Obfuscation Through Entry:

The Relationship Between Uncontested Incumbents and Incumbent Upsets

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Abstract

While incumbents overwhelmingly win contested reelection bids, a proper understanding of candidate behavior in races with incumbents must account for the less common outcomes that also occur. Analysts risk mischaracterizing behavior by studying only contested races or by ignoring the possibility that an advantaged incumbent may lose due to "rational complacency." We argue below that incorporating incumbent uncertainty about challenger strength is both realistic and necessary to achieve an equilibrium in which an advantaged incumbent may successfully defend her seat, run uncontested, or be upset by a challenger, all with positive probability. Several implications for empirical work arise: comparative statics may switch signs if analyses exclude uncontested elections, a strong potential challenger is less likely to win the higher the likelihood of an upset conditional on entry, and reducing the frequency of uncontested elections may entail the trade-off for voter welfare of producing more extreme platforms in contested elections.

Keywords: Uncontested elections, upsets, voter welfare, incumbency advantage

Word count: 10,125

Introduction

In the 2014 midterm elections, Steve Southerland, a two-term Republican incumbent representing Florida's 2nd Congressional District, faced a challenge from political upstart Gwen Graham. At the outset, the Republican establishment and Southerland himself considered the seat to be safe. Yet Graham ran an energetic and decidedly ideologically moderate campaign, while Southerland did little to distance himself from Republican obstinance during the Obama presidency. When asked about his campaign's slow start and the early lead the centrist Graham had taken in the polls, Southerland explained away his lack of concern by saying, "It's easy to score touchdowns when the defense isn't on the field," reportedly leaning back against his pick-up truck as if to underscore the recognition of his own complacency. Soon after, Southerland and his party would begin pouring resources into the race. It would be too late to make up for early mistakes, though, and Graham would go on to defeat Southerland that November (Sherman 2014).

Southerland should have been well-acquainted with the pitfalls of incumbent complacency, especially around the failure to moderate ideologically. Just four years earlier, then-political neophyte Southerland defeated seven-term incumbent Allen Boyd in the 2010 midterm election in the same district. Boyd, a Blue-Dog Democrat, entered the race with a four-to-one fundraising advantage, but in addition to an unclear stance on the Affordable Care Act, Boyd abandoned past fiscal conservatism and embraced President Obama's stimulus spending. Reflecting his uncharacteristic tack leftwards, signs reading "Blue Dog = Lap Dog" emerged around the district, impugning his credentials as a moderate (Ward 2010).

In neither of these instances of incumbent upsets does it seem right to infer that the current officeholders did not benefit from the well-documented incumbency advantage. The fact that Southerland and his party considered the seat to be safe suggests the candidate had no ex ante liabilities, and Boyd was the picture of an advantaged incumbent, with over a decade in office and a massive warchest. Further, each might have suspected his challenger was likely to be strong from the opponent's decision to contest the race in the first place. Instead, it appears each made a calculated decision to run closer to their own ideological preferences than the district's. While they may appear complacent in retrospect, to treat the decisions as irrational does a disservice both to analyses of these particular elections and to analyses of elections with an incumbent more broadly.

Certainly incumbent upsets are rare. In the 2016 U.S. Congressional elections, 97% of House incumbents won reelection, with upwards of 50 of them running unopposed. This rate of incumbent retention is on the high end but within the range of reelection rates in recent decades, and dozens of incumbents go unchallenged in every election cycle. It is particularly striking that rates of incumbent reelection and uncontested elections remained at their usual levels even in a year in which over 75% of the United States disapproved of the performance of Congress on election day[1] and in which media accounts suggested that many congressional seats might be up for grabs (Martin & Burns 2016).

The seemingly assured, and not infrequently unfettered, path to victory for sitting officials has long been a subject of scholarly attention and even popular concern. Yet in the extensive study of the incumbency advantage, two consistent forms of selecting on the dependent variable threaten the validity of theoretical predictions and empirical analyses. The first is to ignore uncontested elections, which occurs in many theoretical studies and many more empirical analyses and which ignores the potential for entry against an incumbent to signal information about challenger quality. The second is to treat cases in which incumbents lose as indicative of incumbent quality rather than incumbent behavior.

This paper presents a model of elections that addresses both of these omissions. A potential challenger, who may be a strong or weak type, first decides whether to enter the election or not. The incumbent, while possessing an advantage over either strength challenger, is uncertain as to the strength of the potential challenger. If the challenger enters, the two, policy-motivated candidates engage in policy competition. Uncontested elections and incumbent upsets both occur in equilibrium, and importantly the upsets result not from exceptions to the rule of incumbency advantage but rather because the incumbent's uncertainty persists and she may wish to push the limits of her advantage. In addition to better understanding uncontested elections and incumbent upsets, an equilibrium in which these less common events arise endogenously offers new insights into the behavior of candidates even in the more routine event that incumbents successfully defend their seats.

The next section motivates the incorporation of incumbent uncertainty into the model, noting how this approach differs from and contributes to various related literatures. The following section details the model and derives the equilibrium cases. The remainder of the paper distills a number of insights from the model, including often counterintuitive implications for the evaluation of electoral competition and voter welfare. Specifically, results follow on the equilibrium dependence between the probability that incumbents run uncontested and the probability that challengers upset incumbents, the way in which strong challengers suffer even as more strong challengers defeat incumbents, and the tension between reducing uncontested elections and encouraging more moderate policy proposals in contested elections for the sake of voter welfare. A brief conclusion follows.

[1] `http://www.realclearpolitics.com/epolls/other/congressional_job_approval-903.html`

Incumbent Uncertainty About Challenger Strength

The model presented in this paper seeks to understand why an incumbent who could ensure victory for herself with a campaign targeted at the median voter would rationally risk defeat and adopt a more ideologically extreme stance. Further, it seeks to do so without assuming the decision of a potential challenger to contest the election is "as if random" and without consequence for candidate behavior in the election. In so doing, it offers an explanation of the way in which uncontested elections, incumbents defending their seats, and challengers upsetting incumbents may all arise in the same equilibrium, and this unified explanation provides a deeper understanding of the relationship among these outcomes.

In a model with full information, an incumbent who possesses an advantage over any potential challenger would never be upset. A much richer and more realistic set of outcomes emerge in a model in which the incumbent is uncertain about challenger strength. This need not imply that the incumbent does not possess an advantage even over strong challengers, rather only that she be unsure of the extent of her advantage over a given challenger. Simply supposing such uncertainty exists is dubious, as an incumbent would attempt to glean any information she could about about the strength of a challenger she faced. Endowing the potential challenger with private information and incorporating the potential challenger's decision to contest the election allows for some signaling of challenger strength to occur through the entry decision, intentionally or otherwise. From an analytical perspective, this elucidates the connection between a potential challenger's decision to contest the incumbent's seat and subsequent candidate positioning in equilibrium and leads to a range of new insights regarding the evaluation of electoral competition and voter welfare.

Under the plausible assumption that strong challengers benefit if the incumbent underestimates them and weak challengers benefit if the incumbent overestimates them, we find that the initial uncertainty persists. The incumbent will never know with certainty whether she faces a strong challenger or a weak challenger in equilibrium, and this underlies the incumbent's willingness to push the limits of her advantage, i.e., to display "rational complacency." The intuition for this result centers on the interaction of the uncertainty with the challenger's strategic entry decision. If an incumbent believes she will only face strong challengers, a weak challenger will induce significantly more moderation in policy from an incumbent simply by entering than the policy moderation such a challenger might expect based strictly on her viability as a candidate. If, however, an incumbent is quite sure she will only be challenged by weak candidates, a strong candidate becomes a wolf in sheep's clothing, able to capitalize on an incumbent's complacency and potentially achieve an upset victory. In the presence of some uncertainty over how strong a potential challenger may prove to be as a candidate, the potential benefit to a weak challenger of being overestimated and the potential benefit to a strong challenger of being underestimated effectively preclude separating equilibria in which only one type of challenger enters. Rather than signaling, the challenger achieves obfuscation through entry.

Two questions about the decision to challenge an incumbent up for reelection have consistently motivated studies of the significance of upset victories and uncontested elections for voter welfare. First, to what extent and to what effect do strong challengers wait for an open seat to run for office, the so-called "scare-off" effect of incumbency (Abramowitz 1991, Cox & Katz 1996, Jacobson 1989, Krasno & Green 1988, Stone, Maisel & Maestas 2004)? Second, why would weak candidates ever challenge a sitting office-holder (Banks & Kiewiet 1989, Canon 1993)? The reasoning implicit in these questions has a degree of intuitive appeal, and goes something as follows: given the well-documented incumbency advantage, strong challengers ought to increase their comparatively better chances at winning by competing on a more level playing field, i.e., entering an open-seat race;[2] meanwhile, weak challengers only stand to decrease their already low chances of winning by taking on a sitting official. This line of reasoning tacitly makes the strong assumption that an incumbent knows the strength of a challenger she faces.

The model presented in this paper suggests the logic underlying these common arguments about strategic entry may be incomplete in the presence of uncertainty. If potential challengers care about the winning policy even if they lose, and if incumbents must adopt different policies to retain their seats when challenged than if running uncontested, then the incumbent will never know with certainty the strength of a challenger she faces. It is never the case, as the studies above presume, that only strong or weak challengers enter, because each would prefer to be mistaken for the other.

The empirical work on strategic challengers has traditionally, and out of necessity, focused on observable measures of candidate quality, such as previous officeholding or campaign experience. From the theoretical standpoint of this paper, this focus errs in that it ignores the effects of potentially unobservable (or at least less readily observable) determinants of candidate quality that may correlate imperfectly with observable measures. For instance, it ignores the difference in quality within the population of untested candidates. Although some papers recognize the possibility and potential relevance of unobservable measures of quality (e.g., Canon (1993)), none engage seriously with the uncertainty surrounding untested candidates and the implications of the potentially large differences in quality among this group.

The asymmetric nature of the uncertainty that appears in the model below – with the potential challenger possessing better information than the incumbent about the challenger's strength – contributes an element of realism that previous theoretical work has hitherto largely foregone. It is quite likely that a potential challenger should be better informed about his quality, especially a challenger that was previously untested

[2] Hall & Snyder (2015) provide a thorough summary of the development of this received wisdom in a study that finds little empirical evidence to support the notion that scare-off explains the incumbency advantage.

or had only held office at a lower level. A potential challenger will have better knowledge of his motivations for running (genuinely seeking office or just running to keep an incumbent honest), means (personal and political capital), and party support (financial and otherwise, e.g., appearances from high-profile party members on the campaign trail).

The failure to include such asymmetric information also reduces the significance of the challenger's entry decision in the model. (Carson 2005) demonstrates the pitfalls of performing empirical analyses that ignore the strategic interaction and learning that occurs between challengers and incumbents, and a spate of recent, mostly theoretical scholarship examines some element of the strategic interaction and learning that occurs between candidates and voters (Ashworth 2005, Ashworth & Bueno de Mesquita 2006, Ashworth & Bueno de Mesquita 2008, Banks & Sundaram 1998, Bernhardt, Camara & Squintani 2011, Epstein & Zemsky 1995, Gordon, Huber & Landa 2007, Gordon & Landa 2009, Gordon, Huber & Landa 2009). None of these, however, sheds light on the potential signaling that may occur through challenger entry that determines the value an incumbent may extract from her advantage.[3] Earlier models of challenger entry (Feddersen, Sened & Wright 1990, Osborne 1993, Osborne 2000) do not speak to incumbency, and models of simultaneous spatial competition in which candidates possess differing levels of valence as well as some element of uncertainty (Groseclose 2001, Aragones & Palfrey 2002, Aragones & Palfrey 2005, Hummel 2010) do not engage with strategic entry or asymmetric information about quality.

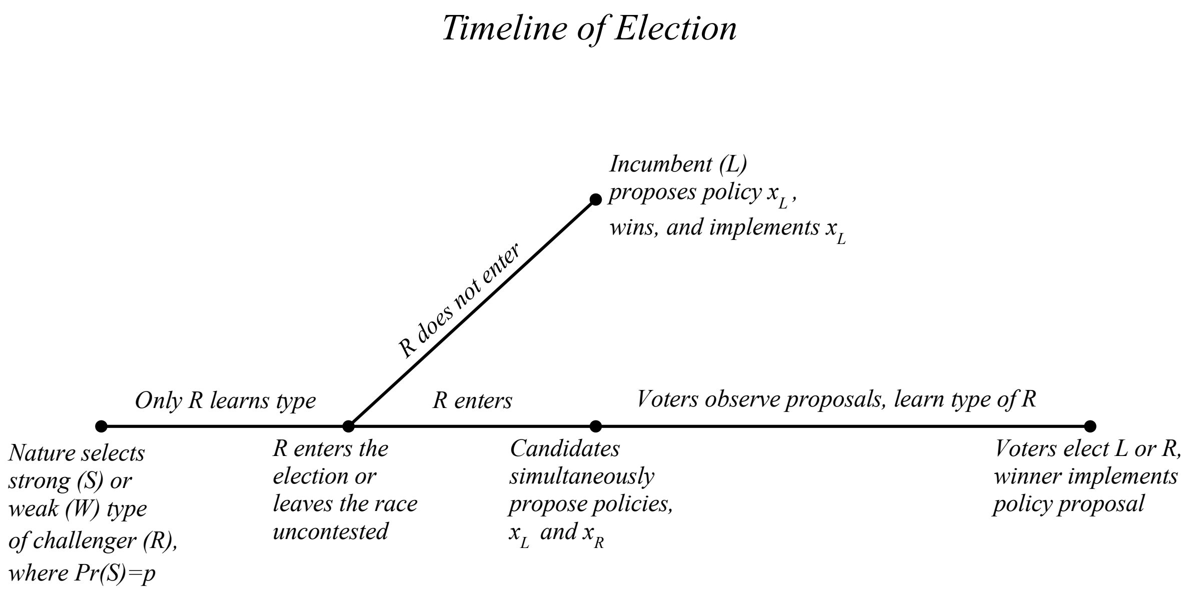
Carter & Patty (2015) study dynamic policy competition where candidates possess different valences and make an effort decision akin to entry. While their model shares several features with the model in this paper, the authors assume candidates to be vote-maximizing, rather than some combination of office- and/or policy-motivated, a significant point of difference (see Patty (2002)). In the context of a model with entry, this distinction is crucial. Their model employs sequential policy announcements and explores the impact of different orders of play. This model here employs simultaneous policy announcements, which seems a more natural assumption, though importantly, the findings of the model below prove robust to sequential (rather than simultaneous) policy announcements.

