A theory of equality and growth*

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February 14, 2024

Abstract

We provide a general framework to study labor mobility, inequality, and aggregate growth. The non-parametric framework encompasses a large class of models and the link between labor mobility frictions and aggregate growth is fully characterized by the well-known Theil inequality index. We also provide parametric decompositions of misallocation and inequality and a general equilibrium model with heterogeneous occupations in an endogenous production network model where workers allocate between sectors and occupations following a nested Roy-Fréchet model. We quantify the decomposition and the size of the various mechanisms in the GE model using data on labor markets and input-output linkages for the US between 1980 and 2019. Misallocation increased over time suggesting a worse allocation in occupations as workers disproportionately allocate towards sectors with lower income shares. Finally, we estimate the maximum cost of a policy that would reallocate workers across occupations while improving aggregate output. This investment ceiling is large, and has grown markedly over time: from 2% of real GDP in 1980 up to 8% in 2019. Any smaller investment would imply a decrease in misallocation and inequality while still being efficiency-improving. This provides a rationale to invest where aggregate gains are the largest and therefore where workers are worst off initially.

Keywords: Income inequality, production networks, wage gaps, mobility of workers.

JEL Codes: D24, D33, D50, D57, D63, E24, J31.

*We would like to thank David Baqaee, Bart Capéau, Hierro Castro, Rafael Dix-Carneiro, Jan Eeckhout, Jozef Konings, Mathieu Leduc, Ernest Liu, Domenico Moramarco, Mathieu Parenti, Julien Sauvagnat, Clément Staner and Ezra Oberfield for enriching discussions and comments, the participants at the National Bank of Belgium Colloquium 2022, the KULeuven-Louvain-la-Neuve trade workshop, the Tinbergen institute seminar, the Annual US Conference on Network Science and Economics, the Oxford university seminar and at the Meeting of the Society for the Study of Economic Inequality. Bastien Bernon gratefully acknowledges financial support from FNRS (FRESH grant). Glenn Magerman gratefully acknowledges financial support from BELSPO (B2/233/P3/HAIOPOLICY).Bastien Bernon: bastien.bernon@ulb.be. Glenn Magerman: glenn.magerman@ulb.be (corresponding author).

1 Introduction

Inequality is a central concern for policymakers and citizens around the world. Understanding better how inequality emerges and evolves in market economies is therefore a central subject to be studied and better understood. In this paper, we demonstrate that in a very large class of growth accounting models, more production is linked to less inequality through the allocation of workers across tasks. Once workers' allocation is improved, the economy reaches a point where both efficiency and equity are improved. We then provide several theoretical and empirical results on how efficiency, equity and workers' allocation are linked in complex economies.

We start with an envelope theorem result with very few assumptions on the production and consumption sides that provides a positive relationship between equity and efficiency in a very large class of economic models where workers are imperfectly mobile across labor tasks (we first use the word task interchangeably for occupation, sector or skill). This result follows from the first welfare theorem implying that for the same factor of production, the efficient allocation is the one equalizing prices across allocations. The key observation is that if workers do not only allocate given their expected wage but also given their preferences and the implied mobility or training cost to work in the task, then the workers' allocation is inefficient and wage inequality across tasks is an indicator of this inefficiency.

Building on this first result, we then study several parametric and non-parametric implications for growth accounting, policy cost-benefit analysis, social welfare functions, inequality indices, misallocation measurement and TFP mis-measurement. First, assuming that indeed workers are imperfectly mobile across tasks, we provide a direct negative link between growth and the Theil inequality index which appears naturally from making only two general assumptions, namely that the production functions in the economy feature constant returns-to-scale and that final demand is homothetic. For this first set of results, we take the allocation as exogenously given, making no assumptions on its determination. Measuring misallocation using this exogenous allocation implies assuming that workers are homogeneous. A strong assumption that we relax later by assuming that they differ in preferences or mobility costs.

Second, we use this result to estimate the size of the joint gain in efficiency and equity from improving workers' allocation. We do not specify a policy that would improve this allocation, but we estimate the maximum level of resources that could be invested in such a policy while still being beneficial in terms of efficiency and equity. If improving workers' allocation allows to increase real GDP by 100$ while decreasing inequality, then a social planner that values both equity and efficiency would decide to invest up to 100$ in this policy while funding it through non-distortive lump-sum taxes. We then estimate this maximum cost in the empirical section of the paper.

Third, a direct consequence of the alignment between equity and efficiency is that social welfare functions are redundant. Indeed, if the policy cost to reach perfect equality is smaller than or equal to the real GDP gains from improving workers' allocation, then any social welfare function valuing real GDP and equality can be reduced to a welfare function maximizing only real GDP. We provide this theoretical result and its consequences for welfare analysis.

Finally, we study parametric economies, using Cobb-Douglas and CES production functions, and we analyze the relationship between inequality, misallocation, and productivity in these frameworks. Several key results emerge. First, we provide a positive inequality index which is jointly an inequality index, respecting the classical inequality indices' properties, and a measure of efficiency. We show how this measure is related to misallocation and how the measure of inequality evolves as its consequences on efficiency differ, giving an inequality index directly derived from efficiency concerns. Second, we provide several results on the link between our positive measure of inequality and misallocation and the existing measures of productivity provided by (Olley and Pakes, 1996) and (Melitz and Polanec, 2015). We show that our measure is exactly decomposable in the same way as the productivity measures and in other ways that we also provide. Furthermore, these decompositions allow to provide an additional margin of growth accounting decomposition considering the entry and exit of occupations through time.

In the second part of this paper, we study a non-parametric and a parametric general equilibrium model featuring jointly multiple factors of production and occupations, multiple sectors interlinked through an endogenous production network, arbitrary elasticity of substitution across factors of production, intermediate goods and occupations within each sector, multiple households with different incomes and endogenous allocation of workers across occupations and sectors following a nested Roy-Fréchet model. The non-parametric general equilibrium model provides all the aggregate and inequality results holding for any economy with constant returns-to-scale production functions, homothetic final demand, and constant returns-to-scale workers' allocation functions. These results are the basis of our study in the parametric general equilibrium model that we calibrate using nested CES production functions similar to (Baqaee and Farhi, 2019) on the production side and with heterogenous workers supplying their labor across occupations and sectors given their preferences and mobility costs drawn from a nested Fréchet probability distribution on the consumption and worker side. Using this model, we provide different non-parametric decompositions of the impact of shocks on inequality and aggregate values when workers are imperfectly mobile. Then we turn to the parametric model and study polar results to delineate the mechanisms occurring in the model before moving to the general characterization that we use in our comparative statics exercises.

First, in the non-parametric GE model, we provide comparative statics for the wages in each sector and occupation. When a productivity shock occurs, the sectors within the production network adjust their demand for all types of labor (labor demand channel). This first channel is composed of a substitution and a scale effect. The substitution effect captures how sectoral and occupational labor use adjusts in response to a productivity shock, as prices for goods and labor in all sectors and occupations react to the shock. The scale effect captures how the change in output in that sector modifies the demand for labor in that sector. Intuitively, as the sector grows in size, the demand for the types of labor used in that sector increase accordingly.

In response to these adjustments, workers adjust their supply to each labor type conditional on their preferences and mobility/training costs (labor supply channel). The reallocation towards an occupation in a specific sector depends on the change in wage of that specific type of labor relative to the preference/cost-weighted change in wages in all sectors and occupations. If labor mobility is low, productivity shocks trigger stark changes in income inequality, as workers cannot move and need to take the wage shock resulting from the productivity shock as given. Conversely, high labor mobility dampens the effect of wage inequality as people move more freely across sectors and occupations in response to the shock.

Finally, the aggregate real GDP of the economy changes in equilibrium following the shock (aggregate channel). The first two channels generate inequality changes while real GDP changes only impacts welfare. We are able to separate the impact of these different channels on each wage both in our non-parametric and in our parametric GE models and further decompose each of these channels between different effects using only observables, we name this first exercise the wage decomposition. Using only the parametric model, we are also able to separate the impact of the within-sector shock from the shocks in other sectors (direct vs indirect decomposition), the within-labor type shock from the shocks in other labor types (labor direct vs indirect decomposition) and the direct impact of the shock from the impact of the endogenous price changes (shock vs GE decomposition).

Second, considering calibrated models, we first document an inequality-neutral result to build further intuition on inequality formation. In a benchmark Cobb-Douglas production network and consumption economy, there is only an effect of the real GDP change, independent of the amount of labor mobility. Indeed, while real wages change in accordance with the shift in GDP, all wages move in tandem, resulting in no changes in income inequality in response to a shock. The intuition is that labor use and wages offset each other perfectly to keep labor shares in production constant. In general however, shocks are not inequality-neutral, and are governed by the sign and size of the other two channels.

To further highlight the functioning of the channels in our model, we provide results for some simple economies that differ in their input-output structure and in workers' mobility. In a horizontal economy, households provide labor to one of the sectors which is the single input of production, and all sectors sell to final demand. In this structure, there is only a scale effect, and no substitution effect: the relative change in wages in two sectors is only determined by the final demand elasticity, and how consumers reallocate their expenses across sectors in response to the productivity shock. In another example, households supply one type of heterogeneous labor to a single sector in the economy. In this roundabout economy, used in classic models of wage gaps (e.g. (D. H. Autor, Levy, and Murnane, 2003) and (Acemoglu and D. Autor, 2011)), one sector produces everything in the economy and uses all types of heterogeneous labor and its own output in production. In this special case, there is only a substitution effect, but no scale effect. The scale effect is moot as this one sector generates all output in the economy. Wage inequality appears because of a reallocation of the use of heterogeneous labor in input use. Finally, in a vertical economy, firms only sell to downstream producers up to final demand, combining labor with upstream inputs. In this case, both substitution and scale effects shape income inequality. We then turn to two additional cases where workers are imperfectly mobile across sectors to illustrate the labor supply channel. Intuitively, all the results on substitution and scale hold, with a dampening effect on income inequality which is governed by the structure of workers' decision-making in terms of allocation.

We conclude this parametric GE model section by presenting the general characterization of the model that we calibrate using US data. To understand the amplitude of all the theoretical results presented above, we use the IPUMS census data from the US and the BEA input-output data to compute all the parametric and non-parametric results provided in the first section of the theory and to calibrate our general equilibrium model and proceed to some comparative statics analysis when shocks occur in this economy.

First, we study the non-parametric envelope theorem result and we find that the first and second order approximations of the workers' allocation impact on real GDP are of crucial importance for the total impact. The first order approximation of the workers' allocation impact, which only considers change in wages while keeping income shares fixed, gives a positive and strong effect of workers' allocation on real GDP of approximately 10% of real GDP growth per decade. This means that between 1980 and 2019, the relative wages in more important sectors have actually decreased, implying an improved workers' allocation and a positive effect on real GDP. Because we know that wage inequality increased in that period, this comes as a surprise as we would anticipate a deterioration of workers' allocation, implying an increase in wage inequality and a negative effect on real GDP. This is exactly what we observe when studying the second-order approximation. This second result comes from considering the adjustment of income shares of the different occupations jointly with the wage changes. This second effect entirely counters the first effect and pushes the total effect of workers' reallocation towards negative grounds. The amplitude of these two contradictory effects is massive, more than 10% of real GDP growth per decade, and growing in time. Nevertheless, the net effect of these two forces is much more negligible, around 1% of real GDP growth per decade.

Second, we study the parametric theoretical results using occupational data for the US. This allows us to study in levels the misallocation impact on real GDP and several decompositions of it. First, we observe that the misallocation increased in time, in accordance with the results for wage inequality across occupations, and that this increase is entirely due to a stark increase in the covariance between income shares and relative wages which reflects our second-order approximation result emphasized above. This result implies that the occupations with higher importance for real GDP had an increase in their relative wage through time, suggesting a worse workers' allocation where workers disproportionately allocate towards less important sectors as measured by income shares.

Moving to the analysis in changes, we exploit the (Melitz and Polanec, 2015) decomposition in our context to study changes in decomposition and the importance of entry and exit of occupations, which is a new exercise to the literature. We observe, as expected from the analysis in levels, that the change in covariance between income shares and relative wages lead the total changes in misallocation. The largest changes are concentrated in the two decades leading up to 2000 and decreases afterwards. Using the occupations codes from 1990, we find a limited effect from occupations' entry and exit on the general level of misallocation while we emphasize the sensitivity of these results to the occupations codes used as newer codes would virtually increase the number of entry while older codes would virtually increase the number of exits as occupations considered as deserving a code are time-dependent, and some occupations may not even exist yet.

Finally, we study our newly defined decomposition of misallocation changes which isolates the effect of wage changes, income shares changes and the joint effect. We find that the isolated income shares effect is crucial, echoing the second-order approximation result and the results in levels, and is as large as the combined effect of wage changes and joint changes. Finally, we estimate the maximum cost of policy that would be invested to reallocate workers across occupations. We find that this maximum cost is substantial and increasing in time between 1980 and 2019. Starting from 2% of real GDP in 1980, it moved up to an astonishing 8% of real GDP in 2019. Considering the large GDP growth experienced between 1980 and 2019, the maximum cost of this policy that would still allow it to be beneficial has been multiplied by more than 7. Starting from around 100 billion 1980 US$ in 1980 to more than 700 billion 1980 US$ in 2019. Any smaller amount of money that would be used in this type of policies would imply a joint increase in equity and efficiency. Two additional facts come make this case even stronger. First, we assume that the money invested in these policies is lost while if it is used to create employment for the reallocation of workers, this could also be beneficial to the economy as a whole. Second, there are different gains depending on the worker that is reallocated, and each gain is always perfectly aligned with the gain in equality. Indeed, if a worker from a badly paid occupation is trained to be reallocated in a well-paid occupation, the gain for real GDP and the gain in equality are aligned. This would be an efficient rationale to invest where the aggregate gains, both in efficiency and equity, are the largest and therefore where the people are the poorest.

Finally, we study the impact of different shocks on inequality and aggregate variables in our calibrated GE model conditional on the assumed workers' mobility. We observe that the misallocation effect tend to be as large, and sometimes bigger, than the linear effect of productivity shocks defined by (Hulten, 1978). Furthermore, these productivity shocks have large heterogeneous effect on inequality as measured by the standard deviation of real wage changes across occupations and industries. These effects also vary dramatically as workers' mobility change.

2 Related literature

We participate to several literatures at the conjunction between economic growth, wage inequality and the different ways these matters are intertwined, theoretically and empirically, through imperfect workers' mobility.

First, we consider differently the equity-efficiency trade-off which was initially developed in the public finance literature where seminal papers include (Diamond and Mirrlees, 1971) and (Anthony Barnes Atkinson and Stiglitz, 1976) and more recently (Saez and Stantcheva, 2016). In these works, the authors consider the relative importance of growth and inequality when considering public policies such as distortive taxation. Our work also relates to the consideration of equitable allocation of resources across agents and their relationship with welfare theorems and Pareto efficient allocations to which (Dworczak, Kominers, and Akbarpour, 2021) contributed recently.

Second, we participate to the several literatures linking growth accounting and misallocation. On the growth accounting side, we extend a very large class of growth accounting models including the seminal works of (Solow, 1956) and (Hulten, 1978) and the more recent work of (Baqaee and Farhi, 2020) by introducing imperfect workers' mobility and its impact on growth and inequality. On the misallocation side of the literature, one of the first author to consider workers' misallocation is (Kuznets, 1961) specifically considering the allocation of workers between rural and urban areas. This work was developed in a very large strand of literature in development economics but only started to be exploited in developed economies when considering the allocation of workers to firms in (Hsieh and Klenow, 2009). Their results were further generalized in endogenous network economies by (Baqaee and Farhi, 2020). Finally, our results also relate closely to the work of (Hsieh, Hurst, et al., 2019) where the authors use a GE model to estimate the impact of talent misallocation due to discrimination on GDP growth in the US. Our work is also more indirectly linked to the large literature considering human capital accumulation and its impact on growth where seminal papers include (Romer, 1986) and (Lucas Jr, 1988).

Third, our paper relates to the works studying the relationship between inequality and growth both theoretically and empirically. In the macro literatures, different types of models have attempted to explain the correlation and causality between economic growth and income inequality observed at different stages of development on the path opened by the well celebrated (Kuznets, 1955) curve. Several papers explained that more inequality would lead to more demand for distortive redistributive policies and therefore less efficiency (Alesina and Rodrik, 1994), other papers considered the importance of capital accumulation for growth and therefore the necessity of a minimum level of inequality (Kaldor, 1957). Finally, other papers considered the link between inequality, social instability and economic growth (Benhabib and Rustichini, 1996). On the other hand, several empirical works have emphasized both positive and negative correlations between economic growth and inequality through time (see e.g. (Forbes, 2000) and (Banerjee and Duflo, 2003)). The novelty of our work is to consider inequality and growth to be negatively correlated due to the negative causal impact of misallocation on growth where the measure of inequality is simultaneously a measure of misallocation. Other authors considered the link between the income and wealth distribution and economic growth through risk heterogeneity and imperfect capital markets (Bewley, 1986), (Huggett, 1993) and (Aiyagari, 1994). These works ultimately led to the more recent models of heterogeneous agent new Keynesian models (Kaplan, Moll, and Violante, 2018) where the link between heterogeneous incomes and marginal propensity to consume and growth were emphasized in (Bilbiie, 2020) and (Auclert and Rognlie, 2018) in the pure spirit of the Keynesian multiplier (Keynes, 1936) where more consumption fuels income growth while savings slows it down. We abstract from these last considerations by assuming a representative final consumer.

Fourth, considering wage inequality in general equilibrium models, several seminal papers in labor economics, studied the relationship between skill demand and supply and wage gaps (see e.g. (D. H. Autor, Levy, and Murnane, 2003), (Acemoglu and D. Autor, 2011) and (D. H. Autor, 2014)) but also occupational mobility and wage inequality (Kambourov and Manovskii, 2009). On the other hand, several more recent papers studied the impact of automation shocks on income inequality both when tasks are allocated to demographic groups through GE effects (Acemoglu and Restrepo, 2022) and in production networks (Jackson and Kanik, 2019). Finally, (Huneeus, Kroft, and Lim, 2021) developed a GE production network model based on the work of (Lamadon, Mogstad, and Setzler, 2022) on the labor side to isolate the effect of exposure to shocks through the firm-to-firm network from the other determinants of workers' earnings. The framework developed in the present paper is closer to the framework presented in (Baqaee and Farhi, 2020) in its attempt to incorporate general and non-parametric components of GE models but with the added intuition provided from workers' imperfect mobility.

Fifth, we also relate to the literature in trade which studies how imperfect workers' mobility impacts the gains from trade. This literature takes imperfect workers' mobility and costs to move as given and attempts to explain how gains from trade are impacted by these constraints (see e.g. (Dix-Carneiro, 2014) and (Galle and Lorentzen, 2021) among several other works).

Finally, two strands of literature to which we relate consider the measurement of inequality and productivity. On the one hand, several papers suggested inequality indices based on properties such as in (Theil, 1967), (Anthony B Atkinson, 1970) and (Shorrocks, 1982). Here we suggest an inequality index that respects the classical properties suggested in these papers while being based on efficiency concerns which was not yet considered to our knowledge. Second, our paper relates to productivity measurement and decomposition developed in (Olley and Pakes, 1996) and (Melitz and Polanec, 2015) among others which we directly link to our measures and decompositions of misallocation and inequality.

3 Growth accounting

3.1 Envelope theorem

We consider a general production function with two factors of production, labor L and capital K, where households supply their labor to different tasks $t \in \mathcal{T}$:

$$Y = zF(l_1,...,l_{\mathcal{T}},\bar{K}) \quad with \quad \sum_{t\in\mathcal{T}} l_t = \bar{L}$$

where Y is real GDP, z is the productivity shifter, F is a production function with constant returns-to-scale, l_t is the number of households/workers allocated to task t and $\mathcal{T}$ is the set of tasks in the economy. A tasks can be an occupation in a specific sector requiring a specific diploma for example. We give more structure to these tasks in the general equilibrium model and in the empirics section. We solve the following value function for real GDP when the social planner faces restrictions to the workers' allocation between tasks:

$$Y\left(z,\bar{L},\bar{l}_1,...,\bar{l}_{\mathcal{T}},\bar{K}\right) = \max_{\{l_t\}_{t\in\mathcal{T}}} zF(l_1,...,l_{\mathcal{T}},K) - \tau_K\left(\bar{K}-K\right) - \sum_{t\in\mathcal{T}}\mu_t\left(\bar{l}_t - l_t\right)$$

where $\bar{l}_t$ is the imposed labor allocation for task t, $\frac{\partial F}{\partial l_t}$ is equal to w_t in a competitive equilibrium, the average marginal productivity $\sum_{t\in\mathcal{T}}\left(\frac{l_t}{\bar{L}}\right)\frac{\partial F}{\partial l_t}$ is equal to mean wage $\tilde{w}$, τ_K is the lagrange multiplier of capital and the vector μ corresponds to the lagrange multipliers of the mobility constraints.

This maximization is trivial because the allocations are already given as a constraint in the problem. Nevertheless, it can still teach us something about workers' misallocation. Because the vector μ is the vector of the shadow prices of the imposed workers' allocation on Y, the value of these Lagrange multipliers tell us what happens to Y when the constraints on workers' allocations change.

At the competitive equilibrium, we have that $\mu_t = w_t$. If the imposed allocation corresponds to the unconstrained allocation of workers across tasks, the Lagrange multipliers are all equal to the unique wage $\tilde{w}$. In that case, changing the vector of imposed allocations lowers Y as marginal productivities across tasks would not be equal anymore. Some tasks would feature greater marginal productivities than others. This means that the movement of a worker from the less productive to the more productive task increases real GDP, implying a misallocation of workers and a non-null distance to the efficiency frontier.

