \( \kappa > 0 \).

Evaluating (24) requires us to establish whether the consumer is constrained given an \( \varepsilon \) shock and for what shock an individual chooses to liquidate bonds and stocks. The following help us characterize the demand for assets.

**Lemma 4** There is an interval \([0, \varepsilon_m]\) in which an individual can reach the first best using just money and an interval \((\varepsilon_m, \varepsilon_s]\) in which an individual can reach the first best using stocks and bonds but not money alone.

**Proof.** From the Inada condition on \( v(\cdot) \), as \( \varepsilon \to 0 \), \( pv^{\varepsilon-1}(p/\varepsilon) \to 0 < a^m \). For \( \varepsilon \) sufficiently large, \( pv^{\varepsilon-1}(p/\varepsilon) > a^m \). By continuity, there is a value \( \varepsilon_m \) such that \( pv^{\varepsilon-1}(p/\varepsilon_m) = a^m \), which satisfies \( \varepsilon_m = p/v'(a^m/p) \). By a similar argument, there is an \( \varepsilon_s \) such that an individual can just afford the first best using stocks and bonds, given by \( \varepsilon_s = p/(v'(a^m + A^g + \psi nJ)/p)) \). It immediately follows that \( \varepsilon_s > \varepsilon_m \). \( \blacksquare \)

**Lemma 5** \( \varepsilon_m < \tilde{\varepsilon} \). The preference shock at which an individual ceases to be able to finance the first best using cash alone is strictly below the threshold at which the individual chooses to liquidate stocks and bonds.

**Proof.** Suppose instead that \( \varepsilon_m \geq \tilde{\varepsilon} \). An individual in the range \([\tilde{\varepsilon}, \varepsilon_m]\) choose to liquidate stocks and bonds at cost \( \kappa \) but can afford the first best with cash. Since liquidating stocks and bonds does not change the consumption profile and imposes a cost, it is suboptimal. Hence, \( \varepsilon_m < \tilde{\varepsilon} \). \( \blacksquare \)
Lemmas 4 and 5 imply that there are only two possibilities: $\varepsilon_m < \varepsilon_s < \bar{\varepsilon}$, or $\varepsilon_m < \bar{\varepsilon} < \varepsilon_s$. In the first case, an individual liquidates stocks and bonds only after being unable to finance the first best even with stocks and bonds. In the second case, the individual liquidates stocks and bonds and is able to finance the first best until the preference shock rises sufficiently. The following lemma characterizes which case holds.

**Lemma 6** Provided that

$$\frac{p}{v'(a^m + \psi nJ + A^g)} < \frac{\psi nJ + A^g + \kappa}{v([a^m + \psi nJ + A^g]/p) - v(a^m/p)}$$

then $\varepsilon_m < \varepsilon_s < \bar{\varepsilon}$. Under CRRA preferences, (25) simplifies to

$$(a^m + \psi nJ + A^g)^\gamma < \frac{(\psi nJ + A^g + \kappa)(1 - \gamma)}{(a^m + \psi nJ + A^g)^{1-\gamma} - (a^m)^{1-\gamma}}$$

which depends only on holdings of liquid assets (independent of the price).

**Proof.** In order to check whether $\varepsilon_m < \varepsilon_s \leq \bar{\varepsilon}$, note that once the consumer liquidates stocks and bonds, then he will use all his assets. Accordingly,

$$\bar{\varepsilon}v(a^m/p) - a^m = \bar{\varepsilon}v([a^m + \psi nJ + A^g]/p) - (a^m + \psi nJ + A^g) - \kappa$$

Rearranging (27) for $\bar{\varepsilon}$ yields the right hand side of (25). The left hand side of (25) follows immediately from Lemma 4. Equation (26) results from substitution of the CRRA form. \[\blacksquare\]

**Further details in model with money**

**Pricing relationship from market clearing**

Market clearing implies that

$$npc^{\gamma-1}(p) = \alpha \int_0^\infty \min \{pv^{-1}(p/\varepsilon), \chi(a^g + \psi a^s) + a^m \} dF(\varepsilon)$$
If Case 1 holds, then \( \varepsilon_m < \varepsilon_s < \bar{\varepsilon} \) and

\[
np^{(\sigma+1)/\sigma} = \alpha \int_0^{\varepsilon_m} p \left( \frac{\varepsilon}{\varepsilon_s} \right)^{-1/\gamma} dF(\varepsilon) + \alpha \int_{\varepsilon_m}^{\bar{\varepsilon}} dF(\varepsilon) + \alpha \int_{\varepsilon_m}^{\varepsilon_s} l + \psi n J + A^g dF(\varepsilon)
\]