The Model

The set of players consists of an incumbent, L, a (potential) challenger R, and a continuum of voters whose ideal points are strictly ordered in $\mathbb{R}$. Let $c \in \{L, R\}$ denote an arbitrary candidate, v denote an arbitrary

[3] It is noteworthy that Gordon and Landa as well as Epstein and Zemsky uncover a tendency for incumbents of different types to pool in the equilibria of their models, as is found for challengers of different strengths in this paper.

Figure 1: Order of Play



voter, and V denote the median voter. The potential challenger ("he") is either a strong type (S) or a weak type (W). The Incumbent ("she") takes only one type, which may be thought of as strong.

Nature first selects a type of challenger, $t \in \{S, W\}$, choosing a strong type ($t = S$) with probability $p \in (0, 1)$, where p reflects the strength of the pool of potential candidates. The challenger learns his type and then decides to enter the race or leave the seat uncontested. The entry decision of the challenger is immediately observable to all players, but the incumbent does not know the challenger's type. The incumbent and the challenger (if contesting the election) simultaneously propose a policy, $x_c \in \mathbb{R}$. Finally, all voters observe the policy proposals, learn the challenger's type, and cast a vote for one of the candidates that entered the election. The candidate that garners a majority of votes wins the election and implements his/her platform.

Each player's utility from an implemented policy is strictly decreasing in the distance of the policy from the player's ideal point, $M_v, \forall v$, and $M_c, c = L, R$. Specifically, the disutility individual i receives from a winning policy x is given by $-|M_i - x|$. We assume that $M_L < M_V < M_R$. Both types of challenger share the same ideal point. For convenience, we set $M_V = 0$, hence $M_L < 0 < M_R$.[4]

Candidates incur a strictly positive cost upon entering the election, $C > 0$. Many interpretations are

[4]As an extension below, we allow the candidates to be office-motivated as well as policy-motivated.

compatible with this quantity, including the signatures required to appear on a ballot, prices in the relevant media market, and the geographic span or population of the constituency, all of which increase the cost of running.[5] The cost of running in the race enters additively into the candidates' utility. Payoffs to the candidates are then as follows:

$$U_L = \begin{cases} -|M_L - x_L| - C & \textit{if } L \textit{ wins} \\ -|M_L - x_R| - C & \textit{if } R \textit{ wins} \end{cases} \tag{1}$$

$$U_R = \begin{cases} -|M_R - x_R| - C & \textit{if } R \textit{ wins} \\ -|M_R - x_L| - C & \textit{if } L \textit{ wins and } R \textit{ enters} \\ -|M_R - x_L| & \textit{if } L \textit{ wins and } R \textit{ does not enter} \end{cases} \tag{2}$$

Voters derive non-policy utility specific to each (type of) candidate, in addition to disutility from the implementation of winning policies not at their ideal point. Some degree of valence is associated with each (type of) candidate and accrues to the voter if that candidate is elected, entering additively into each voter's utility function. Denote the incumbent's valence by A_L, the strong type of challenger's valence by $A_{R|S}$, and the weak type of challenger's valence by $A_{R|W}$. A voter's payoffs are specified below:

$$U_v = \begin{cases} -|M_v - x_L| + A_L & \textit{if } L \textit{ wins} \\ -|M_v - x_R| + A_{R|t} & \textit{if } R \textit{ wins and is of type } t \in \{S, W\} \end{cases} \tag{3}$$

Granting voters full information when deciding whom to vote for captures the idea that the challenger's type is revealed over the course of the campaign but after the candidates stake policy positions.

Imposing the ordering $A_L \geq A_{R|S} > A_{R|W}$ embodies the assumption that the incumbent possesses an advantage over either type of challenger, but a strictly larger advantage over the weak type, W, than over the strong type, S. Defining $\underline{a} := A_L - A_{R|S}$ and $\overline{a} := A_L - A_{R|W}$, the incumbent then possesses a

[5] All (types of) candidates face the same cost of entry. Including some costs to entry are necessary if we want to see strategic entry (and strategic "non-entry"), as discussed in Callander (2008). Introducing different costs, especially between the types of challengers, might be a realistic nuance to add to the model, but the model embeds the varying levels of incumbent advantage entirely in the candidates' valences. More to the point, any separation that resulted from assuming greater costs on weaker challengers would not be strategic but rather an artifact of the different costs each type of R faced. We do not include the possibility for the incumbent not to run, and thus forgo C, assuming instead that she has already announced her intention to seek reelection.

valence advantage of $\underline{a}$ over the strong type of challenger and a valence advantage of $\overline{a}$ over the weak type of challenger, such that $\underline{a} < \overline{a}$. We refer to an arbitrary such valence advantage as $a \in \{\underline{a}, \overline{a}\}$. Given the conception of challenger strength as equivalent to the extent of the incumbent's valence advantage, nature is effectively choosing the size of L's net valence advantage, a, over R in the game's first move. To streamline the notation for the sake of exposition, in text the incumbent's valence is set equal to a strong challenger's valence ($A_L = A_{R|S} =: A$), and the weak challenger's valence is set equal to zero ($A_{R|W} = 0$).[6] Under this assumption, $\underline{a} = 0$ and $\overline{a} = A$.

Throughout, we assume that the incumbent cannot be assured to win – against either type of challenger – by simply proposing her ideal point, M_L. This supposes that incumbents must propose a different platform to win the election if challenged than they would if they ran uncontested. It requires that the incumbent's ideal point not be too moderate (close to the median voter's) and/or that the incumbent's valence advantage not be too large, specifically, $M_L < -A$.

Preliminary Results

The solution concept for this sequential-move game with uncertainty is weak perfect Bayesian equilibrium. Only equilibrium strategies that do not include weakly dominated actions receive consideration. This implies that voters will vote sincerely, which permits restricting attention to the median voter's decision (see Lemma 1 in Appendix A).[7] Henceforth, "the voter" will refer to the median voter, V.

Given the assumed ordering of ideal points, it is also weakly dominated for candidates to propose any policies that do not lie between their own ideal point and the ideal point of the median voter (see Lemma 2 in Appendix A). Neither type of R will take a position to the left of the median voter's ideal point ($x_R < M_V = 0$) or to the right of his own ideal point ($x_R > M_R$), nor will L propose a policy to the left of her ideal point ($x_L < M_L$) or to the right of the median voter's ideal point ($x_L > M_V = 0$). As such, it will always be the case that $x_L \in [M_L, 0]$ and $x_R \in [0, M_R]$.

The voter's decision rule if R does challenge L is straightforward, except in cases of indifference. Recall that the incumbent possesses a valence advantage of a over her challenger, and that while the valence advantage

[6] The appendix derives equilibrium in the more general case, where $A_L \geq A_{R|S}$ and $A_{R|W} \geq 0$. To assume $A_L = A_{R|S}$ does make a degree of sense, though, as only strong challengers ascend to office in the model, so tomorrow's valence-advantaged incumbent may be today's high-valence potential challenger. The assumption that the weak challenger's valence is zero does not sacrifice any insight.

[7] The addition of valence to the model does nothing to affect the standard result that, in uni-dimensional policy settings, the support of the median voter is necessary and sufficient to achieve the support of a majority of voters.

is known to R but not to L when each proposes a policy, it is revealed over the course of the election, so V will take the size of L's valence advantage over R into account when casting a vote. Given the payoffs specified above, V votes for L if

$$x_L + a > -x_R \tag{4}$$

and for R if the inequality were reversed.

The only cases of voter indifference that will occur with positive probability in equilibrium in a contested election involve the challenger converging entirely to the median voter's ideal point, 0. If R of type t proposes $x_R = 0$ and L proposes $x_L = A_{R|t} - A_L$ such that V is indifferent, equilibrium existence requires that V vote for L (see Lemma 3 in Appendix A). Such tie-breaking rules arise endogenously, in the sense of "endogenous sharing rules" (Simon & Zame 1990).[8]

Remark Recalling the assumption that voters must vote for exactly one of the candidates that entered the election, if the incumbent is not challenged, she will propose her ideal point, M_L, and receive V's vote.

As such, if a potential challenger leaves the race uncontested, he receives the policy disutility associated with the incumbent's ideal point, M_L, as the winning platform, but he does not incur the cost of entry, C. Accordingly, let the "opportunity cost of contesting the election" be given by $C + M_L$. This quantity increases as the incumbent's ideal point becomes more moderate (recall $M_L < -A < M_V = 0$), which makes remaining out of the race more attractive, and as the cost of contesting the election increases, which makes contesting the election less attractive.

If the incumbent knew which type she faced, she would only moderate her policy proposal as much as necessary to win, and she would have to converge farther when facing a strong type of challenger than when facing a weak type. The incumbent lacks this knowledge, however, unless the two types of potential challenger fully separate with respect to entry. Proposition 1 states that, in fact, the incumbent will never know exactly which type of challenger she faces. The two types at least partially pool on the entry decision, either both entering with strictly positive probability or neither entering at all. The central tension in the model, then, is that the incumbent is uncertain how much she must moderate in policy towards the median to win the election. If the incumbent guesses incorrectly which type she faces, she runs the risk of moderating

[8] These tie-breaking rules may depend both on the policies leading to voter indifference and the type of challenger. For instance, while the result above indicates that V must vote for L if $x_L = -A$ and a weak challenger proposes $x_R = 0$, V need not vote for L if, against the same policy $x_L = -A$, a strong challenger proposes $x_R = A$, a pair of policies at which, given the valences, V is again indifferent between the two candidates. Indeed, this is the only other case of indifference for which we need specify a voting rule to establish equilibrium existence, and V must vote for R in this case.

more than she needed to in order to win against a weak challenger (incurring disutility from a policy farther from her ideal point) or losing the election to a strong type by not converging enough. In and of itself, the impossibility that the two types of challenger ever fully separate with respect to entry is an important, albeit straightforward, implication of the model.

Proposition 1. *The incumbent is never certain of the strength of a challenger who contests the election.*

Underlying this result is the benefit to weak challengers from being overestimated if the incumbent adopts a more moderate policy than she needs to in order to win and the benefit to strong challengers from being underestimated if they upset an incumbent that did not moderate enough in policy. Essential to this line of argument is the principle that, in equilibrium, the incumbent must possess correct beliefs about the type of challenger she faces. If one type of candidate would be willing to enter, both would, so separation cannot occur on entry. The proof, found in Appendix B, follows this same line of argument.

The fact that, in equilibrium, the two types of potential challengers cannot fully separate with respect to entry follows from a feature of electoral settings not present in standard signaling models. Namely, in classic signaling models (e.g., Rothschild & Stiglitz 1976), one type wishes to reveal itself truthfully, while the other type wishes to be mistaken. In the context of potential challengers weighing entry against an incumbent, both strong and weak types wish to be mistaken for the other, so obfuscation rather than revelation of types ensues.

Characterizing Equilibrium Strategies and Cases

Lemmas 4 – 5 (see Appendix C), along with the results thus far, establish that three equilibrium cases may obtain and Proposition 2' establishes that all three cases do obtain uniquely for some open set in the parameter space. Figure 2 labels the region of the parameter space corresponding to each case. Specifically, either both types of potential challenger will enter with some strictly positive probability (cases 1-2) or neither type will enter (case 3). In cases where the potential challenger contests the election with positive probability, strong challengers will always enter, and weak challengers will either strictly randomize between entering and leaving the race uncontested (case 1) or enter with certainty (case 2).

Proposition 2 follows immediately from Proposition 2', which may be found in Appendix D.[9] The proposition highlights how the probabilities of a contested election and an upset victory vary in response to changes

[9]Proposition 2' specifies the exact parameter values, the distributions of policy proposals over which the candidates randomize, and the voter strategies that characterize the equilibrium cases. The distributions of proposals take a somewhat complex form and as such are relegated to the appendix. The intuition behind these policy-proposal distributions follows Proposition 2 below.

in the opportunity cost of contesting the election ($C + M_L$) and the probability the potential challenger is a strong type (p). For the latter outcome, the proposition highlights how the probability that an upset occurs responds differently to an increase in C or M_L depending on whether it is measured unconditionally and conditionally on the election being contested.

Proposition 2. *Suppose the opportunity cost of contesting the election is neither too high nor too low relative to the probability that the potential challenger is a strong type (i.e., region 1 in Figure 2), such that both contested and uncontested elections may occur in equilibrium.*

An increase in the probability that the potential challenger is a strong type (p) increases the probability that the election is contested and the probability that the incumbent is defeated.