However, if the imposed allocation is not equalizing marginal productivities of workers across tasks, a change in imposed allocations might increase or decrease Y conditional on the new distribution of imposed allocations. If the new imposed allocations decrease differences in marginal productivities across tasks, then Y would increase. On the other hand, if the new vector of allocations increases differences in marginal productivities across tasks, then Y would decrease. Take the first-order change across the initial equilibrium:

$$d\ln Y = d\ln z + \left(\frac{rK}{GDP}\right)d\ln\bar{K} + \sum_{t\in\mathcal{T}}\left(\frac{w_t l_t}{GDP}\right)d\ln\bar{l}_t$$

where we take the price index P as the numeraire and therefore $Y = GDP$. Rearranging terms and showing wages in this equation using the dual approach between quantities and prices allows to recover the intuition presented above:

$$d\ln Y = \underbrace{d\ln z}_{\text{Technology}} + \underbrace{\left(\frac{\tilde{w}L}{GDP}\right) d\ln \bar{L}}_{\text{Labor supply}} + \underbrace{\left(\frac{rK}{GDP}\right) d\ln \bar{K}}_{\text{Capital supply}} - \underbrace{\left(\frac{wL}{GDP}\right) \sum_{t\in\mathcal{T}} \left(\frac{l_t}{L}\right)\left(\frac{w_t}{\tilde{w}}\right) d\ln\left(\frac{w_t}{\tilde{w}}\right)}_{\text{Inequality}}$$

where $\frac{w_t l_t}{\tilde{w}L}$ is the share of labor income accrued to workers allocated to task t,
$\tilde{w} = \sum_{t\in\mathcal{T}} \left(\frac{l_t}{L}\right) w_t$ is the average wage in the economy and the inequality change corresponds to the change in workers' allocation across tasks. Intuitively, an increase of the wedge between the task-specific marginal productivity w_t and the average marginal productivity $\tilde{w}$ reflects an increasing misallocation of workers.
We now demonstrate that this change in inequality corresponds to a first-order change in the first Theil index which is a well-known inequality index. Start with the definition of the Theil index in our setting:

Definition 1 (Theil index) *The first Theil inequality index is defined as follow:*

$$I = Q(\mathcal{H}) - Q(\check{\Theta})$$

where I is the first Theil index, $Q(\mathcal{H})$ is the maximum entropy in the distribution of income shares given by equal earnings to each household which implies an income share for each task that would be proportional to the share of households that is allocated to this task. The distribution of these shares of workers' allocation is given by $\mathcal{H}$. Then, we have that $Q(\check{\Theta})$ is the current level of entropy given the distribution of labor income shares $\check{\Theta}$ (where the labor income share of households supplying labor type i is given by $\check{\Theta}_t = \frac{\Theta_{t,L}}{\Theta_L}$). Replacing these entropy levels by their value in our setup, we have that:

$$\begin{aligned} I &= \sum_{t\in\mathcal{T}} \left(\frac{1}{\bar{L}}\right) \ln\left(\frac{\bar{L}}{1}\right) - \sum_{t\in\mathcal{T}} l_t \left(\frac{1}{\bar{L}}\frac{w_t}{\tilde{w}}\right) \ln\left(\frac{\bar{L}}{1}\frac{\tilde{w}}{w_t}\right) \\ &= \ln \bar{L} - \sum_{t\in\mathcal{T}} \left(\frac{\Theta_t}{\Theta_L}\right) \ln\left(\frac{\bar{L}}{1}\frac{\tilde{w}}{w_t}\right) \\ &= \sum_{t\in\mathcal{T}} \left(\frac{\Theta_t}{\Theta_L}\right) \ln\left(\frac{w_t}{\tilde{w}}\right) \end{aligned}$$

Labelling the wage ratio of households working in labor task t with respect to average wage $\Gamma_t = \frac{w_t}{\tilde{w}}$, we have that the Theil index is equal to the entropy of wage ratios while taking the labor income shares as probability weights:

$$I = \sum_{t\in\mathcal{T}} \left(\frac{\Theta_{t,L}}{\Theta_L}\right) \ln\left(\frac{w_t}{\tilde{w}}\right) = Q\left(\check{\Theta}, \Gamma\right)$$

With this result in mind, we develop the following measure of a change in income inequality:

Proposition 1 (First-order change in Theil index) *Start by taking the following equation:*

$$d\,I = d\,Q(\mathcal{H}) - d\,Q(\check{\Theta})$$

Then, we compute the change in the entropy by keeping the income shares weights as fixed

$$\begin{aligned} d\,I(\bar{\check{\Theta}}) &= \sum_{t\in\mathcal{T}} \left(\frac{\Theta_{t,L}}{\Theta_L}\right) d\ln \bar{L} - \sum_{t\in\mathcal{T}} \Theta_{t,L} d\ln\left(\frac{\bar{L}}{1}\frac{\tilde{w}}{w_t}\right) \\ &= \sum_{t\in\mathcal{T}} \left(\frac{\Theta_{t,L}}{\Theta_L}\right) d\ln\left(\frac{w_t}{\tilde{w}}\right) \end{aligned}$$

which corresponds to a change in income inequality in the envelope theorem as the income shares are also considered as fixed.

As a result, our model gives a simple expression of the relationship between growth and income inequality in the form of a semi-elasticity:

Corollary 1 (Relationship between growth and inequality) *In a competitive economy where workers are exogenously allocated to tasks, the semi-elasticity of real GDP with respect to income inequality is given as follow:*

$$\frac{d\ln Y}{d\,I(\bar{\check{\Theta}})} = -\Theta_L$$

where Θ_L is the share of factors' total income that is exogenously allocated.

3.2 Implications for growth accounting

Now, we transfer this envelope result to a growth accounting framework to understand better the implication of workers' imperfect mobility on growth. Take a constant returns-to-scale production function with one capital factor and $\mathcal{T}$ labor types:

$$Y = zF(l_1, ..., l_{\mathcal{T}}, \bar{K}) \quad with \quad \sum_{t\in\mathcal{T}} l_t = \bar{L}$$

where Y is the production of the economy's representative good, l_t is the t-th labor type, $\bar{K}$ is the capital factor and z is the Hicks-neutral productivity shifter. Production takes place in perfect competition with constant returns-to-scale. Hence, the price of the representative good in the economy is equal to its cost and is given by:

$$P = \frac{1}{z} C(w_1, ..., w_{\mathcal{T}}, r)$$

where P is the price of the representative good, C is its cost, w_t is the wage paid to workers supplying labor's type t and r is the cost of capital. To illustrate the point, take the following cases where workers are either perfectly mobile or immobile between workers' types.

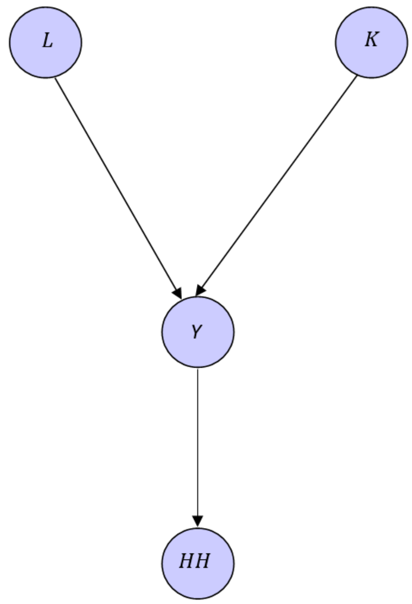


Figure 1: Perfect mobility

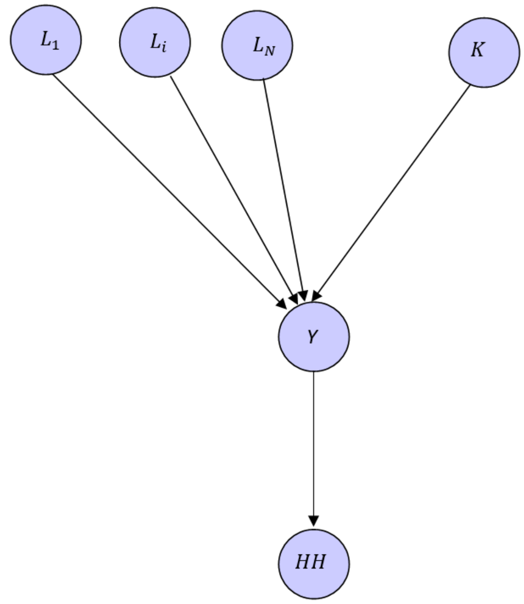


Figure 2: Perfect immobility

In the first case, we obtain a static growth accounting model with two factors of production, labor and capital (also sometimes called Solow model). In the second case, we have a slightly extended model with $\mathcal{T}+1$ factors of production, $\mathcal{T}$ labor types and one capital factor. In between these two extreme cases, workers are imperfectly mobile between labor types. The mobility of workers between labor types is determined by underlying frictions. In this simple growth accounting model, we assume that these frictions are exogenous, we relax this assumption in the general model. These frictions create a wedge between the perfect allocation of workers and the actual allocation of workers between labor types. We represent this case as follow:

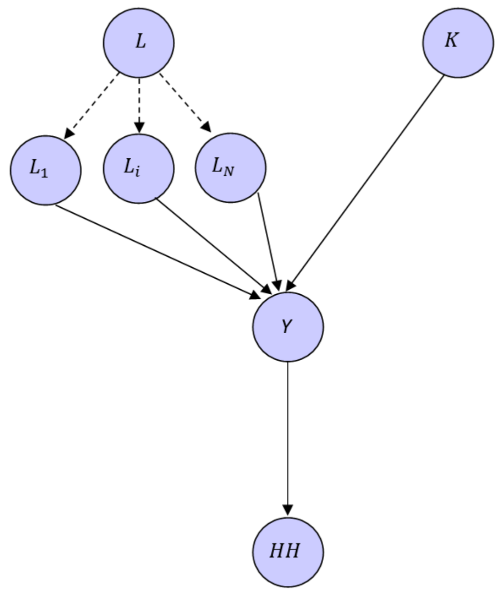


Figure 3: Imperfect mobility

where workers are part of the same working age population $\bar{L}$ but are only able to imperfectly move between labor types $(l_1, ..., l_{\mathcal{T}})$. We call the vector of the exogenous underlying frictions Γ (we explicit these frictions later). This simple extension of the Solow model gives the following change in real GDP Y:

$$d\ln Y = \underbrace{d\ln z}_{\text{Technology change}} + \underbrace{\Theta_L d\ln \bar{L} + \Theta_K d\ln \bar{K}}_{\text{Factor supply change}} \underbrace{- \Theta_L \sum_{t\in\mathcal{T}} \left(\frac{l_t}{\bar{L}}\right)\left(\frac{w_t}{\tilde{w}}\right) d\ln\left(\frac{w_t}{\tilde{w}}\right)}_{\text{Inequality change}}$$

where Θ_L is the share of labor in the GDP, Θ_K is the share of capital, $\tilde{w} = \sum_{t\in\mathcal{T}} \left(\frac{l_t}{\bar{L}}\right) w_t$ is the mean wage and the change in inequality is a change in labor income inequality. This result draws a direct line between fairness and efficiency in growth accounting. The only extension necessary in the Solow model is imperfect mobility of workers. This model can be generalized with multiple sectors, input-output linkages, arbitrary elasticities between factors, intermediates and types of labor and with endogenous imperfect workers' mobility. This is what we do in this paper.

3.3 A social welfare function irrelevance theorem

Take a social welfare function where welfare is increasing in individual's real incomes and decreasing in aggregate income inequality. Then we have that if real incomes are maximized when income inequality is minimized, we have that maximizing real incomes is necessary and sufficient to maximize welfare.

Proposition 2 (Social welfare function irrelevance theorem) *Following the classical Atkinsonian welfare function literature, take a welfare function increasing in individuals' real incomes and decreasing in aggregate income inequality:*

$$\mathscr{W}(\overset{+}{Y}, \overset{-}{I})$$

Now, by the demonstration above, we have that aggregate real income increases as income inequality decreases:

$$Y = G(\overset{-}{I})$$

Hence, if we proceed to the following maximization under constraint:

$$max\, \mathscr{W}(\overset{+}{G(\overset{-}{I})}, \overset{-}{I})$$

Hence, to maximize $\mathscr{W}$, I has to be minimized in both arguments of the maximization implying that there is no trade-off between real incomes and inequality and that welfare $\mathscr{W}$ is at its peak when inequality I is at its minimum. A dual statement is that $\mathscr{W}$ is maximized when only Y is maximized:

$$max\, \mathscr{W}(\overset{+}{Y}, \overset{-}{I}) \iff max\, Y$$

which proves the irrelevance. Hence, maximizing real incomes is necessary and sufficient to maximize the social welfare function.

Intuitively, this theorem means that if there are frictions to mobility in an economy, a benevolent government that wants to maximize real incomes and minimize inequality only needs to maximize real incomes as income inequality would be minimized at the same equilibrium. This maximum is reached when all constraints to workers' mobility are removed.

3.4 Distance to the frontier and no trade-off cost-benefit analysis

To understand better the potential gains from removing frictions, we provide a new measure of the distance to the efficiency frontier and an analytical framework to study the cost-benefits from removing frictions. We start with the distance to the frontier. The efficiency frontier is the point where the wage ratio is equal to 1. Hence, we have that:

$$\Delta = \ln\left(\frac{Y^*}{Y}\right) \approx -\Theta_L \sum_{t \in \mathcal{T}} \left(\frac{\Theta_{t,L}}{\Theta_L}\right) \left(\ln 1 - \ln\left(\frac{w_t}{\tilde{w}}\right)\right) = \Theta_L \sum_i \left(\frac{\Theta_{t,L}}{\Theta_L}\right) \ln\left(\frac{w_t}{\tilde{w}}\right) = \Theta_L \times I$$

where I is the first Theil index of the labor incomes in the economy as defined above. Hence, in this framework, the distance to the efficiency frontier is a direct mapping from a well-known income inequality measure.

Having established a no trade-off result in the previous section, we know that diminishing the frictions $\ln\left(\frac{w_t}{\tilde{w}}\right)$ allows to simultaneously increase real output and decrease income inequality. In the result section, we establish the maximum cost that could be invested in removing entirely the frictions while keeping real output fixed and removing entirely income inequality. We also provide results for the heterogenous gains from removing frictions depending on between which occupations the workers are moved.
To specify this maximum cost policy, assume that a government levy a lump-sum tax to fund the frictions' diminution and that the gain in allocation dY has to be greater than the government spendings G to diminish frictions:

$$dY > G \iff d\ln Y > G/Y = \tau Y$$

where τ is a lump-sum tax collected to fund the cost of removing frictions. Note that we assume here that the cost is entirely lost to remove frictions which implies that the policy would be a lower bound of the real output gains from the policy as workers could also be paid to move across regions or occupations, implying no loss in real output on the cost side but a gain in allocation improving real output.

3.5 Relation to (Baqaee and Farhi, 2020)

Compared to the existing literature, the misallocation we uncover here is independent from the one presented in (Baqaee and Farhi, 2020) and therefore both types of misallocation could be accounted for in a more general model (extending their model and extending ours) by introducing exogeneous markups:

$$d\ln Y = \underbrace{d\ln z}_{\text{Technology}} + \underbrace{\grave{\Theta}_L d\ln L + \grave{\Theta}_K d\ln K}_{\text{Factor supply}} - \underbrace{\sum_{t\in\mathcal{T}} \grave{\Theta}_{t,L} d\ln \Theta_{t,L}}_{\text{Allocative efficiency}} - \underbrace{\grave{\Theta}_L \sum_{t\in\mathcal{T}} \left(\frac{\grave{\Theta}_{t,L}}{\grave{\Theta}_L}\right) d\ln\left(\frac{w_t}{\tilde{w}}\right)}_{\text{Inequality}}$$

where the $\grave{\Theta}$ are the markup-corrected income shares of the different groups of households.

3.6 Relation to multiple factors' results

Introducing multiple labor factors of production in a growth accounting model is not new and has been the subject of numerous papers before ours. However, each of these papers consider labor factors as immobile, this implies that the economy is at the efficiency frontier but also that any aggregation or disaggregation of labor factors is impossible without a measurement error. Hence, the introduction of more factors implies less measurement error and the worst case is when only one labor factor is assumed, no decomposition is possible without measurement error as workers are assumed to be perfectly mobile within each factor.
Our result is different along two dimensions. First, we allow for a decomposition between

changes in more aggregated factors' supply and the reallocation between different groups within a factor. And second, we demonstrate that this reallocation is actually a change in the distance to the efficiency frontier due to changes in labor allocation. The multiple factors' framework is as follow:

$$d\ln Y = d\ln z + \underbrace{\Theta_L d\ln L}_{\text{aggregated labor}} + \Theta_K d\ln K \neq d\ln z + \underbrace{\sum_{t\in\mathcal{T}} \Theta_{t,L} d\ln l_t}_{\text{disaggregated labor}} + \Theta_K d\ln K = d\ln Y'$$

where there is no misallocation because workers in different labor factors are considered as different as labor and capital could be. Our framework allows to have both at the same time:

$$d\ln Y =$$

$$d\ln z + \underbrace{\sum_{t\in\mathcal{T}} \Theta_{t,L} d\ln l_t}_{\text{disaggregated labor}} + \Theta_K d\ln K = d\ln z + \underbrace{\Theta_L d\ln L}_{\text{aggregated labor}} + \underbrace{\sum_{t\in\mathcal{T}} \Theta_{t,L} d\ln\left(\frac{l_t}{L}\right)}_{\text{misallocation}} + \Theta_K d\ln K$$

where:

$$\sum_{t\in\mathcal{T}} \Theta_{t,L} d\ln\left(\frac{l_t}{L}\right) = -\sum_{t\in\mathcal{T}} \Theta_{t,L} d\ln\left(\frac{w_t}{\tilde{w}}\right)$$

and where different variants of the misallocation between urban and rural workers have been studied extensively in the development economics literature (see e.g.).

4 Aggregate misallocation

In section 2, we defined a general model with misallocation and studied the implications for general accounting, the distance to the efficiency frontier and we compared it to other results from the literature. In this section, we assume parametric structures for the aggregate production function and we study what these assumptions imply for misallocation, productivity and inequality.

First, we show that we are able to develop a *positive measure of inequality* that is directly related to misallocation and to well-known inequality indices from the literature. We provide some intuition on the behavior of this measure, how it relates to misallocation and to the distance to the efficiency frontier in the general case.

Second, we provide a direct link between our measure of misallocation and inequality across tasks and the measures and decomposition of productivity across firms emphasized in the literature. This allows us to provide several new decomposition of inequality across tasks and to incorporate a new concept which we denominate entry and exit of labor tasks and its contribution to misallocation, inequality and economic growth.

Finally, we switch back to the general results from previous section and provide measures of TFP mis-measurement up to a second-order approximation using the Tornqvist adjustment and we show how these approximations relate to the general results and to specific parts of the parametric decomposition we do.

4.1 A positive measure of inequality

Theory

To understand better the relationship between misallocation, production and inequality indices, we study different parametric production functions and their misallocation and inequality counterparts. Start with the Cobb-Douglas production function. There are one capital factor and $\mathcal{T}$ labor tasks:

$$Y = \frac{z}{\chi} \prod_{t \in \mathcal{T}} l_t^{\Theta_{t,L}} K^{\Theta_K}$$

where $\chi = \Pi_i \Theta_{t,L}^{\Theta_{t,L}} \Theta_K^{\Theta_K}$. By the first welfare theorem, we know that there exists one allocation of workers across tasks that maximizes output. This situation occurs when there is only one wage $\tilde{w}$ across tasks. Optimal real output can be represented as follow:

$$Y = z \prod_{t \in \mathcal{T}} \left(\frac{w_t}{\tilde{w}}\right)^{-\left(\frac{\Theta_{t,L}}{\Theta_L}\right)\Theta_L} L^{\Theta_L} K^{\Theta_K}$$

where:

$$0 < \exp\left(\Delta^{CD}\right) = \left(\prod_{t \in \mathcal{T}} \left(\frac{w_t}{\tilde{w}}\right)^{-\left(\frac{\Theta_{t,L}}{\Theta_L}\right)}\right)^{\Theta_L} < 1$$

is a measure of the misallocation of workers across tasks. When this measure is close to 1, there is little misallocation and dispersion of wages and marginal productivities and the production function is close to the perfectly mobile workers' case. On the contrary, when this measure is close to zero, workers are very poorly allocated across tasks, there is a large dispersion of wages and marginal productivities and the production function is very far away from the perfectly mobile workers' case. $1 - \exp\left(\Delta^{CD}\right)$ gives the percentage loss of production due to the misallocation of workers.
Surprisingly, this measure of misallocation shares several properties with inequality indices. Indeed, a simple transformation of $\exp\left(\Delta^{CD}\right)$ gives the exact first Theil index:

$$\ln\left(\exp\left(\Delta^{CD}\right)\right) = \Delta^{CD} = \Theta_L \times I$$

where I is the first Theil index:

$$I = \sum_{t \in \mathcal{T}} \left(\frac{l_t}{L}\right) \left(\frac{w_t}{\tilde{w}}\right) \ln\left(\frac{w_t}{\tilde{w}}\right)$$

In the more general CES case, we have the following production function:

$$Y = \left(\sum_{t \in \mathcal{T}} \left(\frac{\Theta_{t,L}}{\Theta_L} \right)^{1/\rho} l_t^{\frac{\rho-1}{\rho}} \right)^{\frac{\rho}{\rho-1}\Theta_L} K^{\Theta_K}$$

where labor and capital are related through a Cobb-Douglas production function. In this case, we have the following relationship with the misallocation of workers:

$$Y = \left(\sum_{t \in \mathcal{T}} \left(\frac{\Theta_{t,L}}{\Theta_L} \right) \left(\frac{w_t}{\tilde{w}} \right)^{1-\rho} \right)^{\frac{\rho}{\rho-1}\Theta_L} L^{\Theta_L} K^{\Theta_K}$$

where:

$$0 < \exp\left(\Delta^{CES}\right) = \left(\left(\sum_{t \in \mathcal{T}} \left(\frac{\Theta_{t,L}}{\Theta_L} \right) \left(\frac{w_t}{\tilde{w}} \right)^{1-\rho} \right)^{\frac{-1}{1-\rho}} \right)^{\rho\Theta_L} < 1$$

This misallocation function recovers the same properties than in the Cobb-Douglas case. Furthermore, it also shares properties with inequality indices as it is related to the Atkinson index A in the following way:

$$1 - \left(\exp\left(\Delta^{CES}\right)\right)^{-\frac{1-\rho}{2-\rho}\frac{1}{\rho\Theta_L}} = A$$

where $\rho - 1$ is the equivalent of the inequality aversion ε used by the Atkinson index.

Definition 2 (Positive measure of inequality) *A positive measure of income inequality is defined as an inequality index which is simultaneously a measure of income inequality respecting usual properties of inequality indices:*

(1) Normalisation

(2) Symmertry

(3) Population replication

(4) The principle of transfers

(5) Continuity/Differentiability

(6) Scale independence

(7) Additive decomposability

and a measure of workers' misallocation across labor tasks:

$$\mathscr{P} = \begin{cases} 1 - \exp\left(\Delta^{CES}\right) = 1 - \left(\left(\sum_{t \in \mathcal{T}} \left(\frac{\Theta_{t,L}}{\Theta_L}\right) \left(\frac{w_t}{\tilde{w}}\right)^{1-\rho}\right)^{\frac{-1}{1-\rho}}\right)^{\rho \Theta_L} & \text{when } \rho \neq 1 \\ 1 - \exp\left(\Delta^{CD}\right) = 1 - \left(\prod_{t \in \mathcal{T}} \left(\frac{w_t}{\tilde{w}}\right)^{-\left(\frac{\Theta_{t,L}}{\Theta_L}\right)}\right)^{\Theta_L} & \text{when } \rho \to 1 \end{cases}$$

where $\mathscr{P}$ is the positive measure of inequality which varies between 0 and 1 where 0 means perfect equality and allocation and 1 means total inequality and misallocation. The level of inequality that matters for misallocation depends on the complementarity/substitution between labor tasks. As this elasticity increases, implying a greater substituability between labor tasks, inequality matters less for misallocation. Conversely, when labor tasks are more complementary, inequality matters more for misallocation.

We give intuition on how this positive inequality measure behaves in the next section using numerical examples.

