Stone Duality for Separation Logic
Part 1: A Stone Duality Primer

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Outline

What Is Duality?
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Algebra, Logic and Topology
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Stone Duality
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Algebra, Logic and Topology

Stone Duality

Duality in Logic and Computer Science
What Is Duality?
A Really Informal Description of Duality
Duality relates two types of mathematical structure in a strong way.

1. Every structure of type A can be systematically transformed into a structure of type B (and vice versa).
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2. Every structure preserving map of type A can be systematically transformed into a structure preserving map of type B in the other direction.
An Slightly Less Informal Description of Duality

Duality relates two types of mathematical structure in a strong way.

1. Every structure of type A can be systematically transformed into a structure of type B (and vice versa).
2. Every structure preserving map of type A can be systematically transformed into a structure preserving map of type B in the other direction.
3. These transformations are (essentially) inverse to each other.
A Formal Definition of Duality

A **dual equivalence of categories** is

- a pair of functors $F : C \rightarrow \mathcal{D}^{\text{op}}$ and $G : \mathcal{D}^{\text{op}} \rightarrow C$
A Formal Definition of Duality

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- a pair of functors $F : C \rightarrow \mathcal{D}^{op}$ and $G : \mathcal{D}^{op} \rightarrow C$
- together with natural transformations $\epsilon : Id_{\mathcal{D}^{op}} \rightarrow GF$ and $\eta : Id_C \rightarrow GF$
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A \textbf{dual equivalence of categories} is

- a pair of functors $F : C \rightarrow \mathcal{D}^{\text{op}}$ and $G : \mathcal{D}^{\text{op}} \rightarrow C$
- together with natural transformations $\epsilon : \text{Id}_{\mathcal{D}^{\text{op}}} \rightarrow FG$ and $\eta : \text{Id}_C \rightarrow GF$
- such that every component $\epsilon_D : D \rightarrow FG(D)$, $\eta_C : C \rightarrow GF(C)$ is an isomorphism.
Algebra, Logic and Topology
Classical Propositional Logic

Syntax

Formulas generated by grammar

\[ p, T, F, \neg, \lor, \land, \rightarrow \]

Expressions \( \varphi \vdash \psi \) derived by rules

\[
\begin{align*}
\eta \vdash \phi & \quad \eta \vdash \psi \\
\hline
\eta \vdash \phi \land \psi
\end{align*}
\]

\[
\begin{align*}
\eta \land \phi \vdash \psi \\
\hline
\eta \vdash \phi \rightarrow \psi
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\[
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\eta \vdash \phi \rightarrow \psi & \quad \eta \vdash \phi \\
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\end{align*}
\]

Semantics

A valuation \( v : \text{Prop} \rightarrow \{0, 1\} \) assigns truth values to each propositional variable. An expression \( \hat{v}(\varphi) \) is defined recursively:

\[
\begin{align*}
\hat{v}(p) &= v(p) \\
\hat{v}(\top) &= 1 \\
\hat{v}(\bot) &= 0 \\
\hat{v}(\varphi \lor \psi) &= \max(\hat{v}(\varphi), \hat{v}(\psi)) \\
\hat{v}(\varphi \land \psi) &= \min(\hat{v}(\varphi), \hat{v}(\psi)) \\
\hat{v}(\neg \varphi) &= 1 - \hat{v}(\varphi) \\
\hat{v}(\varphi \rightarrow \psi) &= \max(1 - \hat{v}(\varphi), \hat{v}(\psi))
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\( \varphi \vdash \psi \) if every \( v \) satisfying \( \varphi \) satisfies \( \psi \).
Classical Propositional Logic

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Formulas generated by grammar

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Semantics

A valuation \( \nu : \text{Prop} \rightarrow \{0, 1\} \) assigns truth values to each propositional variable.
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Semantics
A valuation \( \nu : \text{Prop} \rightarrow \{0, 1\} \) assigns truth values to each propositional variable.