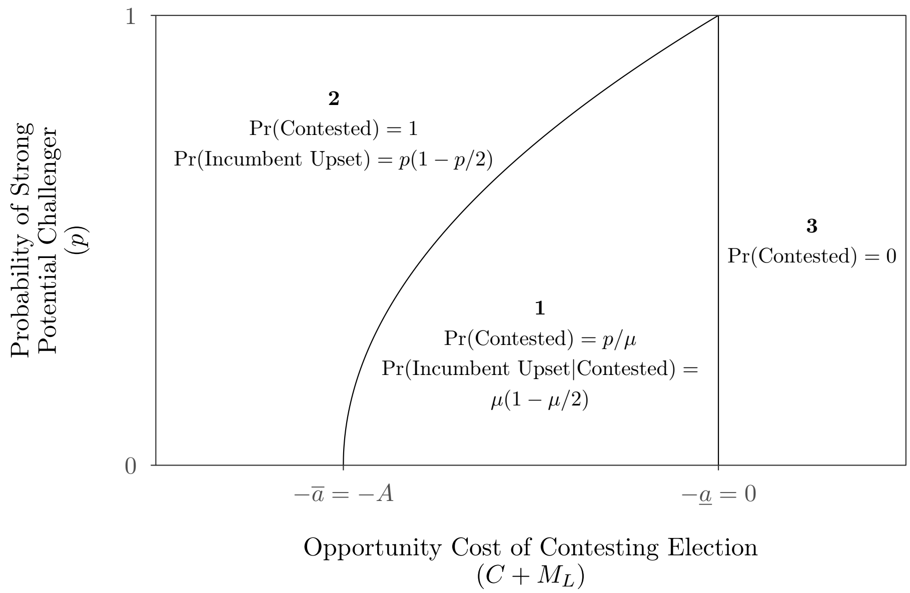
An increase in the opportunity cost of contesting the election ($C + M_L$) lowers the probability that the incumbent is challenged and the probability that the incumbent is defeated but raises the probability that the incumbent is defeated conditional on having been challenged.

In the equilibrium case of greatest interest (corresponding to region 1 in Figure 2), contested and uncontested elections both occur with strictly positive probability. While the strong type of potential challenger always contests the election, the weak type of potential challenger randomizes between entering the election and leaving the race uncontested. As such, the incumbent believes she faces a strong type with greater probability, denoted by μ, than the latent probability that the potential challenger is of strong type, p. Specifically, when challenged, the incumbent believes she faces a strong type with probability $\mu = \sqrt{(A + C + M_L)/A}$. The weak type of potential challenger's randomization with regard to the entry decision leaves the incumbent indifferent when challenged between proposing policies as though she faces a weak type and proposing policies as though facing a strong type.

It is useful to consider not only the candidates' equilibrium policy proposals but also the utility offered to the median voter by the candidates' policy proposals and their respective valences. For instance, if the weak challenger enters, he proposes $x_R = 0 = M_V$, a platform that offers the median voter a utility of 0. The incumbent (if challenged) and the strong challenger both randomize over an interval of policies. The most extreme policies (vis-à-vis the median voter's ideal point) in the support of their respective distributions of policy proposals are $x_L = -A$ and $x_R = A$, which offer the voter a utility of 0 given the candidates' valence. The incumbent proposes this policy with probability $1 - \mu$ when challenged, and she adopts a distribution of policy proposals that mirrors the strong challenger's with probability μ, the probability that she faces a strong type of challenger in equilibrium.[10] The "degree of moderation" of these distributions of policy

[10]The distributions of policy proposals leave the strong type of challenger and the incumbent indifferent among all of the proposals over which they randomize and leave the weak type of potential challenger

Figure 2: Equilibrium Cases



Notes: The opportunity cost of contesting the election, consisting of the cost of entry C (not incurred unless the potential challenger enters) and the policy M_L (the incumbent's ideal point) at which the incumbent wins if the potential challenger does not contest the election, increases along the horizontal axis. The probability p that the potential challenger is of strong type is increasing along the vertical axis.

Within region 1, the equilibrium case of greatest interest, challengers both win upon entering and choose not to contest the election with positive probability. In the event that the potential challenger contests the election, the incumbent's belief that her opponent is of strong type is given by $\mu = \sqrt{(A + C + M_L)/A}$. The unconditional probability that the incumbent is upset is given by $p(1 - \mu/2)$.

proposals will be said to have increased if, for each policy in the support, the probability of proposing a policy at least that far from the median voter's ideal point has weakly decreased.[11]

Across all of the cases, the weak type of challenger never wins.[12] Within case 1, the weak challenger is willing to enter only if the incumbent is sufficiently likely to propose policies as though she faces a strong

indifferent between entering and not.

[11]Alternatively, the degree of moderation of a candidate's distribution of proposals may be said to have increased if and only if the distribution of utilities offered to the voter (a function of the policy proposal and the candidate's valence) is increasing in the sense of first-order stochastic dominance.

[12]A weak challenger can offer at most 0 in utility to the median voter, which is the minimum the incumbent will ever offer to the voter in equilibrium. Recall, the voter breaks ties in favor of the incumbent over the weak challenger.

type. Given the incumbent's distribution of policy proposals, the strong challenger wins half of the time that the incumbent proposes policies as though she faces a strong type ($\mu/2$) and wins all of the time the incumbent plays as though she faces a weak type ($1-\mu$). Conditional on the election being contested, there is a μ probability that the challenger is of strong type, so there is a $\mu(1-\mu/2)$ chance that the incumbent loses. The probability that the election is contested is p/μ, which increases when increased entry by weak potential challengers lowers μ.

Within case 1, if the probability nature chose a strong type were to increase from p to p', the incumbent would propose more moderate policies, as she believes she more likely faces a stronger type. This would induce weak challengers to enter more frequently, however, which would mitigate the incumbent's incentive to converge. To maintain equilibrium, then, the weak challenger must enter somewhat more frequently, but all other strategies remain the same. The probability of a contested election rises, but there are no additional effects. An increase in p does not necessitate a change in μ in order to maintain equilibrium.

If the opportunity cost of contesting the election were to increase from $(C+M_L)$ to $(C+M_L)'$,[13] given the proposal strategy of the incumbent under $(C+M_L)$, weak challengers would strictly prefer to remain out of the race. In turn, this would lead to an increase in the moderation of the incumbent's distribution of proposals, who would be sure she faces a strong type, which would lead to entry by weak types again. Only through an increase in the moderation of the proposal distributions and a reduction in the frequency with which the weak challenger enters can equilibrium be maintained when the opportunity cost of contesting the election increases. The decrease in the likelihood of a contested election serves in equilibrium to increase the incumbent's belief that she faces a strong type (by way of reduced entry by weak types), and so more protective policy proposal strategies by the incumbent and strong types of challenger must accompany this change. Because it affects the incumbent's belief that she faces a strong type, an increase in the opportunity cost of contesting the election leads to fewer contested elections and fewer upsets, but a higher likelihood of an upset conditional on the election being contested.

Proposition 2 summarizes these mechanisms as predictions for the change in the frequency of contested elections and incumbent defeats as a result of changes in the opportunity cost of entry or the strength of the pool of potential challengers. Perhaps more importantly, though, it establishes that the probability of the upsets conditional on the election being contested and the probability that the election is contested are not independent of one another. In fact, they are inversely related through the incumbent's equilibrium belief that she faces a strong type, which is a function of the opportunity cost of entry in equilibrium case 1.

[13]Via the cost of entry rising from C to C' or the incumbent's ideal point moving from M_L to a more moderate M_L'.

The implication for empirical work is stark: the decision to restrict attention to only contested elections is not without consequence. Estimates of the effect of changes in either component of the opportunity cost of entry (the cost of entry or the winning policy in an uncontested race) on the likelihood that an incumbent retains her seat could be incorrectly signed. The bias would be so severe as to mischaracterize the most fundamental descriptor of a relationship. We explore the issues this poses for estimation in greater depth below when considering voter welfare.

Challenger Quality and Upset Victories

Animating the previous result is the role the incumbent's equilibrium belief that a challenger is strong, μ, plays in determining the frequency of various outcomes in equilibrium. While this is an endogenous quantity, itself a function of parameters A, C, and M_L, it most parsimoniously demonstrates the connections among contested elections, upset victories, and the degree of moderation in candidate proposals in contested elections. Consider the following decomposition of the probability that the incumbent, L, is upset by a potential challenger, R, who is a strong type S with probability p.

$$\begin{aligned} \Pr(R \text{ upsets } L) &= \Pr(R \text{ wins}|t=S, \text{contested}) \cdot \Pr(t=S|\text{contested}) \cdot \Pr(R \text{ challenges } L)^{14} \\ &= (1-\mu/2) \cdot \mu \cdot p/\mu \end{aligned} \tag{5}$$

A contested election is of course a necessary condition for an upset to occur. When the election is contested, only strong challengers defeat incumbents in the model, and then only when the Incumbent underestimates a strong challenger, believing him to be of weak type.

The next proposition highlights that the probability of a given strong challenger winning is inversely related to the incumbent's belief that a challenger is of strong type, while the probability of incumbent defeat depends positively on the equilibrium probability she faces a strong type. The result establishes that if the likelihood of an upset victory occurring in a contested election rises, the probability that a given strong challenger will win actually falls. The proposition applies to all cases in which entry occurs, which requires only that the opportunity cost of entry not be too high (regions 1 and 2 in Figure 2).[15]

Proposition 3. *Suppose the opportunity cost of contesting the election is not too high such that entry occurs.*

[14] $\Pr(R \text{ wins}|t=W, \text{contested})) \cdot \Pr(t=W|\text{contested}) = 0 \cdot (1-\mu)$, so Equation 5 omits these terms.

[15] Note that if the probability that a potential challenger is too high relative to the opportunity cost of contesting the election (region 2 in Figure 2), potential challengers of both types always enter, so $\mu = p$, and the probability of an upset conditional on entry is then equal to the unconditional probability of an upset: $p(1-p/2)$.

While an increase in the opportunity cost of contesting the election ($C + M_L$) or the likelihood that a potential challenger is of strong type (p) raises the probability that in a contested election the challenger defeats the incumbent increases, the probability that a given strong challenger wins decreases.

The greater the likelihood that the incumbent is facing a strong challenger, the less likely she will be to underestimate her opponent. As μ increases, then, strong challengers are less likely to win. Although a higher equilibrium probability that a challenger is of strong type leads to a lower probability that a given strong challenger wins, the first effect outweighs the second. The probability of an upset conditional on entry is still increasing in μ.

Empirical work on challenger quality employs observable measures, such as previous office-holding experience, while the model presented here is predicated on challenger strength being unobservable. The preceding result is nonetheless relevant to these studies. Suppose that observable signs of strength (e.g., experience) are positively correlated with unobservable qualities. A challenger with experience is likely drawn from a pool of challenger types with a higher p than the pool from which a challenger without experience was drawn. Either may be strong or weak types on unobserved factors, but it is more likely the experienced candidate is a strong type.

A higher probability of nature choosing a strong type increases increase the frequency of upset victories. If unobservable aspects of strength are correlated with observable measures, then studies would find (as they have) that experienced challengers more frequently defeat incumbents. Yet this finding may miss that an individual strong challenger could have worse prospects, as the expected strength of the pool of experienced candidates causes the incumbent to be more cautious about diverging from the median voter's preferences. A strong challenger with previous experience holding office may be less likely to win than a strong challenger who has no previous office-holding experience to signal his unobservable quality.

Voter Welfare

Although the probability of an upset victory rises at a muted rate relative to the share of challengers contesting the election that are strong types, the moderation of the candidates' equilibrium proposal distributions is increasing. As such, the results above both suggest a nuanced relationship of voter welfare to the frequency of contested elections and upset victories. The next result delves more deeply into the implications of the model for voter welfare, taking into account the competing effects of changes in the equilibrium probability that the incumbent faces a strong type.[16]

[16]Specifically, "voter welfare" denotes the utility of the median voter.

Proposition 4. *Suppose the opportunity cost of contesting the election is neither too high nor too low relative to the probability that the potential challenger is a strong type (i.e., region 1 in Figure 2), so contested and uncontested elections may occur in equilibrium.*

Voter welfare in contested elections is increasing in C and M_L, however, the probability of an uncontested election is increasing in C and M_L, which exerts downward pressure on voter welfare.

Voter welfare is strictly increasing in p as long as the opportunity cost of contesting the election is not too high (i.e., regions 1 and 2).

An implication arises in this proposition that runs particularly counter to popular narratives: voter welfare in contested elections is decreasing as the cost of entering the election falls. This effect is only tempered by a corresponding increase in the likelihood that the election is contested. Conversely, raising the cost of entry to positively affect voter welfare in contested elections entails an increase in uncontested elections, which negatively impacts voter welfare. Which of these two effects dominates varies across the equilibrium case of interest in which uncontested elections and upsets occur with positive probability. Fundamentally, however, manipulating the cost of entry entails a trade-off in voter welfare with respect to the likelihood of an uncontested election and the expected moderation of the winning policy in contested elections.

A similar analysis applies when considering changes in the location of the incumbent's ideal point relative to the median voter's, with one key difference. Analogous to an increase in the cost of entry, a decrease in the distance between the incumbent's ideal point and the median voter's ideal point leads to more competitiveness in contested elections – a boon for voter welfare – but also more uncontested elections – the bane of voter welfare. As M_L grows closer to M_V, however, voter welfare in uncontested elections becomes less negative (closer to the utility the voter receives from electing an incumbent that proposed $x_L = -A$). Because of this additional implication of increasing M_L, the downward pressure on voter welfare of increasing the frequency of uncontested elections is less than when increasing C. If an increase in C led to a net increase in voter welfare, then a corresponding increase in M_L would certainly lead to a net increase in voter welfare.

The most likely means by which the distance between the incumbent's and the median voter's ideal points would decrease is through an increase in the homogeneity of the constituency. Even though the frequency of uncontested elections increases, the incumbent's ideal point is more palatable to the median voter, and the distribution of policy proposals in contested elections is more moderate, so geographic sorting may contribute positively to the representation individuals receive. With fewer weak potential challengers contesting the election, the incumbent's belief that she faces a strong type when the election is contested increases, leading to more moderate distributions of policy proposals in contested elections.

Only an increase in p, the latent quality of the pool of potential challengers, always has the net effect of raising voter welfare. When the opportunity cost of contesting the election is large enough such that weak potential challengers randomize between entering the race and not (case 1), the probability that the election is contested increases without affecting the moderation of proposals in contested elections. When the opportunity cost of contesting the election is too small relative to the probability the potential challenger is of strong type – such that all potential challengers always enter (case 2) – the competitiveness of contested elections is increasing in p, and all elections are contested. The reduced probability of an upset is of secondary concern for voter welfare as it stems from the incumbent believing she more likely faces a strong challenger and, accordingly, proposing more moderate policies in expectation.