Intuition

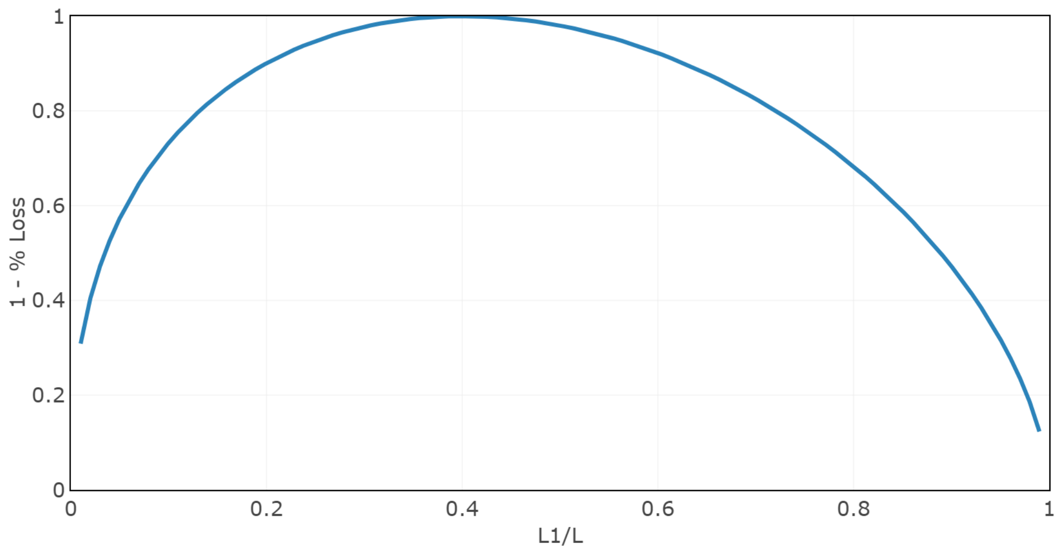
To understand better the concepts developed above, we consider the following simple examples. Start with the Cobb-Douglas case, we have the following production function with two labor factors:

$$Y = \frac{1}{\chi}\left(\frac{L_1}{L}\right)^{0.4}\left(\frac{L_2}{L}\right)^{0.6} \times L \quad \text{with } L = 1 \quad \text{and} \quad \chi = (0.4)^{0.4}(0.6)^{0.6}$$

Imposing the share of labor allocated to task 1, we observe the following misallocation:

$$\exp\left(\Delta^{CD}\right) = \left(\frac{w_1}{\tilde{w}}\right)^{-0.4}\left(\frac{w_2}{\tilde{w}}\right)^{-0.6} = \left(\frac{0.4}{L_1/L}\right)^{-0.4}\left(\frac{0.6}{L_2/L}\right)^{-0.6}$$

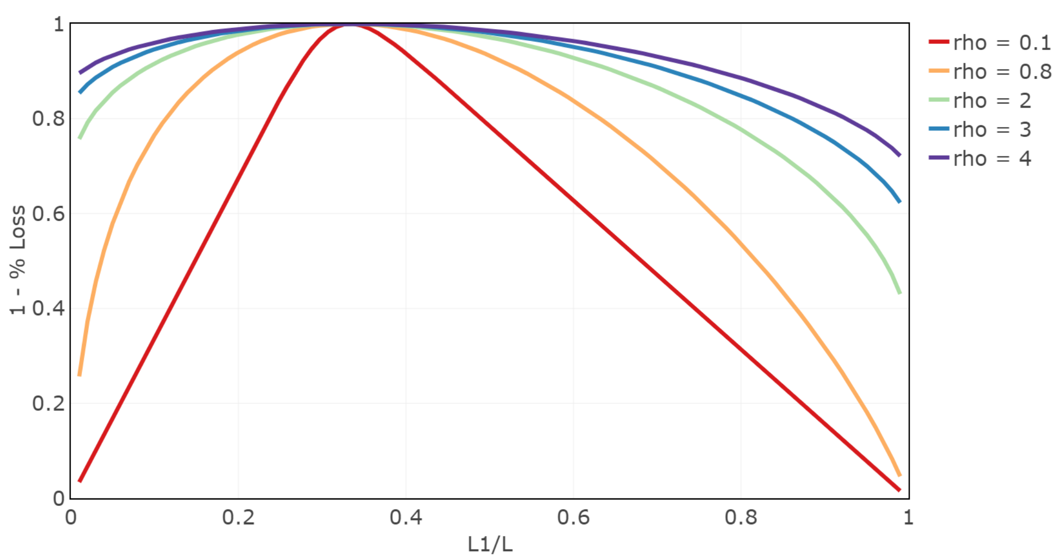
which gives the following plot when varying the allocation to the first labor factor:



This confirms what the theory implied that the misallocation level $\exp\left(\Delta^{CD}\right)$ stands between 0 and 1. $\exp\left(\Delta^{CD}\right) = 1$ at the optimal allocation where $L_1/L = \Theta_1 = 0.4$ and tends towards 0 when the allocation diverges from the optimal allocation.
This intuition can be extended to the CES case where we use the following simple production function:

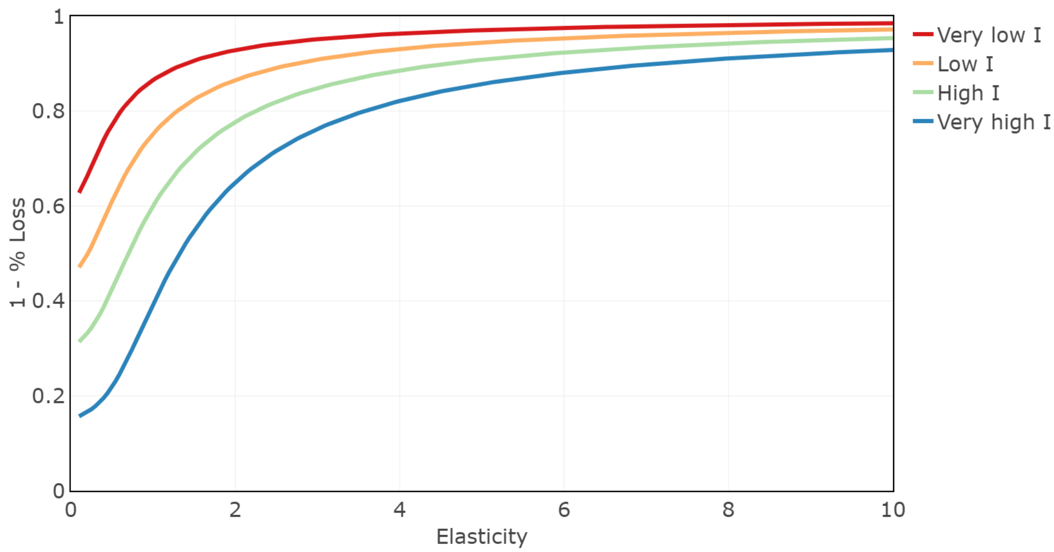
$$Y = \left(\left(\frac{1}{3}\right) \times \left(\frac{w_1}{\tilde{w}}\right)^{1-\rho} + \left(\frac{2}{3}\right) \times \left(\frac{w_2}{\tilde{w}}\right)^{1-\rho}\right)^{\frac{\rho}{\rho-1}} \times L \quad \text{with } L = 1$$

For different level of the constant elasticity ρ, we find the following evolution of the misallocation:



A second intuition appears here. We have that as the elasticity ρ decreases, the increase in complementarity between tasks imply a stronger cost of misallocation for out-

put. Hence, as ρ decreases, we have that inequality becomes more harmful for output. This also implies that our measure of misallocation increases as ρ decreases while in the classical Atkinson inequality indices, inequality increases as the elasticity of inequality aversion increases.
The intuition expressed above also appears when we considered fixed level of misallocation across different elasticity levels:



Here, the level of misallocation increases with the level of inequality and the ranking is respected across levels of elasticity. However, we observe that as the elasticity increases, all levels of misallocation tends to one when tasks become more and more substituable.

4.2 Productivity and misallocation measurement

The positive measure of inequality presented in the previous section also relates to the literature on productivity which studies aggregate changes in productivity when firms are heterogeneous and there is entry and exit of firms. We compare our measure of misallocation and inequality to seminal decomposition of productivity and we also provide a new decomposition. The application of these decompositions to our measures allow to develop new decompositions and study a new concept, the impact of entry and exit of labor tasks on inequality, misallocation and economic growth.

Relation to (Olley and Pakes, 1996)

Using the Cobb-Douglas result above, we have that:

$$Y = z \prod_{t \in \mathcal{T}} \left(\frac{w_t}{\tilde{w}} \right)^{-\left(\frac{\Theta_{t,L}}{\Theta_L}\right)\Theta_L} L^{\Theta_L} K^{\Theta_K}$$

Taking the logarithm of this result, we have that:

$$\ln Y = \ln\left(zL^{\Theta_L}K^{\Theta_K}\right) - \Delta^{CD}$$

where $\Delta^{CD} = \Theta_L \times I$ can be decomposed using classical methods in productivity measurements. Start with the results from (Olley and Pakes, 1996), we can decompose the level of misallocation/inequality as follow:

$$\Delta^{CD} = \Theta_L \times I = \Theta_L \times \left(\widetilde{\ln\Gamma} + Cov\left(\check{\Theta}, \ln\Gamma\right)\right)$$

where Γ is the ratio of wages and $\check{\Theta}$ is the labor income share allocated to task t, $\widetilde{\ln\Gamma}$ is the average log wage ratios and $\mathrm{Cov}\left(\check{\Theta}, \ln\Gamma\right)$ is the covariance between the labor incomes and the log wage ratios.

Relation to (Melitz and Polanec, 2015)

From this first result in levels, we can study the changes of the different components using the result of (Melitz and Polanec, 2015):

$$d\,I = d\,\widetilde{\ln\Gamma} + d\,\mathrm{Cov}\left(\check{\Theta}, \ln\Gamma\right)$$

To go from the change in inequality dI to the change in misallocation $d\Delta^{CD}$, we have to adjust for the labor share. This gives the following result in changes:

$$d\Delta^{CD} = -\left(d\ln Y - d\ln\left(zL^{\Theta_L}K^{\Theta_K}\right)\right) = d\Theta_L \times I + d\Theta_L \times dI + \Theta_L \times \left(\underbrace{d\widetilde{\ln\Gamma} + d\mathrm{Cov}\left(\check{\Theta}, \ln\Gamma\right)}_{dI}\right)$$

where the total change in misallocation $d\Delta^{CD}$ also takes into account the change in aggregate labor share and the joint change in labor share and in Theil index.

A new decomposition

We know that the Theil index I is given by:

$$I = \sum_{t\in\mathcal{T}} \left(\frac{\Theta_{t,L}}{\Theta_L}\right) \ln\left(\frac{w_t}{\tilde{w}}\right)$$

hence, we decompose the change in the Theil index dI between changes of each element and their joint change:

$$dI = \underbrace{\sum_{t\in\mathcal{T}} \left(\frac{\Theta_{t,L}}{\Theta_L}\right) d\ln\left(\frac{w_t}{\tilde{w}}\right)}_{\text{wage changes}} + \underbrace{\sum_{t\in\mathcal{T}} d\left(\frac{\Theta_{t,L}}{\Theta_L}\right) \ln\left(\frac{w_t}{\tilde{w}}\right)}_{\text{income share changes}} + \underbrace{\sum_{t\in\mathcal{T}} d\left(\frac{\Theta_{t,L}}{\Theta_L}\right) d\ln\left(\frac{w_t}{\tilde{w}}\right)}_{\text{joint changes}}$$

where we decompose the change in the Theil index between a change in log wage ratios keeping the labor income share fixed - which corresponds to the first-order change in the envelope theorem result -, a change in labor income shares keeping the log wage ratios fixed and a joint change in labor income shares and log wage ratios - which corresponds to the double of the second-order change which we present in the following section about TFP mis-measurement in general economies.
Finally, we adjust for labor share changes to obtain the change in misallocation:

$$d\Delta^{CD} = d\Theta_L \times I + d\Theta_L \times dI + \Theta_L \times \left(\sum_{t \in \mathcal{T}} \check{\Theta}_t d \ln \Gamma_t + \sum_{t \in \mathcal{T}} d\check{\Theta}_t \ln \Gamma_t + \sum_{t \in \mathcal{T}} d\check{\Theta}_t d \ln \Gamma_t \right)$$

where $\check{\Theta}_t = \frac{\Theta_{t,L}}{\Theta_L}$ and $\Gamma_t = \frac{w_t}{\tilde{w}}$.

Introducing entry and exit of labor tasks

We now develop a new concept of study in growth accounting. The entry and exit of labor tasks and its joint contribution to inequality, misallocation and economic growth. From a growth accounting perspective, the entry/exit of labor tasks implies gains from new, more productive tasks appearing and older, less productive tasks disappearing. Take the following simple decomposition of the Theil index change:

$$I_{t+1} - I_t = \underbrace{\left(\sum_{j \in \mathcal{C}} \check{\Theta}_{j,t+1} \ln \left(\frac{w_{j,t+1}}{\tilde{w}_{t+1}} \right) - \sum_{j \in \mathcal{C}} \check{\Theta}_{j,t} \ln \left(\frac{w_{j,t}}{\tilde{w}_t} \right) \right)}_{\text{Continuing tasks}} + \underbrace{\sum_{k \in \mathcal{E}} \check{\Theta}_{k,t+1} \ln \left(\frac{w_{k,t+1}}{\tilde{w}_{t+1}} \right)}_{\text{Entering tasks}} - \underbrace{\sum_{m \in \mathcal{X}} \check{\Theta}_{m,t} \ln \left(\frac{w_{m,t}}{\tilde{w}_t} \right)}_{\text{Exiting tasks}}$$

where $j \in \mathcal{C}$ is the set of continuing tasks, $k \in \mathcal{E}$ the set of entering tasks and $m \in \mathcal{X}$ the set of exiting tasks. We use the decomposability of the Theil index to separate the contribution to the Theil index from the continuing, entering and exiting tasks. We are then able to decompose these contributions between changes in sub-group Theil indices and the changes in aggregate wages using the additive decomposability property of Theil indices:

$$I_{t+1} - I_t = \underbrace{(I_{c,t+1} - I_{c,t}) + \left(\ln \left(\frac{\tilde{w}_{c,t+1}}{\tilde{w}_{t+1}} \right) - \ln \left(\frac{\tilde{w}_{c,t}}{\tilde{w}_t} \right) \right)}_{\text{Continuing tasks}} + \underbrace{\check{\Theta}_{e,t+1} \left(\ln \left(\frac{\tilde{w}_{e,t+1}}{\tilde{w}_{t+1}} \right) - \ln \left(\frac{\tilde{w}_{c,t+1}}{\tilde{w}_{t+1}} \right) \right)}_{\text{Entering tasks}}$$

$$\underbrace{+\check{\Theta}_{e,t+1} \left(I_{e,t+1} - I_{c,t+1} \right)}_{\text{Entering tasks}} + \underbrace{\check{\Theta}_{x,t} \left(\ln \left(\frac{\tilde{w}_{c,t}}{\tilde{w}_t} \right) - \ln \left(\frac{\tilde{w}_{x,t}}{\tilde{w}_t} \right) \right) + \check{\Theta}_{x,t} \left(I_{c,t} - I_{x,t} \right)}_{\text{Exiting tasks}}$$

where $\tilde{w}_{c,t}$, $\tilde{w}_{e,t}$ and $\tilde{w}_{x,t}$ are respectively the average wage of continuing, entering and exiting tasks in time t. Using these decompositions, we are able to separate the decomposition of the labor tasks derived before between continuing, entering and exiting tasks between two periods $t \in \{1,2\}$:

$$\Delta_2 - \Delta_1 = -\underbrace{(\Theta_{L,2} - \Theta_{L,1}) \times I_1}_{\text{Labor share change}} - \underbrace{(\Theta_{L,2} - \Theta_{L,1}) \times (I_2 - I_1)}_{\text{Cross labor-Theil change}} - \Theta_{L,1} \times$$

$$\left[\underbrace{\left(\sum_{j \in \mathcal{C}} \check{\Theta}_{j,2} \ln\left(\frac{w_{j,2}}{\tilde{w}_2}\right) - \sum_{j \in \mathcal{C}} \check{\Theta}_{j,1} \ln\left(\frac{w_{j,1}}{\tilde{w}_1}\right) \right)}_{\text{Continuing tasks}} + \underbrace{\sum_{k \in \mathcal{E}} \check{\Theta}_{k,2} \ln\left(\frac{w_{k,2}}{\tilde{w}_2}\right)}_{\text{Entering tasks}} - \underbrace{\sum_{m \in \mathcal{X}} \check{\Theta}_{m,1} \ln\left(\frac{w_{m,1}}{\tilde{w}_1}\right)}_{\text{Exiting tasks}} \right]$$

Using the tools developed above to apply (Melitz and Polanec, 2015) decomposition results to misallocation, we introduce entry and exit of occupations in that framework:

$$\Delta_2 - \Delta_1 = -\underbrace{(\Theta_{L,2} - \Theta_{L,1}) \times I_1}_{\text{Labor share change}} - \underbrace{(\Theta_{L,2} - \Theta_{L,1}) \times (I_2 - I_1)}_{\text{Cross labor-Theil change}}$$

$$- \Theta_{L,1} \times \left[\underbrace{\iota_{c,2} - \iota_{c,1}}_{\text{Continuing tasks}} + \underbrace{\check{\Theta}_{e,2} (\iota_{e,2} - \iota_{c,2})}_{\text{Entering tasks}} + \underbrace{\check{\Theta}_{x,t} (\iota_{c,1} - \iota_{x,1})}_{\text{Exiting tasks}} \right]$$

where $\iota_{c,1} = \sum_{j \in \mathcal{C}} \check{\Theta}_{j,1} \ln\left(\frac{w_{j,1}}{\tilde{w}_1}\right)$ is the contribution of continuing tasks to the change in misallocation and inequality. The difference with the Theil index is the mean wage used in the ratio. We can relate this decomposition to a decomposition uniquely in Theil indices as follow:

$$\Delta_2 - \Delta_1 = -\underbrace{(\Theta_{L,2} - \Theta_{L,1}) \times I_1}_{\text{Labor share change}} - \underbrace{(\Theta_{L,2} - \Theta_{L,1}) \times (I_2 - I_1)}_{\text{Cross labor-Theil change}}$$

$$- \Theta_{L,1} \times \left[\underbrace{(I_{c,2} - I_{c,1}) + \left(\ln\left(\frac{\tilde{w}_{c,2}}{\tilde{w}_2}\right) - \ln\left(\frac{\tilde{w}_{c,1}}{\tilde{w}_1}\right) \right)}_{\text{Continuing tasks}} \underbrace{\check{\Theta}_{e,2} \left(\ln\left(\frac{\tilde{w}_{e,2}}{\tilde{w}_2}\right) - \ln\left(\frac{\tilde{w}_{c,2}}{\tilde{w}_2}\right) \right)}_{\text{Entering tasks}} \right.$$

$$\left. \underbrace{+ \check{\Theta}_{e,2} (I_{e,2} - I_{c,2})}_{\text{Entering tasks}} + \underbrace{\check{\Theta}_{x,1} \left(\ln\left(\frac{\tilde{w}_{c,1}}{\tilde{w}_1}\right) - \ln\left(\frac{\tilde{w}_{x,1}}{\tilde{w}_1}\right) \right) + \check{\Theta}_{x,1} (I_{c,1} - I_{x,1})}_{\text{Exiting tasks}} \right]$$

which corresponds to a decomposition of a between-group and a within-group contribution to inequality changes. Then, we decompose the change in the Theil index of the continuing tasks $d\,I_c$ using the (Melitz and Polanec, 2015) decomposition and our own decomposition. First, decomposing the Theil index between average inequality and the covariance between labor income shares and log ratio wages, we can decompose the change in the Theil index of continuing tasks between the change in average inequality and the change in covariance:

$$d\,I_c = \left(\widetilde{\ln\Gamma_{c,2}} - \widetilde{\ln\Gamma_{c,1}}\right) + \left(\mathrm{Cov}\left(\check{\Theta}_{c,2}, \Gamma_{c,2}\right) - \mathrm{Cov}\left(\check{\Theta}_{c,1}, \Gamma_{c,1}\right)\right)$$

which is a direct application of the decomposition presented for all tasks.
We can also decompose the change in the continuing tasks' Theil index along the three dimensions that we presented to decompose the Theil index, wage changes, income shares changes and joint changes.

$$d\,I_c = \sum_{j\in\mathcal{C}} \check{\Theta}_{j,1}\left(\ln\Gamma_{j,2} - \ln\Gamma_{j,1}\right) + \sum_{j\in\mathcal{C}}\left(\check{\Theta}_{j,2} - \check{\Theta}_{j,1}\right)\ln\Gamma_{j,1} + \sum_{j\in\mathcal{C}}\left(\check{\Theta}_{j,2} - \check{\Theta}_{j,1}\right)\left(\ln\Gamma_{j,2} - \ln\Gamma_{j,1}\right)$$

We study the results of these different decompositions of misallocation and inequality changes for the United States in the section 6.

4.3 TFP mis-measurement

Now, moving from the Cobb-Douglas assumption back to the general case, we have that a change in real GDP can be approximated using Tornqvist adjustment which allows us to recover several changes implied in the Cobb-Douglas assumption and give a 1st and 2nd order approximation to the change in Y and to the TFP mis-measurement. We had that, up to a first-order approximation, the change in the loss to efficient output - which is also the TFP mis-measurement - is given by:

$$\begin{aligned} d\Delta &\overset{1}{\approx} d\ln Y - d\ln z - \Theta_L d\ln L - \Theta_K d\ln K \\ &\overset{1}{\approx} -\sum_{t\in\mathcal{T}} \Theta_{t,L} d\ln\left(\frac{w_t}{\tilde{w}}\right) = -\Theta_L \sum_{t\in\mathcal{T}} \left(\frac{\Theta_{t,L}}{\Theta_L}\right) d\ln\left(\frac{w_t}{\tilde{w}}\right) \end{aligned}$$

which is also the first-order change in the first Theil index $dI\left(\bar{\check{\Theta}}\right)$ weighted by $-\Theta_L$ and where $\overset{1}{\approx}$ means that the result is from a first-order approximation. Using the Tornqvist adjustment, we study the second-order approximation to the change in loss to real GDP:

$$\begin{aligned} d\Delta &\overset{2}{\approx} d\ln Y - d\ln z - \left(\Theta_L + \frac{d\Theta_L}{2}\right) d\ln L - \left(\Theta_K + \frac{d\Theta_K}{2}\right) d\ln K \\ &\overset{2}{\approx} -\sum_{t\in\mathcal{T}} \left(\Theta_{t,L} + \frac{d\Theta_{t,L}}{2}\right) d\ln\left(\frac{w_t}{\tilde{w}}\right) \end{aligned}$$

where we further decompose the second line between changes in labor share and in labor task shares:

$$d\Delta \overset{2}{\approx} \underbrace{-\frac{d\Theta_L}{2}\sum_{t\in\mathcal{T}}\left(\frac{\Theta_{t,L}}{\Theta_L}\right)d\ln\left(\frac{w_t}{\tilde{w}}\right)}_{\text{labor share change}} \underbrace{-\Theta_L\sum_{t\in\mathcal{T}}\left(\frac{\Theta_{t,L}}{\Theta_L}\right)d\ln\left(\frac{w_t}{\tilde{w}}\right)}_{\text{1st order change}} \underbrace{-\frac{\Theta_L}{2}\sum_{t\in\mathcal{T}}d\left(\frac{\Theta_{t,L}}{\Theta_L}\right)d\ln\left(\frac{w_t}{\tilde{w}}\right)}_{\text{2nd order change}}$$

when keeping the factor supply constant and where $\overset{2}{\approx}$ means that the result is from a second-order approximation. Hence, we have that even in the most general non-parametric case, several inequality mechanisms can be intertwined when we reach higher-order approximations to real GDP changes. In section 6, we show that both 1st order and 2nd order approximations to the change in inequality matter significantly for real GDP and TFP mis-measurement in the United States.

5 General equilibrium

Even though we can measure the amount of misallocation directly from the data, we still do not understand how these changes in income inequality occur. In this section, assuming we observe the underlying productivity and mobility shocks, we provide a direct mapping from the set of factor-specific productivity shocks and mobility shocks to the changes in factor prices and in real GDP in a very general economy. First, we provide a description of the model, then aggregate results and disaggregated results.

5.1 Model description

To study in detail the consequences of frictions on growth, we formulate an economy with sectoral input-output linkages, arbitrary elasticities of factors and intermediates and between skilled workers, and heterogeneous workers that specialize in an occupation and supply their labor in different sectors.

Production

The production side consists of $\mathcal{N}$ sectors and $\mathcal{O}$ occupation, we denote the number of units $\mathcal{N}$ and $\mathcal{O}$ in the same way as the set of sectors $\mathcal{N}$ and of occupations $\mathcal{O}$. A representative firm in each sector produces a homogeneous good, using imperfectly mobile labor, perfectly mobile capital and goods from other sectors to produce their output following a constant returns-to-scale technology:

$$y_i = F_i\left(z_{io,L}, z_K, \{l_{io}\}_{o\in\mathcal{O}}, k_i, \{x_{ij}\}_{j\in\mathcal{N}}\right)$$

where y_i is sectoral output in sector i , F_i is the production function that transforms inputs into outputs, $z_{io,L}$ is a Harrod-neutral productivity shock specific to occupation o in sector

i, z_K is a Harrod-neutral productivity shock specific to capital, l_{io} is the labor used for occupation o used in i, k_i is the capital use in sector i and x_{ij} is the use of the good produced in sector j in sector i. We denote the variables with lowercase letters and the index variables with uppercase letters. Production takes place in perfect competition. Hence, sector prices are equal to marginal costs, so that:

$$p_i = f_i\left(z_{io,L}, z_K, \{w_{io}\}_{o\in\mathcal{O}}, r, \{p_j\}_{j\in\mathcal{N}}\right)$$

where p_i is the price of the good in sector i, f_i is the cost function, w_{io} is the wage of workers in occupation o in sector i, r is the capital cost and p_j is the price of input j.