\[ \nu \text{ extends to } \hat{\nu} : \text{Form} \rightarrow \{0, 1\} : \]
\[ \hat{\nu}(p) = \nu(p) \quad \hat{\nu}(\top) = 1 \quad \hat{\nu}(\bot) = 0 \]
\[ \hat{\nu}(\varphi \lor \psi) = \max(\hat{\nu}(\varphi), \hat{\nu}(\psi)) \]
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\( \phi \vdash \psi \) if every \( \nu \) satisfying \( \phi \) satisfies \( \psi \).
Algebraizing Classical Propositional Logic

- Define $\varphi \equiv \psi$ iff $\varphi \vdash \psi$ and $\psi \vdash \varphi$ provable.
- $[[\varphi]] = \{\psi \mid \varphi \equiv \psi\}$
- $\mathbb{A}_{\text{Form}} = \{[[\varphi]] \mid \varphi \in \text{Form}\}$.
- $\mathbb{A}_{\text{Form}}$ has the structure of a **Boolean algebra** when $\lor, \land, \neg$ interpreted as join, meet and negation.
- If $\varphi \vdash \psi$ not provable, then $[[\varphi \rightarrow \psi]] < \top$ in $\mathbb{A}_{\text{Form}}$.
Ultrafilters

A **filter** on a Boolean algebra $\mathbb{A}$ is a subset $F \subseteq \mathbb{A}$ satisfying the following properties

1. $\bot \notin F$;
2. $x, y \in F$ implies $x \land y \in F$;
3. $x \in F$ and $x \leq y$ implies $y \in F$.
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An **ultrafilter** additionally satisfies

4. $x \lor y \in F$ implies $x \in F$ or $y \in F$. 

**Theorem**

On a Boolean algebra $A$, a filter $F$ is an ultrafilter iff $F$ is maximal wrt $\subseteq$ iff for all $a \in A$, $a \in F$ or $\neg a \in F$. 

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Another Perspective on Ultrafilters

- For any Boolean algebra $A$, ultrafilters are in bijective correspondence with homomorphisms $f : A \to \{0, 1\}$.
- Every homomorphism $f : A_{\text{Form}} \to \{0, 1\}$ uniquely corresponds to a valuation $v : \text{Prop} \to \{0, 1\}$ with $f = \hat{v}$.
- Ultrafilters on the formula algebra $A_{\text{Form}}$ are in bijective correspondence with valuations.
Stone’s Representation Theorem

Theorem (Stone 1936)

1. Every Boolean algebra is isomorphic to a Boolean algebra of sets.

Proof.

- Let $\mathbb{A}$ be a Boolean algebra and $Uf(\mathbb{A})$ its set of ultrafilters.
- Consider the power set algebra $(\mathcal{P}(Uf(\mathbb{A})), \cap, \cup, \setminus, Uf(\mathbb{A}), \emptyset)$.
- The map $h : \mathbb{A} \to \mathcal{P}(Uf(\mathbb{A}))$, defined

$$h(a) = \{F \in Uf(\mathbb{A}) \mid a \in F\},$$

is an embedding.

---

From Representation, Completeness

**Theorem (Completeness Theorem for Propositional Logic)**
\[ \varphi \models \psi \implies \varphi \vdash \psi. \]

**Proof.**

- The representation theorem gives us an embedding 
  \[ h : \mathbb{A}_{\text{Form}} \to \mathcal{P}(\text{Val}) \]
  where \( \mathbb{A}_{\text{Form}} \) is the set of all valuations and 
  \[ h(\varphi) = \{ v \mid \hat{v}(\varphi) = 1 \}. \]
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- Suppose \( \varphi \vdash \psi \) not provable. Then \( \varphi \to \psi \not< \top \) in \( \mathbb{A}_{\text{Form}} \). Hence \( h(\varphi \to \psi) \not\in \text{Val} \).
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Theorem (Completeness Theorem for Propositional Logic)

\( \varphi \vdash \psi \) implies \( \varphi \models \psi \).

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From Representation, Completeness

**Theorem (Completeness Theorem for Propositional Logic)**

ϕ ⊨ ψ implies ϕ ⊢ ψ.

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  \( h(\varphi \rightarrow \psi) \not\subseteq \text{Val} \).
- Thus there exists a valuation \( v \) such that \( \hat{v}(\varphi \rightarrow \psi) = 0 \).
- Hence \( \hat{v}(\varphi) = 1 \) and \( \hat{v}(\psi) = 0 \). So \( \varphi \vdash \psi \) does not hold.
What about semantics?