Earlier results suggested that excluding uncontested elections in studies of the frequency of challengers upsetting incumbents could suffer from severe bias, to the point of incorrectly estimating the sign of the coefficient on the cost of entry or the location of the incumbent's ideal point. Similarly, if an analysis excluded uncontested elections, then the effects on voter welfare of an increase in the cost of contesting an election or the incumbent's ideal point would be unambiguously positive, but this would ignore the negative effects on voter welfare of having more uncontested elections. The exclusion of uncontested races could significantly bias inference.

Suppose, for instance, that one wished to understand the effect on the distance of the winning policy from the median voter's ideal point as a function of the distance of the incumbent's ideal point from the median voter's. To simplify the thought experiment greatly, the analyst might run the following model:

$$|x - M_V| = \alpha + \beta \cdot |M_L - M_V| + \epsilon. \tag{6}$$

This analysis would be in the spirit of studies such as McCarty, Poole & Rosenthal (2006), who investigate the effect of gerrymandering on polarization, i.e., the effect of changes in the distribution of ideologies within a district on the extremism of the policies espoused by the representative of that district.

Were the sample used to test the model given in equation 6 to include only contested elections, the estimate of β would be negative. This would be a somewhat surprising conclusion, suggesting greater district-level polarization leads to less extreme policies. The model offers an explanation, though, namely that a more homogenous district would lead to less entry. This in turn gives the incumbent a higher belief that she faces a strong type and thus leads to more moderate, i.e., protective policy proposals (and thus safer seats, a plausible intention of gerrymandering). However, this inference would be incomplete, as it excludes the relatively extreme winning policies that occur in uncontested elections, a crucial part of the mechanism by which greater moderation came about. If the ideological dispersion were sufficiently great in the constituency, the bias could even result in incorrect signing of the effect of interest.

Incorporating a Non-Policy Benefit to Holding Office

To this point, the model has supposed only policy-motivated candidates. In what ways would the analyses change if holding office also motivated the candidates to win the election? As the spoils of office increase, what is the effect on the frequency of uncontested elections, the likelihood of challenger upsets, and the degree of moderation of policy proposal distributions in elections that are contested?

We suppose candidates are office-motivated as well as policy-motivated, where winning the election confers a benefit of $B > 0$, representing the salary, professionalization, or prestige of holding office. As demonstrated by the more general version of the model derived in the appendix, the structure of the equilibrium cases takes essentially the same form.[17] In the equilibrium case of interest, which features both uncontested elections and incumbent upsets, the benefit to holding office operates through the same mechanism as the opportunity cost of contesting the election $(C + M_L)$, although in the opposite direction.

An increase in B lowers the equilibrium probability μ that the incumbent faces a strong challenger in a contested election. At equilibrium, a higher non-policy benefit to holding office would provide an incentive for the incumbent to converge fully, which would lead to greater entry by weak types of potential challengers. Equilibrium requires greater entry by weak types, which lowers the incumbent's belief that she faces a strong type, leaving her indifferent among the interval of proposals over which she randomizes. The strong type of challenger, however, proposes policies according to a more moderate distribution of platforms. The incumbent's distribution of policy proposals is more moderate when she proposes policies as though she faces a strong type, but she more often proposes $x_L = -A$, as though she faces a weak type, so her entire distribution of proposals cannot be said to be more moderate.[18]

Because a decrease in B leads to an increase in μ, and as Proposition 5 in Appendix G makes clear, an increase in B leads to an increase in the probability of a contested election, increasing the overall probability of an upset but decreasing the probability of an upset conditional on the election being contested. As in Proposition 3, a given strong challenger fares better even as the likelihood of an upset in a contested election falls. Although though the equilibrium probability increases that the incumbent faces a weak type increases and thus proposes policies as though she faces a weak type, the increased moderation of proposals from

[17]Regions 1 and 2 change somewhat vis-à-vis their appearance in Figure 2 to accommodate subcases in which the incumbent converges entirely to the median with positive (even full) probability; this provides a greater incentive for the weak type of potential challenger to enter, and it reflects increased incumbent defensiveness over an increasingly valuable office as the probability that the potential challenger is of strong type increases.

[18]The distribution of utilities offered to the voter is not first-order stochastically increasing.

strong challengers and thus the incumbent when she proposes policies as though she faces a strong challenger leads to a net increase in voter welfare in contested elections. The expected winning policy in a contested election, however, is closer to the median voter's ideal point. Combined with the accompanying decrease in uncontested elections, an increase in B results in increased voter welfare.

The non-policy benefits of holding office have long been subject to scholarly attention. Squire (2000) examines uncontested elections at the level of state legislatures. He hypothesizes that increased professionalization within a legislature encourages entry by challengers, and he uncovers exactly this relationship. Further, the frequency of uncontested seats at the national level is even lower, where the prestige of holding office is even greater (Wrighton & Squire 1997). Within the model, and in line with these patterns, an increase in the benefit of holding office leads to greater entry. The increase in entry, however, is not a result of potential challengers finding the prospect of holding office more valuable. Instead, the increase in entry is driven by the weak challenger contesting the election with greater frequency. The weak type will never win, but in equilibrium the increase in entry serves to temper the incumbent's desire to converge enough to defend her seat and secure the benefit B with certainty.

A fall in uncontested congressional elections in the period leading up to their research prompts Wrighton and Squire to conclude that according to this metric, "American democracy is healthier than at any other time [in the 20^{th} century]" (p. 452). Does the logic by which an increase in B might actually lead to more contested elections support or subvert this conclusion? While it is true within the model that the incumbent will more often adopt her most divergent policy, expected voter welfare in contested elections does increase in B, so overall voter welfare increases, as well. If an increase in B underlies the uptick in contested elections that the authors observed, policy proposals would have grown more moderate in expectation, supporting the authors' claim of democratic "health." That polarization grew over this same time period, however, contradicts the notion that an increasing non-policy benefit to holding office lies behind the rise in contested elections. Instead, fewer uncontested elections and more polarized policies are consistent with a decrease in the opportunity cost to contesting the election, either through reduced cost of entry or (more likely) more extreme incumbent ideal points.

Finally, a number of papers in recent years have used a regression discontinuity design in which small population shifts trigger discontinuous increases in the salary awarded to politicians to draw inference about electoral accountability and the incumbency advantage (Eggers, Freier, Grembi & Nannicini 2016). The theoretical quantity of interest, elected officials' salaries, corresponds the quantity B in our model. It bears noting again that whether or not such analyses include only contested elections could have dramatic effects on inference about the effect of B on outcomes such as incumbent retention and voter welfare.

Conclusion

This paper presented a model of challenger entry followed by electoral competition, where the incumbent begins uncertain as to the strength of a potential challenger. Under the plausible assumptions that strong challengers have something to gain if underestimated and weak challengers have something to gain if overestimated, the model revealed that the incumbent will never know with certainty whether she faces a strong or a weak opponent until after she has laid down a policy stake. In the equilibrium cases of greatest interest, uncontested elections, upset victories, and varying levels of convergence in the winning policy all occur. The rich set of outcomes stem from what might be termed the incumbent's "rational complacency." The incumbent, valence-advantaged as she is, could moderate enough in policy to ensure victory. She is willing, though, to adopt less moderate positions, weighing the chance of winning at policies she finds more attractive against the possibility of losing to a strong challenger.

A key set of relationships emerges from the equilibrium strategies, namely, the connection between the probability that the election is contested and the behavior of candidates in contested elections. Specifically, exogenous changes that affect the equilibrium probability that the incumbent faces a strong type of challenger affect both the likelihood of contested elections as well as the frequency of upsets, which itself hinges on the extent to which candidates moderate in policy proposals towards the median voter. These parameters include the cost of contesting the election, the incumbent's ideal point (which is the winning policy in an uncontested election), and the non-policy benefit to holding office.

This complex web of effects becomes particularly salient when considering voter welfare. A decrease in the cost of entry or a more moderate incumbent ideal point lead the probability of a contested election to increase (a benefit to the median voter) while causing the expected policy proposals in contested elections to be farther from the median voter's ideal point (a loss for the median voter). A number of implications for empirical work followed, notably the potential bias induced by dropping uncontested elections from studies of incumbent retention or the polarization of winning policies. Comparative statics may change in magnitude or even in sign depending on the decision to consider all elections or only contested elections.

An increase in the prior probability that the challenger is strong is the only parameter change that does not entail trade-offs in voter welfare (although an increase in the non-policy benefit to holding office does result in a net increase in voter welfare). Contested elections increase, and proposals in contested elections become increasingly moderate. As such, recruiting stronger pools of potential candidates emerges as a recommendation of the model for those seeking to improve election outcomes by producing winning policies that are closer in expectation to the median voter in each constituency.

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Appendix

Results and Proofs Referenced In-Text

A Preliminary Results: Lemmas 1-3

In the context described in-text, the elimination of weakly dominated strategies implies that voters will vote sincerely, which in turn allows us to invoke the median voter theorem and restrict attention to the median voter's decision. The first result formalizes this, establishing that it is as though the incumbent and the challenger, L and R, compete only for the support of the median voter, V. The elimination of weakly dominated strategies will also help narrow the range of policy proposals we need to consider as we develop the players' equilibrium strategies below, which the second lemma makes explicit.

Lemma 1. *The majority preference relation is equivalent to the median voter's preference relation.*

Proof of Lemma 1. The assumption that players, including voters, do not play weakly dominated actions implies that If L is challenged by R, no voter votes for a candidate whose policy proposal, if implemented, and valence, if elected, would provide the voter strictly lower utility than the opposing candidate. Voting is sincere, with voters only voting for their most preferred candidate.

Voter preferences are supermodular in ideal points and the winning policy proposal, and this property is preserved with valence entering additively into voter utility. This is a sufficient condition to apply the results from Gans & Smart (1996), and the result follows immediately. ■

Lemma 2. *In any equilibrium, $x_L \in [M_L, 0]$ and $x_R \in [0, M_R]$.*

Proof of Lemma 2. We show that for any policy $x_L \notin [M_L, 0]$ or $x_R \notin [0, M_R]$, the candidates could weakly improve their chances of winning and/or the utility they would derive from winning by proposing a policy within those intervals.

Consider $x_L < M_L$. $\hat{x}_L = M_L$ increases the utility the incumbent offers the voter, thus weakly increasing L's chance of winning, and provides L more utility from the winning policy should she win. $\hat{\hat{x}}_L = 0$ similarly weakly dominates any proposal $x_L > 0$. A symmetric argument applies to R.

As such, the assumption to eliminate weakly dominated strategies narrows the domain of proposals available to L and R to the intervals given above. ■

A strategy for the incumbent is a policy to propose when unchallenged as well as a (possibly degenerate) distribution of policies according to which to propose when challenged, where in this latter case x_L is distributed according to the cumulative distribution function ξ_L. The support of this distribution will be a subset of the interval $[M_L, 0]$. A strategy for the challenger is a pair for each type consisting of a probability of entry, $\sigma_{R|t} = \Pr(enter|t)$, and a distribution of policy proposals, $x_{R|t}$, with distribution function $\xi_{R|t}$. The support of this distribution will be a subset of the interval $[0, M_R]$.[19] Let the incumbent's belief that she faces a strong type be denoted by μ, which by consistency of beliefs must equal $\frac{p\sigma_{R|S}}{p\sigma_{R|S}+(1-p)\sigma_{R|W}}$ when at least one of the types of challenger enters with strictly positive probability.

The possibility of voter indifference affects a candidate's expected utility from proposing x_c only to the extent that, given ξ_{-c}, there is a strictly positive probability that V will be indifferent the two candidates. However, two cases of voter indifference will be of particular importance going forward, and the next result concerns these cases. Specifically, in the event that a challenger of type t converges in policy entirely to the median voter's ideal point, 0, and the incumbent proposes $x_L = A_{R|t} - A_L$ (converging just enough to leave V indifferent), the next lemma says V must vote for L.

Lemma 3. *No equilibrium can exist in which a challenger of type t proposes $x_R = 0$ with strictly positive probability and the support of ξ_L includes $x_L = A_{R|t} - A_L$, but in which V does not vote for L in the event that the incumbent proposes $x_L = A_{R|t} - A_L$ and a challenger of type t proposes $x_R = 0$.*

Proof of Lemma 3. For $\xi_{R|t}$ such that $\Pr(x_R = 0|t) = \alpha > 0$, suppose that V votes for type t of R with probability $\beta > 0$ when $x_L = A_{R|t} - A_L$ and type t of R has proposed $x_R = 0$.

Consider $\hat{x}_L = A_{R|S} - A_L + \epsilon$, where $\epsilon > 0$. Then $\mathbb{E}U_L(\hat{x}_L) > \mathbb{E}U_L(A_{R|S} - A_L)$ if $-M_L + A_L - A_{R|S} + B - \epsilon > -M_L + (1 - \mu\alpha\beta)(A_L - A_{R|S} + B) + \mu\alpha\beta(0) \Rightarrow \epsilon < A_L - A_{R|S} + B$. Such an ϵ clearly exists, so it cannot be the case that $x_L = A_{R|S} - A_L$ was in the support of ξ_L in any equilibrium in which ties are not broken entirely in L's favor when $x_L = -\underline{a}$ and a strong challenger has proposed $x_R = 0$.