Workers

In this general model, we assume that workers are allocated between occupations and sectors by maximizing an aggregate allocation function G with constant returns-to-scale and where G increases in the labor allocated to each occupation:

$$L = G\left(\nu_1 l_1, \cdots, \nu_{\mathcal{O}} l_{\mathcal{O}}\right)$$

where L is total labor supply, ν_o is a mobility cost/preference shock specific to occupation o and l_o is the labor allocated to occupation o. Within each of these occupation, workers are allocated to sectors following a similar function G_o:

$$l_o = G_o\left(\phi_{1o} l_{1o}, \cdots, \phi_{\mathcal{N}o} l_{\mathcal{N}o}\right)$$

where ϕ_{io} is the mobility cost/preference shock specific to occupation o in sector i and where l_{io} is the labor allocated to occupation o in sector i. Maximizing these aggregate allocation functions with respect to the households' budget constraint, we have that the optimal allocation of workers across occupations and sectors is a function of wages and preferences/mobility costs:

$$\left(\frac{l_{1o}}{l_o}\right) = g_o\left(\phi_{1o} w_{1o}, \cdots, \phi_{\mathcal{N}o} w_{\mathcal{N}o}\right); \qquad \left(\frac{l_o}{L}\right) = g\left(\nu_1 W_1, \cdots, \nu_{\mathcal{O}} W_{\mathcal{O}}\right)$$

which gives the following joint allocation per sector and occupation:

$$\left(\frac{l_{io}}{L}\right) = \left(\frac{l_{io}}{l_o}\right) \times \left(\frac{l_o}{L}\right) = g_o\left(\phi_{1o} w_{1o}, \cdots, \phi_{\mathcal{N}o} w_{\mathcal{N}o}\right) \times g\left(\nu_1 W_1, \cdots, \nu_{\mathcal{O}} W_{\mathcal{O}}\right)$$

Which follows the results in (Horvath, 2000), (Kim and Kim, 2006) and (Hoynck, 2020) for CES aggregate allocation functions.
We assume that total labor supply is determined by an aggregate utility function where workers enjoy a positive utility of consumption and suffer disutility from working:

$$\mathscr{U} = C - h(L)$$

Consumption

On the consumption side, workers share identical homothetic preferences:

$$Y = C\left(c_1, \cdots, c_{\mathcal{N}}\right)$$

where Y is the real output of this economy, c_i is the final demand for the good from sector i and C is an aggregator of consumption equal to welfare and real GDP in this economy and features constant returns-to-scale. Total income of the group of workers specialized in occupation o and supplying labor to sector i is equal to $w_{io}l_{io}$, which pins down the budget constraint of each household.

Input-output definitions

In this economy, the technical coefficient matrix Ω_I is defined as a $\mathcal{N} \times \mathcal{N}$ matrix whose ij-th element is :

$$\Omega_{ij,I} = \frac{p_j x_{ij}}{p_i y_i}$$

The technical coefficients represent the cost share of the use of intermediate good j in total output of i. The $\mathcal{N} \times \mathcal{N}$ Leontief matrix Ψ is built on the technical coefficient matrix:

$$\Psi = \left(I - \Omega_I\right)^{-1}$$

where I is the identity matrix. The ij-th element Ψ_{ij} of this matrix denotes the direct and indirect use of sector j as a supplier to sector i. We define the labor matrix Ω_L as a vector of dimensions $\mathcal{N} \times \mathcal{O}$ and the capital matrix Ω_K as a vector of dimension $1 \times \mathcal{N}$, whose respectively io-th and i-th element are the labor compensation and capital cost allocated to workers specialized in occupation o and capital in sector i:

$$\Omega_{io,L} = \frac{w_{io} l_{io}}{p_i y_i}; \qquad \Omega_{i,K} = \frac{r k_i}{p_i y_i}$$

Similarly, we define the vectors of final consumption Υ, whose i-th element is:

$$\Upsilon_i = \frac{p_i c_i}{GDP}$$

which gives the final demand for good i as a share of GDP. These variables allow to define the importance of sectors and groups of households for the production of real GDP, Y. By accounting definition, real GDP is equal to the final demand homothetic function C. Therefore the importance of sectors and households for production is given by their direct and indirect importance in production of final demand goods. The sector sales shares of nominal GDP, $\vartheta_i = \frac{p_i y_i}{GDP}$ which are the Domar weights, and the income shares of households supplying labor, $\Theta_{io,L} = \frac{w_{io} l_{io}}{GDP}$, embody this intuition as follows:

$$\Theta_L' = \Upsilon' \Psi diag(\Omega_L); \qquad \vartheta' = \Upsilon' \Psi$$

where the i-th element of vectors ϑ and the io-th element of Θ_L gives the direct and indirect use of sector i and labor o supplied to sector i for the production of final goods:

$$\Theta_{io,L} = \sum_j \Upsilon_j \Psi_{ji} \Omega_{io,L} \ ; \qquad \vartheta_i = \sum_j \Upsilon_j \Psi_{ji} \qquad \text{(supply centrality)}$$

Hence, the importance of a household supplying labor for the real output Y is given by the direct and indirect importance of the labor it supplies to the production of final demand goods. This importance is given respectively by their income shares and their sales shares of nominal GDP. This importance is supply-oriented as it considers the importance of a sector or a labor type as a supplier to downward sectors up to final demand. The same intuition follows if we consider other sectors as the final buyer. This gives the following statistics:

$$\theta_{mi} = \sum_j \Omega_{mj,I} \Psi_{ji} \ \ with \quad \theta_{mi,k} = \Omega_{mk,I} \Psi_{ki} \qquad \text{(generalized Domar weights)}$$

where θ_{mi} is to be interpreted as ϑ_i where the final demand is replaced by sector m as the final consumer. The specific element $\theta_{mi,k}$ is the backward exposure of sector m to sector i through sector k. Considering now the sectors as buyers, we find the following statistics for the sector importance as a buyer to other sectors and as a demander of labor:

$$\xi_m = \sum_j \Psi_{mj} \qquad \text{(Leontief multipliers)}$$

$$\Xi_{moL} = \sum_j \Psi_{mj} \Omega_{jo,L} \ \text{ with } \quad \Xi_{mo,k} = \Psi_{mk} \Omega_{ko,L} \qquad \text{(demand centrality)}$$

where ξ_m is the celebrated Leontief multiplier which gives the total importance of sector m as a direct and indirect buyer to the other sectors. On the other hand, the statistic Ξ_{moL} gives the importance of sector m as a direct and indirect buyer of labor type o. The specific element $\Xi_{moL,k}$ gives the direct and indirect importance of sector m as a buyer of labor type o through sector k. By Sheppard's lemma, we observe that these statistics assume a central importance for price changes in perfect competition. Hence, we have that:

$$d \ln p_i = \sum_j \sum_o \Xi_{ioL,j} d \ln \left(\frac{w_{jo}}{z_{jo,L}} \right) + \sum_j \Xi_{iK,j} d \ln r$$

Intuitively, when a factor price changes, it impacts sector i's price through i's direct and indirect importance as a buyer of that factor. When a sector is a major buyer of a specific factor, when this factor's price increases, it leads to a large increase in that sector price in perfect competition. Hence, in perfect competition, sectoral price changes are the direct results of these sectors' importance as buyers through the production network. We come back to this intuition while presenting the general characterization results.

Equilibrium

We now define the general equilibrium in the model described above.

Definition 3 (General equilibrium) *In this model, we have that a general equilibrium is given by a vector of prices* $\{p_i\}\,\forall i \in \mathcal{N}$*, a vector of wages* $\{w_{io}\}\,\forall i \in \mathcal{N}, \forall o \in \mathcal{O}$*, a vector of intermediate good use* $\{x_{ij}\}\,\forall i,j \in \mathcal{N}$*, a vector of labor use and allocation* $\{l_{io}\}\,\forall i \in \mathcal{N}, \forall o \in \mathcal{O}$*, a vector of capital use* $\{k_i\}\,\forall i \in \mathcal{N}$*, a vector of output* $\{y_i\}\,\forall i \in \mathcal{N}$*, a price of capital* r *and a vector of consumption levels* $\{c_i\}\,\forall i \in \mathcal{N}$ *which are determined jointly by the following conditions:*

(1) Firms maximize their profit

(2) The profit of firms is equal to zero

(3) Workers maximize their utility

(4) Goods, labor and capital markets clear:

$$L = \sum_o \sum_i l_{io}; \qquad \bar{K} = \sum_i k_i; \qquad y_i = c_i + \sum_j x_{ji}$$

Where the general equilibrium is solved when given a vector of productivity shocks $\{z_{io,L}\}\,\forall i \in \mathcal{N}, \forall o \in \mathcal{O}$ *and of mobility costs/preferences* $\{\nu_o\}\,\forall o \in \mathcal{O}$ *and* $\{\phi_{io}\}\,\forall i \in \mathcal{N}, \forall o \in \mathcal{O}$*.*

We solve for the comparative statics of this general equilibrium model in section 5 where we assume a parametric form for each general function defined above. Nevertheless, we are already able to give general results that hold for any parametric form of this general model, this is what we do in sections 4.2 and 4.3 for aggregate and inequality results.

5.2 General aggregate results

We study two types of shocks. A Hicks-neutral or Harrod-neutral shock to productivity and a shock to workers' mobility cost/preferences. First we study the first-order approximation of the shocks around the initial equilibrium.

Proposition 3 (General result) *Take an initial equilibrium defined by vectors of technology* z *and of mobility costs/preferences* ν *and* ϕ*. Then the impact on growth of an exogenous change in technology or mobility frictions is given by:*

$$\begin{aligned} d\ln Y = & \sum_{i\in\mathcal{N}} \sum_{o\in\mathcal{O}} \frac{d\ln Y}{d\ln z_{io,L}} d\ln z_{io,L} + \frac{d\ln Y}{d\ln z_K} d\ln z_K + \sum_{i\in\mathcal{N}} \sum_{o\in\mathcal{O}} \frac{d\ln Y}{d\ln \phi_{io}} d\ln \phi_{io} \\ & + \sum_{o\in\mathcal{O}} \frac{d\ln Y}{d\ln \nu_o} d\ln \nu_o + \frac{d\ln Y}{d\ln L} d\ln L + \sum_{j\in\mathcal{N}} \sum_{p\in\mathcal{O}} \frac{d\ln Y}{dI} \frac{dI}{d\ln \Gamma_{jp}} d\ln \Gamma_{jp} \end{aligned}$$

where the equilibrium levels of labor supply and labor allocations change in response to the exogenous shocks.

Intuitively, the equation above means that not only technology and factor supply matter for real GDP but also how factors are allocated across sectors and occupations. By taking only the change in total factor supply, the growth accounting literature has implicitly assumed that these factors are always allocated to their best use. Either because they are perfectly immobile, implying that factors are different and could not be reallocated or either by assuming that factors are perfectly mobile, implying that they always allocate to their best use.
Futhermore, we already know some of these elasticities from (Hulten, 1978). First, we know that the elasticity of a total factor supply is given by the total income share of this factor, even in a production network:

$$\frac{d\ln Y}{d\ln L} = \Theta_L$$

and we know that following a Hicks-neutral or Harrod-neutral productivity shock, the elasticity of real GDP is given by the sales share of that sector for Hicks-neutral and by the sales multiplied by the input share for Harrod-neutral:

$$\frac{d\ln Y}{d\ln z_i} = \vartheta_i; \qquad \frac{d\ln Y}{d\ln z_{io,L}} = \vartheta_i \Omega_{io,L}$$

Using the results developed in this paper, we know that a change in labor allocation's wedge implies the following real GDP change:

$$\frac{d\ln Y}{dI} = -\Theta_L; \qquad \frac{dI}{d\ln \Gamma_{jp}} = \frac{\Theta_{jp,L}}{\Theta_L}$$

The results given by this equation depend on the shock that is occuring. We also know that a mobility cost/preference shock to labor allocation impacts growth proportionally to the income share of the household group that is shocked:

$$\frac{d\ln Y}{d\ln \phi_{io}} = \Theta_{io,L} \qquad \frac{d\ln Y}{d\ln \nu_o} = \sum_{i\in\mathcal{N}} \Theta_{io,L} = \Theta_{o,L}$$

Take first the Harrod-neutral productivity shocks:

Proposition 4 (productivity shock) *Take an initial equilibrium defined by vectors of technology z and of mobility costs/preferences ν and ϕ. Then an exogenous change in all the elements of the technology vector is given by:*

$$d\ln Y = \underbrace{\sum_i \sum_o \Theta_{io,L} d\ln z_{io,L}}_{\text{Direct effect}} + \underbrace{\sum_i \sum_o \Theta_L \frac{d\ln L}{d\ln z_{io,L}} d\ln z_{io,L}}_{\text{Indirect through labor supply}} - \underbrace{\sum_i \sum_o \sum_j \sum_p \Theta_{jp,L} \frac{d\ln \Gamma_{jp}}{d\ln z_{io,L}} d\ln z_{io,L}}_{\text{Indirect through inequality}}$$

This type of shock has both direct and indirect effect on real GDP. First, by (Hulten, 1978) theorem, we know that a Harrod-neutral productivity shock to labor productivity has an impact on real GDP which is proportional to the income share of this labor type. Second, the shock impacts Y through the change in labor supply and the reallocation of workers across sectors and occupations following endogenous change in wages.
A similar intuition follows when we study a shock to preferences or mobility costs:

Proposition 5 (preference/mobility shock) *Take an initial equilibrium defined by vectors of technology z and of preferences/mobility costs ϕ and ν. Then an exogenous change in all the elements of the technology vector is given by:*

$$d\ln Y = \underbrace{\sum_i \sum_o \Theta_{io,L} d\ln\phi_{io} + \sum_o \Theta_{o,L} d\ln\nu_o}_{\textit{Direct effect}} + \underbrace{\sum_i \sum_o \Theta_L \frac{d\ln L}{d\ln z_{io,L}} d\ln\phi_{io} + \sum_o \Theta_L \frac{d\ln L}{d\ln\nu_o} d\ln\nu_o}_{\textit{Indirect through labor supply}}$$
$$- \underbrace{\left(\sum_i \sum_o \sum_j \sum_p \Theta_{jp,L} \frac{d\ln\Gamma_{jp}}{d\ln\phi_{io}} d\ln\phi_{io} + \sum_o \sum_j \sum_p \Theta_{jp,L} \frac{d\ln\Gamma_{jp}}{d\ln\nu_o} d\ln\nu_o \right)}_{\textit{Indirect through inequality}}$$

where the shocks to preferences/mobility costs have both a direct effect on labor allocation and an indirect effect through changes in labor supply and changes in allocation of workers following the endogenous change in wages at the new equilibrium in the economy.

5.3 General inequality results

We now analyze comparative statics of the network economy developed above, starting from the initial equilibrium where the labor-specific productivity variables $z_{sp,L}$, are set to 1 for all sectors s and all occupations p. Hence, taking the log-changes of income shares $\frac{w_{io}l_{io}}{GDP}$ with respect to a Harrod-neutral shock in sector s to occupation p and isolating the log-change in wage, we have that:

$$\frac{d\ln w_{io}}{d\ln z_{sp,L}} = \underbrace{\frac{d\ln\Theta_{io,L}}{d\ln z_{sp,L}}}_{\text{Labor demand channel}} - \underbrace{\frac{d\ln l_{io}}{d\ln z_{sp,L}}}_{\text{Labor supply channel}} + \underbrace{\frac{d\ln GDP}{d\ln z_{sp,L}}}_{\text{Aggregate channel}} \quad \forall i \in \mathcal{N}, \forall o \in \mathcal{O}$$

This equation is central to the understanding of income inequality in network economies and provides the basic intuition for the rest of the paper.
First, the *labor demand channel* represents the change in the importance of labor supplied in sector i for final demand and therefore for GDP. It is given by:

$$\frac{d\ln\Theta_{io,L}}{d\ln z_{sp,L}} = \underbrace{\frac{d\ln\vartheta_i}{d\ln z_{sp,L}}}_{\text{scale effect}} + \underbrace{\frac{d\ln\Omega_{io,L}}{d\ln z_{sp,L}}}_{\text{substitution effect}}$$

where the change in income shares $\Theta_{io,L}$ appear through a change in labor demand $d\ln\Omega_{io,L}$ and/or through a change in market size $d\ln\vartheta_i$. These effects are respectively the substitution and scale effects. When a productivity shock occurs in one or more sectors of the economy, intermediate goods and labor have new equilibrium prices and wages that clear the markets. Firms and consumers update their optimal decision and new labor shares and sector sizes result in equilibrium.
Second, the *labor supply channel* indicates the impact of workers' reallocation between sectors on labor supply in one sector. This channel is given by :

$$\frac{d\ln l_{io}}{d\ln z_{sp,L}} = \underbrace{\frac{d\ln\Phi_{io}}{d\ln z_{sp,L}}}_{\text{sectoral share}} + \underbrace{\frac{d\ln\Phi_{o}}{d\ln z_{sp,L}}}_{\text{Occupational share}} + \underbrace{\frac{d\ln L}{d\ln z_{sp,L}}}_{\text{Total labor supply}}$$

Frictions' parameters (preferences and mobility costs) are fixed, but wages in each sector change in response to the labor-sector productivity shock, creating an incentive for workers to move. When the labor allocation functions are not very sensitive to wages, workers do not move, even if wages differ a lot between sectors and occupations. Hence, productivity shocks will trigger stark changes in wage inequality, as workers do not respond to the shock by moving across sectors and occupations. This coincides with a specific factors model. In the other limit, when the labor allocation functions are very sensitive to wages, workers immediately adapt to small changes in wages and therefore wage inequality is not be able to take shape. This is the standard case in which wages clear in general equilibrium with perfect mobility.
Finally, the *aggregate channel* specifies the importance of the change in GDP for the sectoral wage. We have that :

$$\frac{d\ln GDP}{d\ln z_{sp,L}} = \underbrace{\Theta_{sp,L}}_{\text{Productivity}} + \underbrace{\Theta_L\frac{d\ln L}{d\ln z_{sp,L}}}_{\text{Total labor supply}} \underbrace{-\Theta_L\sum_j\sum_o\left(\frac{\Theta_{jo,L}}{\Theta_L}\right)\frac{d\ln\Gamma_{jo}}{d\ln z_{sp,L}}}_{\text{Inequality}}$$

which is an implication of Hulten's theorem for factors of production. This channel produces a scale effect on wage inequality as it impacts all sectoral wages in the same way. We have that if the two first channels are muted and the shock only impacts wages through the aggregate channel then there is no change in wage inequality. We elaborate on this intuition in the next section to develop an *inequality-neutrality result* which is central for the understanding of income inequality in production networks.

6 Parametric general equilibrium models

In this section, we provide different parametric forms for the general model presented in the previous section and we explain the resulting aggregate and disaggregated results using simple economies and then a theorem. Finally, we provide a general characterization for the different shocks that we study in the empirical section of the model.

6.1 Parametric form

Production and consumption

Firms use a sector-specific nested CES technology and there are $\mathcal{O}$ occupations within each sector:

$$y_i = \left(\omega_{i,L}^{\frac{1}{\sigma}} L_i^{\frac{\sigma-1}{\sigma}} + \omega_{i,K}^{\frac{1}{\sigma}} k_i^{\frac{\sigma-1}{\sigma}} + \sum_j \omega_{ij}^{\frac{1}{\sigma}} x_{ij}^{\frac{\sigma-1}{\sigma}} \right)^{\frac{\sigma}{\sigma-1}} \quad \text{with} \quad L_i = \left(\sum_o \left(\frac{\omega_{io,L}}{\omega_{i,L}} \right)^{\frac{1}{\eta}} (z_{io,L} l_{io})^{\frac{\eta-1}{\eta}} \right)^{\frac{\eta}{\eta-1}}$$

where $\omega_{i,L}$ is the parameter governing labor productivity in sector i, $\omega_{i,K}$ governs capital productivity in sector i, ω_{ij} governs the j-th intermediate good productivity in sector i, $\omega_{io,L}$ governs occupation o-th productivity in sector i, $z_{io,L}$ is the productivity shifter for occupation o in sector i, L_i is the aggregate labor use in sector i, k_i is the capital use, x_{ij} is the use of intermediate good j in sector i, l_{io} is the use of occupation o in sector i, σ is the constant elasticity of substitution between factors and intermediate goods and η is the constant elasticity of substitution between occupations.
On the consumption side, all households have the same CES preferences across sectoral goods:

$$\mathscr{U}_c = \left(\sum_j v_j^{\frac{1}{\rho}} c_j^{\frac{\rho-1}{\rho}} \right)^{\frac{\rho}{\rho-1}}$$

where $\mathscr{U}_c$ is the utility from consumption, v_j is the parameter governing the preference for good j, c_j is the consumption of good j and ρ is the constant elasticity of substitution across goods. The indirect utility function from consumption in this case where preferences are homothetic and identical across consumers is given by:

$$\mathscr{V}_{c,io} = \frac{w_{io}}{P}$$

where the indirect utility from consumption $\mathscr{V}_c$ depends on wage w_{io} which differs between sectors and occupations. P is the representative consumer price index common across workers because of their identical homothetic preferences. To obtain the total indirect utility of workers, we multiply the indirect utility from consumption by the idiosyncratic friction that each specific worker has drawn for that combination of industry and occupation:

$$\mathscr{V}_{io}^h = \varphi_{io}^h \times \mathscr{V}_{c,io}$$

where $\mathscr{V}_{io}^h$ is the indirect utility of household h specialized in occupation o and working in sector i. This indirect utility differs for each household because each household draws a vector of idiosyncratic frictions $\left\{ \varphi_{jp}^h \right\}$ where $j \in \mathcal{N}$ and $p \in \mathcal{O}$ that is household-specific and drawn from a nested joint Fréchet distribution. These frictions can be understood as a mix of mobility costs that workers face to move in one occupation/industry or another and the preferences that they have across these occupations/industries. Household choose the combination of occupation and industry that maximizes their indirect utility:

$$\mathscr{V}_{io}^{h} = \max_{j \in \mathcal{N}, p \in \mathcal{O}} \left\{ \varphi_{jp}^{h} \times \frac{w_{jp}}{P} \right\}$$

where the vector $\{\varphi\}$ is drawn from a nested joint Fréchet distribution G:

$$G \sim \exp\left(\sum_{o \in \mathcal{O}} \nu_o \left(\sum_{i \in \mathcal{N}} \phi_{io} \left(\varphi_{io} \right)^{-\kappa} \right)^{\frac{\lambda}{\kappa}} \right)$$

where ν is the vector of location parameters of the upper nest of the Fréchet distribution that vary across occupations $o \in \mathcal{O}$, λ is the dispersion parameter of the upper nest Fréchet distribution, ϕ is the vector of location parameter of the lower nest of the Fréchet distribution that vary across occupations $o \in \mathcal{O}$and $i \in \mathcal{N}$and where κ is the dispersion parameter of the lower nest of the Fréchet distribution. Each worker in the economy draws a vector $\{\varphi\}$ of frictions from this distribution and choose the occupation and industry that maximize their indirect utility. Below, we explain what is the allocation of workers across industry and occupations resulting from this distribution.

Workers' allocation

For a clearing understanding of workers' allocation, we present separately the choice of occupation and industry but the actual draw in the Fréchet distribution is joint and therefore the only resulting allocation is the one of workers in a specific industry and a specific occupation l_{io}. Nevetheless, we first present them as separate allocations resulting from separate Fréchet distribution for more clarity. Then we present the result for the nested joint Fréchet.