We’ve generalized the **syntax** of propositional logic to obtain a category of algebras; can we similarly generalize the **semantics**?
What about semantics?

We’ve generalized the **syntax** of propositional logic to obtain a category of algebras; can we similarly generalize the **semantics**?

"A cardinal principle of modern mathematical research may be stated as a maxim: 'One must always topologize’" - Marshall H. Stone.
Topologizing Classical Propositional Logic

A **topological space** is a pair \((X, \mathcal{O})\) s.t. \(X\) is a set and \(\mathcal{O} \subseteq \mathcal{P}(X)\) such that

- \(\emptyset, X \in \mathcal{O}\),
- \(\mathcal{O}\) closed under finite intersections,
- \(\mathcal{O}\) closed under arbitrary unions.
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For a \(A \subseteq X\), \(A\) is...

- **Open** if \(A \in O\)
- **Closed** if \(X \setminus A \in O\)
- **Clopen** if both open and closed.
Topologizing Classical Propositional Logic

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Define \(h_\varphi = \{v \in \text{Val} \mid \hat{v}(\varphi) = 1\}\). The set \(\mathcal{B} = \{h_\varphi \mid \varphi \in \mathcal{L}\}\) generates a topology on the set of valuations. We call this the **Valuation Space**.
Properties of the Valuation Space

A topological space is...

- **compact** iff for every set of closed sets \( \{ C_i \mid i \in I \} \), if every finite subset has non-empty intersection then \( \bigcap_i \{ C_i \mid i \in I \} \neq \emptyset \).

- **Hausdorff** iff every pair of distinct points is separated by disjoint open sets.

- **zero-dimensional** iff it is generated by clopen sets.

- a **Stone space** if it is **compact**, **Hausdorff** and **zero-dimensional**.

**Theorem**

*The valuation space is a Stone space.*
Compactness begets Compactness

Theorem (Compactness Theorem)

*Given a (possibly infinite) set of propositional formulas* $\Sigma$, $\Sigma$ is satisfiable iff every finite subset of $\Sigma$ is satisfiable.*
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- Let $\Sigma$ be finitely satisfiable. Define $C = \{h_\varphi \mid \varphi \in \Sigma\}$: each member is a clopen, thus closed, set.
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- Given $\varphi_1, \ldots, \varphi_n \in \Sigma$, by finite satisfiability there exists $v$ satisfying $\varphi_1, \ldots, \varphi_n$ so $v \in \bigcap_i^n h_{\varphi_i}$. 
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- Given $\varphi_1, \ldots, \varphi_n \in \Sigma$, by finite satisfiability there exists $v$ satisfying $\varphi_1, \ldots, \varphi_n$ so $v \in \bigcap_i h_{\varphi_i}$.
- Hence by topological compactness there exists $v' \in \bigcap C$, thus $v'$ satisfies $\Sigma$. 

$\square$
Connecting The Perspectives

- Boolean algebras are an abstraction of the **syntax** of classical propositional logic.
- Similarly, Stone spaces are an abstraction of the **semantics** of classical propositional logic.
- Syntax and semantics are connected by **soundness and completeness theorems**.
- **Stone duality** is the generalization of soundness and completeness to the level of Boolean algebras and Stone spaces.
Stone Duality
Recap

- Formulas quotiented by logical equivalence give Boolean algebra $\mathbb{A}_{\text{Form}}$.
- Representation Theorem: $h : \mathbb{A} \to \mathcal{P}(Uf(\mathbb{A}))$ gives an embedding where $h(a) = \{F \mid a \in F\}$.
- $h_\varphi = \{v \mid \hat{v}(\varphi) = 1\}$ generates Stone topology on set of valuations.
- $h(\varphi) = h_\varphi$ when $\mathbb{A} = \mathbb{A}_{\text{Form}}$. 
The Stone Space of a Boolean Algebra

**Theorem**

*Given a Boolean algebra $\mathbb{A}$, the set $Uf(\mathbb{A})$ topologised by $\{h(a) \mid a \in \mathbb{A}\}$ is a Stone space.*

We denote the Stone dual of a Boolean algebra by $S(\mathbb{A})$. 
The Boolean Algebra of a Stone Space

**Theorem**

Given a Stone space $X$, the set of clopen sets of $X$ carries the structure of a Boolean algebra.