This same logic applies to $\xi_{R|W}$ s.t. $\Pr(x_R = 0|W) > 0$, except that we must account for the fact that it is possible that the strong type could be proposing $A_{R|S} - A_{R|W}$. This only makes the argument above stronger, however, as $\hat{\hat{x}}_L = A_{R|W} - A_L + \epsilon$ also reduces the likelihood of the strong challenger winning. Setting this added benefit aside, we have $\mathbb{E}U_L(\hat{\hat{x}}_L) > \mathbb{E}U_L(A_{R|W} - A_L)$ if $\epsilon < A_L - A_{R|W} + B$. It is possible to find such an ϵ, so $x_L = -\overline{a}$ cannot be in L's equilibrium distribution of proposals if she loses at that proposal to a weak type proposing $x_R = 0$ with any positive probability. ■

[19] We do not assume the distributions are continuously differentiable, but results below establish that the interiors of non-degenerate strategy distributions lack mass points and gaps and so are, in fact, in C^1.

B No Entry Separation: Proposition 1

Proposition 1. *The two types of challengers never fully separate with respect to entry in equilibrium.*

The proof merely provides extra detail around the argument in text.

Proof of Proposition 1. If the incumbent believes that only weak types of challengers enter in equilibrium, she would propose $x_L = A_{R|W} - A_L$, converging enough so that there is no policy available to the weak type such that voting for R offers the voter more utility than would voting for L would provide.[20] However, it would not be sequentially rational for the strong type to remain out of the election. Any $x_R < A_{R|S} - A_{R|W}$, which makes voting for S more attractive for V than voting for L, would win with certainty.

If the incumbent believes she faces only a strong type, then she would converge to $x_L = A_{R|S} - A_L$. As established V must break ties in favor of L for an equilibrium to exist, so R would be losing with certainty. If the strong type remains willing to contest the race, i.e., $C + M_L < A_{R|S} - A_L$, the weak type would also prefer to enter (getting more convergence from the incumbent than his low valence "deserves"). If the strong type is not willing to incur the cost of entry only to lose to an incumbent converged to $x_L = A_{R|S} - A_L$, then neither would the weak type. ■

C Constructing Equilibria in Randomized Strategies: Lemmas 4-6

In developing the policy proposal strategies in this model, it is useful to consider the utility that a candidate offers to the voter with the candidate's policy proposal and associated valence. For instance, if L proposes x_L, the voter's utility from voting for L would be $x_L + A_L$. As such, let $z_L : [M_L, 0] \to \mathbb{R}$ simply be an affine transformation given by $z_L(x_L) = x_L + A_L$, the utility the voter would receive from electing L. Similarly, we characterize the utility offered to the median voter by the strong type with the function $z_{R|t} : [0, M_R] \to \mathbb{R}$ where $z_{R|t}(x_R) = -x_R + A_{R|t}$.

Since $x_L \sim \xi_L$, we write $z_L \sim \zeta_L$, where $\zeta_L(z) = \xi_L(z - A_L)$. Similarly, write $z_{R|t} \sim \zeta_{R|t}$, where $\zeta_{R|t}(z) = 1 - \xi_{R|t}(A_{R|t} - z)$. Denoting L's belief (correct on the equilibrium path) that the probability she faces a strong challenger is $\mu \in [0, 1]$, we may write $z_R \sim \zeta_R$ with $\zeta_R = \mu \cdot \zeta_{R|S} + (1 - \mu) \cdot \zeta_{R|W}$.

Definition We refer to z_c as comprising candidate c's *policy-valence offer* (PVO), and ζ_c as c's *distribution of PVOs*.[21]

[20] Per Lemma 3, even if the weak type proposes $x_R = 0$, V must break the tie in favor of L in any equilibrium.

[21] The strategies and their derivation hold much in common with those from the theory of asymmetric

If c offers a PVO of $z_c > z_{-c}$, then the voter (voting sincerely by Lemma 1) will vote for candidate c. Note that conditional on winning, each candidate would prefer to do so at a PVO that offers the voter a lower level of utility, thus winning at a more divergent policy (i.e., a policy closer to the candidate's ideal point). Candidate c's utility, then, is decreasing in z_c. As a first step in characterizing the equilibrium distributions of PVOs, the next lemma establishes that L and R's maximum PVO must be the same.

Lemma 4. *The support of both candidates' equilibrium distribution of PVOs must have the same maximum,* $\overline{z}$.

Recall that "candidate" here refers to L or R without drawing distinction between the possible types of challenger, i.e., both types' strategies need not include $\overline{z}$. Conditional on R winning, the type of challenger behind a given winning PVO certainly has implications for L's utility. For the sake of this result, however, it only matters to L whether or not some type of R is offering a higher PVO (i.e., more utility to the voter) than she is.

Proof of Lemma 4. Suppose that the candidates' maximum PVOs are such that $\overline{z}_c > \overline{z}_{-c}$. Then $\hat{z}_c = \frac{\overline{z}_c + \overline{z}_{-c}}{2}$ would yield c as high a probability of winning as z_c but at a policy closer to M_c. As such, it cannot be that $\overline{z}_c$ is part of c's distribution of PVOs in equilibrium, and so $\overline{z}_c = \overline{z}_{-c}$. ■

The challenger's highest possible PVO is $z_R = A_{R|S}$, if the challenger is of strong type and has converged in his policy proposal all the way to the voter's ideal point. As both candidates' distributions of PVOs must share the same maximum, $A_{R|S}$ is clearly an upper bound on $\overline{z}$.

Indeed, if L's equilibrium strategy entails a degenerate distribution of PVOs, her pure strategy must consist of offering $z_L = A_{R|S}$. For any $\hat{z}_L < A_{R|S}$ offered with full mass, any candidate in whose favor V does not break ties at $\hat{z}_L$ would benefit from proposing a slightly more moderate policy and offering $z_c > \hat{z}_L$. Such a beneficial deviation exists for L or (at least the strong type of) R if L plays a pure strategy unless $z_L = A_{R|S}$ and V awards L the vote in the event of indifference.

Remark Given Lemma 2, there must also exist finite minimum PVOs, $\underline{z}_c$. L's minimum offer cannot be less than $z_L = M_L - A_L$, while R's cannot be less than $z_R = -M_R - A_{R|S}$.

A related result is that if L is not converging to $x_L = A_{R|S} - A_L$ ($z_L = A_{R|S}$) with probability one, then the minimum of the support of both candidates' distributions of PVOs is equal to $\underline{z} = A_{R|W}$. Before

auctions (see Maskin & Riley (2000)). The context of electoral politics, however, features a complexity absent from auctions: namely, if a candidate loses, he/she cares about the policy at which his/her opponent wins. In auctions, if a bidder does not win, she does not care about her opponent's bid.

proceeding to this lemma, consider that if L plays a mixed strategy, she must win with certainty at the maximum offer of utility, $\overline{z}$, that the candidates share. Either $\overline{z} = A_{R|S}$, and so L must win by Lemma 3, or $\overline{z} < A_{R|S}$ and $\Pr(z_R < \overline{z}) = 1$, by Lemma 4. Further, if L is randomizing in equilibrium, the strong type of R must win with at least some strictly positive probability at all policies (PVOs) over which she randomizes. For the candidates to be willing to randomize over lower PVOs, they trade off a lower probability of winning and a smaller policy loss if they do win with winning more often at $\overline{z}$, a less attractive policy.[22]

Lemma 5. *If both candidate's equilibrium strategies entail non-degenerate distributions of PVOs, z_c, then the minimum of the supports of these distributions, $\underline{z}_c$, must be the same, namely $\underline{z} = A_{R|W}$.*

Proof of Lemma 5. Any (type of) candidate whose equilibrium strategy entails a non-degenerate distribution of PVOs must either win with some strictly positive probability at the minimum of the distribution's support, or lose at all points in the support of the distribution. If $\overline{z} < A_{R|S}$, then both candidates win with certainty if they offer $z_c = A_{R|S}$. Even if $\overline{z} = A_{R|S}$ and L offers this PVO with positive mass, if she does not place full mass on it (i.e., $\Pr(z_L < \overline{z}) \in (0, 1)$), then R must clearly win with some probability if he offers $z_{R|S} = \overline{z} = A_{R|S}$, by the fact that the support of $\zeta_{R|S}$ must include $\overline{z}$ and because $\zeta_L(\overline{z}) > 0$.

As candidates randomize over less divergent policy proposals, they are trading off some probability of winning for the prospect of winning at a more appealing policy. In fact, they not only trade-off the probability of winning, though, but also incur greater expected disutility in expectation from the winning policy if they lose as they diverge. Nonetheless, for both of the candidates' equilibrium distributions to be non-degenerate, the candidates must be indifferent among all policies (or offers, z_c) over which they randomize, and so they must be winning with at least some strictly positive probability at the lowest PVO in their distributions. It must be, then, that $\underline{z}_L = \underline{z}_R$. If not, e.g., $\underline{z}_c < \underline{z}_{-c}$, then candidate c would lose with certainty at all $z \in [\underline{z}_c, \underline{z}_{-c})$, which we have argued cannot occur if ζ_c, ζ_{-c} are non-degenerate. ■

For candidate L and at least one type of R to be randomizing in equilibrium, it must be true that $\zeta_R(\underline{z}) = \mu\zeta_{R|S}(\underline{z}) + (1 - \mu)\zeta_{R|W}(\underline{z}) > 0$ and $\zeta_L(\underline{z}) > 0$. This implies that $\Pr(z_L = \underline{z}) > 0$. Suppose L and at least one type t of R both place positive mass on $\underline{z}$ and both win with some strictly positive probability against the other. One or both of L and R of type t would find it worthwhile to offer slightly above $\underline{z}$ instead of ever offering $\underline{z}$. So it must be the case that whichever type of R is placing mass on $\underline{z}$ is losing with certainty, but this means that it cannot be the type of R that is randomizing. It must be that S randomizes, does not place positive mass on $\underline{z}$, and W loses with certainty, but the only $\underline{z}$ for which there could be no

[22]Complicating this trade-off is that, the more divergent the policy position at which a candidate loses, the greater the expected disutility from the winning policy, i.e., the more divergent the opposing candidate's policy could have been while still offering the voter the greater level of utility, $z_{-c} > z_c$.

deviation is $\underline{z} = A_{R|W}$, where by Lemma 3, we know V must vote for L if indifferent between L and type W of R. Finally, note that it must be true that V would vote for type S of R if the strong challenger proposes $x_R = A_{R|S} - A_{R|W}$ and the incumbent $x_L = A_{R|W} - A_L$, but the strong challenger will not offer $\underline{z}$ with positive probability, and so from L's perspective, she need only take into account the probability with which she will face a weak type of challenger, degenerately offering $\underline{z}$.

The final lemma asserts that the PVO distributions when the candidates are mixing are continuously differentiable over their interiors. The proof establishes that, if L is mixing, R must be as well, and that each will randomize over an interval of PVOs, $[A_{R|W}, \overline{z}]$. Furthermore, when mixing, neither candidate will place any positive mass on any offer except $A_{R|W}$ and, for L, the maximum, $\overline{z}$, but only if $\overline{z} = A_{R|S}$.

Lemma 6. *In any equilibrium, the incumbent does not put strictly positive mass on any PVO* $z_L \in (\underline{z}, A_{R|S})$ *with strictly positive probability. If* ζ_L *is a non-degenerate distribution,* $\zeta_{R|S}$ *will not place strictly positive mass on any PVO* $z_R \in (A_{R|S}, A_{R|S}]$.

If the candidates' equilibrium distributions of PVOs, $\zeta_c, c = L, R$, *are both non-degenerate, the distributions will also have no gaps, i.e.,* $\forall a, b \in [\underline{z}, \overline{z}]$ *s.t.* $b > a, \zeta_c(b) - \zeta_c(a) > 0$.

Proof. Note that if $\overline{z} < A_{R|S}$, which we know by the discussion above cannot occur unless L's strategy is non-degenerate, then neither candidate will offer the voter utility of $\overline{z}$ with strictly positive probability in equilibrium. Only if $\overline{z} = A_{R|S}$ may L place positive mass on $\overline{z}$ in an equilibrium involving non-degenerate distributions of PVOs.

Suppose by way of contradiction that candidate c has placed positive mass on $\hat{z} \in (\underline{z}, \overline{z})$ (i.e., either L placing strictly positive mass on some $\hat{z} \in (\underline{z}, \overline{z})$ or R placing strictly positive mass on some $\hat{z} \in (\underline{z}, \overline{z})$ if ζ_L is a non-degenerate distribution). By the definition of mass point, $\Pr(z_c \in (z^-, z^+)) > 0$, where $z^- < \hat{z} < z^+$, and so regardless of how V breaks a tie at $\hat{z}$, there exists a discontinuous increase in the probability of $-c$ winning by increasing her PVO from z^- to z^+. As such, $\exists \gamma$ s.t. $\mathbb{E}U_{-c}(z^-) < \mathbb{E}U_{-c}(z^+), \forall z^- \in (\hat{z} - \gamma, \hat{z}), z^+ \in (\hat{z}, \hat{z} + \gamma)$. It cannot be a best response if $\exists z_{-c} \in (\hat{z} - \gamma, \hat{z})$, implying that c should instead offer $\hat{z} - \alpha\gamma, \alpha \in (0, 1)$, so $\hat{z}$ cannot be part of c's best response. This proves the first part of the lemma.

To prove that there will not be gaps in the distributions of PVOs when they are both non-degenerate, suppose by way of contradiction that c's distribution lacks support over $(a, b) \in [\underline{z}, \overline{z}]$, but where a, b are in the support of ζ_c. Note that $b \leq \overline{z}$. Also note that if both candidates propose according to non-degenerate distributions, each trades winning more often at higher PVOs with losing more often at lower PVOs, which are more attractive to the candidate.

Because $z_{-c} = a$ offers $-c$ a strictly higher expected payoff than any $z_{-c} \in (a, b)$ (the same winning probability but at a PVO that is more attractive to $-c$), $\nexists z_{-c} \in (a, b)$. As we established that the offer distributions share the same minimum if both are non-degenerate, then the two distributions must share the same gaps, if any exist. However, $z_c = a$ offers a strictly higher payoff than $z_c = b$, so it cannot be that b is in the support of either candidate's distribution, a contradiction of the assumption that there exists a gap in the distributions. ■

D Characterizing the Equilibrium Cases: Proposition 2

Before proceeding to fully characterize the equilibrium cases, we recall and establish some notation: $\underline{a} := A_L - A_{R|S}$, $\overline{a} := A_L - A_{R|W}$, the opportunity cost of contesting the election is given by $C + M_L =: \kappa$, $\sigma_{R|t}$ denotes the probability that a challenger of type t enters the election, μ denotes L's belief that she faces a strong challenger,[23] and $\overline{z}$ is the maximum of the distribution of platform-valence offers (PVO) made by the candidates in equilibrium.