If Case 2 holds, then \( \varepsilon_m < \bar{\varepsilon} < \varepsilon_s \) and

\[
np^{(\sigma+1)/\sigma} = \alpha \int_0^{\varepsilon_m} p \left( \frac{\varepsilon}{\varepsilon_s} \right)^{-1/\gamma} dF(\varepsilon) + \alpha \int_{\varepsilon_m}^{\bar{\varepsilon}} dF(\varepsilon) + \alpha \int_{\varepsilon_m}^{\bar{\varepsilon}} p \left( \frac{\varepsilon}{\varepsilon_s} \right)^{-1/\gamma} dF(\varepsilon) + \alpha \int_{\varepsilon_m}^{\varepsilon_s} l + \psi n J + A^g dF(\varepsilon)
\]

Letting \( \varepsilon_{\text{max}} = \max\{\varepsilon_s, \bar{\varepsilon}\} \), we can express both cases as

\[
np^{(\sigma+1)/\sigma} = \alpha \int_0^{\varepsilon_m} p \left( \frac{\varepsilon}{\varepsilon_s} \right)^{-1/\gamma} dF(\varepsilon) + \alpha \int_{\varepsilon_m}^{\varepsilon_s} l + \psi n J + A^g dF(\varepsilon)
\]

We assume that the preference shock follows a Pareto distribution, so that

\[
F(\varepsilon) = 1 - \left( \frac{b}{\varepsilon} \right)^\lambda, \quad \varepsilon \geq b
\]

We use

\[
\int \varepsilon^{1/\gamma} dF(\varepsilon) = \int \varepsilon^{1/\gamma} \frac{\lambda b^\lambda}{\varepsilon^{1+\lambda}} d\varepsilon = \lambda b^\lambda \frac{\varepsilon^{1/\gamma-\lambda}}{1/\gamma - \lambda}
\]

so that

\[
np^{(\sigma+1)/\sigma} = \alpha b^\lambda \left[ \lambda p(\gamma-1)/\gamma \varepsilon_m^{1/\gamma-\lambda} - b^{1/\gamma-\lambda} + \varepsilon_m^{1/\gamma-\lambda} - \bar{\varepsilon}^{1/\gamma-\lambda} + \frac{l(\varepsilon_m - \bar{\varepsilon} - \varepsilon_{\text{max}})}{\bar{\varepsilon}^{1/\gamma-\lambda}} + (\psi n J + A^g) \varepsilon_{\text{max}}^{1/\gamma-\lambda} \right]
\]
Equation (28) implicitly defines the price $p(n, J, l)$ implicitly as a function of $n, J,$ and $l$.

**Firm revenue**

Firm revenue $z$ satisfies

$$z = \bar{z} + py_s - c(y_s)$$
$$= \bar{z} + y_s^{1+\sigma} - \frac{1}{1 + \sigma}y_s^{1+\sigma}$$
$$= \bar{z} + \frac{\sigma}{1 + \sigma}y_s^{1+\sigma}$$
$$= \bar{z} + \frac{\sigma}{1 + \sigma}p(n, J, l)^{\frac{\sigma+1}{\sigma}}$$

**Liquidation threshold $\bar{\varepsilon}$**

The liquidation threshold $\bar{\varepsilon}$ satisfies

$$\max_{py_s \leq a_m} \{\bar{\varepsilon}v(y) - py_s\} = \max_{py_s \leq a_m + a^g + \psi a^s} \{\bar{\varepsilon}v(y) - py_s\} - \kappa$$