First, in the upper nest of the Fréchet distribution, the frictions implied by mobility costs and preferences follow a joint Fréchet distribution, with location parameters ν_o that vary across occupations $o \in \mathcal{O}$, and a dispersion parameter λ which governs the mobility of workers between occupations. The equilibrium share of workers specializing in occupation o is given by:

$$\Phi_o = \frac{l_o}{L} = \frac{\nu_o \left(W_o^s \right)^{\lambda}}{\left(W^s \right)^{\lambda}}$$

where W_o^s is the wage for workers with education o and W^s is the wage index of workers equal to $\left(\sum_o \nu_o \left(W_o^s \right)^{\lambda} \right)^{\frac{1}{\lambda}}$. Within each of these occupations, workers then decide to supply their labor in one of the sector. In that lower nest of the nested joint Fréchet distribution, frictions follow a joint Fréchet distribution with location parameters ϕ_{io} that vary across sectors $i \in \mathcal{N}$, and a dispersion parameter κ. The share of workers supplying their labor to sector i is equal to:

$$\Phi_{io} = \frac{l_{io}}{l_o} = \frac{\phi_{io} \left(w_{io} \right)^{\kappa}}{\left(W_o^s \right)^{\kappa}}$$

where w_{io} is the wage in sector i for workers specialized in occupation o, $l_o = \sum_i l_{io}$ and W_o^s is the wage index of workers equal to $\left(\sum_j \phi_{jo} \left(w_{jo}\right)^{\kappa}\right)^{\frac{1}{\kappa}}$.
We have that the equilibrium share of workers specialized in occupation o and working in sector i following the nested joint Fréchet distribution is given by:

$$\frac{l_{io}}{l_o}\frac{l_o}{L} = \Phi_{io}\Phi_o = \frac{\phi_{io}\left(w_{io}\right)^{\kappa}}{\left(W_o^s\right)^{\kappa}}\frac{\nu_o\left(W_o^s\right)^{\lambda}}{\left(W^s\right)^{\lambda}}$$

We give now the extreme cases on how the equilibrium allocation of workers across sectors - and similarly across occupations - behaves. When $\kappa \rightarrow 0$, workers are immobile between sectors, we have that:

$$\lim_{\kappa \rightarrow 0}\frac{l_{io}}{l_o} = \frac{\phi_{io}}{\sum\limits_j \phi_{jo}}$$

which depends only on the frictions faced by workers regarding the sector in which they want to work. Then labor supply does not depend on wages. Conversely, when $\kappa \rightarrow +\infty$, workers are perfectly mobile between sectors, and we have that:

$$\begin{cases} \lim\limits_{\kappa \rightarrow +\infty} \frac{l_{io}}{l_o} = 0 & \text{if } \exists w_{io} < w_{jo} \text{ with } j \in \mathcal{N} \\ \lim\limits_{\kappa \rightarrow +\infty} \frac{l_{io}}{l_o} = 1 & \text{if } w_{io} > w_{jo} \forall j \in \mathcal{N} \\ \lim\limits_{\kappa \rightarrow +\infty} \frac{l_{io}}{l_o} = \frac{\phi_{io}}{\sum\limits_j \phi_{jo}} & \text{if } w_{io} = w_{jo} \forall j \in \mathcal{N} \end{cases}$$

which implies that workers would all move to the same sector where the wage is the highest. This is a partial equilibrium analysis where wages are exogeneous. Once they are endogeneous to the model, the labor supply would converge to the third case where all wages are equal which is the classical result for perfect workers' mobility.
Compared to this actual labor allocation, the perfect labor allocation occurs when wages are equal and the labor allocation is entirely determined by the relative productivity of different sectors. The ratio of perfect labor allocation to actual labor allocation is called the observed labor frictions and is given by the vector Γ_o^s:

$$\Gamma_{io}^s = \frac{\left(l_{io}/l_o\right)^*}{\left(l_{io}/l_o\right)} = \frac{w_{io}}{\tilde{w}_o}$$

where $\left(l_{io}/l_o\right)^*$ is the perfect labor allocation equal to $\left(\Theta_{io,L}/\Theta_{o,L}\right)$ and $\tilde{w}_o$ is the average wage of workers in occupation o. Note that we have now endogenized the underlying frictions in the economy. When the perfect allocation corresponds to the actual allocation, the sectoral wage is equal to the mean wage in the economy.

6.2 An Inequality-Neutral Result

To understand how changes in income inequality occur in network economies, we start with a benchmark framework where production functions and household preferences are Cobb-Douglas, there is only one type of occupation, total labor is fixed and workers are immobile between sectors ($\kappa = 0$):

$$y_i = l_i^{\omega_{i,L}} \Pi_j x_{ij}^{\omega_{ij,I}}; \qquad \mathscr{U} = \Pi_j c_i^{v_i}; \qquad \frac{l_i}{L} = \frac{\phi_i}{\sum_j \phi_j}$$

By the first-order conditions, we have that the shares of expenses for intermediate good and labor of the firms with respect to their total sales and the share of expenses by good of households are fixed. Hence in Cobb-Douglas economies we have that:

$$\omega_{i,L} = \Omega_{i,L} \forall i \in \mathcal{N}; \quad \omega_{ij,I} = \Omega_{ij,I} \forall i,j \in \mathcal{N}; \quad v_i = \Upsilon_i \forall i \in \mathcal{N}$$

which means that the change in expenses and in total sales (or in total incomes for households) always compensate in order to keep a share of expenses fixed in this simple model. In this case where the changes in intermediate good, factor shares and consumption shares are always equal to zero and knowing that the change in Domar weights ϑ_i only depend on the change in these shares, we also have that:

$$d\vartheta_i = \sum_m \Psi_{mi} \left(\underbrace{d\Upsilon_m}_{=0} + \sum_j \underbrace{d\Omega_{jm,I}}_{=0} \vartheta_j \right) = 0 \forall i \in \mathcal{N} \text{ in this model}$$

which conveys the intuition that the change in the importance of a sector i compared to GDP is given by the change in final sales and intermediate good sales of the other sectors $j \in \mathcal{N}$ weighted by the importance of sector i as a supplier to these other sectors. Therefore, building on the first-order conditions, we have that $d\vartheta_i = 0 \forall i \in \mathcal{N}$ in Cobb-Douglas economies which implies that the relative size of each sector with respect to GDP does not change.

These results imply that the *labor demand channel* is silent in this model. Then, following the assumption that workers are immobile and therefore cannot change sectors, we have that the *labor supply channel* is also silent in this model. Both these results imply that the only channel impacting wages in this model is the *aggregate channel* which is the same for each sectoral wage. Hence, we have that each worker is impacted in the same way, and therefore any labor-specific productivity shock will only make each worker richer or poorer in real terms, but will not impact wage inequality. We formalize this intuition in the following theorem for any model implying inequality-neutral results.

Proposition 6 (Inequality-neutrality) *Assume a Harrod-neutral productivity shock specific to labor occurs in the economy. Then the two following propositions are correct if and only if the third is:*

(1) $\frac{d\ln\Theta_{i,L}}{d\ln z_{s,L}} = 0 \forall i \in \mathcal{N}$

(2) $\frac{d\ln l_i}{d\ln z_{s,L}} = 0 \forall i \in \mathcal{N}$

(3) $d\ln\left(\frac{w_i}{w_j}\right) = \frac{d\ln GDP}{d\ln z_{s,L}} \quad \forall i,j \in \mathcal{N}$

Corollary 2 *A corollary of the inequality-neutrality theorem is its negative statement. At least one of the two following propositions is correct if and only if the third is:*

(1) $\exists i \in \mathcal{N} : \frac{d\ln\Theta_{i,L}}{d\ln z_{s,L}} \neq 0$

(2) $\exists i \in \mathcal{N} : \frac{d\ln l_i}{d\ln z_{s,L}} \neq 0$

(3) $\exists i,j \in \mathcal{N} : d\ln\left(\frac{w_i}{w_j}\right) \neq 0$

This result can easily be extended to a network economy with occupations as the cost share of each occupation within each sector would not change following a productivity shock due to the Cobb-Douglas assumption.
In the next section, we build on this benchmark result to study separately the channels creating wage inequality in equilibrium using simple network economies.

6.3 Elementary economies

In this section, we study simple economic structures to specify the different channels through which changes occur in wage inequality, following labor-specific productivity shocks. To do so, we study the case where firms use a sector-specific CES technology, there is only one occupation level, total labor is fixed and households have CES preferences:

$$y_i = \left(\omega_{i,L}^{\frac{1}{\sigma}} l_i^{\frac{\sigma-1}{\sigma}} + \sum_j \omega_{ij,I}^{\frac{1}{\sigma}} x_{ij}^{\frac{\sigma-1}{\sigma}}\right)^{\frac{\sigma}{\sigma-1}} \quad ; \quad \mathcal{U} = \left(\sum_j v_j^{\frac{1}{\rho}} c_j^{\frac{\rho-1}{\rho}}\right)^{\frac{\rho}{\rho-1}}$$

where σ is the elasticity of substitution between inputs in the production process, ρ is the elasticity of substitution between consumption goods for households and the ω parameters give the importance of inputs and consumption goods in the aggregation processes. In this setup, the inequality-neutrality theorem no longer holds, and productivity shocks can generate labor inequality effects.
We proceed to the study of different stylized economies and what basic production structure can teach us about changes in wage inequality when **1)** labor is specific to each sector, **2)** when labor is imperfectly mobile and **3)** when there are different occupations.
We start with the case where labor is specific to a sector ($\kappa \to 0$). Each of the following

economies represent a different combination of channels impacting wage inequality. The first, the horizontal economy, only opens the change in market channel but keeps the labor demand channel closed. Second, in the roundabout economy, only the labor demand channel is opened and the market size does not change. Finally, in the vertical economy, both channels are opened. We study what is the sign and size of the effects implied by these channels for wage inequality.

Horizontal economy

See Figure 4. Households supply one type of labor specific to one sector, while sectors only use labor to produce the output that they sell directly to final consumers:

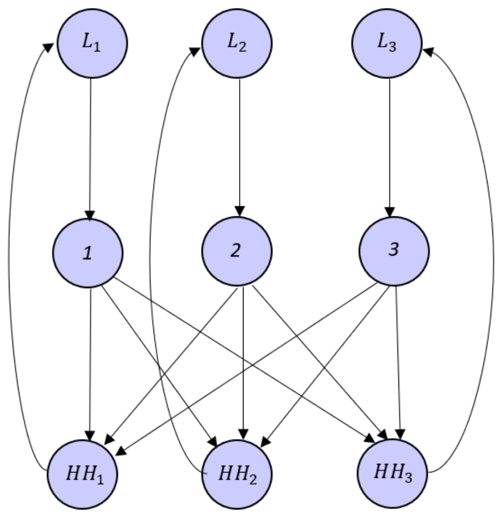


Figure 4: Horizontal economy

In this economy, firms only sell to final demand and their sales shares only depend on final consumers. As sectors only use labor, sales shares and income shares coincide ($\vartheta_i = \Theta_i$) and household supplying labor in sector i is the only one impacted by the change in sales share of sector i. Hence, the sign of the effect on wage inequality depends on final demand's elasticity of substitution between consumption goods and how final consumers reallocate their expenses between one sector and another. The change in sales shares is the only effect impacting wage inequality, as labor is the only input and cannot be substituted by another input. This implies that there are only scale effects but no substitution effect:

$$
\begin{aligned}
d\ln\left(\frac{w_1}{w_2}\right) &= d\ln\left(\frac{\vartheta_1}{\vartheta_2}\right) = d\ln\left(\frac{U_1}{U_2}\right) \\
&= \left(\frac{\rho-1}{\rho}\right) d\ln\left(\frac{z_{1,L}}{z_{2,L}}\right)
\end{aligned}
$$

where $\left(\frac{\rho-1}{\rho}\right)$ is the elasticity of the wage ratio to the labor-specific productivity ratio. This is the same result as in (Acemoglu and Autor, 2011) where there is only one sector of production and the elasticity of interest is the one of the production. Hence, the only difference between the two setups is that in the horizontal economy, the elasticity that matters is the one from final demand aggregation as the production process in each sector only uses labor. In the roundabout economy, we reproduce an almost identical economy as the one in (Acemoglu and Autor, 2011) to understand the difference.

Roundabout economy

In this example, each household supplies one type of labor, see Figure 5. All labor is used by one sector, which subsequently sells its output to itself and to final consumers.

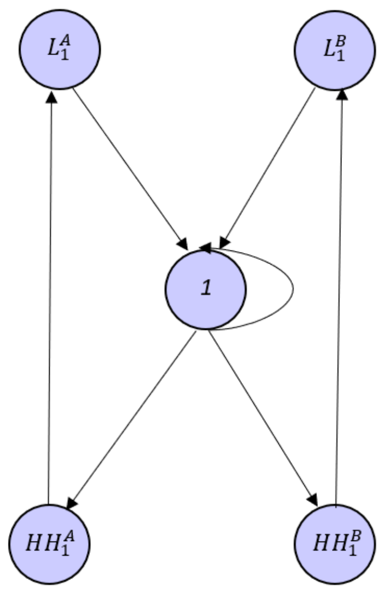


Figure 5: Roundabout economy

One sector produces everything in the economy and uses two sorts of labor and its own output in production. In this setup, only substitution matters as sector 1 will always produce the entirety of real output in this economy. Therefore, when a labor-specific productivity shock occurs, wage inequality changes as follows:

$$\begin{aligned} d\ln\left(\frac{w_{1s}}{w_{1u}}\right) &= d\ln\left(\frac{\Omega_{1s,L}}{\Omega_{1u,L}}\right) \\ &= \left(\frac{\sigma-1}{\sigma}\right) d\ln\left(\frac{z_{1s,L}}{z_{1u,L}}\right) \end{aligned}$$

Here, we have that the elasticity is identical to the one in (Acemoglu and Autor, 2011). This result follows from the fact there is only one sector and therefore the households do not choose between different goods, hence their decision-making do not matter for income inequality.

Three tastes of vertical economy

Each household supplies its labor to a different sector at a different position in the supply chain in the simple vertical economy, see Figure 6.The most upstream sector in the supply chain sells its output to the second sector, which in turn sells its output to the third and so forth. Each sector, except the last one, use labor and the output from the upstream sector as input. Firms in the last sector only use the output from its upstream sector as input and sell their output to the most downstream final consumers.

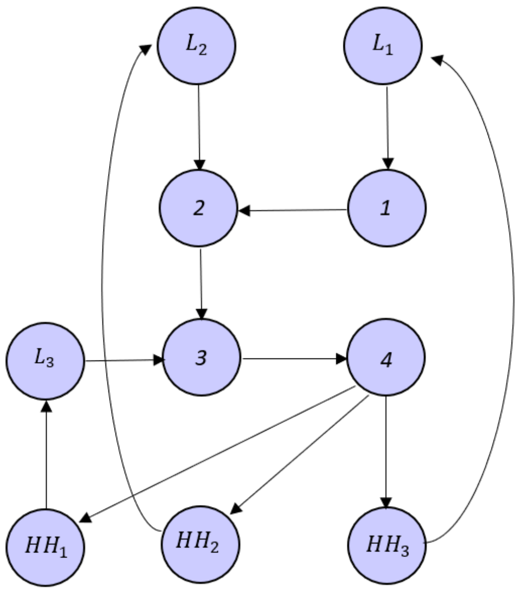


Figure 6: Simple vertical economy

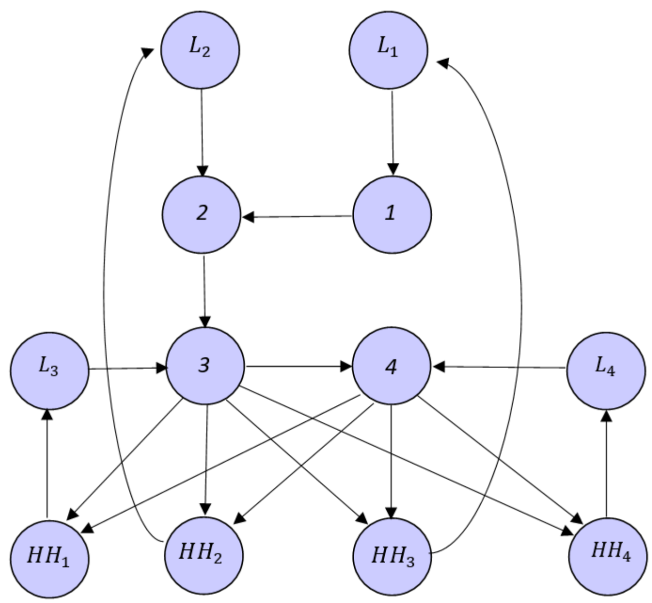


Figure 7: Double vertical economy

The intuition developed in the horizontal and roundabout economies applies jointly as there are now substitution and scale effects when a shock occurs. The change in wage inequality follows:

$$\begin{aligned} d\ln\left(\frac{w_2}{w_3}\right) =& d\ln\left(\frac{\vartheta_1}{\vartheta_2}\right) + d\ln\left(\frac{\Omega_{1,L}}{\Omega_{2,L}}\right) = d\ln\Omega_{32} + d\ln\left(\frac{\Omega_{2,L}}{\Omega_{3,L}}\right) \\ =& \left(\frac{\sigma-1}{\sigma}\right) d\ln\left(\frac{z_{2,L}}{z_{3,L}}\right) \end{aligned}$$

Here, we have that the elasticity of wage inequality to labor-specific productivity is the same as in the roundabout economy. This results from the fact that each sector only sells to one other sector, hence, the ratio of sales can be reduced to the expense share of the downstream sector on the upstream sector as an input. We see in the next examples that this polar result breaks down when sectors have different elasticity of substitution or when a sector sells to more that one other sector.
First, we stay in the simple vertical economy but we now assume that:

$$\sigma_2 \neq \sigma_3$$

hence, the price changes' effect do not cancel out anymore. This implies that wage inequality now takes the following form:

$$\begin{aligned} d\ln\left(\frac{w_2}{w_3}\right) &= (\sigma_3-\sigma_2)\, d\ln p_2 + (1-\sigma_2)\, d\ln\left(\frac{w_2}{z_{2,L}}\right) - (1-\sigma_3)\, d\ln\left(\frac{w_3}{z_{3,L}}\right) \\ &= (\sigma_3-\sigma_2)\, \Omega_{21,I} d\ln\left(\frac{w_1}{z_{1,L}}\right) + (1-(1+\Omega_{2,L})\,\sigma_2 + \Omega_{2,L}\sigma_3)\, d\ln\left(\frac{w_2}{z_{2,L}}\right) \\ &\qquad - (1-\sigma_3)\, d\ln\left(\frac{w_3}{z_{3,L}}\right) \end{aligned}$$

where wage inequality changes now take a recursive form where the wage inequality cannot be solved linearly. This implies that we now need to solve for all wage changes to find the change in inequality. A first takeaway from this result is that the classical wage inequality results depend heavily on the perfect homogeneity of elasticities of substitution. Once there exists a heterogeneity, wage inequality changes take a recursive form depending on potentially all other wage changes in the economy. This implies that wage inequality changes now also depends on the heterogeneous elasticities of substitution but also on the structure of the production network and on potentially all the labor-specific productivity shocks in the economy.
Second, take now the case of the double vertical economy in 7. There we study the wage inequality between sectors 3 and 4 while now sector 3 also sells to final demand and sector 4 also uses labor as an input. Hence, we now have that:

$$\begin{aligned} d\ln\left(\frac{w_3}{w_4}\right) &= d\ln\left(\frac{\vartheta_3}{\vartheta_4}\right) + d\ln\left(\frac{\Omega_{3,L}}{\Omega_{4,L}}\right) \\ &= \left(\frac{\sigma-1}{\sigma}\right) d\ln\left(\frac{z_{3,L}}{z_{4,L}}\right) + \left(\frac{\sigma-\rho}{\sigma}\right)\left(\frac{p_3c_3}{p_3y_3}\right) d\ln\left(\frac{p_3}{p_4}\right) \\ &= \left(\frac{\sigma-1}{\sigma}\right) d\ln\left(\frac{z_{3,L}}{z_{4,L}}\right) + \left(\frac{\sigma-\rho}{\sigma}\right)\left(\frac{p_3c_3}{p_3y_3}\right)\left(\sum_j \left(\Xi_{3l,j}-\Xi_{4l,j}\right) d\ln\left(\frac{w_j}{z_{jl}}\right)\right) \end{aligned}$$

where wage inequality still depends on the ratio of labor-specific productivity but also on all the previous wages in the value chain weighted by the difference in elasticity of substitution between production and consumption, the share of final demand in the gross production of sector 3 and the net importance of other wage changes for sector 3 and 4.

Imperfectly mobile labor

We now turn to an economic structure when κ does not tend to zero and therefore labor is (im)perfectly mobile between sectors in 8. The households supply their labor to each sector with preferences to work in one sector or another. Their labor supply is readjusted as wages adapt and dampen the impact of the shock. Firms in each sector only use labor as input and sell their output directly to final consumers.

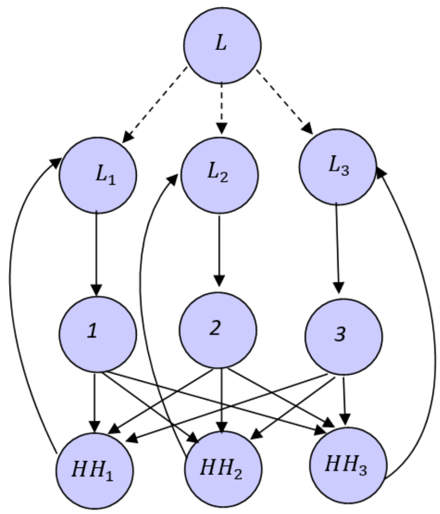


Figure 8: Allocation economy

This structure can be compared to the horizontal economy where κ^s does not tend to zero and where therefore the labor channel of wage inequality has been unlocked. Hence, wage inequality follows:

$$\begin{aligned} d\ln\left(\frac{w_1}{w_2}\right) &= d\ln\left(\frac{\Upsilon_1}{\Upsilon_2}\right) - d\ln\left(\frac{\Phi_1}{\Phi_2}\right) \\ &= \left(\frac{\rho-1}{\rho+\kappa}\right) d\ln\left(\frac{z_{1,L}}{z_{2,L}}\right) \end{aligned}$$

where the case of the horizontal economy is included when $\kappa = 0$ and where the change in wage inequality is equal to zero when labor is perfectly mobile ($\kappa \to +\infty$). The labor supply channel works as a dampening effect applied to the channels already developed above. The more mobile are workers, the more mitigated will be the change in wage inequality.

Heterogeneous occupational labor

In this last scenario, we assume multiple sectors with no interdependence through intermediate good use but each of these sectors use different occupations to produce their good:

$$y_i = \left(\sum_o \omega_{io,L}^{\frac{1}{\eta}} l_{io}^{\frac{\eta-1}{\eta}}\right)^{\frac{\eta}{\eta-1}} \quad ; \quad \mathscr{U} = \left(\sum_j v_j^{\frac{1}{\rho}} c_j^{\frac{\rho-1}{\rho}}\right)^{\frac{\rho}{\rho-1}}$$

Hence, we have that workers now differentiate between sectors *and* occupations. This new feature has consequences for wage inequality as now workers mobility impacts wages on two levels:

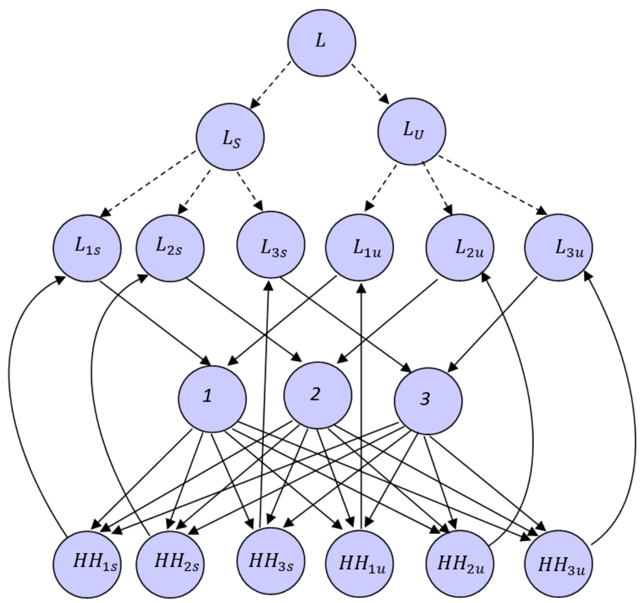


Figure 9: Skilled economy

$$\begin{aligned} d\ln\left(\frac{w_{1s}}{w_{2u}}\right) &= d\ln\left(\frac{\Upsilon_1}{\Upsilon_2}\right) + d\ln\left(\frac{\Omega_{1s,L}}{\Omega_{2u,L}}\right) - d\ln\left(\frac{\Phi_s}{\Phi_u}\right) - d\ln\left(\frac{\Phi_{1s}}{\Phi_{2u}}\right) \\ &= \left(\frac{\eta-1}{\eta+\kappa}\right) d\ln\left(\frac{z_{1s,L}}{z_{2u,L}}\right) + \left(\frac{\eta-\rho}{\eta+\kappa}\right) d\ln\left(\frac{W_1^d}{W_2^d}\right) + \left(\frac{\kappa-\lambda}{\eta+\kappa}\right) d\ln\left(\frac{W_s^s}{W_u^s}\right) \end{aligned}$$

where $d\ln W_1^d = \sum_j \Omega_{1j,L} d\ln\left(\frac{w_{1j}}{z_{1j}}\right)$ and $d\ln W_s^s = \sum_j (l_{js}/l_s) d\ln w_{js}$. Wage inequality now depends on the labor cost on the production side W^d but also on how workers consider the outside option of the other wages on the labor supply side W^s.