Consider a function of the opportunity cost of contesting the election that specifies what the maximum of the distribution of PVOs must be, $Z : [-\overline{a}, \overline{\kappa}] \longrightarrow [A_{R|W}, A_{R|S}]$, to support the indifference conditions necessary for a mixed-strategy equilibrium, i.e., one involving non-degenerate distributions of policy proposals. As is verified below, this is a strictly increasing function in κ, so we may define $\overline{\kappa} := Z^{-1}(A_{R|S})$ to be the value of κ at which $Z(\kappa)$ is equal to $A_{R|S}$, the upper bound of $\overline{z}$.

Further, let $\tilde{P} : [A_{R|W}, A_{R|S}] \longrightarrow [0, 1]$ give the highest proportion of strong potential challengers in the population for which, given $\overline{z} \in [A_{R|W}, A_{R|S}]$, there exists some level of entry by weak types, $\sigma_{R|W} \in [0, 1]$, that could support beliefs held by L that would sustain a mixed-strategy equilibrium. Entry by weak challengers decreases μ so that L remains willing to randomize. Note that P is a strictly increasing function in $\overline{z}$, and set $\hat{p} := P(A_{R|S})$.

Then let $P : [-\overline{a}, \overline{\kappa}] \longrightarrow [0, \hat{p}]$ be a function defined by $\tilde{P}(Z(\kappa))$ and represented by the curve in Figure 3 separating cases 1 and 2.[24] We refer to this function below simply by $P(\kappa)$.

Proposition 2'. *The following strategies, by case, constitute an equilibrium of the model:*[25]

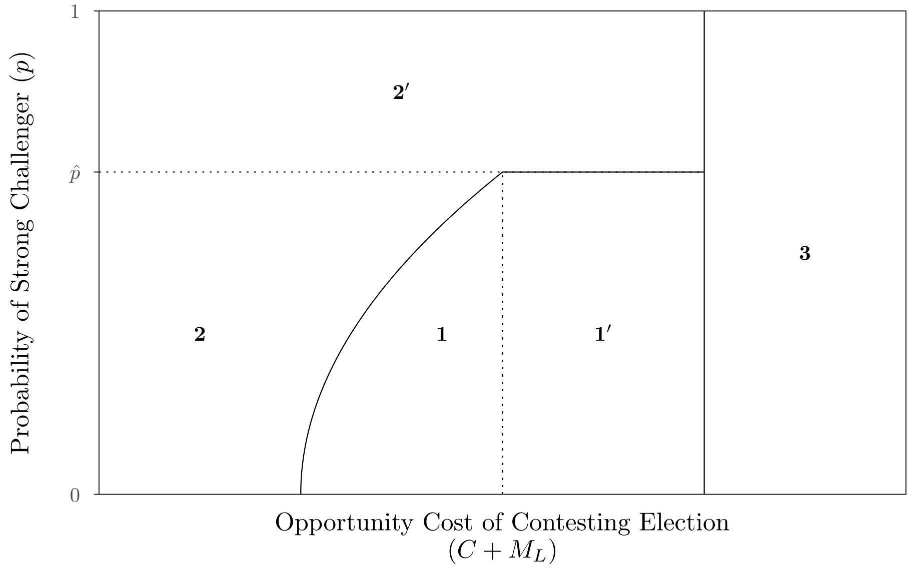
In all cases, if unchallenged, the incumbent offers $z_L = M_L + A_L = M_L = A$ (i.e., proposes $x_L = M_L$).

[23] From Lemma 3, we know that $\sigma_{R|W} > 0 \Rightarrow \sigma_{R|S} = 1$, so we know $\mu = \frac{p}{p+(1-p)\sigma_{R|W}}$ when the election is contested.

[24] With regards to uniqueness, the cases indicated below must obtain, but R's strategy in case 2' need not be as stated in the proposition. Up to this element, the equilibrium is unique. R's strategy in case 2' is chosen because it works throughout the entire case, has intuitive appeal, and holds even as $A_L \to A_{R|S}$ and $B \to 0$.

[25] Corresponding to Figure 3.

Figure 3: Equilibrium Cases in a More General Version of the Model



1 & 1'. $(\kappa, p) \in \{(\kappa, p) | \kappa \in (-\overline{a}, \overline{\kappa}], p \in (0, P(\kappa)]\} \cup \{(\kappa, p) | \kappa \in (\overline{\kappa}, -\underline{a}], p \in (0, \hat{p}]\}$:

$$\overline{z} = \begin{cases} Z(\kappa) & \kappa \in [-\overline{a}, \overline{\kappa}] \\ A_{R|S} & \kappa \in (\overline{\kappa}, -\underline{a}] \end{cases}$$

$$Z(\kappa) = A_{R|W} + \sqrt{A_L - A_{R|W} + \kappa}\left(\sqrt{2(A_L + A_{R|S} - 2A_{R|W} + B)} - \sqrt{A_L - A_{R|W} + \kappa}\right)$$

$$\mu = 1 - \frac{\sqrt{A_L + A_{R|S} - 2\overline{z} + B}}{\sqrt{A_L + A_{R|S} - 2A_{R|W} + B}} = \begin{cases} \frac{\sqrt{2(A_L - A_{R|W} + \kappa)}}{\sqrt{A_L + A_{R|S} - 2A_{R|W} + B}} & \kappa \in [-\overline{a}, \overline{\kappa}] \\ 1 - \frac{\sqrt{A_L - A_{R|S} + B}}{\sqrt{A_L + A_{R|S} - 2A_{R|W} + B}} & \kappa \in (\overline{\kappa}, -\underline{a}] \end{cases}$$

$$z_L \sim \begin{cases} \zeta_L(z_L) = \begin{cases} 0 & z_L < A_{R|W} \\ (1-\mu)\frac{\sqrt{A_L + A_{R|S} - 2A_{R|W} + B}}{\sqrt{A_L + A_{R|S} - 2z_L + B}} & z_L \in [A_{R|W}, \overline{z}] \end{cases} & \forall \kappa \in [-\overline{a}, \overline{\kappa}] \\ \hat{\zeta}_L(z_L) = \begin{cases} 0 & z_L < A_{R|W} \\ \frac{(A_{R|S} - A_L - C - M_L)/\sqrt{A_L + A_{R|S} - 2z + B}}{\left(\sqrt{A_L + A_{R|S} - 2A_{R|W} + B} - \sqrt{A_L - A_{R|S} + B}\right)} & z_L \in [A_{R|W}, A_{R|S}) \\ 1 & z_L \geq A_{R|S} \end{cases} & \forall \kappa \in (\overline{\kappa}, -\underline{a}] \end{cases}$$ [26]

$$z_{R|S} \sim \frac{(1-\mu)\left(\sqrt{A_L + A_{R|S} - 2A_{R|W} + B} - \sqrt{A_L + A_{R|S} - 2z + B}\right)}{\mu\sqrt{A_L + A_{R|S} - 2z + B}}, \; z_{R|S} \in [A_{R|W}, \overline{z}]$$ [27]

$$\sigma_{R|S} = 1$$

---

[26] At $\overline{\kappa}$, $\hat{\zeta}_L = \zeta_L$, and for higher values of κ, $\hat{\zeta}_L$ is just a mix of ζ_L and (increasingly) a distribution placing all mass on $A_{R|S}$.

[27] The truncated distribution of z_L over $(A_{R|W}, \overline{z})$ is $\frac{\mu}{1-\mu}\zeta_L(z) = \zeta_{R|S}(z)$, which is to say that the truncated distribution of z_L over $[A_{R|W}, \overline{z})$ is $\zeta_L(z) = \zeta_R(z)$, even when $\kappa \in (\overline{\kappa}, -\underline{a}]$ (see the previous footnote).

$z_{R|W} = A_{R|W}$

$$\sigma_{R|W} = \frac{p(1-\mu)}{(1-p)\mu} = \begin{cases} \frac{p(\sqrt{A_L+A_{R|S}-2A_{R|W}+B}-\sqrt{2(A_L-A_{R|W}+\kappa)})}{(1-p)\sqrt{2(A_L-A_{R|W}+\kappa)}} & \kappa \in [-\overline{a}, \overline{\kappa}] \\ \frac{p\sqrt{A_L+A_{R|S}-2A_{R|W}+B}}{(1-p)(\sqrt{A_L+A_{R|S}-2A_{R|W}+B}-\sqrt{A_L-A_{R|S}+B})} & \kappa \in (\overline{\kappa}, -\underline{a}] \end{cases}$$

2. $(\kappa, p) \in \{(\kappa, p) | \kappa \leq -\overline{a}, p \in (0, \hat{p}]\} \cup \{(\kappa, p) | \kappa \in (-\overline{a}, \overline{\kappa}], p \in (P(\kappa), \hat{p}]\}$*:*

$\overline{z} = P^{-1}(p) = A_{R|W} + p(1 - \frac{p}{2})(A_L + A_{R|S} - 2A_{R|W} + B)$

$\mu = p$

$$z_L \sim \zeta_L(z_L) = \begin{cases} 0 & z_L < A_{R|W} \\ (1-\mu)\frac{\sqrt{A_L+A_{R|S}-2A_{R|W}+B}}{\sqrt{A_L+A_{R|S}-2z_L+B}} = (1-p)\frac{\sqrt{A_L+A_{R|S}-2A_{R|W}+B}}{\sqrt{A_L+A_{R|S}-2z_L+B}} & A_{R|W} \in [A_{R|W}, \overline{z}] \end{cases}$$

$$z_{R|S} \sim \frac{(1-\mu)(\sqrt{A_L+A_{R|S}-2A_{R|W}+B}-\sqrt{A_L+A_{R|S}-2z+B})}{\mu\sqrt{A_L+A_{R|S}-2z+B}} = \frac{(1-p)(\sqrt{A_L+A_{R|S}-2A_{R|W}+B}-\sqrt{A_L+A_{R|S}-2z+B})}{p\sqrt{A_L+A_{R|S}-2z+B}},$$

$z_{R|S} \in [A_{R|W}, \overline{z}]$

$\sigma_{R|S} = 1$

$z_{R|W} = A_{R|W}$

$\sigma_{R|W} = 1$

2'. $(\kappa, p) \in \{(\kappa, p) | \kappa \leq \overline{\kappa}, p \in (\hat{p}, 1)\}$*:*

$\overline{z} = A_{R|S}$

$\mu = p$

$z_L = A_{R|S}$

$$z_{R|S} \sim \zeta_{R|S}(z_{R|S}) = \begin{cases} \frac{(1-p)(\sqrt{A_L+A_{R|S}-2A_{R|W}+B}-\sqrt{A_L+A_{R|S}-2z+B})}{p\sqrt{A_L+A_{R|S}-2z+B}} & z_{R|S} \in [A_{R|W}, A_{R|S}) \\ 1 & z_{R|S} \geq A_{R|S} \end{cases}$$

$\sigma_{R|S} = 1$

$z_{R|W} = A_{R|W}$

$\sigma_{R|W} = 1$

3. $(\kappa, p) \in \{(\kappa, p) | \kappa > \overline{\kappa}, p \in (0, 1)\}$*:*

$\mu = 1$ *(such that off-path challengers are believed to be strong types)*

$z_L = A_{R|S}$ *(when challenged, which would only occur off-path)*

$z_{R|S} = A_{R|S}$

$\sigma_{R|S} = 0$

$z_{R|W} = A_{R|W}$

$\sigma_{R|W} = 0$

If the incumbent is unchallenged, V *votes for* L*.*

If a challenger of type t has proposed $x_R = 0$ and $x_L = A_{R|t} - A_L$ such that V is indifferent between the candidates, V votes for L.

If $x_L = A_{R|W} - A_L$ and a strong challenger proposes $x_R = A_{R|S} - A_{R|W}$, V votes for R. In any other case in which V would be indifferent between the two candidates, V may break the tie in favor of L with any probability.

Proof of Proposition 2'. Given Lemma 5, several cases may obtain (numbered according to the labels applied in Figure 3): 1 & 1') L and R could both propose policies according to non-degenerate distributions and the weak type of challenger could leave the election uncontested with some strictly positive probability, 2) L and R could both propose policies according to non-degenerate distributions and the weak type of challenger could always enter, 2') L could propose a degenerate distribution in which $x_L = A_{R|S} - A_L$ with R possibly still proposing policies according to a non-degenerate distribution, or 3) neither type of R could enter the election.

To preclude entry by both types, costs must be sufficiently high, $\kappa \geq -\underline{a}$, such that even if L converges as though she is certain she faces the strong type, a challenger would not find it worthwhile to incur the cost of entry. The beliefs which support equilibrium case 3 are $\mu = 1$, such that L is certain she faces a strong type of challenger off the equilibrium path and would respond to entry with a policy convergent enough to dissuade entry.

Lemma 6 guarantees that $\zeta_c, c = L, R$ will be either degenerate or continuously differentiable with pdf ζ'_c. If the latter is true, we may write the probability that any candidate wins at a PVO $z_c \in (A_{R|W}, \overline{z})$ as $\zeta_{-c}(z_c)$. Recall that if $\overline{z} < A_{R|S}$, $\zeta_c(\overline{z}) = 1$ and $\Pr(z_c = \overline{z} | \overline{z} \neq A_{R|S}) = 0$.