We denote the Boolean dual of a Stone space $X$ by $\mathbb{A}(X)$.
Transforming Morphisms

Theorem

1. If \( f : \mathbb{A} \to \mathbb{A}' \) is a Boolean homomorphism, then

\[
 f^{-1} : S(\mathbb{A}') \to S(\mathbb{A})
\]

is a continuous map of Stone spaces.

2. If \( g : \mathcal{X} \to \mathcal{X}' \) is a continuous map of Stone spaces, then

\[
 g^{-1} : \mathcal{A}(\mathcal{X}') \to \mathcal{A}(\mathcal{X})
\]

is a Boolean homomorphism.

Note the change in direction of arrows!
Retreiving the Original Structures

Theorem

1. For all Boolean algebras $\mathbb{A}$, the map

$$a \mapsto \{ F \in Uf(\mathbb{A}) \mid a \in F \}$$

is an isomorphism between $\mathbb{A}$ and $\mathbb{A}S(\mathbb{A})$.

2. For all Stone spaces $\mathcal{X}$, the map

$$x \mapsto \{ C \mid C \text{ clopen and } x \in C \}$$

is an isomorphism between $\mathcal{X}$ and $\mathcal{S} \mathbb{A}(\mathcal{X})$. 
Theorem

The categories of Boolean algebras and Stone spaces are dually equivalent.
Duality in Logic and Computer Science
A Cornucopia of Logical Dualities

Stone duality has been generalized to a variety of logical settings.

Logic
Positive
Intuitionistic
Classical
Modal
Relevant
Many-valued
Quantum action
Nominal Classical
Hybrid
Modal $\mu$-calculus
Markovan
A Cornucopia of Logical Dualities

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<table>
<thead>
<tr>
<th>Logic</th>
<th>Algebra</th>
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<tr>
<td>Positive</td>
<td>Distributive lattice</td>
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<td>Heyting algebra</td>
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<td>Boolean algebra</td>
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<td>Modal</td>
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<td>Many-valued</td>
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<td>Modal $\mu$-calculus</td>
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<tbody>
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<td>Positive</td>
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<td>Many-valued</td>
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<td>Markovan</td>
<td>Aumann algebra</td>
<td>Markov process</td>
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Applications of Logical Dualities

Dualities can be used to prove serious results about logics.

**Theorem**

2. Every *Sahlqvist formula* of modal logic is *canonical* and corresponds to a *first-order frame condition*.

**Theorem**

3. *Interpolation* fails in relevant logic.

**Proof Sketch.**

In both cases: reduce problem to an algebraic or topological property and transfer along the dual equivalence.

---


Automata Theory
Coalgebra

- Coalgebra give a common mathematical framework for investigating state-based dynamics
- Ingredients: state space $X$ and transition structure $TX$.
- A coalgebra is a map $\alpha : X \rightarrow TX$.
- Examples: automata, transition systems, Kripke frames, petri nets, etc.
Stone-type duality gives a machine to generate logics of coalgebraic structures\(^4\)

\[^4\text{M. M. Bonsangue and A. Kurz. Duality for Logics of Transition Systems. FOSSACS 2005}\]
Program Semantics

- Plotkin/Smyth\(^5\): Duality between state transformation semantics and predicate-transformer semantics.
- Kozen\(^6\): Duality between two kinds of probabilistic program semantics.
- Abramsky\(^7\): Duality between domain theoretic semantics and operational semantics. Framework for outputting program logics.

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\(^6\)D. Kozen, A Probabilistic PDL, STOC ’83, ACM, 1983
The Moral of The Story

- Dualities are everywhere in computer science.
- Dualities provide a general framework for syntax, semantics, soundness and completeness in logic.
- Dualities provide new perspectives to prove things with.
- Dualities provide a mathematical foundation for established theory.
- Dualities facilitate the transfer of theory from one field to another.