6.4 General characterization

We now characterize the behavior of wage changes in this economy when there is respectively a shock to labor-specific productivity and to workers' mobility.

Inequality results for productivity shocks

To find the system of equations solving the wage elasticities with respect to productivity shocks, start with the following equation for the change in income share of the *io*-th group of households:

$$\frac{d\ln \Theta_{io,L}}{d\ln z_{sp,L}} = \underbrace{(\eta-\sigma)\frac{d\ln\left(\frac{w_{io}}{z_{io,L}}\right)}{d\ln z_{sp,L}} + (1-\eta)\frac{d\ln W_i}{d\ln z_{sp,L}} - (1-\sigma)\frac{d\ln p_i}{d\ln z_{sp,L}}}_{\text{Substitution effect}}$$

$$+ \underbrace{\frac{1}{\vartheta_i}(1-\rho)\sum_o \vartheta_{i,o}\frac{d\ln\left(\frac{p_o}{P}\right)}{d\ln z_{sp,L}} + \sum_k \frac{\vartheta_k}{\vartheta_i}(1-\sigma)\sum_o \theta_{ki,o}\frac{d\ln\left(\frac{p_o}{p_k}\right)}{d\ln z_{sp,L}}}_{\text{Scale effect}}$$

where $d\ln p_i = \sum_j \sum_o \Xi_{io,j} d\ln\left(\frac{w_{jo}}{z_{jo,L}}\right) + \sum_j \Xi_{ik,j} d\ln r$, $d\ln P = \sum_j U_j d\ln p_j$ and $d\ln W_i^d = \sum_o \Omega_{io} d\ln\left(\frac{w_{io}}{z_{io,L}}\right)$. The sign of the change in income shares depends on the elasticities of substitution in each sector σ, the elasticity of substitution between labor types η and in final demand ρ and on the comparative price change of sector i and the labor used in i compared to all the other sectors and to all the other labor types.

For example, take a positive occupation p-specific shock in sector s in an economy where all the elasticities of substitution are strictly smaller than 1. In this economy with complementarities, the more competitive is sector i following the shock, the more it loses in market share and the more the group of households *io* loses in income share. In the first part of the equation, when the equilibrium wage paid to households *io* decreases with respect to the cost of all the other inputs of sector i as reflected by p_i, $\Theta_{io,L}$ is impacted negatively. In the second part of the equation, the importance of sector i as a supplier kicks in. If the downstream sectors to i become cheaper, as a result of a smaller price of i or because they are more exposed to the positive shock, then the importance of i as a supplier decreases as the expenses of final consumers and downstream sectors are allocated elsewhere.

These consequences of the shock are conditional on the value of the elasticities compared to one as explained in the simple structure economies. When the elasticities are strictly greater than one, the mechanisms are reversed.

Study now the change in labor supply:

$$\frac{d\ln l_{io}}{d\ln z_{sp,L}} = \underbrace{\kappa\left(\frac{d\ln w_{io}}{d\ln z_{sp,L}} - \sum_j \left(\frac{l_{jo}}{l_o}\right)\frac{d\ln w_{jo}}{d\ln z_{sp,L}}\right)}_{\text{Sectoral labor}} + \underbrace{\lambda\left(\frac{d\ln W_o^s}{d\ln z_{sp,L}} - \sum_j \left(\frac{l_j}{L}\right)\frac{d\ln W_j^s}{d\ln z_{sp,L}}\right)}_{\text{Occupational labor}}$$

where $d\ln W_o = \sum_j (\frac{l_{jo}}{l_o}) d\ln w_{jo}$. From the perspective of the households, what matters is the wage earned in the sector/education level in which they are currently working (here sector i and occupation o) compared to all the other sectors/education levels weighted by frictions' parameters. When the wage in sector i increases, workers specialized in occupation o from other sectors move to sector i conditional on their possibility to move governed by elasticity κ. The same occurs between occupations conditional on the mobility between education levels λ. When the distance between $\ln w_{io}$ and $\ln W_o$ increases, workers specialized in occupation o move to sector i with an elasticity of κ. Hence in our setup, the log-changes in wages are given by:

$$\begin{aligned}\frac{d\ln w_{io}}{d\ln z_{sp,L}} = & \underbrace{(\eta-\sigma)\frac{d\ln\left(\frac{w_{io}}{z_{io,L}}\right)}{d\ln z_{sp,L}} + (1-\eta)\frac{d\ln W_i^d}{d\ln z_{sp,L}} - (1-\sigma)\frac{d\ln p_i}{d\ln z_{sp,L}}}_{\text{Labor demand}} \\ & \underbrace{+\frac{1}{\vartheta_i}(1-\rho)\sum_o \vartheta_{i,o}\frac{d\ln\left(\frac{p_o}{P}\right)}{d\ln z_{sp,L}} + \sum_k \frac{\vartheta_k}{\vartheta_i}(1-\sigma)\sum_o \theta_{ki,o}\frac{d\ln\left(\frac{p_o}{p_k}\right)}{d\ln z_{sp,L}}}_{\text{Labor demand}} \\ & \underbrace{-\kappa\left(\frac{d\ln w_{io}}{d\ln z_{sp,L}} - \sum_j\left(\frac{l_{jo}}{l_o}\right)\frac{d\ln w_{jo}}{d\ln z_{sp,L}}\right) - \lambda\left(\frac{d\ln W_o^s}{d\ln z_{sp,L}} - \sum_j\left(\frac{l_j}{L}\right)\frac{d\ln W_j^s}{d\ln z_{sp,L}}\right)}_{\text{Labor supply}} \\ & \underbrace{+\Theta_{sp,L} - \Theta_L\sum_j\sum_o\left(\frac{\Theta_{jo,L}}{\Theta_L}\right)\frac{d\ln\Gamma_{jo}}{d\ln z_{sp,L}}}_{\text{Aggregate channel}}\end{aligned}$$

We solve this system of $(\mathcal{N}\times\mathcal{O})+1$ linear equations where the $(\mathcal{N}\times\mathcal{O})+1$ unknowns are the log-changes in wages and in the capital cost.

General shock to mobility frictions

On the other hand, a shock to workers' mobility frictions only triggers change in equality efficiency. We have that:

Proposition 7 (workers' mobility shock) *Take an initial equilibrium defined by mobility frictions κ and λ. Then an exogenous change in these frictions is given by:*

$$\begin{aligned}d\ln Y = & \underbrace{\Theta_L\frac{d\ln L}{d\ln\lambda}d\ln\lambda}_{\textit{Indirect through labor supply}} + \underbrace{\Theta_L\frac{d\ln L}{d\ln\kappa}d\ln\kappa}_{\textit{Indirect through labor supply}} \\ & \underbrace{-\sum_j\sum_p\Theta_{jp,L}\frac{d\ln\Gamma_{jp}}{d\ln\lambda}d\ln\lambda}_{\textit{Indirect through inequality}} \underbrace{-\sum_j\sum_p\Theta_{jp,L}\frac{d\ln\Gamma_{jp}}{d\ln\kappa}d\ln\kappa}_{\textit{Indirect through inequality}}\end{aligned}$$

Hence, no matter if workers become more or less mobile compared to the initial situation, what matters for real GDP is if they allocate better or worse as a consequence of this change. It might seem intuitive to think that if workers were more mobile, real output would increase. But this would be the case only if wages were initially higher in sectors that are more productive. Thus, more mobile workers would move to these sectors and would improve the labor allocation. On the contrary, if the initial wages are lower in more productive sectors, an increase in workers' mobility would *decrease* real output. We test this theory in the exercise section of this paper.

Calibrated shocks to mobility between occupations

Study the case of a shock to workers' mobility between occupations $o \in \mathcal{O}$, we observe that the labor supply reaction to the shock is different. Nevertheless, the rest of the full characterization is very similar to the productivity shock, hence we only present the labor supply elasticity to a workers' mobility shock:

$$\frac{d\ln l_{jo}}{d\ln\lambda} = \underbrace{\left(\ln W_o - \sum_m \left(\frac{l_m}{L}\right)\ln W_m^s\right)}_{\text{Occupational labor}} + \underbrace{\kappa\left(\frac{d\ln w_{io}}{d\ln\lambda} - \sum_j \left(\frac{l_{jo}}{l_o}\right)\frac{d\ln w_{jo}}{d\ln\lambda}\right)}_{\text{Sectoral labor}}$$

where this new characterization is the result of the change in what was before the labor supply elasticity to a relative change in wages. This implies, in the case of a shock to workers' mobility between occupations that we have to take into account the initial level of wages for the labor supply elasticity. This means that the labor supply elasticity is *different* depending on the initial equilibrium. Intuitively, this result means that when workers become more mobile, they tend to move more if the initial difference in wages is bigger. To understand this better, take the allocation economy presented above: But now take a change in labor supply that is triggered by a change in workers' mobility. Then we have that:

$$d\ln\left(\frac{w_1}{w_2}\right) = \left(\frac{\rho-1}{\rho}\right) d\ln\left(\frac{z_{1,L}}{z_{2,L}}\right) - \left(\frac{\ln w_1 - \sum_j \left(\frac{l_j}{L}\right)\ln w_j}{\ln w_2 - \sum_j \left(\frac{l_j}{L}\right)\ln w_j}\right)$$

where the constant second term depends on the initial difference in wages. The intuition is the following, if the wage in sector 1 is initially relatively larger than in other sectors, then if workers' mobility increases, it leads more workers to go to sector 1, which increases labor supply to that sector and therefore weight negatively on the wage. The second term is simply the ratio of these two initial situations. The sign and the amplitude of these initial differences jointly dictate the change in wage inequality in that economy when a shock to workers' mobility occurs. Concerning the general characterization for a workers'

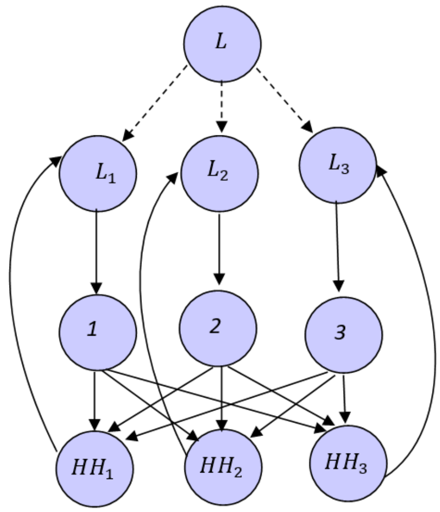


Figure 10: Allocation economy

mobility shock, we find the following result:

$$\frac{d\ln w_{io}}{d\ln\lambda} = \underbrace{(\eta-\sigma)\frac{d\ln w_{io}}{d\ln\lambda} + (1-\eta)\frac{d\ln W_i^d}{d\ln\lambda} - (1-\sigma)\frac{d\ln p_i}{d\ln\lambda}}_{\text{Labor demand}}$$

$$+\underbrace{\frac{1}{\vartheta_i}(1-\sigma)\sum_o \vartheta_{i,o}\frac{d\ln\left(\frac{p_o}{P}\right)}{d\ln\lambda} + \sum_k \frac{\vartheta_k}{\vartheta_i}(1-\sigma)\sum_o \theta_{ki,o}\frac{d\ln\left(\frac{p_o}{p_k}\right)}{d\ln\lambda}}_{\text{Labor demand}}$$

$$\underbrace{-\left(\ln W_o^s - \sum_m \left(\frac{l_m}{L}\right)\ln W_m^s\right) - \kappa\left(\frac{d\ln w_{io}}{d\ln\lambda} - \sum_j \left(\frac{l_{jo}}{l_o}\right)\frac{d\ln w_{jo}}{d\ln\lambda}\right)}_{\text{Labor supply}}$$

$$\underbrace{-\Theta_L\sum_j\sum_o\left(\frac{\Theta_{jo,L}}{\Theta_L}\right)\frac{d\ln\Gamma_{jo}}{d\ln\lambda}}_{\text{Aggregate channel}}$$

We solve this system of $(\mathcal{N}\times\mathcal{O})+1$ linear equations where the $(\mathcal{N}\times\mathcal{O})+1$ unknowns are the log-changes in wages and in the capital cost.

7 Data description and patterns

In this section of the paper, we start by describing the datasets that we use to apply our decompositions and calibrate our model. We then provide some patterns and descriptive statistics that we observe in the data.

7.1 Data description

To measure our growth accounting statistics and to calibrate the GE model developed in the previous sections, we use data from IPUMS for wages, employment, occupation, industry and demographics and we use data from the BEA to construct the IO table from 2007. More precisely concerning the IPUMS data, we selected samples approximately one decade appart from each other between 1980 and 2019. For 1980 and 1990, we use the 5% census data at the state level and for the years 2000, 2007 and 2019, we use the ACS data respectively taking a sample of 0.13%, 1% and 1% of the US population. We chose the years 2007 and 2019 on purpose to be able to recover hourly wages thanks to a precise value of the number of weeks worked the previous year, the number of hours usually worked per week and the total wage income from the previous year for each individual.
In these census data, we also have access to individual weights to be able to reconstruct the US population from the sample. We also use these data to reconstruct the labor income share allocated to different subgroups of the working population based on occupations, industry and education level. Finally, we also construct the total number of hours worked for each of these subgroups which we use a labor factors in the growth accounting exercises.
On the other hand, we use the BEA IO supply data to build the IO tables for the year 2007 and we develop a crosswalk between the industries in the IPUMS data and the major industries from the BEA data. We also develop a crosswalk between the IPUMS occupations and major occupations from the census data to have a reasonable amount of subgroups to track for the GE exercises (10 occupations $\times$ 13 industries, removing the army).

7.2 Patterns

Income shares, wages and labor supply

Before to dive into the exercises, we provide descriptive statistics on the correlations and the general behavior of employment, wages, wage ratios and labor income shares across decades between 1980 and 2019.
First, we observe that, even across decades, the total number of hours does not increase dramatically when the wage in that sector increases compared to other wages. If anything the correlation seems slightly positive but most occupations seem randomly plotted in each decade as can be seen on the graphs below:

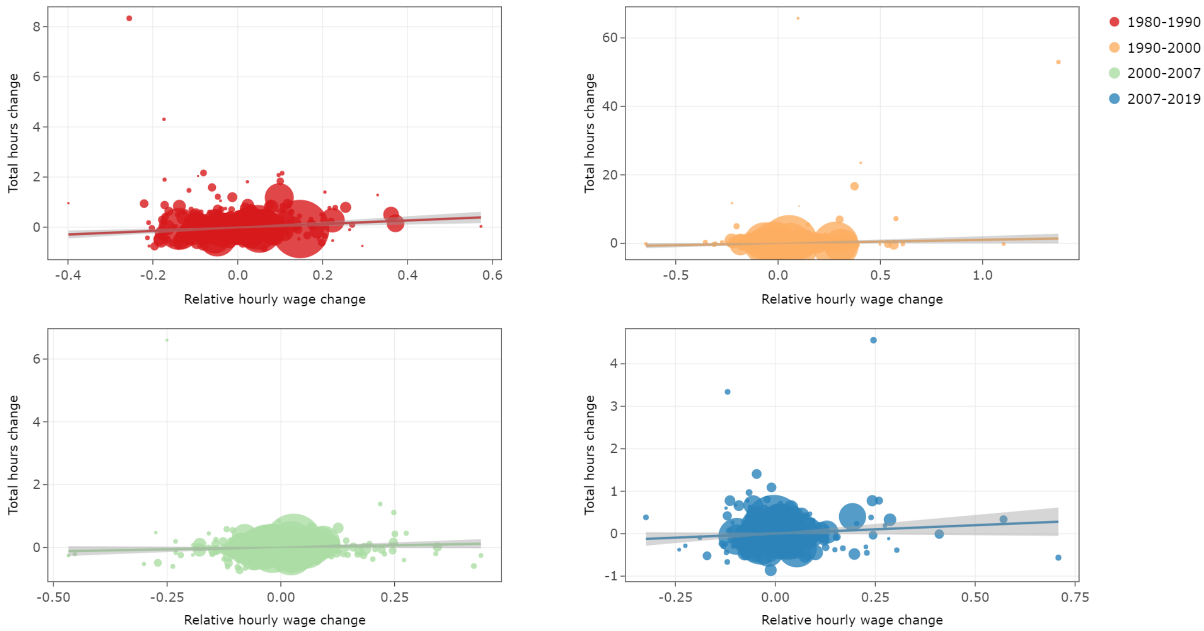


Figure 11: On the x-axis are the relative percent changes in hourly wage diminished by the percent changes in hourly wages in other occupations weighted by their share in total labor supply. The occupations are the recoded from 1990 occupations as developed by IPUMS. The total hours change are the percent change in total hours for the same occupations. The dot sizes are the total number of hours supplied in each of these occupations at the beginning of the decade.

This would therefore imply that workers only marginally decide for which occupation to work depending on the change in wages implying an imperfect workers' mobility with a low elasticity κ which confirms other empirical studies (e.g. (Galle and Lorentzen, 2021) and (Rodriguez-Clare et al., 2021)).
This means that changes in income inequality are in great part induced by changes in labor demand. In our model, this labor demand is summarized within a sufficient statistics which is labor income share. This income share incorporates the demand for a specific occupation o in each sector depending on the changes in sector size and the changes in occupation o's demand in that specific sector.
First, observe the following correlation between changes in income share levels and the initial income share at the beginning of the decades:

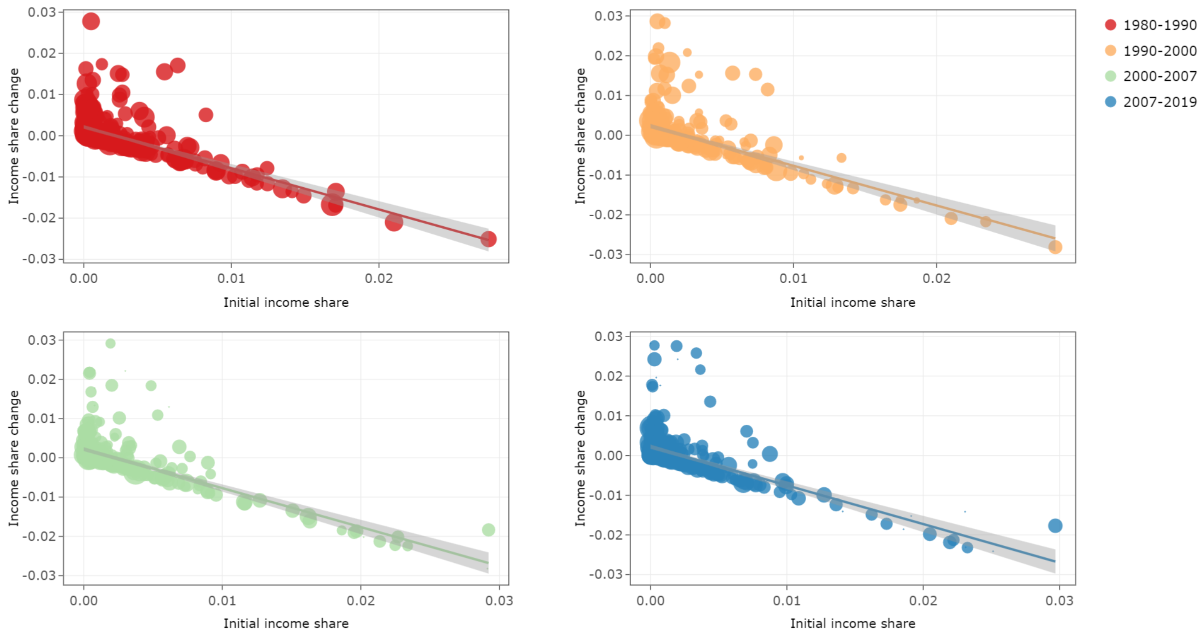
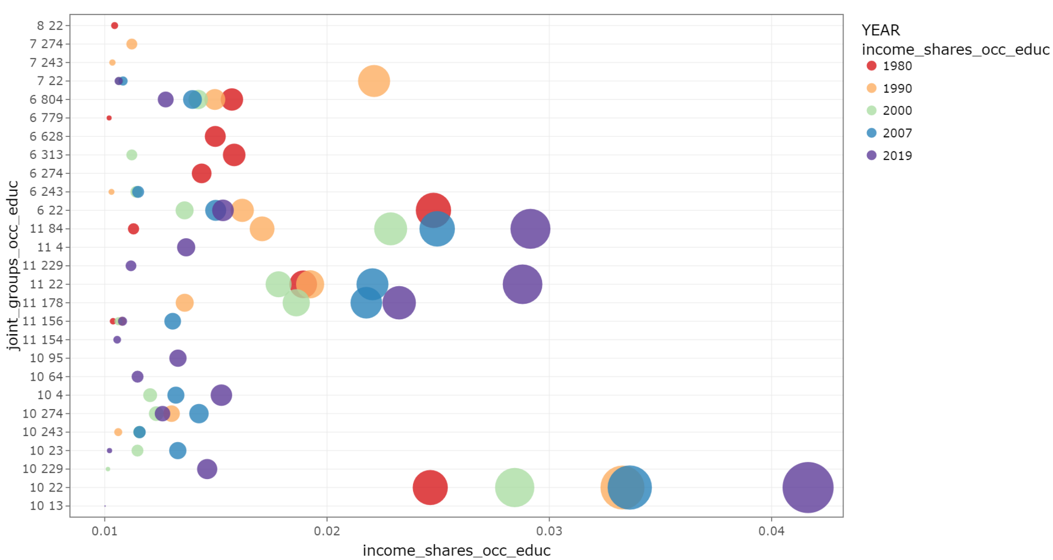


Figure 12: On the x-axis are the level changes in income shares on each decade and on the y-axis are the level of income share of each occupation at the beginning of the decade. The size of the points is given by the average hourly wage at the beginning of the decade.

We see that greater income shares tend to decrease in size while smaller income shares tend to increase. This implies strong dynamics at play in the production network formation. Below, we present the income shares of occupation $\times$ education groups with an income share greater than 1% of labor income at some point in time:



- Replace the number above by group names

Second, when studying the relationship between hourly wage levels at the beginning of each decade and the percent hourly wage changes, we observe a positive relationship. This implies that as wages are higher, they tend to increase more, contributing to the increase in income inequality.