Next, we consider the possibility that the two candidates' distributions of PVOs are both non-degenerate (cases 1 and 2). Arguments above establish it could either be the case that $\overline{z} < A_{R|S}$, where neither puts positive mass on $\overline{z}$, or that $\overline{z} = A_{R|S}$, with L offering $\overline{z}$ with potentially positive (but not full) probability but still randomizing (along with the challenger). We first suppose that $\overline{z} < A_{R|S}$. Indifference conditions are as follows, where $V(\cdot)$ denotes utility in terms of PVOs, z, such that $V_c(z_c(x_c)) = U_c(x_c)$.

$$
\begin{aligned}
\mathbb{E}V_L(z_L) &= M_L + [1 - \mu + \mu\zeta_{R|S}(z_L)] \cdot [A_L - z_L + B] + \mu \int_{z_L}^{\overline{z}} (z_{R|S} - A_{R|S})\zeta'_{R|S}(z_R) dz_{R|S} - C \\
&= M_L + A_L - \overline{z} + B - C, \;\; \forall z_L \in [A_{R|W}, \overline{z}] \\
\mathbb{E}V_{R|S}(z_{R|S}) &= -M_R + \zeta_L(z_{R|S}) \cdot [A_{R|S} - z_{R|S} + B] + \int_{z_{R|S}}^{\overline{z}} (z_L - A_L)\zeta'_L(z_L) dz_L - C \\
&= -M_R + A_{R|S} - \overline{z} + B - C, \;\; \forall z_{R|S} \in [A_{R|W}, \overline{z}] \\
\mathbb{E}V_{R|W}(A_{R|W}) &= -M_R + \zeta_L(A_{R|W})[A_{R|W} - A_L] + \int_{A_{R|W}}^{\overline{z}} (z_L - A_L)\zeta'_L(z_L) dz_L - C \\
&= -M_R + M_L
\end{aligned}
$$

We have three degrees of freedom, in a manner of speaking, namely: μ, $\overline{z}$, and $\zeta_L(A_{R|W})$.[28] These may be adjusted to satisfy the three equalities, where it must also be the case that ζ_{-c} must leave c's expected utility constant for all $z_c \in [A_{R|W}, \overline{z}]$, $\zeta_c(\overline{z}) = 1$, and $\zeta_{R|S}(A_{R|W}) = 0$.

Examining the second equality (evaluated at $z_{R|S} = A_{R|W}$) and the third equality, we see it must be the case that $\kappa - \zeta_L(A_{R|W})[A_{R|W} - A_L] = A_{R|S} - \overline{z} + B - \zeta_L(A_{R|W})[A_{R|S} - A_{R|W} + B] \Rightarrow \zeta_L(A_{R|W}) = \frac{(A_{R|S} - \overline{z} + B) - \kappa}{A_L + A_{R|S} - 2A_{R|W} + B}$.

The conditions on the distributions then yield the following:

$\zeta_L(z) = \frac{A_{R|S} - \overline{z} + B - \kappa}{\sqrt{A_L + A_{R|S} - 2A_{R|W} + B}\sqrt{A_L + A_{R|S} - 2z + B}}$, $\zeta_{R|S}(z) = \frac{(1-\mu)\left(\sqrt{A_L + A_{R|S} - 2A_{R|W} + B} - \sqrt{A_L + A_{R|S} - 2z + B}\right)}{\mu\sqrt{A_L + A_{R|S} - 2z + B}}$

The imposition that $\zeta_L(\overline{z}) = 1 = \zeta_{R|S}(\overline{z})$ and $\zeta_{R|S}(A_{R|W}) = 0$ yields relationships between μ and $\overline{z}$ as well as $\overline{z}$ and κ:

Let the function specifying the highest value of p for which the belief μ could be achieved by modulating entry by the weak type is given by: $P(\overline{z}) = 1 - \frac{\sqrt{A_L + A_{R|S} - 2\overline{z} + B}}{\sqrt{A_L + A_{R|S} - 2A_{R|W} + B}}$.

Let the function specifying the value of $\overline{z}$ for which randomization by the incumbent could leave the weak type of challenger indifferent between entering and staying out of the election is given by:

$Z(\kappa) = -A_L + 2A_{R|W} - \kappa + \sqrt{2(A_L + A_{R|S} - 2A_{R|W} + B)(A_L - A_{R|W} + \kappa)}$.[29]

These conditions suggest a few natural limits on case 1. First, if $\kappa < -\underline{a}$ and $p \in (0, \hat{p}]$, or $\kappa \in (-\overline{a}, \overline{\kappa}]$ and $p \in (P(Z(\kappa)), \hat{p}]$, then the cost is too low for the weak challenger to remain out of the election given the extent to which the incumbent is willing to converge in light of the (relatively high) probability of facing a strong challenger. Second, $p \in (0, \hat{p}]$ and $\kappa > B - \sqrt{(A_L - A_{R|S} + B)(A_L + A_{R|S} - 2A_{R|W} + B)} =: \overline{\kappa} =: Z^{-1}(A_{R|S}) \Rightarrow \overline{z} > A_{R|S}$, which cannot occur.[30] The second scenario is easily grouped within case 1; the first will comprise case 2.

Returning to the parameter values for which the indifference conditions can all simultaneously hold, namely $\kappa \in (-\overline{a}, \overline{\kappa}], p \in (0, P(\kappa)]$ (such that $Z(\kappa) \leq A_{R|S}$),[31] then we may write the probability with which the weak type enters as $\sigma_{R|W} = \frac{p(1-\mu)}{(1-p)\mu}$, where L's belief that she faces a strong type is given by $\mu = \frac{\sqrt{A_L + A_{R|S} - 2A_{R|W} + B} - \sqrt{\left(\sqrt{A_L + A_{R|S} - 2A_{R|W} + B} - \sqrt{2(A_L - A_{R|W} + \kappa)}\right)^2}}{\sqrt{A_L + A_{R|S} - 2A_{R|W} + B}}$.[32]

---

[28] $\zeta_L(A_{R|W}) \geq 0$

[29] Note that at $\kappa = -A_L + A_{R|W}$, $Z(\kappa) = A_{R|W}$, and $P(A_{R|W}) = 0$, such that only if the incumbent believes she faces a weak challenger with certainty would she be willing to adopt this "randomization," which involves playing as though only weak types exist.

[30] Note that $\overline{\kappa} \in (-\overline{a}, -\underline{a})$, so κ s.t. $\overline{z} > A_{R|S}$, does not imply that neither type of challenger would be willing to enter even against L converging to $x_L = -\underline{a}$, as though only facing strong types.

[31] i.e., $\kappa \leq \hat{\kappa} := B - \sqrt{(A_L + A_{R|S} - 2A_{R|W} + B)(A_L - A_{R|S} + B)}$

[32] $\kappa < \overline{\kappa} \Rightarrow \sqrt{A_L + A_{R|S} - 2A_{R|W} + B} > \sqrt{2(A_L - A_{R|W} + \kappa)} \Rightarrow$

If $\kappa > \overline{\kappa}$ while $p \in (0, \hat{p}]$, then the upper bound of the PVOs over which the candidates randomize is too low to simultaneously keep S indifferent among $z_R \in [A_{R|W}, A_{R|S}]$ while also keeping W indifferent among staying out of the election and entering and making a PVO of $z_{R|W} = A_{R|W}$. However, by placing positive mass on $A_{R|S}$, L can satisfy these conditions. Specifically, modifying the indifference conditions accordingly yields the distribution $\hat{\zeta}_L(z) = \frac{A_{R|S}-A_L-\kappa}{\sqrt{A_L+A_{R|S}-2z+B}\left(\sqrt{A_L+A_{R|S}-2A_{R|W}+B}-\sqrt{A_L-A_{R|S}+B}\right)}$, $\forall z \in [A_{R|W}, A_{R|S}]$, where L proposes a PVO of $z_L = A_{R|W}$ with probability $\hat{\zeta}_L(A_{R|W})$ and a PVO of $z_L = A_{R|S}$ with probability $1 - \hat{\zeta}_L(A_{R|S})$.[33] This covers the remainder of case 1.

Turning to case 2, consider $\kappa \leq -\overline{a}$ and $p \in (0, \hat{p}]$, or $\kappa \in (-\overline{a}, \overline{\kappa}]$ and $p \in (P(\kappa), \hat{p}]$. Due to the low likelihood of facing a strong type, the incumbent is not willing to increase the extent of her convergence enough that the weak type would be willing to randomize. In this case, the weak type enters will full probability (so no uncontested elections occur), and the incumbent and strong challenger still randomize (so upset victories still occur). The strategies arise from the same indifference conditions for L and the strong type of R that led to case 1, and indeed their proposal strategies are as in case 1. We have lost a degree of freedom in setting $\mu = p$, but the weak type of R does not need to be indifferent. As such, $\overline{z} = \tilde{P}^{-1}(p) = A_{R|W} + p\left(1 - \frac{p}{2}\right)(A_L + A_{R|S} - 2A_{R|W} + B)$ is the highest PVO in the candidates' distributions.

Finally, we consider the scenario in which $\kappa \leq -\underline{a}$ and $p \in (\hat{p}, 1)$, which constitutes case 2'. Here the probability of facing a strong type is so large that it is impossible to sustain any set of strategies in which the incumbent is randomizing in equilibrium. While the incumbent must then converge to offer a PVO of $z_L = A_{R|S}$ with full mass, the strong type of challenger's strategy must be such that she is willing to do so. Unless $p > \frac{A_{R|S}-A_{R|W}}{A_L-A_{R|W}+B} > 1 - \frac{\sqrt{A_L-A_{R|S}+B}}{\sqrt{A_L+A_{R|S}-2A_{R|W}+B}} = \hat{p}$, the incumbent is not willing to offer a PVO of $A_{R|S}$ if the strong challenger is also doing so with full probability. Finding a strategy for S that does not entail full convergence to $x_{R|S} = A_{R|S}$ but still leaves the incumbent willing to propose $x_L = A_{R|S} - A_L$ is both necessary and desirable.[34]

We propose a strategy similar to L's as κ grows large in case 1. S offers PVOs $z_{R|S} \in [A_{R|W}, A_{R|S})$ according to the same $\zeta_{R|S}$, where $\mu = p$, and offers $z_{R|S} = A_{R|S}$, i.e., $x_R = 0$, with probability $1 - \zeta(A_{R|S})$. This strategy is thus a continuation of case 2. This new distribution is given by $\hat{\zeta}_{R|S}(z) = \frac{\hat{p}(1-p)}{p(1-\hat{p})} \cdot \zeta_{R|S}(z) \oplus$

$\mu = \frac{\sqrt{A_L+A_{R|S}-2A_{R|W}+B}-\sqrt{\left(\sqrt{A_L+A_{R|S}-2A_{R|W}+B}-\sqrt{2(A_L-A_{R|W}+\kappa)}\right)^2}}{\sqrt{A_L+A_{R|S}-2A_{R|W}+B}} = \frac{\sqrt{2(A_L-A_{R|W}+\kappa)}}{\sqrt{A_L+A_{R|S}-2A_{R|W}+B}} \Rightarrow \sigma_{R|W} = \frac{p(\sqrt{A_L+A_{R|S}-2A_{R|W}+B}-\sqrt{2(A_L-A_{R|W}+\kappa)})}{(1-p)\sqrt{2(A_L-A_{R|W}+\kappa)}}$

[33] $\hat{\zeta}_L(A_{R|S}) \to 0$ (i.e., $\Pr(z_L = A_{R|S}) \to 1$) as $\kappa \to -\underline{a}$, the level of cost at which neither type of challenger would want to enter if L were to offer $z_L = A_{R|S}$ with probability 1.

[34] While the equilibrium strategies proposed for case 3 are not unique, the outcome is the same across all such cases: L wins with certainty.

$\frac{p-\hat{p}}{p(1-\hat{p})} \cdot A_{R|S}$.[35]

We must verify that this leaves L indifferent among any $z_L \in [A_{R|W}, A_{R|S})$, but with a strictly higher payoff from $z_L = A_{R|S}$. From the indifference condtions when $p = \hat{p}$, we have that $\mathbb{E}(z_{R|S} - A_{R|S}; \zeta_{R|S}) = \frac{A_L - A_{R|S} + B - (1-\hat{p})(A_L - A_{R|S} + B)}{\hat{p}}$. L's expected utility at $z_L = A_{R|W}$ is equal to her expected utility at all $z_L \in (A_{R|W}, A_{R|S})$ by the construction of S's strategy. So $\mathbb{E}V_L(A_{R|W}) = (1-p)(A_L - A_{R|W} + B) + p\frac{A_L - A_{R|S} + B - (1-\hat{p})(A_L - A_{R|S} + B)}{\hat{p}} + \frac{p-\hat{p}}{p(1-\hat{p})} \cdot (A_{R|S} - A_{R|S}), \forall z \in [A_{R|W}, A_{R|S})$ given p. It is easy to verify this yields lower utility for L than $z_L = A_L - A_{R|S} + B$ if $\hat{p} < p$, which defines the present equilibrium case. Having derived strategies for case 2', this completes the proof.[36] ■

Before we recall Proposition 2, we simplify Proposition 2' as per the assumptions made in-text, $A_L = A_{R|S} =: A$, $A_{R|W} = 0$, and $B = 0$, again noting that $\kappa := C + M_L$.

Corollary 1. *(to Proposition 2': With $A_L = A_{R|S} = A$, $A_{R_W=0}$, and $B = 0$) The following strategies, by case, constitute an equilibrium of the model:*[37]

In all cases, if unchallenged, the incumbent offers $z_L = M_L + A$ (i.e., proposes $x_L = M_L$).