In Case 1, $\varepsilon_m < \varepsilon_s < \bar{\varepsilon}$, then the first best is not attained even with stocks and bonds, and then $\bar{\varepsilon}$ satisfies

$$\bar{\varepsilon} \left(\frac{l}{p}\right)^{1-\gamma} - l = \bar{\varepsilon} \left[\frac{(l + \psi nJ + A^g)p}{1 - \gamma}\right] - (l + \psi nJ + A^g) - \kappa \iff$$

$$\bar{\varepsilon}[(l + \psi nJ + A^g)p]^{1-\gamma} - \bar{\varepsilon}(l/p)^{1-\gamma} = (\psi nJ + A^g + \kappa)(1 - \gamma)$$

so that

$$\bar{\varepsilon} = \frac{(\psi nJ + A^g + \kappa)(1 - \gamma)p^{1-\gamma}}{(l + \psi nJ + A^g)^{1-\gamma} - l^{1-\gamma}}$$

Otherwise in Case 2, $\varepsilon_m < \bar{\varepsilon} < \varepsilon_s$; once the household is indifferent between liquidating stocks or not, then liquidating suffices to finance the first best.

$$\bar{\varepsilon} \left(\frac{l}{p}\right)^{1-\gamma} - l = \bar{\varepsilon}v(y_{\bar{\varepsilon}}^*) - p y_{\bar{\varepsilon}}^* - \kappa$$
where \( y^*_t = (\tilde{\varepsilon}/p)^{1/\gamma} \). Substitution yields

\[
\tilde{\varepsilon}^{(l/p)^{1-\gamma}} - l = \frac{\tilde{\varepsilon}^{1/\gamma}}{p^{(1-\gamma)/\gamma}} \frac{\gamma}{1-\gamma} - \kappa
\]

**Portfolio problems**

Cash holdings generally satisfy

\[
i = \alpha \int_0^\infty \left\{ \varepsilon \frac{\psi'(l + \chi(a^s + a^g))/p}{p} - 1 \right\} dF(\varepsilon)
\]

\[
= \alpha \int_0^\infty \left[ \varepsilon(l + \chi(a^s + a^g))^{-\gamma p^{\gamma-1}} - 1 \right] dF(\varepsilon)
\]

In Case 1, \( \varepsilon_m < \varepsilon_s < \tilde{\varepsilon} \), the integral simplifies to

In Case 2 \( \varepsilon_m < \tilde{\varepsilon} < \varepsilon_s \), the integral evaluates to

\[
i = \alpha \int_{\varepsilon_m}^{\tilde{\varepsilon}} [\varepsilon(l)^{-\gamma p^{\gamma-1}} - 1] dF(\varepsilon) + \alpha \int_{\varepsilon_s}^{\tilde{\varepsilon}} [\varepsilon(l + nJ + A^g)^{-\gamma p^{\gamma-1}} - 1] dF(\varepsilon)
\]

\[
= \alpha b^\lambda \left[ \frac{\lambda}{1-\lambda} l^{-\gamma p^{\gamma-1}} (\tilde{\varepsilon}^{1-\lambda} - \varepsilon_m^{1-\lambda}) + \tilde{\varepsilon}^{-\lambda} - \varepsilon_m^{-\lambda} \right]
\]

\[
+ \alpha b^\lambda \left[ \frac{\lambda}{1-\lambda} (l + \psi nJ + A^g)^{-\gamma p^{\gamma-1}} (-\varepsilon_s^{1-\lambda}) - \varepsilon_s^{-\lambda} \right]
\]

We can encompass both cases as follows:

\[
i = \alpha b^\lambda \left[ \frac{\lambda}{1-\lambda} l^{-\gamma p^{\gamma-1}} (\tilde{\varepsilon}^{1-\lambda} - \varepsilon_m^{1-\lambda}) + \tilde{\varepsilon}^{-\lambda} - \varepsilon_m^{-\lambda} + \frac{\lambda}{1-\lambda} (l + \psi nJ + A^g)^{-\gamma p^{\gamma-1}} (-\varepsilon_{max}^{1-\lambda}) - \varepsilon_{max}^{-\lambda} \right]
\]

Similarly, bond holdings satisfy

\[
\rho - r^g = \alpha b^\lambda \left[ \frac{\lambda}{1-\lambda} (l + \psi nJ + A^g)^{-\gamma p^{\gamma-1}} (-\varepsilon_{max}^{1-\lambda}) - \varepsilon_{max}^{-\lambda} \right]
\]

and stock holdings satisfy

\[
\rho - r^s = \psi \frac{\rho - r^g}{1 + r^g}
\]

With these sets of equilibrium conditions, one can proceed as in the main text.