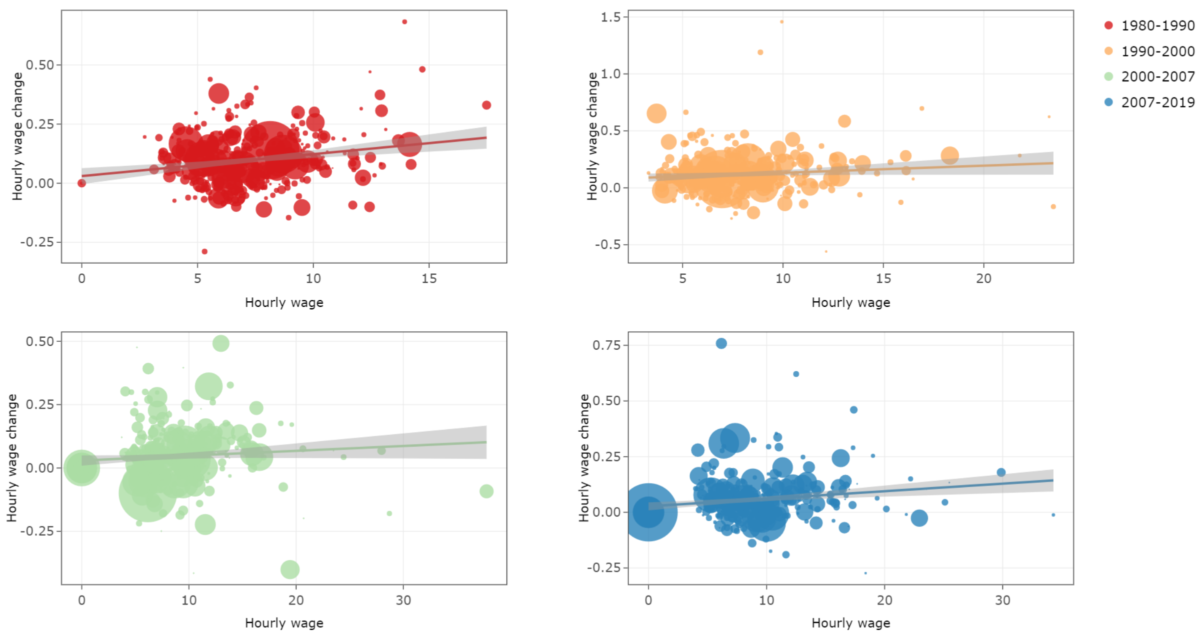


Figure 13: On the y-axis there are the percent changes in hourly wage while on the x-axis are the hourly wages of each occupation at the beginning of the decade. The size of the points are the income shares of each occupation at the beginning of the decade.

However, when it comes to misallocation, this increase in inequality would be less damaging if the increase in wages is greater for occupations that have less importance for the economy - a lower income share - or even a decreasing importance - a lower income share -. We study this correlation in the graph below:

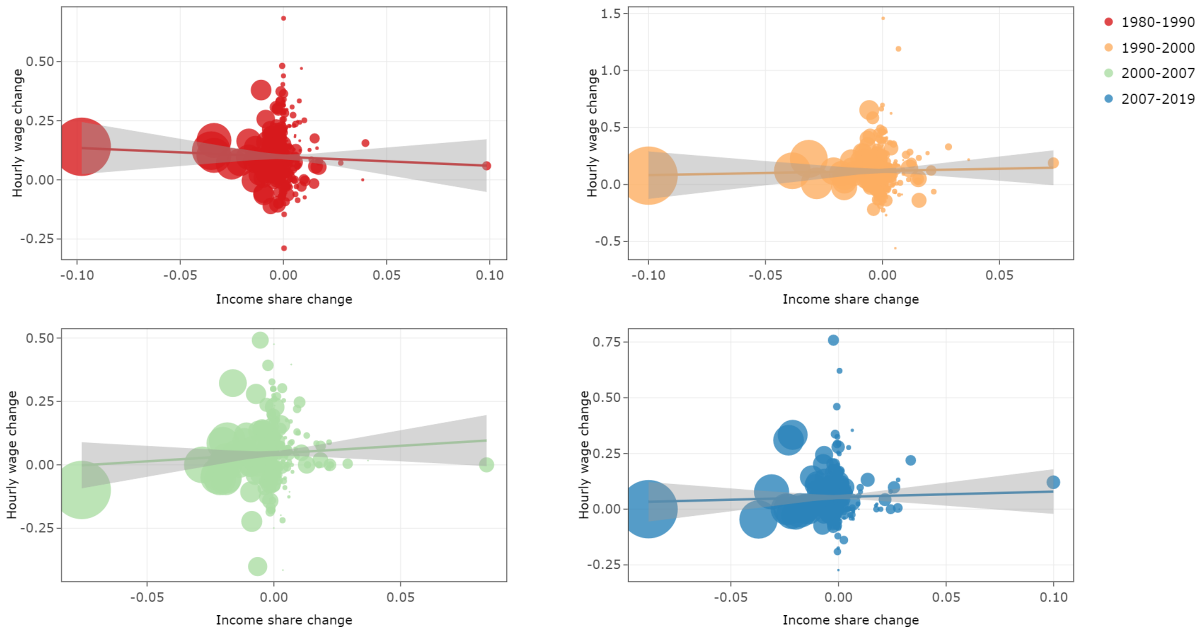


Figure 14: On the x-axis are the level change in income share of each occupation while on the y-axis are the percent change in hourly wage. The size of the points are the income share of the occupation at the beginning of each decade.

No clear pattern seem to emerge from these correlations except that greater income share tend to suffer a decrease in size as already presented in a previous graph. However, it is not because there is no general pattern that there is no change in misallocation and we show in the growth accounting section that actually some specific occupations matter disproportionately for aggregate misallocation in level and in changes.

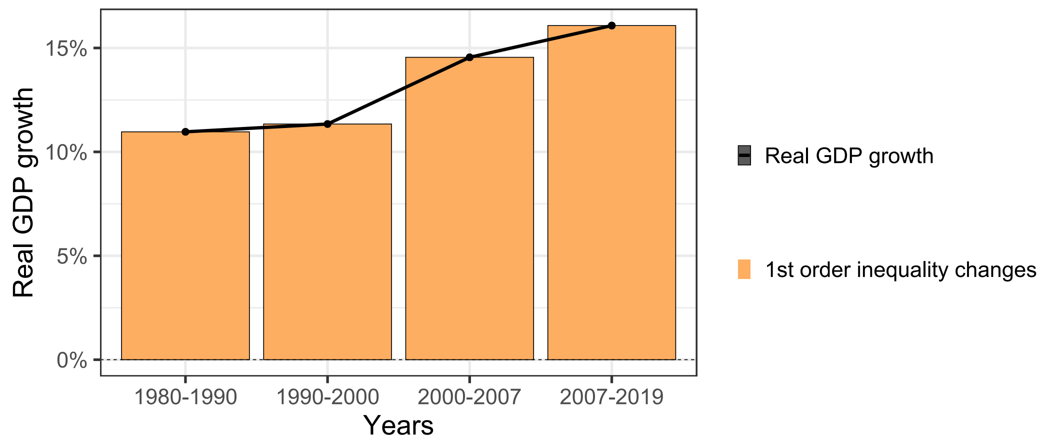
8 Growth accounting/Non-parametric results

8.1 TFP mis-measurement and inequality

We start by studying the results in the US for TFP mis-measurement and the impact of misallocation changes on economic growth. To remind it, we had found that up to a first-order approximation in general economies, we find that:

$$d\ln Y = d\ln z + \Theta_L d\ln L + \Theta_K d\ln K - \underbrace{\Theta_L \sum_{o\in\mathcal{O}} \left(\frac{\Theta_{o,L}}{\Theta_L}\right) d\ln\left(\frac{w_o}{\tilde{w}}\right)}_{dI(\check{\Theta}_L)}$$

where we have shown that $dI\left(\check{\Theta}_L\right)$ is both a change in the Theil index keeping labor income shares fixed and a measure of TFP mis-measurement due to changes in labor allocation across occupations. In the US, we find that the amplitude of this measure is the following:

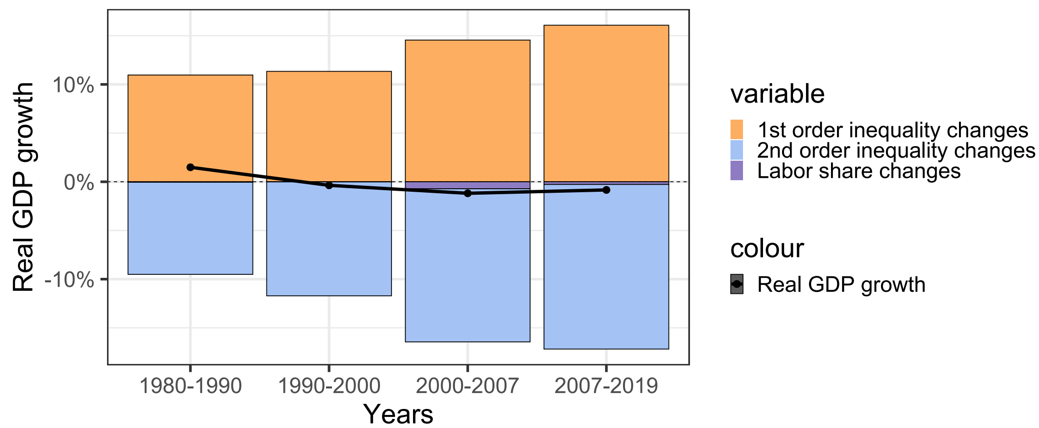


This result means that first order inequality changes $dI(\check{\Theta})$ contributed to real GDP growth at more than 10 percentage points each decade between 1980 and 2019. Furthermore, this contribution actually *increased* in time to reach 16% for the 12 years between 2007 and 2019. This is in direct contradiction with the intuition led by the fact that wage income inequality increased between 1980 and 2019 as presented in the data description section.
Indeed, when we study the first-order change in inequality and we keep income shares fixed, we observe an improvement in workers' allocation as wages in previously important occupations - with large income shares - have decreased.

Nevertheless, the first order change in inequality does not account for the entire change in inequality as the data description imply. Hence, we study the results up to a second-order approximation in general economies using the Tornqvist decomposition and applying it to our measure of misallocation:

$$d\ln Y = d\ln z + \left(\Theta_L + \frac{d\Theta_L}{2}\right) d\ln L + \left(\Theta_K + \frac{d\Theta_K}{2}\right) d\ln K$$
$$\underbrace{-\,\Theta_L \sum_{o\in\mathcal{O}} \left(\frac{\Theta_{o,L}}{\Theta_L}\right) d\ln\left(\frac{w_o}{\tilde{w}}\right)}_{\text{1st order d I}} \underbrace{-\,\frac{d\Theta_L}{2} \sum_{o\in\mathcal{O}} \left(\frac{\Theta_{o,L}}{\Theta_L}\right) d\ln\left(\frac{w_o}{\tilde{w}}\right)}_{\text{Labor share change}} \underbrace{-\,\Theta_L \sum_{o\in\mathcal{O}} d\left(\frac{\Theta_{o,L}}{\Theta_L}\right) d\ln\left(\frac{w_o}{\tilde{w}}\right)}_{\text{2nd order d I}}$$

When we consider the joint change in income share and wages, we find the following result:



Actually, the result for first order changes in inequality is *more than entirely* by the second order changes inequality. Hence, we have that the net contribution of the changes in workers' allocation to real GDP growth across decades is around one percentage point which is an order of magnitude lower than considering only the first order changes. On top of it, the sign flips to become negative which fits more the fact that income inequality increased during that period.

Changes in labor share also contributes negatively to real GDP growth through workers' misallocation (labor share changes also has an impact through factor supply which we do not study) as the labor share decreased between 2000 and 2019 and the contribution of the labor share changes only matters jointly with the first-order change in inequality which impacts positively growth, therefore the decrease in labor share impacted negatively growth up to a second-order approximation.

The second-order changes in inequality mentionned earlier did not only impacted negatively growth but increased in amplitude through decades more than did first-order changes. While the first-order changes in workers' allocation had a positive contribution to real GDP growth from almost 11% in the 80's to 16% in the 2010's, the second-order effect went from a contribution of 9.5% in the 80's to almost 17% in the 2010's. This contributed to the change in sign of the net contribution of workers' allocation on real GDP growth.

8.2 Misallocation decomposition

As a second set of exercises, we study the level of misallocation/inequality and its decomposition in the US for all occupations.

Aggregate misallocation

Following the theory section, we have that the level of misallocation is given by:

$$\exp\left(\Delta^{CD}\right) = (I)^{\Theta_L} = \left(\prod_{o\in\mathcal{O}} \left(\frac{w_o}{\tilde{w}}\right)^{-\left(\frac{\Theta_{o,L}}{\Theta_L}\right)}\right)^{\Theta_L}$$

where a level close to one implies a very good allocation and a level close to zero gives a very bad allocation. Below, we plot the value $1 - \exp\left(\Delta^{CD}\right)$ which is in misallocation terms the % loss to efficient output and in inequality terms it is the positive inequality measure $\mathscr{P}$ that we defined in the theory section:

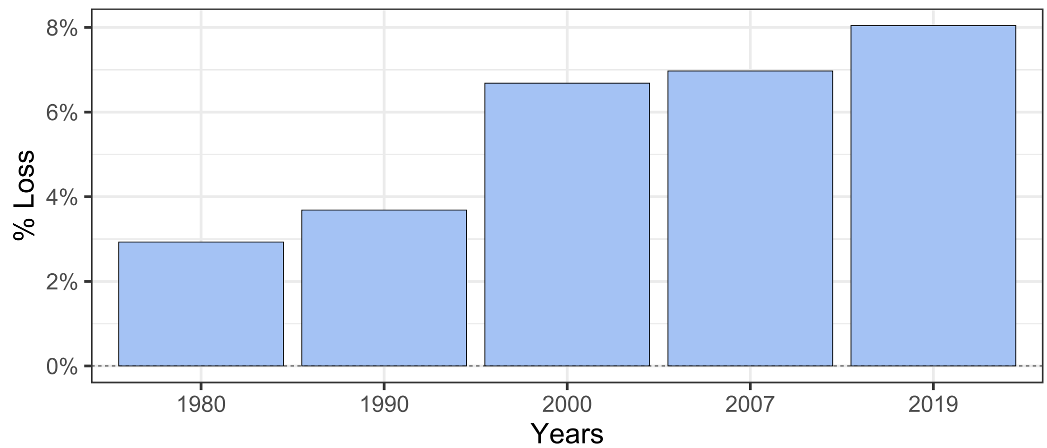


Figure 15: On the x-axis are the % loss to efficient output - equivalent to the level of the positive inequality measure - computed using the labor income shares and the average wage for the 1990 occupations from the IPUMS data while on the y-axis are the different years for which these statistics were computed, usually a decade apart.

We observe that the level of misallocation and inequality more than doubled between 1980 and 2019 and only using the labor income inequality between average wage of occupations for full-time full-year workers. We observe that the most dramatic increase occured during the decade between 1990 and 2000. However, inequality and misallocation increased steadily during all the other decades presented on this graph.

(Olley and Pakes, 1996) and (Melitz and Polanec, 2015) decompositions

Using our application of the (Olley and Pakes, 1996) decomposition, we study how the average log wage ratios between occupations and the covariance between income shares and log wage ratios contributed to misallocation/inequality in each year and how these different parts of the measure evolved through time. First, we present again the decomposition that we take to the data:

$$\Delta^{CD} = \Theta_L \times I = -\Theta_L \times \left(\sum_{o \in \mathcal{O}} \left(\frac{\Theta_{o,L}}{\Theta_L} \right) \ln \left(\frac{w_o}{\tilde{w}} \right) \right) = -\Theta_L \times \left(\widetilde{\ln \Gamma} + \mathrm{Cov}\left(\check{\Theta}, \Gamma\right) \right)$$

which gives the following results using the US data:

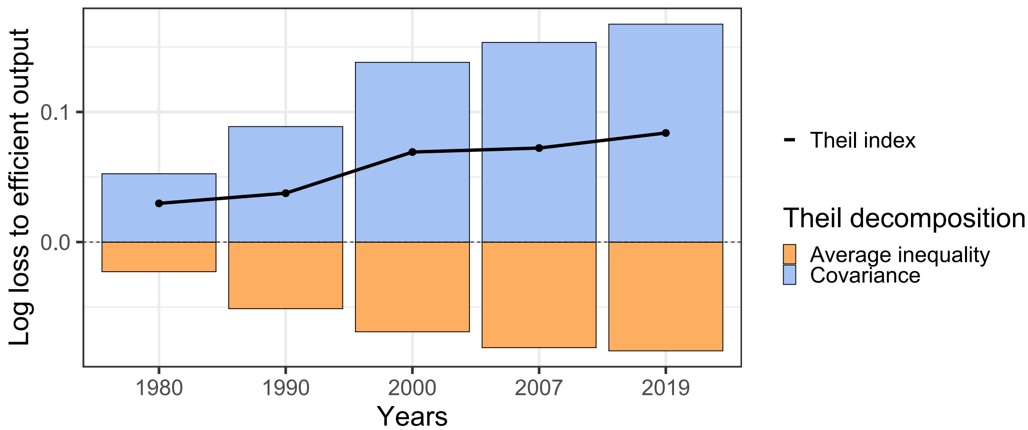


Figure 16: On the x-axis are the % loss to efficient output computed using the labor income shares and the average wage for the 1990 occupations from the IPUMS data while on the y-axis are the different years for which these statistics were computed, usually a decade apart.

We observe first that average inequality has a *negative* contribution to inequality in levels. This is because the average wage ratio across occupations is smaller than one, implying a negative average log wage ratio. We also observe that the contribution of average inequality became more firmly negative through time, implying that more and more occupations existed with a negative log wage ratio. Nevertheless, this does not say anything about the number of workers in each occupation.

The second term which is the covariance between income shares and log wage ratios gives a better indication about it. We observe that the contribution of the covariance more than compensate the negative average inequality and each period. Furthermore, we observe that this overcompensation increases in time. This means that between 1980 and 2019, while the average occupation had a more and more firmly negative log wage ratio, the occupations that matter a lot for the economy, with higher income share and more workers, had a large increase in their log wage ratio. This implies a higher misallocation because the most important occupations had higher wages and it also implies a higher level of inequality because while several important occupations benefited from an increase in their wage compared to the average, a greater number of occupations had an average wage smaller than the average wage in the economy.

To understand better the changes in each year, we use the decomposition from the theory section for misallocation and inequality using the US data. The equation is the following:

$$d\Delta^{CD} = d\left(\Theta_L \times I\right) = -d\Theta_L \times I - d\Theta_L \times dI - \Theta_L \times \left(d\widetilde{\ln \Gamma} + d\mathrm{Cov}\left(\check{\Theta}, \Gamma\right) \right)$$

and we have the following results:

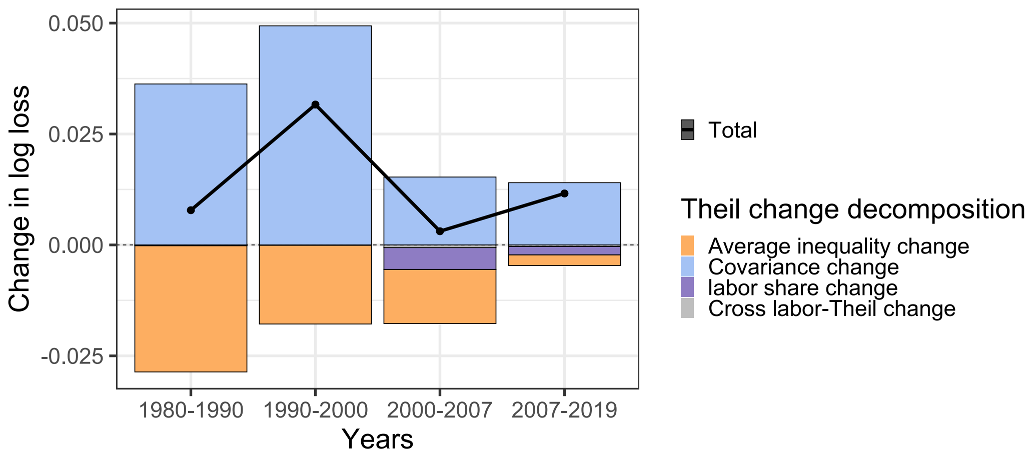


Figure 17: On the x-axis are the changes % loss to efficient output computed using the labor income shares and the average wage for the 1990 occupations from the IPUMS data while on the y-axis are the different decades in which the statistics are computed.

where we observe clearly that the biggest contribution to the increase in misallocation/inequality is the increase in the covariance while the decrease in average inequality was not large enough to compensate. We also observe that the change in the labor share had a slight role played in the decade 2000-2007 for the relatively small change in misallocation. Indeed, the misallocation due to labor mechanically decreases when the labor share decreases.

Our new decomposition

When studying the changes in misallocation/inequality, we can also apply a simple decomposition between the different components of the decomposition that change. We use the following equation for our new decomposition:

$$d\Delta^{CD} = d\left(\Theta_L \times I\right) = -d\Theta_L \times I - d\Theta_L \times dI - \Theta_L \times \left(\sum_{o\in\mathcal{O}} \check{\Theta}_o d\, \ln\Gamma_o + \sum_{o\in\mathcal{O}} d\,\check{\Theta}_o \ln\Gamma_o + \sum_{o\in\mathcal{O}} d\,\check{\Theta}_o d\, \ln\Gamma_o\right)$$

and we apply it using US data on occupations:

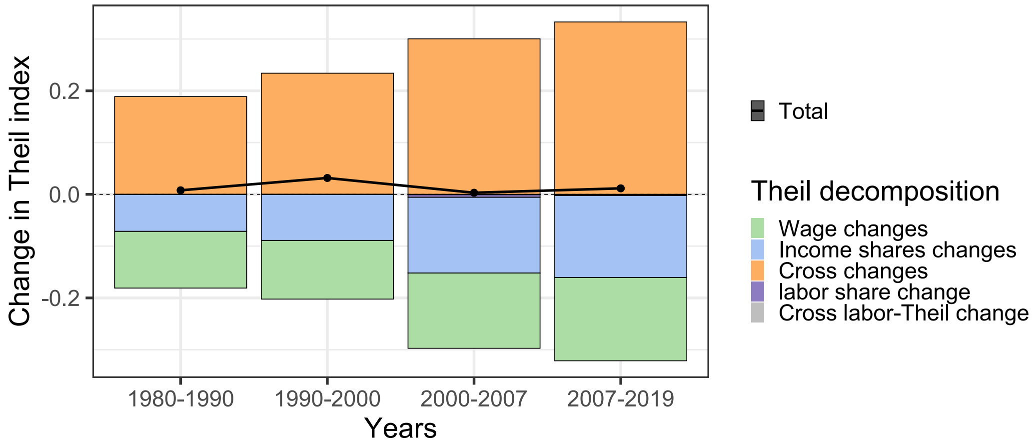


Figure 18: On the x-axis are the changes % loss to efficient output computed using the labor income shares and the average wage for the 1990 occupations from the IPUMS data while on the y-axis are the different decades in which the statistics are computed.

where we observe very large changes in log wages, income shares and cross changes in both log wages and income shares. The amplitude and contribution of these components to aggregate misallocation changes are much larger than the ones highlighted using (Olley and Pakes, 1996) and (Melitz and Polanec, 2015) decompositions. Indeed, we observe very large negative contributions to misallocation changes from wage and income share changes. This means that when wages are high compared to average wage in an occupation at the beginning of a decade, the income share of that occupation tends to decrease a lot. Similarly, when the income share of an occupation is high at the beginning of a decade, the wage ratio in that occupation tends to decrease a lot.
However, the net contribution to misallocation changes is positive and this is due entirely to the joint changes in income shares and wage ratios. This component compensates totally the two others and push the misallocation change in positive grounds. This cross component actually means that even though income shares and wage ratios vary negatively when one is fixed at the decade's initial level, when both change jointly they co-vary very strongly, positively or negatively. This implies that when an occupation becomes more and more important, its wage ratio increases accordingly while when an occupation becomes less and less important for the economy, its wage ratio decreases.