1. $(\kappa, p) \in \{(\kappa, p) | \kappa \in (-A, 0], p \in (0, P(\kappa)]\}$:

$$\overline{z} = Z(\kappa), \kappa \in [-A, 0]$$

$$Z(\kappa) = \sqrt{A+\kappa}\left(2\sqrt{A} - \sqrt{A+\kappa}\right)$$

$$\mu = 1 - \frac{\sqrt{A-\overline{z}}}{\sqrt{A}} = \frac{\sqrt{A+\kappa)}}{\sqrt{A}}, \kappa \in [-A, 0]$$

$$z_L \sim \zeta_L(z_L) = \begin{cases} 0 & z_L < 0 \\ (1-\mu)\frac{\sqrt{A}}{\sqrt{A-z_L}} & z_L \in [0, \overline{z}] \end{cases}, \forall \kappa \in [-A, 0]$$

$$z_{R|S} \sim \frac{(1-\mu)(\sqrt{A} - \sqrt{A - z_{R|S}})}{\mu\sqrt{A - z_{R|S}}},\ z_{R|S} \in [0, \overline{z}]$$[38]

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[35] As $p \to 1$, $\Pr(x_R = 0|S) \to 1$.

[36] We may back out the distributions of policy proposals from the distributions of PVOs given $\xi_L(x_L) = \zeta_L(x_L + A_L)$, and $\xi_{R|S}(x_R) = 1 - \zeta_{R|S}(A_{R|S} - x_R)$, although the distributions of PVOs are in fact all that is necessary to ascertain the frequency of various outcomes and determine voter welfare in equilibrium.

When $p < \hat{p}$, if $\kappa \leq \overline{\kappa}$, then $\xi_L(x_L) = \frac{A_{R|S} - \overline{z} + B - \kappa}{\sqrt{A_L + A_{R|S} - 2A_{R|W} + B}\sqrt{A_{R|S} - 2x_L - A_L + B}}$, and if $\kappa > \overline{\kappa}$, then $\hat{\xi}_L(x) = \frac{A_L - A_{R|S} + \kappa}{\sqrt{A_{R|S} - 2x_L - A_L + B}(\sqrt{A_L + A_{R|S} - 2A_{R|W} + B} - \sqrt{A_L - A_{R|S} + B})}$.

For $\kappa < -\underline{a}$, when $p < \hat{p}$, $\xi_{R|S}(x_R) = 1 - \frac{(1-\mu)\left(\sqrt{A_L + A_{R|S} - 2A_{R|W} + B} - \sqrt{A_L - A_{R|S} + 2x_R + B}\right)}{\mu\sqrt{A_L - A_{R|S} + 2x_R + B}}$, with μ defined as above. When $p > \hat{p}$, $\hat{\xi}_{R|S}(x_R) = 1 - \frac{(1-p)}{p}\frac{\left(\sqrt{A_L + A_{R|S} - 2A_{R|W} + B} - \sqrt{A_L - A_{R|S} + 2x_R + B}\right)}{\sqrt{A_L - A_{R|S} + 2x_R + B}}$, such that additional mass is placed on $x_{R|S} = 0$.

[37] Corresponding to Figure 2 in text.

[38] The truncated distribution of z_L over $(0, \overline{z})$ is $\frac{\mu}{1-\mu}\zeta_L(z) = \zeta_{R|S}(z)$, which is to say that the truncated

$\sigma_{R|S} = 1$

$z_{R|W} = 0$

$\sigma_{R|W} = \frac{p(1-\mu)}{(1-p)\mu} = \frac{p(\sqrt{A}-\sqrt{A+\kappa})}{(1-p)\sqrt{A+\kappa}}, \kappa \in [-A, 0]$

2. $(\kappa, p) \in \{(\kappa, p)|\kappa \leq -A, p \in [0,1]\} \cup \{(\kappa, p)|\kappa \in (-A, 0], p \in (P(\kappa), 1]\}$*:*

$\overline{z} = P^{-1}(p) = p(1 - \frac{p}{2})(2A)$

$\mu = p$

$$z_L \sim \zeta_L(z_L) = \begin{cases} 0 & z_L < 0 \\ (1-\mu)\frac{\sqrt{A}}{\sqrt{A-z_L}} = (1-p)\frac{\sqrt{A}}{\sqrt{A-z_L}} & A_{R|W} \in [0, \overline{z}] \end{cases}$$

$z_{R|S} \sim \frac{(1-\mu)(\sqrt{A}-\sqrt{A-z_{R|S}})}{\mu\sqrt{A-z_{R|S}}} = \frac{(1-p)(\sqrt{A}-\sqrt{A-z_{R|S}})}{p\sqrt{A-z_{R|S}}}$, $z_{R|S} \in [0, \overline{z}]$

$\sigma_{R|S} = 1$

$z_{R|W} = 0$

$\sigma_{R|W} = 1$

3. $(\kappa, p) \in \{(\kappa, p)|\kappa > 0, p \in [0, 1]\}$*:*

$\mu = 1$ *(such that off-path challengers are believed to be strong types)*

$z_L = A$ *(when challenged, which would only occur off-path)*

$z_{R|S} = A$

$\sigma_{R|S} = 0$

$z_{R|W} = 0$

$\sigma_{R|W} = 0$

If the incumbent is unchallenged, V *votes for* L*.*

If a challenger of type t *has proposed* $x_R = 0$ *and* $x_L = A_{R|t} - A_L$ *such that* V *is indifferent between the candidates,* V *votes for* L*.*

If $x_L = -A$ *and a strong challenger proposes* $x_R = A$*,* V *votes for* R*. In any other case in which* V *would be indifferent between the two candidates,* V *may break the tie in favor of* L *with any probability.*

Proposition 2 pertains to equilibrium case 1 and to the probabilities of contested elections, upset victories, and upsets conditional on the election being contested.

Proposition 2. *Suppose the opportunity cost of contesting the election is neither too high nor too low relative to the probability that the potential challenger is a strong type (i.e., region 1 in Figure 2), such that both contested and uncontested elections may occur in equilibrium.*

distribution of z_L over $[0, \overline{z})$ is $\zeta_L(z) = \zeta_R(z)$.

An increase in the probability that the potential challenger is a strong type (p) increases the probability that the election is contested and the probability that the incumbent is defeated.

An increase in the opportunity cost of contesting the election ($C + M_L$) lowers the probability that the incumbent is challenged and the probability that the incumbent is defeated, but raises the probability that the incumbent is defeated conditional on having been challenged.

Proof of Proposition 2. From Corollary 1,[39] we have $\mu = \sqrt{(A + C + M_L)/A} \Rightarrow \frac{\partial \mu}{\partial C} = \frac{\partial \mu}{\partial M_L} = \frac{1}{2\sqrt{(A+C+M_L)A}} > 0$, i.e., μ is increasing in both components of the opportunity cost of contesting the election, C and M_L.

Pr(R contested election) $= p/\mu$ is increasing in p and decreasing in μ, so it is decreasing in C and M_L.

Pr(R upsets L) $= p(1 - \mu/2)$ is increasing in p and decreasing in μ, so it is decreasing in C and M_L.

Pr(L upset by $R|R$ contests the election) $= \mu(1 - \mu/2)$ does not depend on p but is increasing in μ, so it is increasing in C and M_L. ■

E Challenger Quality and Incumbent Upsets: Proposition 3

Proposition 3. *Suppose the opportunity cost of contesting the election is not too high such that entry occurs with positive probability.*

While an increase in the opportunity cost of contesting the election ($C + M_L$) or the likelihood that a potential challenger is of strong type (p) raises the probability that in a contested election the challenger defeats the incumbent increases, the probability that a given strong challenger wins decreases.

Proof of Proposition 3. In cases 1 and 2, Pr(upset|contested) $= \mu(1 - \mu/2)$ and Pr(upset$|t = S$) $= \mu/2 + (1 - \mu) = (1 - \mu/2)$. The former is increasing in μ, the latter is decreasing in μ, where μ is increasing in $C + M_L$ in case 1 and increasing in (in fact, equal to) p in case 2. ■

F Voter Welfare: Proposition 4

Proposition 4. *Suppose the opportunity cost of contesting the election is neither too high nor too low relative to the probability that the potential challenger is a strong type (i.e., region 1 in Figure 2), so contested and uncontested elections may occur in equilibrium.*

Expected voter welfare in contested elections is increasing in C and M_L, however, the probability of an uncontested election is increasing in C and M_L, which exerts downward pressure on voter welfare.

Expected voter welfare is strictly increasing in p as long as the opportunity cost of contesting the election is not too high (i.e., regions 1 and 2).

[39] The proofs of Propositions 3 and 4 also draw on Corollary 1.

Proof of Proposition 4. In cases 1 and 2, conditional on a contested election, the degree of moderation is increasing in μ, as is the probability that L will play as though she faces a strong type. As such,

$$\begin{aligned}\mathbb{E}(U_V|\text{contested election}) &= (1-\mu)^2\cdot 0 + 2\mu(1-\mu)\int_0^{\overline{z}} z\zeta'_{R|S}(z)dz + \mu^2\int_0^{\overline{z}} 2z\zeta'_{R|S}(z)\zeta_{R|S}dz \\ &= A\left((2-\mu)\mu + (1-\mu)^2\ln\left[(1-\mu)^2\right]\right)\end{aligned}$$

is increasing in μ.[40] In case 1, μ is increasing in C and M_L, while in case 2, $\mu = p$, so expected voter welfare in contested elections is increasing in C, M_L, and p.

In case 2, there are no uncontested elections, so overall expected voter welfare is as given above, and increasing in p. In case 1, overall expected voter welfare is given by

$$\mathbb{E}(U_V) = \left(1-\frac{p}{\mu}\right)\underbrace{(M_L+A)}_{-} + \frac{p}{\mu}\underbrace{\mathbb{E}(U_V|\text{contested election})}_{+},$$

which is increasing in p. An increase in C or M_L, however, puts more weight (through an associated increase in μ) on a strictly negative component of expected voter welfare and less weight on the strictly positive component. This trade-off always exists, even though $\mathbb{E}(U_V|\text{contested election})$ grows more positive in μ and even though $M_L + A$ is increasing in M_L, because of the assumption that $M_L < -A$; an increase in M_L or C still takes weight away from a positive term and places is it on a negative term. ■

G Office- and Policy-Motivated Candidates: Proposition 5

Proposition 5. *Suppose the opportunity cost of contesting the election is neither too high nor too low relative to the probability that the potential challenger is a strong type (i.e., region 1 in Figure 3), such that both contested and uncontested elections may occur in equilibrium.*

An increase in the benefit to holding office B raises the probability that the incumbent is challenged and the probability that the incumbent is defeated, but lowers the probability that the incumbent is defeated conditional on having been challenged. The probability a given strong challenger wins, however, is increasing in B.

Voter welfare in contested elections is increasing in B, as is overall voter welfare.

Proof of Proposition 5. We begin by stating case 1 of Proposition 2' with $B > 0$.

Corollary 2. *(to Proposition 2': Case 1, with $B > 0$, $A_L = A_{R|S} = A$, and $A_{R|W} = 0$) Note with $B > 0$, $\overline{\kappa} = B - \sqrt{(2A+B)B} \in (-A, 0)$.*

If unchallenged, the incumbent offers $z_L = M_L + A$ (i.e., proposes $x_L = M_L$).

[40] Mathematica code for verification to be made available with the appendix.

1. $(\kappa, p) \in \{(\kappa, p) | \kappa \in (-A, \overline{\kappa}], p \in (0, P(\kappa)]\}$:

$\overline{z} = Z(\kappa), \kappa \in [-A, \overline{\kappa}]$

$Z(\kappa) = \sqrt{A+\kappa}\left(\sqrt{2(2A+B)} - \sqrt{A+\kappa}\right)$

$\mu = 1 - \frac{\sqrt{2(A-\overline{z})+B}}{\sqrt{2A+B}} = \frac{\sqrt{2(A+\kappa)}}{\sqrt{2A+B}}, \kappa \in [-A, \overline{\kappa}]$

$$z_L \sim \zeta_L(z_L) = \begin{cases} 0 & z_L < 0 \\ (1-\mu)\frac{\sqrt{2A+B}}{\sqrt{2(A-z_L)+B}} & z_L \in [0, \overline{z}] \end{cases}, \forall \kappa \in [-A, \overline{\kappa}]$$

$z_{R|S} \sim \frac{(1-\mu)(\sqrt{2A+B} - \sqrt{2(A-z_{R|S})+B})}{\mu\sqrt{2(A-z_{R|S})+B}}$, $z_{R|S} \in [0, \overline{z}]$[41]

$\sigma_{R|S} = 1$

$z_{R|W} = 0$

$\sigma_{R|W} = \frac{p(1-\mu)}{(1-p)\mu} = \frac{p(\sqrt{2A+B} - \sqrt{2(A+\kappa)})}{(1-p)\sqrt{2(A+\kappa)}}, \kappa \in [-A, \overline{\kappa}]$

If the incumbent is unchallenged, V votes for L.

If a challenger of type t has proposed $x_R = 0$ and $x_L = A_{R|t} - A_L$ such that V is indifferent between the candidates, V votes for L.

If $x_L = -A$ and a strong challenger proposes $x_R = A$, V votes for R. In any other case in which V would be indifferent between the two candidates, V may break the tie in favor of L with any probability.

Per Corollary 2, μ is decreasing in B, so the results from Propositions 2-3 apply with the implications opposite those of C and M_L.

When deriving the analogous results of Proposition 4, we have

$$\mathbb{E}(U_V | \text{contested election}) = \frac{1}{2}(2A+B)\left((2-\mu)\mu + (1-\mu)^2 \ln\left[(1-\mu)^2\right]\right),$$

where the coefficient is increasing in B while the remainder is decreasing in B via μ. The expected voter welfare conditional on the election being contested, though, is net increasing in B. Further, as an increase in B puts more weight on the contested election outcome and less on the uncontested election outcome through μ, overall voter welfare is increasing in B. ■

[41] The truncated distribution of z_L over $(A_{R|W}, \overline{z})$ is $\frac{\mu}{1-\mu}\zeta_L(z) = \zeta_{R|S}(z)$, which is to say that the truncated distribution of z_L over $[A_{R|W}, \overline{z})$ is $\zeta_L(z) = \zeta_R(z)$.