Differentiating between continuing, exiting and entering occupations

Now that we considered the general results for all occupations, we separate occupations dynamically between the continuing occupations - which continued to be used in the economy during the decade -, the exiting occupations - that exited during the decade - , and the entering occupations - that entered during the decade -. As presented in the theory section, we provide two decompositions. In the first case, we adapt the decomposition in (Melitz and Polanec, 2015) to misallocation, we use the additive decomposability of the Theil indices and we decompose the change in Theil indices of continuing occupations between changes in average inequality and in covariance. This gives the following equation:

$$
\begin{aligned}
\Delta_2 - \Delta_1 = -d\,\Theta_L \times I - d\Theta_L \times dI - \Theta_L \times &\\
\Bigg[\underbrace{\left(\widetilde{\ln \Gamma_{c,2}} - \widetilde{\ln \Gamma_{c,1}}\right) + \left(\mathrm{Cov}\left(\check{\Theta}_{c,2}, \Gamma_{c,2}\right) - \mathrm{Cov}\left(\check{\Theta}_{c,1}, \Gamma_{c,1}\right)\right)}_{\text{Continuing occupations}} &\\
+ \underbrace{\left(\ln\left(\frac{\tilde{w}_{c,2}}{\tilde{w}_2}\right) - \ln\left(\frac{\tilde{w}_{c,1}}{\tilde{w}_1}\right)\right)}_{\text{Continuing occupations}} + \underbrace{\check{\Theta}_{e,2}\left(\ln\left(\frac{\tilde{w}_{e,2}}{\tilde{w}_2}\right) - \ln\left(\frac{\tilde{w}_{c,2}}{\tilde{w}_2}\right)\right) + \check{\Theta}_{e,2}\left(I_{e,2} - I_{c,2}\right)}_{\text{Entering occupations}} &\\
+ \underbrace{\check{\Theta}_{x,1}\left(\ln\left(\frac{\tilde{w}_{c,1}}{\tilde{w}_1}\right) - \ln\left(\frac{\tilde{w}_{x,1}}{\tilde{w}_1}\right)\right) + \check{\Theta}_{x,1}\left(I_{c,1} - I_{x,1}\right)}_{\text{Exiting occupations}} \Bigg]&
\end{aligned}
$$

Then, using the data from the US, we obtain the following results:

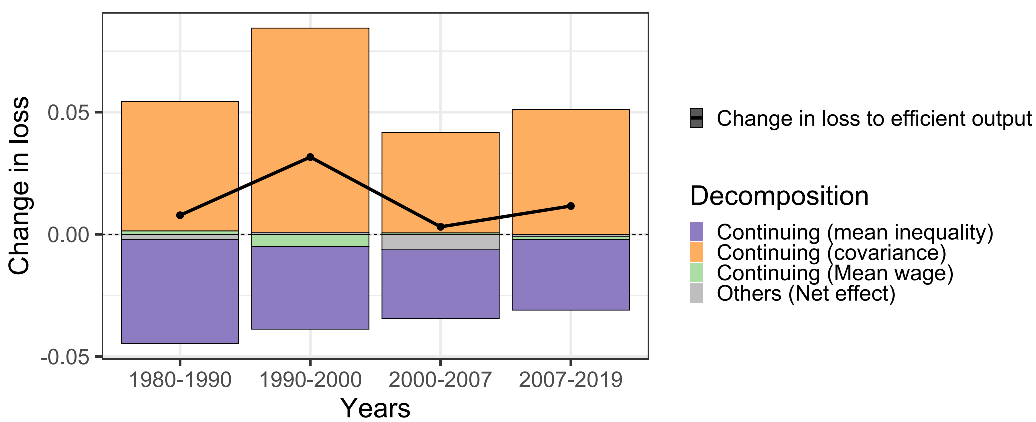
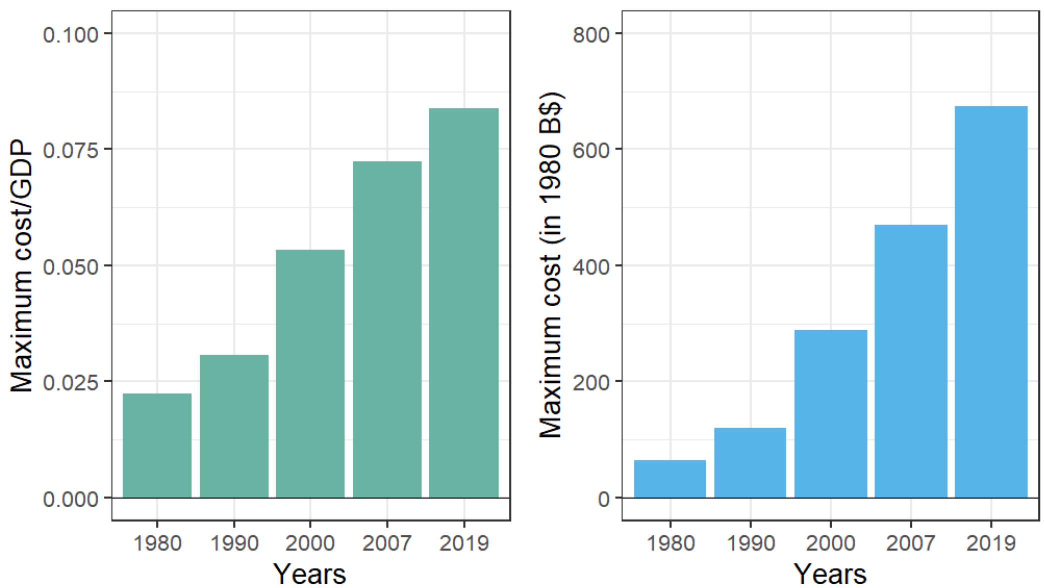


Figure 19: On the x-axis are the changes % loss to efficient output computed using the labor income shares and the average wage for the 1990 occupations from the IPUMS data while on the y-axis are the different decades in which the statistics are computed.

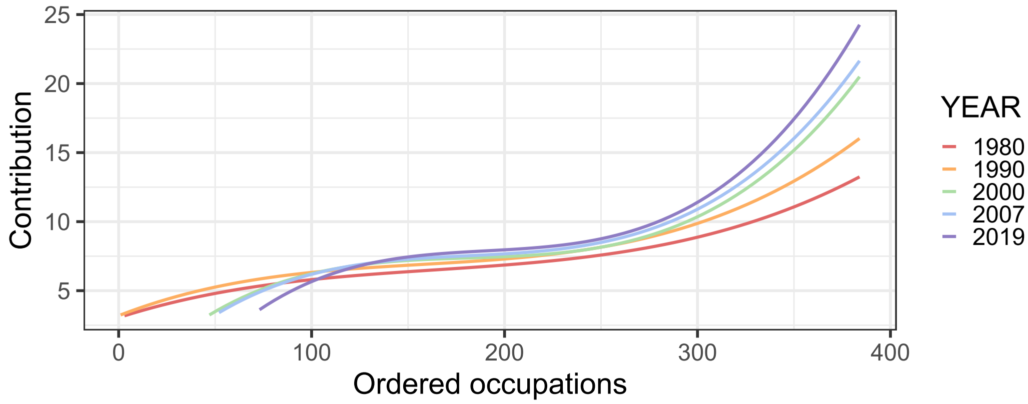
when using the occupations from 1990, the occupations classified as entering or exiting are not numerous enough to weight crucially on the misallocation changes. Hence, we have that the bulk of the contribution comes from the continuing occupations and we recover the same intuition as using the (Melitz and Polanec, 2015) decomposition for all occupations.

8.3 No trade-off results

Finally, we provide results on the maximum amount of resources that could be invested in removing workers' mobility frictions. These results use the non-parametric distance to the frontier result presented in the theory section which takes shape in the form of the Theil index. Using data on wages in occupations in the US between 1980 and 2019, we find the following shares of real GDP that could be invested in each year. The graph on the left does not take into account price changes and economic growth while the graph on the right does. Three effects kick in. First, misallocation in the form of income inequality increased through time, explaining the increase in the maximum cost that could be invested. Second, on top of the misallocation increase, prices increased, implying a large increase in the maximum cost when keeping prices fixed. Finally, real GDP increased exponentially between 1980 and 2019, implying that the absolute maximum amount of money that could be invested increased much more than the maximum share of real GDP that could be invested.



Furthermore, there exists a large heterogeneity in the potential gains in reallocation. On the graph below, we provide a graph of the contribution to real GDP between occupations ordered from the lowest to the highest contribution. We see that the gap between the lowest and highest levels increased in each decade, implying an increase in potential gains from reallocation on top of the aggregate gains in reallocation presented above.



9 General equilibrium results

9.1 Calibration

We use two sets of data to calibrate our model, the IPUMS data from 2007 and the BEA 15 industries input-output tables from 2003 (which we will update using the data from 2007 when it will be available). Then, to calibrate our elasticities, we use estimations from the literature and study different combination of variations of these estimations.

Calibration from the data

To recover the data necessary for the production block of our model, we use input-output data from the BEA. It allows us to recover the gross output from each major 15 sectors in the US economy and the share that is used as an input for other sectors. To match it with the major occupations' classification in the IPUMS data, we reduce it to 13 industries and provide the matching between the two sets of industries in the data appendix.

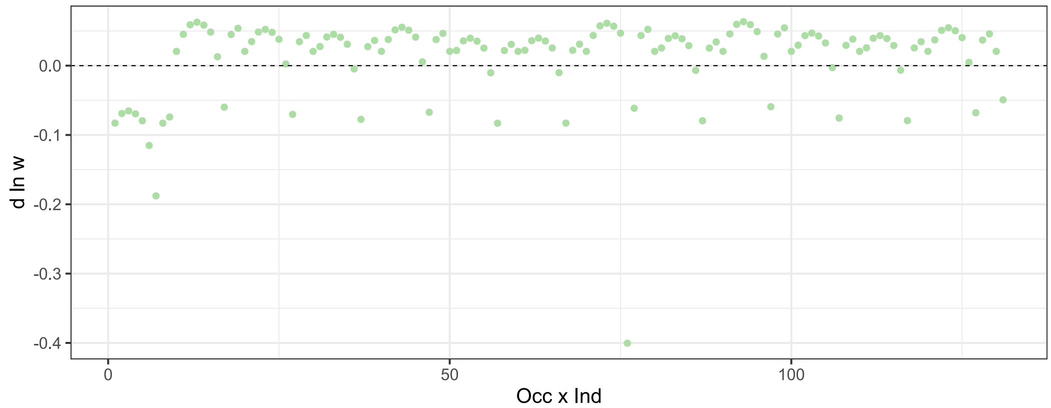
Description	Parameter	Data construction
Technical coefficient matrix	Ω_I	BEA 13×13 matrix divided by sectoral gross output
Leontief matrix	Ψ	$(I - \Omega_I)^{-1}$
Labor share of cost	Ω_L	IPUMS average hourly wage multiplied by the number of hours worked in the sector
Capital share of cost	Ω_K	Value-added per sector from BEA data minus the labor share computed above
Final demand share	Υ	Simple share of total final demand allocated to a specific sector in BEA data
Domar weight	ϑ	Gross output divided by GDP computed as sum of value-added in BEA data
Labor income shares	Θ_L	$\Omega_L^{'}\vartheta$
Capital income share	Θ_K	$\Omega_K^{'}\vartheta$

9.1.1 Calibration from the literature

Description	Parameter	Value	Literature
Occupation elasticity	η	1.2	[Acemoglu and Autor, 2011]
Input elasticity	σ	0.5	[Atalay, 2017]; [Oberfield and Raval, 2021]
Final demand elasticity	ρ	0.9	[Herrendorf et al., 2013]; [Oberfield and Raval, 2021]
Sectoral mobility elasticity	κ	1.4	[Galle and Lorentzen, 2023]

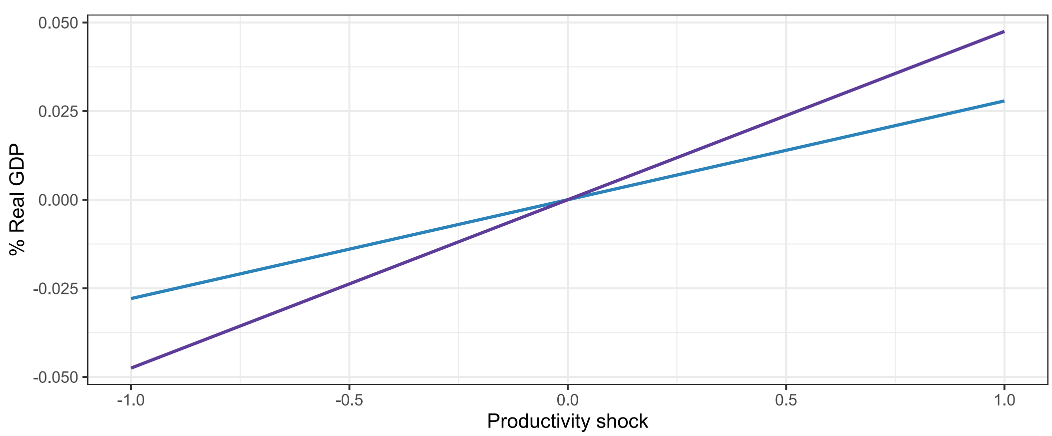
9.2 Comparative statics on the parameters

Benchmark model and shock example To better understand our model output, we start by simulating a simple 1% shock on the production occupation in the manufacturing sector. Using our benchmark calibration, we find the following changes in wages across combinations of occupations and industries:

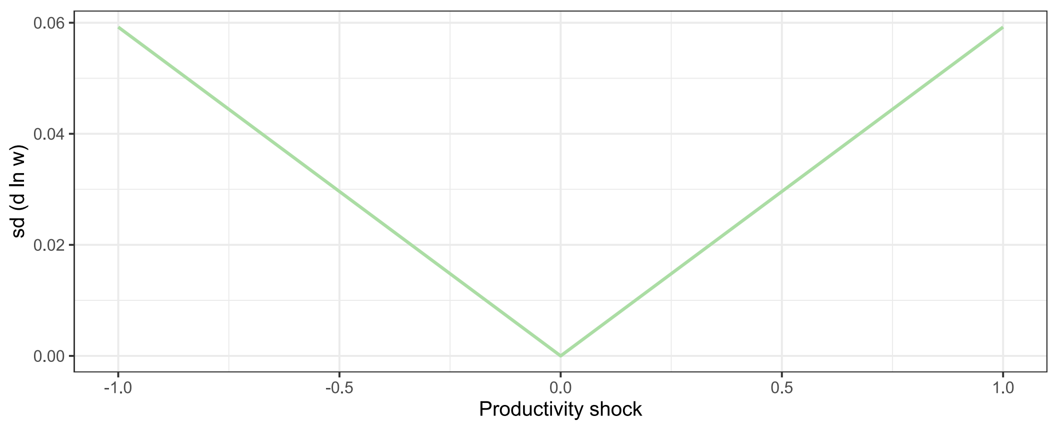


Using these results, we are able to recover the inequality changes implied by the shock and how the changes in misallocation impact real GDP. We study different variations in the shock and in the parameters in the following paragraphs to understand better how our model behaves.

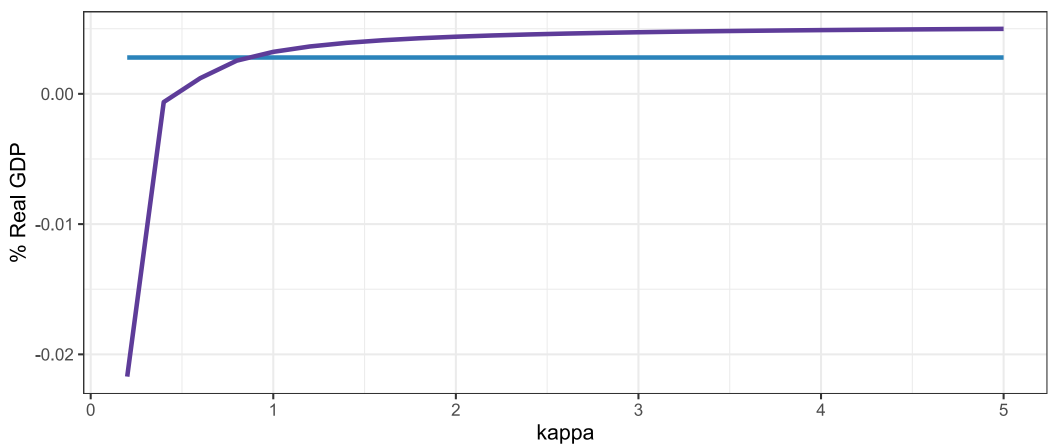
Size of the shock First, we study different levels of productivity shocks on the production occupation in the manufacturing sector. The linear effect is simply given by the income share of that combination of occupation and industry in the economy (blue line) while the total effect is the sum of the linear effect and the misallocation changes in the shape of the first-order change in the Theil index (purple line). We find here that the linear effect is exacerbated by the misallocation changes as the impact of the shock on real GDP is almost doubled at every level of productivity shock.



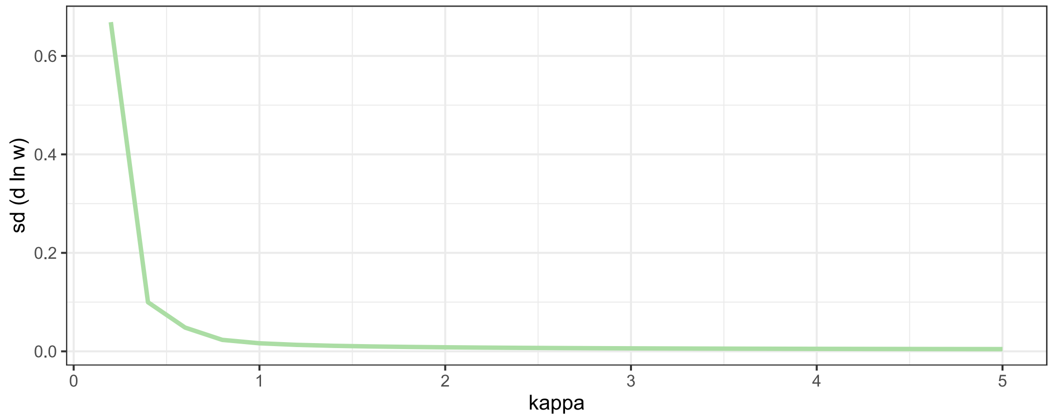
On the inequality side, we observe that the standard deviation of real wage changes across occupations and industries increases linearly in the size of the productivity shocks.



workers' mobility When considering changes in workers' mobility and more specifically changes in κ, we find that, as expected, the linear effect is constant while the misallocation effect rapidly becomes positive as the workers' mobility increases. The sign of the misallocation effect depends on which occupation and industry is shocked but the fact that it rapidly decreases as workers' mobility increases is similar across shock.

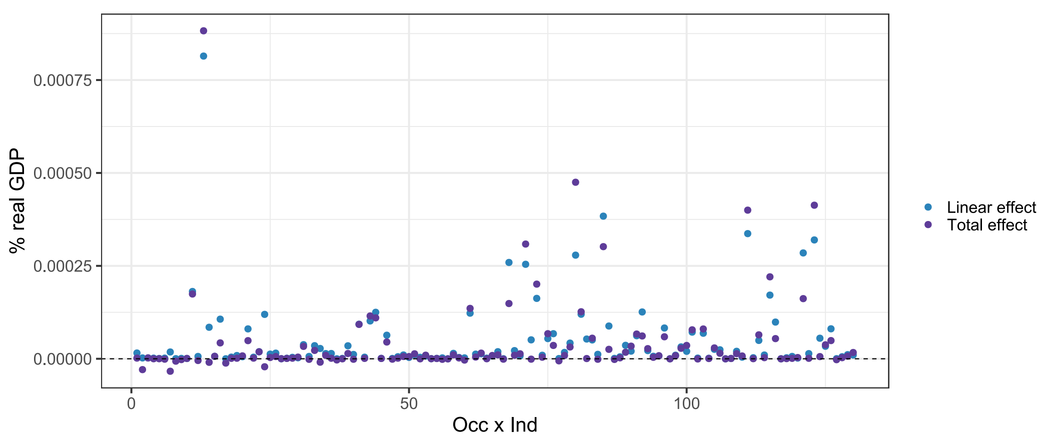


On the inequality side, the standard deviation of wage changes also decreases very rapidly as workers' mobility increases. This implies that, as workers can adjust more easily to the shock by moving between sectors, the impact of the shock on inequality decreases rapidly.

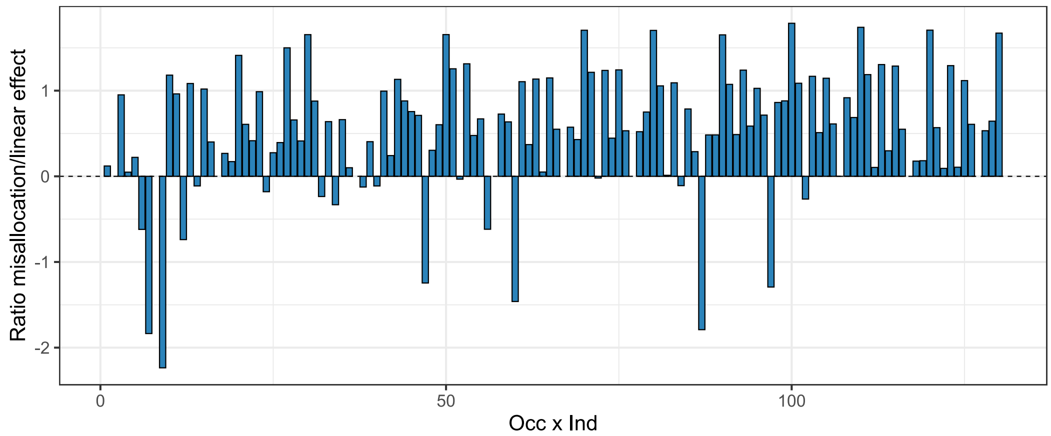


9.3 Comparative statics on the productivity shocks

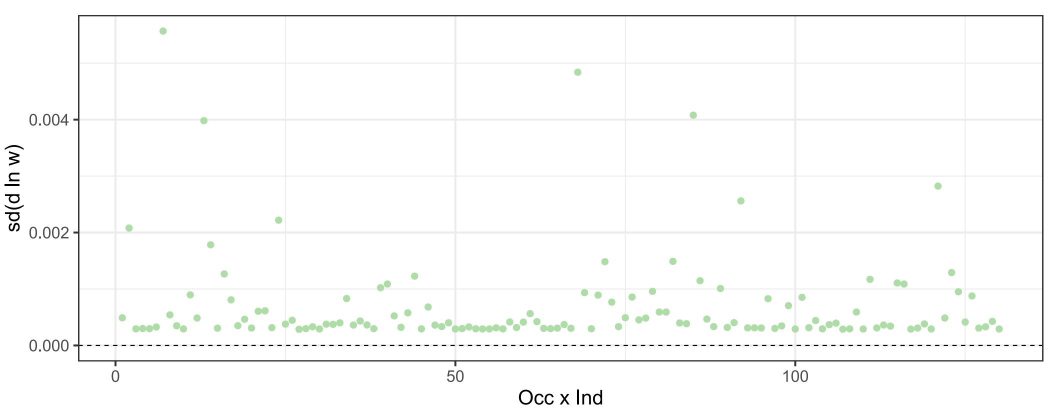
We now move to the study of productivity shocks across occupations and industries. On the graph below, we observe the linear and total effects on real GDP of productivity shocks in each combination of occupation and industries.



The relative amplitude of the linear and misallocation effects on real GDP varies a lot as can be observed on the following graph:



For some combinations of sectors and occupations, the misallocation effect resulting from the productivity shock is almost inexistent while in others the misallocation effect is twice as large as the linear effect. Furthermore, we removed the outliers ratio greater than 5 to increase the readibility of the graph which implies a greater heterogeneity than the one presented in the graph.
Finally, we present the results for the standard deviation of the real wage changes for each shock. In that case too we observe a large heterogeneity across shocks where the standard deviation can be inexistent for some shocks while very large for others.



10 Conclusion

In this paper, we provide several theoretical and empirical results on the relationship between economic growth, wage inequality and workers' mobility. Our results suggest that imperfect workers' mobility matters along three principal dimensions. First, it matters to understand the relationship between economic growth and inequality. Our results provide a new relationship between these two variables through imperfect workers' mobility which takes the shape of a very well-known inequality index. Second, workers' mobility is important to understand theoretical mechanisms on how the economy reacts to shocks. If inequality matters to policymakers, it is important to be able to provide a better understanding on how efficiency and inequality are related in complex economies. Finally, we provide empirical results suggesting that imperfect workers' mobility matters significantly for economic growth and the level of real GDP in the US. We believe that the relationship between workers' mobility and economic growth and specifically the underlying forces behind the frictions that workers face would be a fruitful line of research for future works.

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